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Dynamics of Baby Skyrmions

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Abstract

Baby Skyrmions are topological solitons in a (2+1)-dimensional field theory which resembles the Skyrme model in important respects. We apply some of the techniques and approximations commonly used in discussions of the Skyrme model to the dynamics of baby Skyrmions and directly test them against numerical simulations. Specifically we study the effect of spin on the shape of a single baby Skyrmion, the dependence of the forces between two baby Skyrmions on the baby Skyrmions' relative orientation and the forces between two baby Skyrmions when one of them is spinning.

1 Introduction

The goal of this paper is to study the dynamics of solitons in a (2+1)-dimensional version of the Skyrme model. By a soliton we mean a localised, finite-energy solution of a non-linear field theory. The Skyrme model is a non-linear field theory for pions in 3+1 dimensions with soliton solutions called Skyrmions [1]. Suitably quantised, Skyrmions are models for physical baryons. Skyrme's theory is non-integrable and therefore progress in understanding Skyrmion dynamics has depended on numerical simulations, approximation schemes or a combination of both. This approach has been quite successful in the study of static soliton solutions in Skyrme's theory [2] [3]. However, the interactive dynamics of two Skyrmions, which one needs to understand in order to extract the Skyrme model's predictions for the nuclear two-body problem, is more difficult to describe. Certain scattering processes of two Skyrmions have been simulated numerically [4], but the variety of possible initial conditions that one could consider is so great that it seems impossible to get an overall picture of the scattering behaviour from just a few processes. On the other hand, various approximations have been used which typically involve truncating the field theory with infinitely many degrees of freedom to a finite-dimensional dynamical system [5]. Some approximations have become widely accepted without, however, having been tested against numerical simulations of the full theory. Here we will apply many of the concepts and approximations developed in the Skyrme model to our model and directly compare them with numerical simulations.

Our solitons are exponentially localised in space, a property shared by Skyrmions when the physical pion mass is included in the Skyrme model. In our model a soliton has a fixed size but arbitrary position and orientation. In two spatial dimensions this corresponds to three degrees of freedom, two for the soliton's position and one angle to describe its orientation. This should again be compared with Skyrmions, which have a definite size and six degrees of freedom, three giving its position in space and three parametrising its orientation. The long-range interaction behaviour of our solitons resembles that of Skyrmions in important respects: in both cases the asymptotic forces between two solitons depend on their separation and their relative orientation and are of the dipole-dipole type. Furthermore, there is a bound state of two solitons whose energy distribution is maximal on a ring in both cases. When the solitons are orientated so that the forces are most attractive and then released from rest they scatter through the toroidal configuration and emerge at 90° degrees relative to the initial direction of their motion.

Because of all these similarities we call our solitons baby Skyrmions. This term has been used quite widely to describe solitons in 2+1 dimensions which resemble Skyrmions in certain respects. However, in all the models studied so far the moment of inertia for the rotation of a single soliton is infinite, so that the rotational degrees of freedom are not dynamically relevant. Yet the rotational degrees of freedom are crucial in Skyrmion dynamics. In this paper we are therefore particularly interested in those aspects of our model which depend on the solitons' orientation.

After a description of our model and a quick review of its static solutions, discussed in detail in [6] and [7], we focus on the following questions. Are there exact solutions of the field equations representing spinning baby Skyrmions? How does a baby Skyrmion change its shape when it spins? How do the asymptotic forces between two baby Skyrmions depend on their relative orientation? What are the forces between two baby Skyrmions when one of them is spinning?

While the questions we investigate here are largely motivated by the (3+1)-dimensional Skyrme model we should emphasise that our model is also of interest in the large and growing area of (2+1)-dimensional soliton phenomenology. The solitons in our model have a number of properties which are novel in this context.

2 The Model

The basic field of our model is a map

$$\boldsymbol{\phi} : \mathrm{M}^3 \mapsto S^2_{\mathrm{iso}} \tag{2.1}$$

where M^3 is three-dimensional Minkowski space with the metric diag$(1, -1, -1)$. We set the speed of light to 1 and write elements of M^3 as $(t, \boldsymbol{x})$, where $\boldsymbol{x}$ is a 2-vector with coordinates x^i, i=1,2, also sometimes denoted by x and y. We use the notation x^α, $\alpha = 0, 1, 2$, to refer to both the time and spatial components of $(t, \boldsymbol{x})$, so $t = x^0$. The target space $S^2{}_{\mathrm{iso}}$ is the 2-sphere of unit radius embedded in euclidean 3-space with the Riemannian metric induced by that embedding. Here the suffix 'iso' is used to emphasise the analogy with the target space in the Skyrme model, which is often referred to as iso-space. The field $\boldsymbol{\phi}$ is a scalar field with three components ϕ_a, $a = 1, 2, 3$, satisfying the constraint $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$ for all $x \in \mathrm{M}^3$. Our Lagrangian density, the static part of which was first considered in [6] and discussed further in [7], is

$$\mathcal{L} = F\left(\frac{1}{2}\partial_\alpha\boldsymbol{\phi}\cdot\partial^\alpha\boldsymbol{\phi} - \frac{\kappa^2}{4}(\partial_\alpha\boldsymbol{\phi} \times \partial_\beta\boldsymbol{\phi})\cdot(\partial^\alpha\boldsymbol{\phi} \times \partial^\beta\boldsymbol{\phi}) - \mu^2(1 - \boldsymbol{n}\cdot\boldsymbol{\phi})\right), \tag{2.2}$$

where $\boldsymbol{n} = (0, 0, 1)$ and $\partial_\alpha = \partial/\partial x^\alpha$. The constants F, κ and μ are free parameters: F has the dimension energy and κ and $1/\mu$ have the dimension length. It is useful to think of F and κ as natural units of energy and length respectively and of $1/\mu$ as a second length scale in our model. Here we fix our units of energy and length by setting $F = \kappa = 1$. We have already set the speed of light to 1, so we use geometric units in which all physical quantities are dimensionless. Thus we cannot set μ to 1 by a choice of units, and we will fix its value later after we have discussed its significance. Note also that Planck's constant $\hbar$ will be some number, but not necessarily equal to 1. The first term in (2.2) is familiar from σ-models whose soliton solutions have been studied extensively [8]. The second term, fourth order in derivatives, is the analogue of the Skyrme term in the usual Skyrme model. In [7] we explained in more precise terms in which sense this analogy holds by appealing to a general geometric framework for the Skyrme model due to Manton [9]. Finally, the last term does not contain any derivatives and is often referred to simply as a potential. In three (spatial) dimensions the Skyrme term is necessary for the existence of soliton solutions but the inclusion of a potential is optional from the mathematical point of view. Physically, however, a potential of a certain form is required to give the pions a mass [10]. By contrast, in two dimensions a potential must be included in the above Lagrangian in order to obtain soliton solutions.

To see this, and to understand the Lagrangian $L = \int \mathcal{L} d^2x$ better, we write it in the usual form $L = T - V$. Here T is the kinetic energy

$$T = \int \frac{1}{2}\dot{\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}} + \frac{1}{2}(\dot{\boldsymbol{\phi}} \times \partial_i \boldsymbol{\phi})\cdot(\dot{\boldsymbol{\phi}} \times \partial_i \boldsymbol{\phi}) d^2x, \tag{2.3}$$

where the dot denotes differentiation with respect to time, and V is the potential energy

$$V = \int \frac{1}{2}\partial_i \boldsymbol{\phi}\cdot\partial_i \boldsymbol{\phi} + \frac{1}{4}(\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi})\cdot(\partial_j \boldsymbol{\phi} \times \partial_i \boldsymbol{\phi}) + \mu^2(1 - \boldsymbol{n}\cdot\boldsymbol{\phi}) d^2x. \tag{2.4}$$

We are only interested in fields with finite potential energy and we therefore impose the boundary condition

$$\lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{\phi}(t, \boldsymbol{x}) = \boldsymbol{n} \tag{2.5}$$

for all t. As a result we can formally compactify physical space to a 2-sphere S^2_{space} and regard the fields $\boldsymbol{\phi}$ at a fixed time t as maps from S^2_{space} to S^2_{iso} with an associated integer degree. The analytical formula for the degree is

$$\deg[\phi] = \frac{1}{4\pi} \int \boldsymbol{\phi}\cdot\partial_1 \boldsymbol{\phi} \times \partial_2 \boldsymbol{\phi} d^2x. \tag{2.6}$$

The degree is a homotopy invariant of the field $\boldsymbol{\phi}$ and therefore conserved during time evolution. Moreover, it gives a useful lower bound on the potential energy, the Bogomol'nyi bound

$$V[\boldsymbol{\phi}] \geq 4\pi |\deg[\boldsymbol{\phi}]|. \tag{2.7}$$

It is well known that the usual σ-model Lagrangian, where both the Skyrme term and the potential are omitted, is scale invariant. In particular, this means that a soliton solution of the σ-model can have arbitrary size. The inclusion of the Skyrme term breaks the scale invariance and it follows from a simple scaling argument that the potential energy contributed by the Skyrme term can be lowered by increasing the scale of a configuration. Thus, in order to obtain stable solutions it is necessary to include a potential which, on its own, would favour small scales. It follows that soliton solutions in a Lagrangian with both Skyrme term and potential have a definite size.

The precise form of the potential does not matter in such scaling arguments and potentials other than the one considered here have been studied in the literature [11]. Our potential is the analogue of the potential usually chosen in the Skyrme model. It contains a constant μ which, in the language of quantum field theory, can be interpreted as the inverse Compton wavelength of the mesons in our model. To see this it is best to turn to the equations of motion.

The Euler-Lagrange equations for the Lagrangian L are

$$\partial^\alpha \left(\boldsymbol{\phi} \times \partial_\alpha \boldsymbol{\phi} - \partial_\beta \boldsymbol{\phi} (\partial^\beta \boldsymbol{\phi} \cdot \boldsymbol{\phi} \times \partial_\alpha \boldsymbol{\phi}) \right) = \mu^2 \boldsymbol{\phi} \times \boldsymbol{n}. \tag{2.8}$$

One simple solution is given by $\boldsymbol{\phi}(t, \boldsymbol{x}) = \boldsymbol{n}$. It has degree zero and is called the vacuum configuration. For a physical interpretation of our Lagrangian it is useful to study the equation obeyed by small fluctuation around the vacuum configuration. Decomposing $\boldsymbol{\phi}$ into a component parallel to the vacuum and a component $\boldsymbol{\varphi}$ orthogonal to it

$$\boldsymbol{\phi} = \sqrt{1 - \boldsymbol{\varphi}^2} \boldsymbol{n} + \boldsymbol{\varphi} \approx \boldsymbol{n} + \boldsymbol{\varphi} + \mathcal{O}(\boldsymbol{\varphi}^2), \qquad \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0, \tag{2.9}$$

one checks that the linearised equation for $\boldsymbol{\varphi}$ is the massive Klein-Gordon equation

$$(\Box + \mu^2) \boldsymbol{\varphi} = 0, \tag{2.10}$$

where $\Box = \partial_\mu \partial^\mu$ is the wave operator in 2+1 dimensions. In the language of perturbative quantum field theory, where one quantises small fluctuations around the vacuum, the scalar fields φ_1 and φ_2 therefore correspond to scalar particles or mesons of mass $\hbar\mu$. We now understand μ well enough to fix its value. Our choice is again dictated by the desire to reproduce important features of the Skyrme model. There, as in real nuclear physics, the Compton wavelength of the pion is of the same order as (in fact, slightly larger than) the size of a Skyrmion. Thus, we want to tune μ in our model so that the energy distribution of its basic soliton solution is concentrated in a region of diameter $\approx 1/\mu$. By trial and error we find that this is the case when we set $\mu^2 = 0.1$, which we do for the rest of this paper.

For later use we note a conservation law that can be read off immediately from the equation of motion (2.8). Taking the scalar product with $\boldsymbol{n}$ on both sides of the equation we find that the current

$$\boldsymbol{n} \cdot \boldsymbol{\phi} \times \partial_\alpha \boldsymbol{\phi} - (\boldsymbol{n} \cdot \partial_\beta \boldsymbol{\phi})(\partial^\beta \boldsymbol{\phi} \cdot \boldsymbol{\phi} \times \partial_\alpha \boldsymbol{\phi}) \tag{2.11}$$

has vanishing divergence. The symmetry that leads to this conservation law is $SO(2)$ rotations of the field $\boldsymbol{\phi}$ about $\boldsymbol{n}$, which can be written in terms of an angle χ as

$$(\phi_1, \phi_2, \phi_3) \mapsto (\cos\chi\ \phi_1 + \sin\chi \phi_2, -\sin\chi \phi_1 + \cos\chi \phi_2, \phi_3). \tag{2.12}$$

We call such a transformation an iso-rotation; the corresponding conserved quantity is called isospin and denoted by I:

$$I = \int \boldsymbol{n} \cdot \dot{\boldsymbol{\phi}} \times \boldsymbol{\phi} + (\boldsymbol{n} \cdot \partial_i \boldsymbol{\phi})(\partial_i \boldsymbol{\phi} \cdot \dot{\boldsymbol{\phi}} \times \boldsymbol{\phi}) d^2x. \tag{2.13}$$

For a systematic discussion of the symmetries of our model we refer the reader to [7], but here we need only note that both the Lagrangian L and the degree (2.6) are invariant under simultaneous reflections in space and iso-space

$$P_x : (x, y) \mapsto (-x, y) \quad \text{and} \quad (\phi_1, \phi_2, \phi_3) \mapsto (-\phi_1, \phi_2, \phi_3). \tag{2.14}$$

Later we will also make use of the invariance of both the degree and the Lagrangian under the combination of P_x with a rotation by π in both space and iso-space:

$$P_y : (x, y) \mapsto (x, -y) \quad \text{and} \quad (\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, \phi_3). \tag{2.15}$$

3 Static Solutions Revisited

Time-independent solutions of the equations of motion (2.8), which are stationary points of the potential energy functional V (2.4), were studied in detail in [7]. We briefly recall those results which are relevant here. An important class of static solutions of the equations of motion consists of fields which are invariant under the group of simultaneous spatial rotations by some angle $\alpha \in [0, 2\pi)$ and iso-rotations by $-n\alpha$, where n is a non-zero integer. Such fields are of the form

$$\boldsymbol{\phi}(\boldsymbol{x}) = \begin{pmatrix} \sin f(r) \cos(n\theta - \chi) \\ \sin f(r) \sin(n\theta - \chi) \\ \cos f(r) \end{pmatrix}, \tag{3.1}$$

where (r, θ) are polar coordinates in the $\boldsymbol{x}$-plane and f is a function satisfying certain boundary conditions to be specified below. The angle χ is also arbitrary, but fields with different χ are related by an iso-rotation and therefore degenerate in energy. Thus we concentrate on the standard fields where $\chi = 0$. Such fields are the analogue of the hedgehog fields in the Skyrme model and were also studied in [6] for different values of μ^2.

The function f, which is called the profile function, has to satisfy

$$f(0) = m\pi, \quad m \in \mathbf{Z}, \tag{3.2}$$

for the field (3.1) to be regular at the origin; to satisfy the boundary condition (2.5) we set

$$\lim_{r\to\infty} f(r) = 0. \tag{3.3}$$

Here we will only be interested in profile functions where $m = 1$ (the situation for general m is discussed in [7]). One then finds that the degree of the field (3.1) is

$$\deg[\boldsymbol{\phi}] = n. \tag{3.4}$$

For a field of the form (3.1) to be a stationary point of the energy functional V, f has to satisfy the Euler-Lagrange equation

$$\left(r + \frac{n^2 \sin^2 f}{r}\right) f'' + \left(1 - \frac{n^2 \sin^2 f}{r^2} + \frac{n^2 f' \sin f \cos f}{r}\right) f' - \frac{n^2 \sin f \cos f}{r} - r\mu^2 \sin f = 0 \tag{3.5}$$

It was shown in [7] that the hedgehog fields (3.1) with $n = 1$ and $n = 2$ and profile functions satisfying the equation above for those values of n are the absolute minima of V amongst all field of degree 1 and 2 respectively. We write $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$ for those fields in standard iso-orientation, i.e. with $\chi = 0$ in (3.1), and denote their profile functions by $f^{(1)}$ and $f^{(2)}$. Translations in physical space and iso-rotations act non-trivially on $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$, so there is a three-dimensional family of minima of the energy functional for both $n = 1$ and $n = 2$. We call any field obtained by translating and iso-rotating the field $\boldsymbol{\phi}^{(1)}$ a baby Skyrmion. Baby Skyrmions are the basic solitons of our model; as promised in the introduction they have three degrees of freedom: two translational and one rotational. For the total energy, or mass, of a baby Skyrmion we find $1.564 \cdot 4\pi$; the energy density is rotationally symmetric and peaked at the baby Skyrmion's centre (where $\boldsymbol{\phi}^{(1)} = (0, 0, -1)$). The field $\boldsymbol{\phi}^{(2)}$ (and all those obtained by translating and rotating it) may be thought of as a bound state of two baby Skyrmions and is described in detail in [7]. The energy density is again rotationally symmetric but peaked at a distance $r \approx 1.8$ from the centre. The mass is $2.936 \cdot 4\pi$.

In subsequent sections we will be interested in the field of a spinning baby Skyrmion and the forces between two baby Skyrmions. We recall some simple observation concerning the asymptotic behaviour of $\boldsymbol{\phi}^{(1)}$ from [7] which are the basis of a remarkably accurate model for the dynamical phenomena we will then encounter. For large r, and hence small f, the equation (3.5) for $n = 1$ simplifies to the modified Bessel equation

$$f'' + \frac{1}{r} f' - \left(\frac{1}{r^2} + \mu^2\right) f = 0. \tag{3.6}$$

A solution of this equation which tends to zero at $r = \infty$ is the modified Bessel function $K_1(\mu r)$. Thus, the profile function $f^{(1)}$ (which satisfies (3.5) for $n = 1$) is proportional to K_1 for large r and we can write

$$f^{(1)}(r) \sim \frac{p\mu}{2\pi} K_1(\mu r), \tag{3.7}$$

where p is a constant which we will interpret further below. Since the modified Bessel function has the asymptotic behaviour

$$K_1(\mu r) \sim \sqrt{\frac{\pi}{2\mu r}} e^{-\mu r} \left(1 + \mathcal{O}\left(\frac{1}{\mu r}\right)\right) \tag{3.8}$$

the leading term in an asymptotic expansion of $f^{(1)}$ is proportional to $e^{-\mu r}/\sqrt{r}$, which shows in particular that the (potential) energy distribution of the field $\boldsymbol{\phi}^{(1)}$ is exponentially localised. Most of the analytical results in this paper are based on the observation, made in

[7], that the asymptotic field $\boldsymbol{\varphi}^{(1)}$ of $\boldsymbol{\phi}^{(1)}$ (defined as in (2.9)) can be interpreted in terms of dipole fields. This can be seen as follows. For large r we approximate $\sin f^{(1)} \sim f^{(1)}$ and $\cos f^{(1)} \sim 1$ and, using the asymptotic expression (3.7), we write the field $\boldsymbol{\varphi}^{(1)}$ as

$$\boldsymbol{\varphi}^{(1)}(\boldsymbol{x}) = \frac{p\mu}{2\pi} K_1(\mu r) \begin{pmatrix} \cos(\theta - \chi) \\ \sin(\theta - \chi) \\ 0 \end{pmatrix}. \tag{3.9}$$

Or, introducing the orthogonal vectors

$$\boldsymbol{p}_1 = p(\cos\chi, \sin\chi) \qquad \boldsymbol{p}_2 = p(-\sin\chi, \cos\chi) \tag{3.10}$$

and $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$, we can write

$$\varphi_a^{(1)}(\boldsymbol{x}) = \frac{\mu}{2\pi} \boldsymbol{p}_a \cdot \hat{\boldsymbol{x}} K_1(\mu r) = -\frac{1}{2\pi} \boldsymbol{p}_a \cdot \nabla K_0(\mu r) \qquad a = 1, 2. \tag{3.11}$$

However, since the Green function of the static Klein-Gordon equation is $K_0(\mu r)$,

$$(\Delta - \mu^2) K_0(\mu r) = -2\pi \delta^{(2)}(\boldsymbol{x}), \tag{3.12}$$

we have

$$(\Delta - \mu^2) \varphi_a^{(1)}(\boldsymbol{x}) = \boldsymbol{p}_a \cdot \nabla \delta^{(2)}(\boldsymbol{x}) \qquad a = 1, 2. \tag{3.13}$$

This equation leads to the interpretation of the asymptotic field $\boldsymbol{\varphi}^{(1)}$ as the field produced by a doublet of orthogonal dipoles, one for each of the components $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$, in a linear field theory, namely Klein-Gordon theory. The strength of the dipole can be calculated from the asymptotic form of $f^{(1)}$. One finds, by numerically solving the equation (3.5),

$$p = 24.16. \tag{3.14}$$

Once this single number is calculated from the non-linear equation (3.5) much can be deduced about the dynamics of baby Skyrmions using only the linearised equations of motion.

4 Spinning Baby Skyrmions

How does a soliton in two or three dimensions change its shape when it spins? What is the interactive dynamics of several solitons when some of them are spinning? From the point of view of particle physics these are very natural questions to ask. Yet there are surprisingly few soliton models in which these questions has been addressed seriously and even fewer in which satisfactory answers have been found. This is partly because the questions do not make sense in some of the most popular models. In the much studied abelian Higgs model [12], for example, the soliton solutions, called vortices, are fully characterised by their position and have no rotational degrees of freedom. Lumps in the $\mathbf{CP}^1$ model, on the other hand, can have an arbitrary orientation, but the moment of inertia associated with changes in the orientation is infinite so that the rotational degree of freedom is dynamically frozen out. There is a modification of the $\mathbf{CP}^1$ model [13] in which the solitons, called Q-lumps, necessarily spin, but single soliton solutions have infinite energy and in configurations of several solitons all solitons have to rotate with the same angular frequency. Thus, the effect of relative rotation cannot be investigated.

In the Skyrme model, spin 1/2 quantum states of a single Skyrmion are models for physical nucleons, so the question of spinning Skyrmions has naturally attracted a lot of attention. In the first paper on this subject [14], it was assumed that a Skyrmion would rotate without changing its shape, and, to obtain the quantum states corresponding to the proton, neutron and the Δ-resonance, it was quantised as a rigid body. Although it was quickly pointed out that the classical frequency at which a Skyrmion would have to rotate to have spin 1/2 is so large that centrifugal and relativistic effects are important, there seems to be no quantitative analysis of these effects in the literature. One reason for this is that, in three dimensions, a rotating Skyrmion only has axial symmetry (about the axis of rotation) and is not of the $SO(3)$ symmetric hedgehog form of the static solution. Thus, to find the exact form of a spinning Skyrmion one needs to solve coupled non-linear partial differential equations similar to the ones studied in [15].

In our two-dimensional model, by contrast, there are solutions describing spinning baby Skyrmions which are of the hedgehog form (3.1). Thus we can study the effect of rotation on the soliton's shape, its mass and its moment of inertia by simply solving ordinary differential equations. It follows from the 'principle of symmetric criticality' [16] that we can find time-dependent solutions of the field equations (2.8) by making the time-dependent hedgehog ansatz

$$\boldsymbol{\phi}^\omega(t,\boldsymbol{x}) = \begin{pmatrix} \sin f(r)\cos(\theta-\omega t) \\ \sin f(r)\sin(\theta-\omega t) \\ \cos f(r) \end{pmatrix}, \tag{4.1}$$

where ω, an arbitrary real number, can be interpreted as the field's angular frequency and f satisfies the boundary conditions (3.3) and (3.2) with $m=1$. Then the field (4.1) is a solution of the Euler-Lagrange equation (2.8) if f satisfies the Euler-Lagrange equation obtained from the restriction of the Lagrangian L to fields of the form (4.1). Explicitly this is the ordinary differential equation

$$\begin{aligned}(r+(\frac{1}{r}-\omega^2 r)\sin^2 f)f'' \;&+\; (1-(\omega^2+\frac{1}{r^2})\ \sin^2 f+(\frac{1}{r}-\omega^2 r)f'\sin f\cos f)f' \\ &-\; (\frac{1}{r}-\omega^2 r)\sin f\cos f - r\mu^2\sin f = 0. \end{aligned} \tag{4.2}$$

We call solutions of the form (4.1) spinning baby Skyrmions. The total energy of a spinning baby Skyrmion depends on ω and is given by

$$M(\omega) = \pi \int r \left(f'^2 + (\omega^2 + \frac{1}{r^2})(1 + f'^2)\sin^2 f + 2\mu^2(1 - \cos f) \right) dr. \tag{4.3}$$

$M(0)$ is just the mass of a baby Skyrmion calculated earlier and shall henceforth be denoted M_0. To study the dependence of M and the field $\boldsymbol{\phi}^\omega$ on ω we need to distinguish two regimes, $\omega < \mu$ and $\omega > \mu$. This can be seen from the behaviour of f for large r. In this limit f is small and the equation (4.2) simplifies to

$$f'' + \frac{1}{r}f' - (\frac{1}{r^2} + (\mu^2 - \omega^2))f = 0. \tag{4.4}$$

Thus, for $\omega < \mu$, f decays exponentially for large r while for $\omega > \mu$ it is oscillatory.

It is interesting to compare the asymptotic field of a spinning baby Skyrmion with the field produced by a doublet of scalar dipoles. We then need to consider time-dependent dipoles and note here that the equation for scalar fields φ_a produced by a rotating pair of orthogonal dipoles $\boldsymbol{p}_a(t)$, $a = 1, 2$, is the Klein-Gordon equation with a time-dependent dipole source term:

$$(\Box + \mu^2)\varphi_a(t, \boldsymbol{x}) = -\boldsymbol{p}_a(t) \cdot \nabla\delta^{(2)}(\boldsymbol{x}). \tag{4.5}$$

If the dipoles rotate uniformly at constant angular velocity ω, then $\ddot{\boldsymbol{p}}_a = -\omega^2 \boldsymbol{p}_a$ and the above equation can be solved explicitly in terms of Bessel functions. We will do this below for the two regimes $\omega < \mu$ and $\omega > \mu$ and compare the results with the asymptotic form of $\boldsymbol{\phi}^\omega$.

4.1 The case $\omega < \mu$

If $\omega < \mu$, the equation (4.4) is again the modified Bessel equation of first order. As before we are interested in a solution f which is exponentially small for large r. Such a solution is asymptotically proportional to the modified Bessel function $K_1(\kappa r)$, where $\kappa = \sqrt{\mu^2 - \omega^2}$, i.e.

$$f \sim \frac{\kappa p}{2\pi} K_1(\kappa r) \tag{4.6}$$

for some constant p. Then the asymptotic field $\boldsymbol{\varphi}^\omega$ of the rotating baby Skyrmion (4.1) with that profile function is

$$\boldsymbol{\varphi}^\omega(t, \boldsymbol{x}) = \frac{p\kappa}{2\pi} K_1(\kappa r) \begin{pmatrix} \cos(\theta - \omega t) \\ \sin(\theta - \omega t) \\ 0 \end{pmatrix}. \tag{4.7}$$

Thus, defining the time-dependent dipole moments

$$\boldsymbol{p}_1 = p(\cos\omega t, \sin\omega t) \quad \text{and} \quad \boldsymbol{p}_2 = p(-\sin\omega t, \cos\omega t), \tag{4.8}$$

the first two components of $\boldsymbol{\varphi}^\omega$ can be written

$$\varphi_a^\omega(t, \boldsymbol{x}) = -\frac{1}{2\pi}\boldsymbol{p}_a(t)\cdot\nabla K_0(\kappa r) \qquad a = 1, 2. \tag{4.9}$$

One checks easily that these fields satisfy the linear equation (4.5). Thus, just as in the static case, the asymptotic form of the rotating hedgehog solution (4.1) may be thought of as being produced by a rotating pair of orthogonal dipoles.

We have solved the radial equation (4.2) for various values of $\omega < \mu$. In figure 1 we plot the corresponding profile functions. As ω approaches μ from below, the soliton's energy distribution becomes more and more spread out, which one may interpret as a centrifugal effect. Note, however, that the initial gradient of the profile functions varies very little and that most of the change occurs in the tail, which is well described by the modified Bessel function. This is the first indication that one can understand many of the features of a spinning baby Skyrmion in terms of the field in the asymptotic region where it (approximately) obeys linear equations.

Next we want to understand the dependence of a spinning baby Skyrmion's mass on its frequency. Recalling the asymptotic formula (3.8) we see that the energy distribution of a baby Skyrmion spinning at $\omega < \mu$ is exponentially localised and that the total mass $M(\omega)$ is finite. At the critical angular velocity $\omega = \mu$, however, the equation (4.4) is solved by $f = 1/r$; the corresponding baby Skyrmion is thus only power-law localised and its mass is infinite. For later use we note that the divergent part of the integral in the formula (4.3) is

$$\pi \int r\left(\omega^2 \sin^2 f + 2\mu^2(1 - \cos f)\right) dr. \tag{4.10}$$

Due to the spreading of the energy density as ω approaches μ one needs to integrate to ever larger values of r when computing $M(\omega)$ numerically using the formula (4.3). It is then more practical to find constants r_0 and C such that for $r > r_0$ the profile function is well approximated by $C\exp(-\kappa r)/\sqrt{r}$. Then one integrates the energy density numerically from 0 to r_0 and performs the remaining integral analytically, using $\sin^2 f \approx f^2$and $\cos f \approx 1 - f^2/2$. We plot the function $M(\omega)$ in figure 2.a). It grows rapidly as $\omega \to \mu$ which is consistent with our earlier observation that $M(\mu)$ is infinite.

A further quantity of interest is the conserved charge I discussed at the end of section 2. For fields of the hedgehog form, where spatial rotations and iso-rotations are equivalent, we can interpret I as the angular momentum or spin of the field $\boldsymbol{\phi}$ and we denote it by J in this context. One finds that

$$J(\omega) = \omega\cdot 2\pi \int r \sin^2 f(1 + f'^2) dr. \tag{4.11}$$

The quantity

$$\Lambda(\omega) = J(\omega)/\omega \tag{4.12}$$

may be interpreted as a moment of inertia. Since a spinning baby Skyrmion changes its shape as ω varies, the corresponding moment of inertia changes, too. As mentioned above this effect is customarily ignored in the Skyrme model. To check the validity of this approximation in our model we define $\Lambda_0 = \Lambda(0)$, for later use. Numerically we find $\Lambda_0 = 2\pi\cdot 7.558$

We have calculated $J(\omega)$ for various values of $\omega < \mu$ using the same technique as described above for the computation of $M(\omega)$. Like M, J diverges as $\omega \to \mu$ from below, but one checks that only the term

$$\omega\cdot 2\pi \int r \sin^2 f\, dr \tag{4.13}$$

is a divergent integral when $\omega = \mu$. The divergence is of the same order as that in (4.10) and, comparing coefficients, one is lead to the asymptotic formula

$$M \sim N + \mu J \tag{4.14}$$

for some constant N. In figure 2.b) we plot the precise relation between $M(\omega)$ and $J(\omega)$. Clearly the graph is well described by the linear formula (4.14) already for quite small values of ω. Note that a linear formula of this form holds *exactly* for Q-lumps [13], where the constant N is the Q-lumps topological charge. One may interpret it in words as

"mass of a spinning soliton = constant + meson mass × angular momentum".

For small ω the mass $M(\omega)$ depends quadratically on $J(\omega)$ as one would expect for the rotation of a rigid body. It is instructive to compare our exact results with the non-relativistic rigid body formula

$$\tilde{M} = M_0 + \frac{J^2}{2\Lambda_0}. \tag{4.15}$$

We plot the graph of $\tilde{M}(J)$ is figure 2.b) as well. Note that the rigid body formula is only a good approximation to the true mass-spin relation for small spins and small mass differences $M(\omega) - M$. This observation might be relevant for baryon phenomenology in the Skyrme model. There a non-relativistic formula like (4.15) is used to calculate the theoretical predictions for the masses of the nucleons and the Δ-resonance. However, the nucleon mass is about 10% larger than the mass of a Skyrmion and the Δ is about 40% heavier than a Skyrmion. Our calculations indicate that the formula (4.15) is a poor approximation for a relative mass difference as large as 40% and that it will generally give too large a value for the mass of a spinning soliton at a given angular momentum.

4.2 The case $\omega > \mu$

When $\omega > \mu$ the equation (4.4) is the (unmodified) Bessel equation of first order, all solutions of which are oscillatory for large r. Thus, in terms of $k = \sqrt{\omega^2 - \mu^2}$ we can write the asymptotic form of the solution in terms of the Bessel functions of first and second kind

$$J_1(kr) = -\frac{1}{k}\frac{dJ_0}{dr}(kr) \sim \sqrt{\frac{2}{\pi kr}}\sin(kr - \frac{1}{4}\pi) \tag{4.16}$$

and

$$Y_1(kr) = -\frac{1}{k}\frac{dY_0}{dr}(kr) \sim -\sqrt{\frac{2}{\pi kr}}\cos(kr - \frac{1}{4}\pi). \tag{4.17}$$

Both these functions may occur, so we write the asymptotic form $\boldsymbol{\varphi}^\omega$ as

$$\boldsymbol{\varphi}^\omega(t, \boldsymbol{x}) = -\frac{k}{4}(pY_1(kr) + qJ_1(kr))\begin{pmatrix} \cos(\theta - \omega t) \\ \sin(\theta - \omega t) \\ 0 \end{pmatrix}, \tag{4.18}$$

where q and p are constants whose meaning we will explain later. A hedgehog field with the asymptotic form (4.18) has infinite energy (4.3) and is thus rather unphysical. Nevertheless this solution has a natural interpretation in terms of the dipole model which we will give below.

First, however, we want to study the time evolution of a baby Skyrmion which is given an *initial* angular velocity $\omega > \mu$. To investigate this question we have solved the field equations (2.8) numerically with initial values $\boldsymbol{\phi}(t=0) = \boldsymbol{\phi}^{(1)}$ and $\dot{\boldsymbol{\phi}}(t=0) = -\omega \boldsymbol{n} \times \boldsymbol{\phi}^{(1)}$. for $\omega = 0.5$ and $\omega = 0.9$. The grid for our simulations is a square of 250×250 points, extending in both the x and y direction from -25 to 25. At the boundary we set the field to the vacuum value $\boldsymbol{n}$ and absorb any incident kinetic energy. For both of the initial values of ω we find that the baby Skyrmion radiates. In figure 3 we display the field of the baby Skyrmion whose initial angular velocity was $\omega = 0.5$, 10 units of time after the start of the simulation. The picture clearly shows the sort of spiral pattern which is familiar from dipole radiation in linear relativistic field theories [17].

The radiation carries away both energy and angular momentum. As a result the baby Skyrmion slows down until the angular velocity has dropped to a value below μ in both simulations. Figure 4 shows how the total energy for the two simulations decreases with time. We have also checked that after 1000 units of time the field has settled down to a uniformly rotating field of the form (4.1), and we have extracted the angular frequency. In the simulation where initially $\omega = 0.5$ we now find $\omega \approx 0.28$ and in the simulation where initially $\omega = 0.9$ we now find $\omega \approx 0.3$.

It is instructive to interpret both the solution of the hedgehog form and the numerically found solution with the spiral pattern shown in figure 3 in terms of the dipole model. For this purpose it is best to combine the asymptotic fields φ_1 and φ_2 into the complex field $\Phi = \varphi_1 + i\varphi_2$. Then, the Klein-Gordon equation (4.5), with dipole moments given by (4.8) (where now $\omega > \mu$), can be written

$$(\Box + \mu^2)\Phi(t, \boldsymbol{x}) = -pe^{-i\omega t}(\partial_1 + i\partial_2)\delta^{(2)}(\boldsymbol{x}). \tag{4.19}$$

To solve this equation we separate the time dependence in the form

$$\Phi(t, \boldsymbol{x}) = e^{-i\omega t} g(\boldsymbol{x}), \tag{4.20}$$

so that g has to satisfy the static equation

$$(\Delta + k^2)g(\boldsymbol{x}) = p(\partial_1 + i\partial_2)\delta^{(2)}(\boldsymbol{x}), \tag{4.21}$$

with k as defined above. Next we need suitable Green functions G of the Helmholtz equation in two dimensions, normalised so that

$$(\Delta + k^2)G(kr) = \delta^{(2)}(\boldsymbol{x}). \tag{4.22}$$

A solution which describes an outgoing wave at infinity can be expressed in terms of the first Hankel function

$$G^+(kr) = \frac{1}{4i} H_0^{(1)}(kr) \sim -i\sqrt{\frac{1}{8\pi kr}} e^{i(kr - \frac{\pi}{4})}, \tag{4.23}$$

and a solution which describes an incoming wave at infinity is given in terms of the second Hankel function

$$G^-(kr) = \frac{i}{4} H_0^{(2)}(kr) \sim i\sqrt{\frac{1}{8\pi kr}} e^{-i(kr - \frac{\pi}{4})}. \tag{4.24}$$

A particular real solution of (4.22) is thus given by

$$\frac{1}{2}(G^+(kr) + G^-(kr)) = \frac{1}{4} Y_0(kr), \tag{4.25}$$

but to obtain the general solution we should add an arbitrary multiple of the real solution of the homogeneous Helmholtz equation

$$\frac{1}{2}(G^+(kr) - G^-(kr)) = \frac{1}{4} J_0(kr). \tag{4.26}$$

Thus, converting to polar coordinates

$$\partial_1 + i\partial_2 = e^{i\theta}(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}) \tag{4.27}$$

and choosing the real Green function $\frac{1}{4}Y_0(kr) + \frac{q}{4p}J_0(kr)$ for some real number q we obtain a solution of (4.19)

$$\Phi^{\mathrm{r}}(t, \boldsymbol{x}) = -\frac{k}{4}(pJ_1(kr) + qY_1(kr))e^{i(\theta - \omega t)}, \tag{4.28}$$

which is just the asymptotic form $\boldsymbol{\varphi}^\omega$ (4.18) of the hedgehog field written in complex notation. Thus the hedgehog spinning at $\omega > \mu$ represents a solution with a radiation field that consists of both incoming and outgoing radiation. This is the physical origin of the infinite energy of the hedgehog solution.

It is not difficult to guess which Green function will lead to a solution of (4.19) displaying the spiral pattern observed in our simulation of the spinning baby Skyrmion. Consider the solution constructed from the Green function G^+. Using

$$\frac{dH_0^{(1)}}{dr}(kr) = -kH_1^{(1)}(kr) \tag{4.29}$$

that solution is

$$\Phi^+(t, \boldsymbol{x}) = i\frac{kp}{4}e^{i(\theta-\omega t)}H^{(1)}(kr) \sim p\sqrt{\frac{k}{8\pi r}}e^{i(kr+\theta-\omega t-\frac{\pi}{4})}. \tag{4.30}$$

Remembering that the real and imaginary part of Φ^+ should be interpreted as the first two components of the asymptotic field $\boldsymbol{\varphi}^+$ of a spinning baby Skyrmion we see that, for sufficiently large r and a fixed value of t, the direction of $\boldsymbol{\varphi}^+$ is constant along the spirals $kr = -\theta$. This is precisely the spiral pattern we observed in the numerical simulation of a baby Skyrmion spinning with $\omega > \mu$. The dipole picture shows that it can be accounted for in terms of the Green function G^+.

The dipole picture can be used to make sense of many of the qualitative properties of a spinning baby Skyrmion. In principle one could also check whether the energy loss through radiation plotted in figure 4 can be quantitatively modelled in terms of dipole radiation. However, the centrifugal effects in spinning baby Skyrmions change the dipole strength, which therefore depends on the baby Skyrmion's angular frequency. This considerably complicates the calculations and we therefore have not pursued this path.

5 The Dipole Model for the Interaction of Baby Skyrmions

In section 3 we saw that a baby Skyrmion acts like the source of a doublet of scalar dipole fields. In [7] it was shown, assuming a certain superposition procedure for well-separated baby Skyrmions, that a baby Skyrmion also reacts to the field of a distant baby Skyrmion like a doublet of scalar dipoles. There is a similar correspondence between a superposition procedure and a linear model for the forces between solitons in the Skyrme model. In [18] it was shown that the product ansatz in the Skyrme model (without the pion mass term) leads to the same forces between well-separated moving and spinning Skyrmions as a simple dipole model for Skyrmions, provided relativistic corrections such as retardation effects are included in both approximations. In this section we will ignore such relativistic effects and describe the dipole model for slowly moving baby Skyrmions.

Thus consider two well-separated baby Skyrmions, the first centred at $\boldsymbol{R}_1$ and rotated relative to the standard hedgehog $\boldsymbol{\phi}^{(1)}$ by an angle χ_1 and the second centred at $\boldsymbol{R}_2$ and rotated by an angle χ_2. From [7] we know that, at large separation, the leading term in the potential describing the interaction of two baby Skyrmions is the interaction energy of two doublets of scalar dipoles in the plane, one situated at $\boldsymbol{R}_1$ and the other at $\boldsymbol{R}_2$ such that $|\boldsymbol{R}_1 - \boldsymbol{R}_2|$ is large compared to $1/\mu$. The dipole moments of the first dipole are $\boldsymbol{p}_a$, $a = 1, 2$, where

$$\boldsymbol{p}_1 = p(\cos\chi_1, \sin\chi_1) \qquad \boldsymbol{p}_2 = p(-\sin\chi_1, \cos\chi_1) \tag{5.1}$$

and the dipole moments of the second are $\boldsymbol{q}_a$, $a = 1, 2$, where

$$\boldsymbol{q}_1 = p(\cos\chi_2, \sin\chi_2) \qquad \boldsymbol{q}_2 = p(-\sin\chi_2, \cos\chi_2). \tag{5.2}$$

Then, if the dipoles $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$ and the dipoles $\boldsymbol{p}_2$ and $\boldsymbol{q}_2$ interact via a scalar field obeying the Klein-Gordon equation with mass μ, the interaction energy between the doublets $\boldsymbol{p}_a$ and $\boldsymbol{q}_a$ is

$$W = \sum_{a=1,2} \frac{1}{2\pi}(\boldsymbol{p}_a \cdot \nabla)(\boldsymbol{q}_a \cdot \nabla) K_0(\mu R), \tag{5.3}$$

where $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$ and $R = |\boldsymbol{R}|$. Introducing also the relative angle $\psi = \chi_1 - \chi_2$ one finds

$$W(\psi, R) = \frac{p^2}{\pi} \cos\psi \Delta K_0(\mu R) = \frac{p^2\mu^2}{\pi} K_0(\mu R)\cos\psi. \tag{5.4}$$

In the last step we have used (3.12) and have omitted the $\delta^{(2)}$-function term because we are only interested in large separations $R > 1/\mu$.

Thus we obtain the first prediction of the dipole model. The force between two baby Skyrmions depends on their relative orientation. In particular, two baby Skyrmions in the same orientation repel each other; if one is rotated relative to the other by 90° there are no static forces; if one is rotated relative to the other by 180° the forces are attractive. The forces always act along the line joining the two baby Skyrmions, but in addition there is a torque which tends to rotate the relative angle to 180°. In analogy with the terminology used in discussing Skyrmion dynamics we call this configuration the most attractive channel.

To obtain a more quantitative picture we must take into account the mass and the moment of inertia of the baby Skyrmions. We assume that the rotations of the individual baby Skyrmions are sufficiently slow so that we can approximate the functions $M(\omega)$ and $\Lambda(\omega)$ by the constants M_0 and Λ_0. Hence our model for the asymptotic dynamics of two baby Skyrmions has the Lagrangian

$$L_{\text{dipole}} = \frac{1}{2}M_0\dot{\boldsymbol{R}}_1^{\,2} + \frac{1}{2}M_0\dot{\boldsymbol{R}}_2^{\,2} + \frac{1}{2}\Lambda_0\dot{\chi}_1^2 + \frac{1}{2}\Lambda_0\dot{\chi}_2^2 - W(\psi, R). \tag{5.5}$$

In fact the centre of mass position $\boldsymbol{S} = (\boldsymbol{R}_1+\boldsymbol{R}_2)/2$ and the angle $\chi = (\chi_1+\chi_2)/2$ are cyclical coordinates and decouple from the remaining coordinates. Thus we work in the centre of mass frame and set $\chi = 0$ and $\boldsymbol{S} = 0$. Moreover we can introduce polar coordinates (R, ϕ) for the relative position vector $\boldsymbol{R}$. Then ϕ is also a cyclical coordinate and it is consistent to set $\dot{\phi} = \phi = 0$. Then we obtain the dynamical system with equations of motion

$$\begin{aligned} \frac{1}{2}\Lambda_0\ddot{\psi} &= \frac{p^2\mu^2}{\pi}K_0(\mu R)\sin\psi \\ \frac{1}{2}M_0\ddot{R} &= \frac{p^2\mu^3}{\pi}K_1(\mu R)\cos\psi. \end{aligned} \tag{5.6}$$

These equations can easily be solved numerically, and in the next section we will compare the soliton trajectories predicted by them with those calculated from the full field equations (2.8). Some readers may then find it useful to think of the equations (5.6) in terms of the coupled motion of a pendulum and a point particle. More precisely the angle ψ may be thought of as characterising the angular position of a physical pendulum. Then the first equation in (5.6) specifies the torque acting on the pendulum: it vanishes at the stable equilibrium point $\psi = \pi$ and the unstable equilibrium point $\psi = 0$ and is maximal when $\psi = \pi/2$. Moreover the strength of the torque depends on the 'external' parameter R and decreases with increasing R. The second equation in (5.6) can be interpreted as the equation for the linear motion of a point particle with position R. The force acting on the particle depends on its position, its strength decreasing with increasing R, but is also controlled by the 'external' parameter ψ. When $\psi = \pi$ (the pendulum's stable equilibrium) the force is attractive, tending to decrease R; when $\psi = 0$ (the pendulum's unstable equilibrium) the force is repulsive, tending to increase R.

6 Numerical Simulations

All the simulations of the field equations to be discussed in this section take place on a square grid of 250×250 points, extending in both the x and y direction from -25 to 25. The initial configurations are constructed from two baby Skyrmion fields using the superposition procedure referred to above; thus, these configurations can be characterised by giving the individual baby Skyrmions' positions and orientations. The baby Skyrmions' centre of mass position and the overall iso-orientation are immaterial for the dynamics, and we take the former to be at the grid's origin and usually choose the latter so that the individual baby Skyrmions' orientations are equal and opposite. For each simulation we will specify the initial relative position and the initial relative orientation, denoted ψ_0. We will also consider initial conditions where both baby Skyrmions have some non-zero initial velocity. We then work in the centre of mass frame, so that the baby Skyrmions' velocities are equal and opposite, and we include the effect of Lorentz contraction in our initial configuration. Unless specified otherwise we will consider initial velocities along the x-axis, so generically one baby Skyrmion is initially in the half plane $x > 0$ with velocity $(-v, 0)$ and the other in the half plane $x < 0$ with velocity $(v, 0)$, where $0 \leq v < 1$.

6.1 Scattering from Rest

Suppose the two baby Skyrmions are initially at rest and well-separated, one centred at $(10, 0)$ and the other at $(-10, 0)$. We have calculated the time evolution of the corresponding field configuration for a variety of initial orientations $\psi_0 \in [0, \pi]$ and in figure 5.a) we plot $R/2$, the separation of either baby Skyrmion from the centre, as a function of time. We only show the time evolution until the baby Skyrmions collide - the actual collision will be discussed in the next section. The qualitative features of the time evolution can easily be understood in terms of the dipole model.

Consider for example the motion when $\psi_0 = \pi$. Then the baby Skyrmions are already in the most attractive channel and remain there; hence there is an attractive force between their centres throughout their interaction. The plot of the actually observed time evolution of $R/2$ shows exactly such an accelerating motion. When $\psi_0 = \pi/2$ the initial torque is maximal, but the initial force vanishes. Thus the relative orientation ψ swings through the attractive channel $\psi = \pi$, at which point the force between the baby Skyrmions is maximally attractive, but then overshoots and approaches $\psi = 3\pi/2$. While ψ is close to this value the force between the baby Skyrmions is again very small or zero, so that we expect the separation parameter R to be a linear function of time here. This is precisely what we see in figure 5.a). When ψ_0 is decreased further the force between the baby Skyrmions is initially repulsive. The baby Skyrmions move apart but at the same time ψ increases so that some time later the baby Skyrmions are in the most attractive channel. The force is now attractive and the baby Skyrmions approach each other again. As the relative orientation oscillates the baby Skyrmions experience alternating attractive and repulsive forces and thus perform the oscillatory motion most clearly seen in the trajectory for $\psi_0 = 0.3{\cdot}\pi$. Finally the baby Skyrmions get trapped in the attractive channel and collide.

When ψ_0 is decreased further the initial repulsive force increases and as a result the baby Skyrmions' separation may initially increase so rapidly that the attractive force which the baby Skyrmions experience once they are in the attractive channel is too weak to invert their relative velocity. The baby Skyrmions then escape to infinity, which in our simulations means that they hit the boundary of the grid. We have investigated boundary effects by sending a single baby Skyrmion towards the boundary with velocity $v = 0.1$ and find weak repulsive forces when the baby Skyrmion is approximately 8 units away from the boundary. The relevant boundary for the present simulation is at $x = \pm 25$, so we interpret simulations where $R/2$ becomes larger than 15 as 'escape to infinity'. This happens for $\psi_0 \leq \pi/4$. In our simulations the smallest value of ψ_0 for which the baby Skyrmions ultimately collide is $\psi_0 = 0.275{\cdot}\pi$, for which the trajectory is also shown in figure 5.a).

We have also solved the equations (5.6) numerically for a range of initial values ψ_0, always setting $R(0) = 20$, $\dot{R}(0) = \dot{\psi}(0) = 0$. Comparing these solutions with the corresponding trajectories found in our simulations of the full field equations we find qualitative agreement in all cases, but quantitative agreement only for the first part of the trajectories (typically $0 \leq t < 100$). Moreover, the dependence of the solutions of (5.6) on ψ_0 is exactly as found in the field theory. There is a critical value ψ_c such that for $\psi_0 \in (\psi_c, 0]$, $R(t)$ tends to infinity as $t \to \infty$, and for $\psi_0 \in (\psi_c, \pi]$, $R(t)$ approaches zero (where the equations are singular) after a number of oscillations which becomes arbitrarily large as $\psi_0 \downarrow \psi_c$. Its numerical value is $\psi_c \approx 0.288545$, which should be compared with the value 0.275 found in the field theory. In figure 5.b) we show R and ψ as a function of time for $\psi_0 = 0.28855{\cdot}\pi$. The qualitative features of the interplay between the angular motion (ψ) and the linear motion (R) discussed earlier are clearly illustrated.

6.2 Head-on Collisions

The dipole model only describes the asymptotic part of the trajectories discussed so far. We have seen, however, that for $\psi_0 > 0.275{\cdot}\pi$ the two baby Skyrmions ultimately adjust their relative orientation so that they are in the attractive channel, and collide head on. In the next set of simulations we study the outcome of such a head-on collision in the attractive channel for a variety of different initial speeds. We fix $\psi_0 = \pi$ and place the baby Skyrmions at $(7.5, 0)$ and $(-7.5, 0)$, giving them initial velocities $(-v, 0)$ and $(v, 0)$ respectively, where $0.1 \leq v \leq 0.6$.

In all the simulations the baby Skyrmions merge into the ring-like structure of the 2-soliton solution and emerge at right angles to their initial direction of motion. After the scattering they move away from each other with their relative orientation still in the attractive channel. This is the $90°$ scattering that is now a familiar and apparently generic feature of topological soliton dynamics. However, in our model this scattering process is accompanied by the emission of a large amount of radiation. In figure 6 we show the energy distribution immediately after the collision with $v = 0.6$: the rings of radiation are clearly visible. The radiation carries away so much energy that the baby Skyrmions only 'escape to infinity' (defined as above) for $v \geq 0.46$. For smaller initial velocities the attractive forces between the baby Skyrmions after the collision pull them back and they perform another head-on collision, again scattering through $90°$ and emerging along their initial direction of motion and in the attractive channel. The second collision is again accompanied by the emission of radiation, so the solitons travel less far than after their first collision before they turn round. This process is repeated, but now the motion remains close to the ring-like

2-soliton solution at all times. The individual baby Skyrmions are no longer distinct, and the motion looks like an oscillatory excitation of the 2-soliton. The emission of radiation, however, continues until the kinetic energy has virtually disappeared. The final configuration is numerically indistinguishable from the 2-soliton solution.

The scattering process described above is much more radiative than any observed in previous simulations of lump scattering in the $\mathbf{CP^1}$ model [8] or in other two-dimensional versions Skyrme models with potentials different from ours [11]. At first sight this is surprising: the radiation in our model is massive whereas there are massless radiation modes in all the comparable models mentioned above. However, our model is also the only one in which there are strong attractive forces. Moreover these forces are short-ranged, so that the potential energy functional V (2.4) has a large gradient at configurations consisting of two nearby baby Skyrmions. This means that the time evolution of a configuration in the vicinity of those points in the configuration space is not adiabatic. It follows in particular that the adiabatic or moduli space approximation proposed for the Skyrme model in [5] is not suitable for describing soliton collisions in our model.

In the next set of simulations we keep the initial velocity v fixed at 0.5 but vary the initial relative orientation ψ_0 between 0 and π. Initially, the baby Skyrmions are again placed at $(7.5, 0)$ and $(-7.5, 0)$. To understand the ensuing interaction processes it is useful to note the symmetries of the initial conditions. Recalling that in our conventions the baby Skyrmions have equal and opposite initial orientations we first observe that the initial configurations are invariant under the reflection P_x (2.14). Since P_x is a symmetry of the Lagrangian, the configurations will remain invariant under that operation during their time evolution. Hence, *assuming* that the baby Skyrmions separate after the collision, we can deduce that they must separate along either the x-axis or the y-axis. In other words, if scattering takes place, it must be scattering through either $0°$ (i.e. trivial), $90°$ or $180°$. We also note that, if the baby Skyrmions separate along the y-axis, the requirement of invariance under P_x allows only two possibilities for their individual orientations: either the standard orientation where, using the conventions of figure 3, the fields point radially outwards, or the standard orientation rotated by π, where the fields point radially inwards. Thus, after the scattering the baby Skyrmions either have the same orientation and are in the most repulsive channel or their orientations differ by π, in which case they are in the most attractive channel.

When $\psi_0 = 0$ or $\psi_0 = \pi$ the initial configurations are additionally invariant under, respectively, the reflection P_y (2.15) or the combination of P_y with an iso-rotation by π. It follows that the configurations after the interaction must have the same invariances. In particular for $\psi_0 = \pi$ we can predict purely on the basis of symmetry that, if the baby Skyrmions scatter through $90°$, they must be equidistant from the origin after the scattering and they must be in the most attractive channel. Of course, this is precisely what we observed in our previous simulation. For the other 'special' initial configuration, where $\psi_0 = 0$, we find, however, that the baby Skyrmions scatter through 180°. More precisely they head towards each other and slow down until they come to a halt at a separation $R \approx 3$. Then they turn round and escape to infinity along the line of initial approach. The relative orientation ψ does not change at all during this process. How, within the constraints imposed by the symmetries described above, does the scattering interpolate between 90° scattering and 180° scattering as we vary ψ_0 from π to 0?

In our simulations we find that there is an interval $I^{90} = [\pi/6, \pi]$ such that the baby Skyrmions scatter through 90° for $\psi_0 \in I^{90}$ and a smaller interval $I^{180} = [0, \pi/20]$ such that they scatter through 180° for $\psi_0 \in I^{180}$. However, as $\psi_0 \downarrow \pi/6$ the baby Skyrmions emerge from the 90°-scattering with different speeds: the baby Skyrmion moving in the positive y-direction moves faster than the one moving in the negative y-direction. The momentum balance is restored by radiation which is emitted predominantly along the negative y-axis. This process is presented schematically in figure 7.b).

When ψ_0 is in the remaining interval $I^{\text{capt}} = (\pi/20, \pi/6)$ the two baby Skyrmions do not separate after the collision but form the oscillatorily excited state of the 2-soliton already encountered in the previous set of simulations. However, this time the excited 2-soliton as a whole moves in the positive y-direction, and radiation is emitted in the opposite direction to restore the momentum balance. This process is sketched in figure 7.c).

The notion of the most attractive channel is useful for summarising our results. For $\psi_0 \in I^{90} \cup I^{\text{capt}}$ the baby Skyrmions merge and scatter through 90°; the closer the two baby Skyrmions are initially to being in the most attractive channel the less radiation is emitted in the scattering process. For $\psi_0 \in I^{90}$ the baby Skyrmions therefore escape to infinity after the collision; for $\psi_0 \in I^{\text{capt}}$ they form a coincident configuration which we interpret as an asymmetrically deformed 2-soliton solution. The increased radiation and its asymmetry is due to that deformation. Finally, for $\psi_0 \in I^{180}$ there is not enough time for the baby Skyrmions' relative orientation to adjust before the collision, and as a result the force between them is always repulsive. We have not discussed negative values for ψ_0 separately here because a scattering process with given negative ψ_0 is related to the processes with $-\psi_0$ by the reflection P_y.

Clearly the precise values of the boundaries of the intervals I^{90}, I^{capt} and I^{180} depend on the baby Skyrmions' initial separation and their speeds. We have repeated the simulations just discussed with a different value of the initial velocity v; the qualitative features discussed above and depicted in figure 7 are unaffected by such a change.

6.3 Scattering with non-zero Impact Parameter

We have also investigated scattering processes with non-zero impact parameter. We define the impact parameter in the usual way as the distance of closest of approach between the two soliton centres in the absence of interaction, and denote it by b.

As in the previous simulations, the nature of the scattering processes to be studied here depends crucially on the value of ψ_0, but, again as before, there are basically two regimes. For ψ_0 smaller than some critical value (which depends on the impact parameter and the initial velocity) the forces are repulsive and the baby Skyrmions escape to infinity after scattering through some non-trivial angle. When ψ_0 is larger than the critical value the scattering is more interesting: now the forces are attractive and the baby Skyrmions may be captured into a bound orbit. The orbiting motion is accompanied by emission of radiation. Since the qualitative nature of this process is independent of the precise value of ψ_0 we have investigated it in detail only for the attractive channel $\psi_0 = \pi$.

For a quantitative study we have also concentrated on a fixed initial speed, $v = 0.4$, and looked at the dependence of the scattering on the impact parameter b. We already know from previous simulations that the the baby Skyrmions will be captured in a bound orbit for $b = 0$ (the oscillatory excitation of the 2-soliton), but it is clear that the baby Skyrmions will escape to infinity after the scattering if b is made sufficiently large. Hence there must be a critical impact parameter which separates the two types of scattering. From our simulation we conclude that this critical value is approximately 1.5.

When the baby Skyrmions get captured in a bound orbit their centres trace out ellipse-like figures whose perihelion rotates slowly and whose diameter and eccentricity decrease as energy is lost through radiation. This is best seen in simulations where the speed v is small and the impact parameter is large. In figure 8 we show a part of the baby Skyrmions' trajectories for the initial velocity $v = 0.1$ and the impact parameter $b = 12$. It is not clear from our simulations what the final state of such an orbiting motion is. It is quite possible that there is a non-radiating solution of the field equations where two baby Skyrmions orbit each other with an angular frequency less than μ. Moreover, our analysis of the spinning baby Skyrmion can be extended to 2-solitons, showing that there are finite-energy, spinning 2-soliton solutions of the hedgehog from provided their angular frequency is less than μ. However, contrary to the spinning baby Skyrmion these solutions may well be unstable. In our simulations of orbiting baby Skyrmions the kinetic energy decreases all the time but is never zero. It is not clear whether either of the periodic but non-radiating solutions described above is eventually realised or whether the system will continue to radiate until it reaches the static 2-soliton solution.

6.4 The Effect of Relative Spin

The last set of simulations we want to discuss addresses the effect of relative spin on the interaction of two baby Skyrmions. If one of two well-separated baby Skyrmions is spinning and the other at rest the dipole forces between the two average to zero over the spinning baby Skyrmion's period of rotation. Thus we expect there to be no net force in this situation. This is indeed what we observe: with the stationary baby Skyrmion placed at $(-7.5, 0)$ and the other one, spinning at $\omega = 0.2$ (with the corresponding profile function) placed at $(7.5, 0)$ the separation oscillates around the initial value 15 with a small amplitude ≈ 0.1 and angular frequency ω.

Next we investigate the combined effect of relative spin and relative motion of the baby Skyrmions' centres. In order to see the influence of *spin* rather than that of the relative *orientation* on the interaction we should make sure that the interaction processes takes much longer than one period of rotation. Thus we keep the frequency of the spinning baby Skyrmion constant at $\omega = 0.2$ and send the two baby Skyrmions towards each other along the x-axis with a small speed v. Performing this simulation for $v = 0.02$ and $v = 0.05$ and a variety of values for ψ_0, we find that the baby Skyrmions always repel each other and scatter through $180°$. The distance of closest approach is > 9 and the spinning Skyrmion's angular frequency does not change during the interaction. This remarkable result can be understood in terms of the dipole model as follows. Since the angular velocity is large and the torque acting on the relative orientation weak as long as the two baby Skyrmions are not too close together we can assume that angular velocity is essentially constant during the interaction process. Thus, as the two baby Skyrmions approach each other they experience alternatively an attractive and a repulsive force for approximately equal durations given by $1/(2\omega)$. This leads to a sequence of alternating attractive and repulsive impulses. Since the strength of the force between two baby Skyrmions increases with decreasing R, it is clear that every attractive impulse will invariably be followed by a *stronger* repulsive one. However, it may happen that at some point a repulsive impulse is strong enough to invert the sign of the relative velocity $\dot{R}$. Then that impulse will be followed by an attractive impulse which is *weaker* than itself. Thus, for a sufficiently small initial speed v the baby Skyrmions will always eventually repel each other and escape to infinity.

This qualitative explanation in terms of the dipole model can be confirmed by numerical solutions of the equations (5.6). In figure 9 we show the trajectories $R(t)$ and $\psi(t)$ for a particular set of initial values. The plots show clearly that $\dot{\psi}$ indeed remains essentially constant during the interaction process, as assumed in our qualitative analysis of the dipole model. There is a further interesting effect, though. By carefully measuring $\dot{R}$ and $\dot{\psi}$ long after the interaction we find that, at $t > 200$, $\dot{R} > v$ and $\dot{\psi} < \omega$ (the difference here is very small). Thus rotational kinetic energy has been converted into translational kinetic energy.

We observe similar effects in our simulations of the full field theory which become more pronounced as v is increased. We will describe them for one particular simulation, where the baby Skyrmions are again initially at $(7.5, 0)$ and $(-7.5, 0)$, $\psi_0 = \pi/2$ and $\omega = 0.2$, but now $v = 0.055$. The scattering is again repulsive, but now the distance of closest approach is only ≈ 7. At the point of closest approach the spinning baby Skyrmion's angular frequency decreases to roughly 0.1 and the other baby Skyrmion begins to spin at that frequency, too. The baby Skyrmions then move away from each other, each travelling at the considerably increased speed 0.16. One checks that the loss in rotational kinetic energy of $\frac{1}{4}(0.2)^2\Lambda_0$ is approximately balanced by the gain in translational kinetic energy of $(0.16)^2M_0-(0.055)^2M_0$.

When v is slightly larger than 0.055 the scattering behaviour becomes sensitive to the precise value of the initial relative orientation ψ_0. For some values of ψ_0 the baby Skyrmions repel as before, but for others their linear momentum is large enough to overcome the repulsive barrier due to their relative spin. Then the baby Skyrmions collide and form the oscillating and radiating 2-soliton configuration already encountered in previous simulations, but now this configuration rotates as a whole. As in the simulation discussed at the end of section 6.3 we are unable to decide on the basis of our numerical results whether all the kinetic energy will eventually be radiated away or whether the system will settle down to a uniformly rotating 2-soliton solution.

As v is increased further the baby Skyrmions continue to get captured after the collision, even for values of v as large as 0.5. Finally, for $v = 0.6$, they scatter through $\approx 50°$ and escape to infinity. The collision is again very radiative. Afterwards both baby Skyrmions spin at the same angular velocity, whose numerical value is less than 0.1. Thus the radiation carries away spin as well as energy. 'Exotic' scattering angles in head-on collisions like the one observed here are a familiar feature of lump scattering in the presence of overall spin, see [8] and [13]. Their occurrence is not surprising. If one or both baby Skyrmions are spinning initially, the initial configuration is, in general, no longer invariant under the reflection operations discussed in section 6.2. Thus, in the presence of spin, there is no reason to expect the scattering angle in a head-on collision to take on special values like 90° or 180°.

7 Conclusion

Thinking of baby Skyrmions in terms of pairs of orthogonal dipoles has proven very useful for understanding their dynamical properties. The dipole picture reproduces the qualitative features of the asymptotic field of a spinning baby Skyrmion, both in the radiative regime where the angular frequency ω is larger than μ and in the non-radiative regime $\omega < \mu$. The dipole model is most successful when applied to the interaction between well-separated baby Skyrmions. Here it gives an accurate, quantitative description. Moreover it allows one to understand in simple qualitative terms the repulsive nature of the forces between baby Skyrmions which are spinning rapidly relative to each other and whose centres move slowly relative to each other. The dipole model also predicts the possibility of transfer of spin and rotational kinetic energy from one baby Skyrmion to the other, in agreement with our simulations of the full field theory. This 'spin-exchange' scattering is rather reminiscent of the electric charge exchange in dyon scattering [19].

The dipole picture is based on the linearisation of the non-linear baby Skyrme model. When studying the change in shape of a spinning baby Skyrmion or the interaction of two baby Skyrmions close together, however, non-linear effects dominate and numerical methods become indispensable. Here our simulations also contain a number of lessons for the discussion of Skyrmion dynamics.

We saw that for a rapidly spinning baby Skyrmion centrifugal effects lead to a linear dependence of the mass on the angular momentum. This should be contrasted with the quadratic dependence predicted by the non-relativistic, rigid body treatment commonly used in the Skyrme model. Our results underline the importance of a more careful treatment of centrifugal effects when extracting the baryon spectrum from the Skyrme model.

The notion of the attractive channel proved useful for understanding a wide range of scattering processes. Even though the forces between baby Skyrmions are complicated due to their dependence on the relative orientation (as it is for Skyrmions) the torque acting on the relative orientation is so strong that interacting baby Skyrmions tend to be in the attractive channel by the time they collide. Then they scatter through 90° as if they had been in the attractive channel initially. However, this process is the more radiative the further the baby Skyrmions were initially from being in the attractive channel.

Finally, our results illustrate the importance of radiation in soliton dynamics. Our simulations show that a particularly large amount of radiation is emitted in a head-on collision of two baby Skyrmions. This means that baby Skyrmions only escape to infinity after the interaction when the initial relative speed or the impact parameter are quite large. Otherwise they get captured and, after further emission of radiation, settle down to the static 2-soliton configuration. We have already pointed out that the radiative nature of this process is probably due to the large gradient of the potential energy functional at configurations which describe two baby Skyrmions close together. It may even be possible to relate the large amount of radiation accompanying most soliton interactions in our model to the fact that the energies of the static soliton solutions are much larger (about 50 %) than the Bogomol'nyi bound (2.7). In any case, it is clear that the adiabatic approximation to soliton dynamics proposed in [5] would not be useful in our model. Moreover, our results show that the inclusion of the pion mass does not necessarily make the adiabatic approximation more applicable, as is often claimed. In our model soliton dynamics is much less adiabatic than in comparable model with massless mesons.

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Figure Captions

Figure 1

Profile functions for a static baby Skyrmion (bottom curve) and a spinning baby Skyrmion with angular frequency (from top to bottom) 0.316,0.3 and 0.2.

Figure 2

a) The mass $M(\omega)$ of a spinning baby Skyrmion in units of 4π as a function of the angular frequency.

b) Mass-spin relationship for a spinning baby Skyrmion. The crosses mark pairs (M, J) calculated via (4.3) and (4.11) for the same value of ω; both M and J are plotted in units of 4π. The solid line is a plot of the function $\tilde{M}$ of J (4.15); here, too, $\tilde{M}$ and J are plotted in units of 4π.

Figure 3

Plot of the field of a baby Skyrmion with initial angular frequency $\omega = 0.5$ at time t=10. At every lattice site in physical space we plot an arrow of unit length whose direction is that of (ϕ_1, ϕ_2) (we identify the 1- and 2-axes in the target space S^2 with those in physical space). At the head of the arrow we put a '+' if ϕ_3 is positive and a '$\times$' if ϕ_3 is negative. If $(\phi_1^2 + \phi_2^2) < 2 \times 10^{-4}$ no arrow is plotted. Thus the vacuum is represented simply by a '+'.

Figure 4

Total energy of spinning baby Skyrmions with initial angular frequencies 0.9 (top) and 0.5 (bottom) in units of 4π.

Figure 5

a) Relative motion of two baby Skyrmions released from rest. $R/2$ is plotted as a function of time for, from top to bottom, $\psi_0 = 0.275{\cdot}\pi, 0.3{\cdot}\pi, 0.4{\cdot}\pi, 0.5{\cdot}\pi$ and $\psi_0 = \pi$.

b) Prediction of the dipole model for the relative motion of two baby Skyrmions released from rest. $R/2$ (top) and ψ (bottom) as a function of time for $\psi_0 = 0.28855{\cdot}\pi$.

Figure 6

Total energy density shortly after the head-on collision of two baby Skyrmions with initial speeds $v = 0.6$.

Figure 7

Head-on Collisions

a) Sketch of baby Skyrmion velocities and radiation emitted shortly after scattering with $\psi_0 = \pi$.

b) Sketch of baby Skyrmion velocities and radiation emitted shortly after scattering with $\psi_0 = \frac{\pi}{6}$.

c) Sketch of 2-soliton velocity and radiation emitted shortly after scattering with $\psi_0 \in I^{\mathrm{capt}}$.

d) Sketch of baby Skyrmion velocities shortly after scattering with $\psi_0 = 0$.

Figure 8

Trajectories of two baby Skyrmions with initial speed $v = 0.1$ and impact parameter $b = 12$.

Figure 9

The effect of spin: trajectories $R(t)/2$ (top) and $\psi(t)/2\pi$ (bottom) calculated from the dipole model with initial values $R(0) = 15$, $\dot{R}(0) = 0.06, \psi_0 = 0.5 \cdot \pi$ and $\dot{\psi}(0) = \omega = 0.2$.