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Multisolitons in a Two-dimensional Skyrme Model

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Abstract

The Skyrme model can be generalised to a situation where static fields are maps from one Riemannian manifold to another. Here we study a Skyrme model where physical space is two-dimensional euclidean space and the target space is the two-sphere with its standard metric. The model has topological soliton solutions which are exponentially localised. We describe a superposition procedure for solitons in our model and derive an expression for the interaction potential of two solitons which only involves the solitons' asymptotic fields. If the solitons have topological degree 1 or 2 there are simple formulae for their interaction potentials which we use to prove the existence of solitons of higher degree. We explicitly compute the fields and energy distributions for solitons of degrees between one and six and discuss their geometrical shapes and binding energies.

1 A Skyrme Model in Two Dimensions

The Skyrme model is a non-linear theory for $SU(2)$ valued fields in 3 (spatial) dimensions which has soliton solutions. Each soliton has an associated integer topological charge or degree which Skyrme identified with the baryon number [1]. A soliton with topological charge one is called a Skyrmion; suitably quantised it is a model for a physical nucleon. Solitons of higher topological charge, called multisolitons, are classical models for higher nuclei.

In this paper we study multisolitons in a two-dimensional version of the Skyrme model. The model was first considered in [2], but the motivation there is somewhat different from the approach taken here. For the present purpose it is important to be clear in what sense our model resembles Skyrme's model. In this section we will therefore briefly review a general framework for the Skyrme model due to Manton [3] and explain how our model fits into that framework. In [3] the usual Skyrme energy functional is interpreted in terms of elasticity theory and a static Skyrme field is a map

$$\pi : S \mapsto \Sigma \tag{1.1}$$

from physical space S to the target space Σ. Both S and Σ are assumed to be Riemannian manifolds with metrics t and τ respectively. The energy of a configuration π is expressed in terms of its strain tensor D. To calculate the strain tensor one introduces coordinates p^i on S and π^α on Σ and orthonormal frame fields $\boldsymbol{s}_m$ on S and $\boldsymbol{\sigma}_\mu$ on Σ ($1 \leq i, m \leq \mathrm{dim}S$, $1 \leq \alpha, \mu \leq \mathrm{dim}\Sigma$). The Jacobian of the map π is, in orthonormal coordinates,

$$J_{m\mu} = s^i_m \frac{\partial \pi^\alpha}{\partial p^i} \sigma_{\mu\alpha} \tag{1.2}$$

and the strain tensor D is defined via

$$D_{mn} = J_{m\mu} J_{n\mu}. \tag{1.3}$$

The energy functional should be a function of the strain tensor but it should not depend on the choice of the orthonormal frame $\boldsymbol{s}_m$. The basic invariants under orthogonal transformations of D, however, are well known: they are the coefficients in the characteristic polynomial $\chi_D(t) := \det(D - t\,\mathrm{id})$. If S is three-dimensional there are three such invariants, namely $\mathrm{tr}D$, $\frac{1}{2}(\mathrm{tr}D)^2 - \frac{1}{2}\mathrm{tr}D^2$ and $\det D$. It is explained in [3] that in the usual Skyrme model, where $S = \mathbf{R}^3$ and $\Sigma = SU(2)$, the energy functional is constructed from the first two:

$$E_{\text{Skyrme}} = \int d^3x \, \left(\mathrm{tr}D + \frac{1}{2}(\mathrm{tr}D)^2 - \frac{1}{2}\mathrm{tr}D^2\right). \tag{1.4}$$

It is now straightforward to construct a Skyrme model in two dimensions in this framework. We will reserve the term Skyrme model for Skyrme's original model and refer to our two-dimensional model as a baby Skyrme model, its soliton solutions baby Skyrmions etc. We want to work in flat space, so we choose $S = \mathbf{R}^2$. The choice of Σ is less clear. In order to obtain solitons with a topological charge we require, as we shall explain, $\pi_2(\Sigma) = \mathbf{Z}$, so the simplest choice is $\Sigma = S^2$. Thus a baby Skyrme field is a map

$$\boldsymbol{\phi} : \mathbf{R}^2 \mapsto S^2, \tag{1.5}$$

where S^2 is the unit 2-sphere in euclidean 3-space with the metric induced by that embedding, and we think of $\boldsymbol{\phi}$ as a three component vector (ϕ_1, ϕ_2, ϕ_3) satisfying $\boldsymbol{\phi}^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$. In two dimensions there are only two invariants of the strain tensor, namely

$$\begin{aligned} e_2 &= \mathrm{tr}D = \partial_1\boldsymbol{\phi}^2 + \partial_2\boldsymbol{\phi}^2 \\ e_4 &= \det D = (\partial_1\boldsymbol{\phi} \times \partial_2\boldsymbol{\phi})^2, \end{aligned} \tag{1.6}$$

where $\partial_i, i = 1, 2$, denotes the partial derivative with respect to the cartesian coordinates x^i of the vector $\boldsymbol{x} \in \mathbf{R}^2$ and $\times$ is the vector product in three dimensions.

The simplest choice of the energy functional is $E_\sigma = \frac{1}{2}\int d^2x\, e_2$ and leads to the much studied two-dimensional σ-model whose static solutions are harmonic maps from $\mathbf{R}^2$ to S^2. However, the σ-model is not a good analogue of the Skyrme model since, unlike Skyrmions, its soliton solutions have an arbitrary scale. The scale invariance is broken by adding the Skyrme term $\frac{1}{2}\int d^2x\, e_4$, but the resulting energy functional is still not satisfactory: it can have no minima since the energy of any configuration can be lowered by rescaling: $\boldsymbol{\phi}(\boldsymbol{x}) \to \boldsymbol{\phi}(\frac{\boldsymbol{x}}{\lambda})$, where $\lambda > 1$. Hence it is necessary to include a term in the energy functional which contains no derivatives of the field $\boldsymbol{\phi}$ and which is often called a potential. Here we will consider the potential $\mu^2(1 - \boldsymbol{n}\cdot\boldsymbol{\phi})$, where $\boldsymbol{n} = (0, 0, 1)$ and μ is a constant with the dimension of inverse length. Our potential is analogous to the extra term included in the Skyrme model to give the pions a mass (thus $1/\mu$ may physically be interpreted as the Compton wavelength of the mesons in our model). Both in the Skyrme model and in our model the potential term reduces the symmetry of the model and is responsible for the soliton solutions being exponentially localised in space. In the Skyrme model the size of Skyrmion is slightly smaller than the pion's Compton wavelength, reflecting the relative magnitudes of nucleon size and the pion's Compton wavelength in nature. To mimic these properties we want to have a basic soliton solution whose size is of order $1/\mu$. This can be achieved by setting $\mu^2 = 0.1$, which we do for the rest of this paper. Note, however, that a different choice of μ results in a different model. While some of the general features to be discussed in this paper are independent of μ, other properties, such as the shape of multisolitons (we give a precise definition further below), may well depend on it.

Some readers may want to bear in mind an alternative physical interpretation the field $\boldsymbol{\phi}$: it can also be thought of as the magnetisation vector of a two-dimensional ferromagnetic substance [4]. Then the potential term describes the coupling of the magnetisation vector to a constant external magnetic field.

To sum up: the general framework for Skyrme models described above, augmented by a potential modelled on the one used in the usual Skyrme model leads to the following energy density for the baby Skyrme model:

$$e = \frac{1}{2}(\partial_1\boldsymbol{\phi}^2 + \partial_2\boldsymbol{\phi}^2) + \frac{1}{2}(\partial_1\boldsymbol{\phi} \times \partial_2\boldsymbol{\phi})^2 + \mu^2(1 - \boldsymbol{n}\cdot\boldsymbol{\phi}), \tag{1.7}$$

from which the energy functional is obtained by integration

$$E[\boldsymbol{\phi}] = \int d^2x\, e. \tag{1.8}$$

We are only interested in configurations with finite energy, so we define our configurations space Q to be the space of all maps $\boldsymbol{\phi} : \mathbf{R}^2 \mapsto S^2$ which tend to the constant field $\boldsymbol{n}$, called the vacuum, at spatial infinity

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{n}. \tag{1.9}$$

As a result, a configuration $\boldsymbol{\phi}$ may be regarded as a map from compactified physical space $\mathbf{R}^2 \cup \{\infty\}$, which is homeomorphic to a 2-sphere, to S^2. Thus every configuration $\boldsymbol{\phi}$ may be regarded as a representative of a homotopy class in $\pi_2(S^2) = \mathbf{Z}$ and has a corresponding integer degree, which can be calculated from

$$\deg[\boldsymbol{\phi}] = \frac{1}{4\pi}\int d^2x\, \boldsymbol{\phi}\cdot\partial_1\boldsymbol{\phi} \times \partial_2\boldsymbol{\phi}. \tag{1.10}$$

Configurations with different degrees cannot be smoothly deformed into each other and hence the configuration space Q is not connected. We write Q_n for the component of Q containing the configurations of degree n. In this paper we are interested in stationary points and, if they exist, minima of E in a given sector Q_n. A configuration $\boldsymbol{\phi}$ is a stationary point of E if the first variation of E under $\boldsymbol{\phi}(\boldsymbol{x}) \mapsto \boldsymbol{\phi}(\boldsymbol{x}) + \boldsymbol{\epsilon}(\boldsymbol{x}) \times \boldsymbol{\phi}(\boldsymbol{x})$ vanishes for any function $\boldsymbol{\epsilon} : \mathbf{R}^2 \mapsto \mathbf{R}^3$ satisfying $\boldsymbol{\phi}(\boldsymbol{x})\cdot\boldsymbol{\epsilon}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$. This requirement leads to the Euler-Lagrange equation

$$\partial_i \boldsymbol{j}_i = \mu^2 \boldsymbol{n} \times \boldsymbol{\phi}, \tag{1.11}$$

where

$$\boldsymbol{j}_i = \boldsymbol{\phi} \times \partial_i\boldsymbol{\phi} + \partial_j\boldsymbol{\phi}(\partial_j\boldsymbol{\phi}\cdot\boldsymbol{\phi} \times \partial_i\boldsymbol{\phi}). \tag{1.12}$$

The degree gives a useful lower bound on the potential energy, the Bogomol'nyi bound

$$E[\boldsymbol{\phi}] \geq 4\pi\cdot|\deg[\boldsymbol{\phi}]|. \tag{1.13}$$

This inequality holds already for the σ-model energy functional E_σ (for a proof see [5]) and since E is bounded below by E_σ it holds for E as well. Assuming that there are finite energy configurations of degree n it follows that the infimum of the restriction $E_{|Q_n}$ of the functional E to Q_n exists, and we call it E_n. It is not clear, however, whether there is a configuration of degree n whose energy is E_n. If such a configuration exists for $n = 1$ we call it a 1-soliton or a baby Skyrmion, and if it exists for $n > 1$ and if moreover its energy satisfies

$$E_n < E_k + E_l \qquad \text{for all integers } 1 < k, l < n \quad \text{such that} \quad k + l = n \tag{1.14}$$

we call it a multisoliton or an n-soliton. If we do not need to specify the degree we simply write soliton for an n-soliton with arbitrary $n \in \mathbf{N}$ (all solitons of negative degree can be obtained from solitons of positive degree via the iso-reflection $(\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, \phi_3)$, so we can restrict attention to positive n without loss of generality). We have included the condition (1.14) in our definition because we want multisolitons to be stable with respect to decay into multisolitons of smaller degree.

The goal of this paper is to show the existence of multisolitons in the baby Skyrme model and to describe their properties. In section 2 we discuss a particular ansatz for finding solitons and use it to calculate the field of a baby Skyrmion. In section 3 we set up a general framework for proving the inequality (1.14) and in section 4 we complete the proof for certain values of k, l and n. Section 5 contains numerical evidence for the existence of n-solitons in our model for $1 \leq n \leq 6$ and a description of their properties.

2 Hedgehog Fields and Baby Skyrmions

A powerful method of finding stationary points of the energy functional E (1.8) exploits the invariance of E and Q_n under the symmetry group

$$G = E_2 \times SO(2)_{\text{iso}} \times P. \tag{2.1}$$

Here E_2 is the euclidean group of translations and rotations in two dimensions which acts on fields via pull-back. $SO(2)_{\text{iso}}$ is the subgroup of the three-dimensional rotation group acting on S^2 which leaves $\boldsymbol{n}$ fixed. We call its elements iso-rotations to distinguish them from rotations in physical space. Elements of $SO(2)_{\text{iso}}$ can be parametrised by an angle $\chi \in [0, 2\pi)$ and act on $\boldsymbol{\phi}$ via

$$(\phi_1, \phi_2, \phi_3) \mapsto (\cos\chi\,\phi_1 + \sin\chi\,\phi_2, -\sin\chi\,\phi_1 + \cos\chi\,\phi_2, \phi_3). \tag{2.2}$$

Finally P is a combined reflection in both space and the target space S^2:

$$P : (x_1, x_2) \mapsto (x_1, -x_2) \quad \text{and} \quad (\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, \phi_3) \tag{2.3}$$

The vacuum field $\boldsymbol{n}$ is invariant under the symmetry group G and clearly minimises the energy in Q_0. However, we are interested in stationary points of degree $\neq 0$ and the maximal subgroups of G under which such a field can be invariant are labelled by a non-zero integer n and consist of spatial rotations by some angle $\alpha \in [0, 2\pi)$ and simultaneous iso-rotation by $-n\alpha$. Fields invariant under such a group are of the form

$$\boldsymbol{\phi}(\boldsymbol{x}) = (\sin f(r)\cos(n\theta - \chi), \sin f(r)\sin(n\theta - \chi), \cos f(r)), \tag{2.4}$$

where (r,θ) are polar coordinates in the $\boldsymbol{x}$-plane, and f is function satisfying certain boundary conditions to be specified below. The angle χ is also arbitrary, but fields with different χ are related by an iso-rotation and therefore degenerate in energy. Hence we concentrate on the fields where $\chi = 0$. Such fields are the analogue of the hedgehog fields in the Skyrme model and we also call them hedgehog fields here. They were also studied in [2] for a different value of μ.

The function f, which we call the profile function, has to satisfy

$$f(0) = m\pi, \quad m \in \mathbf{Z}, \tag{2.5}$$

for the field (2.4) to be regular at the origin and to satisfy the boundary condition (1.9) we set

$$\lim_{r\to\infty} f(r) = 0. \tag{2.6}$$

We can assume without loss of generality that m is positive because changing the sign of f in (2.4) is equivalent to an iso-rotation by 180°. The restriction $\tilde{E}$ of E to fields of the form (2.4) is

$$\tilde{E} = 2\pi \int r dr \left(\frac{1}{2} f'^2 + \frac{n^2 \sin^2 f}{2r^2}(1 + f'^2) + \mu^2 (1 - \cos f) \right). \tag{2.7}$$

It follows from the "principle of symmetric criticality" [6] that a field of the form (2.4) is a stationary point of the energy functional E if f is a stationary point of $\tilde{E}$, i.e. if f satisfies the Euler Lagrange equations first written down in [2]:

$$\left(r + \frac{n^2 \sin^2 f}{r} \right) f'' + \left(1 - \frac{n^2 \sin^2 f}{r^2} + \frac{n^2 f' \sin f \cos f}{r} \right) f' - \frac{n^2 \sin f \cos f}{r} - r\mu^2 \sin f = 0. \tag{2.8}$$

The behaviour of solutions of this equation near the origin and for large r can be deduced analytically and was also discussed in [2]. The result is that, for small r,

$$f(r) \approx m\pi + Cr^n, \tag{2.9}$$

where C is a constant which depends on m and n. For large r, the equation (2.8) simplifies to the modified Bessel equation

$$f'' + \frac{1}{r} f' - \left(\frac{n^2}{r^2} + \mu^2\right) f = 0. \tag{2.10}$$

Solutions of this equation which tend to zero at $r = \infty$ are the modified Bessel functions $K_n(\mu r)$ of order n. Thus a solution f of (2.8) is proportional to K_n for large r and we can write

$$f(r) \sim \frac{c_n \mu^n}{2\pi} K_n(\mu r), \tag{2.11}$$

where c_n is a constant, dependent on n and m, which we will interpret further below. Since the modified Bessel functions have the asymptotic behaviour

$$K_n(\mu r) \sim \sqrt{\frac{\pi}{2\mu r}} e^{-\mu r} \left(1 + \mathcal{O}(\frac{1}{\mu r})\right) \tag{2.12}$$

we know that the leading term in an asymptotic expansion of f is $e^{-\mu r}/\sqrt{r}$. The behaviour of f at infinity guarantees that the energy distribution of the corresponding field $\boldsymbol{\phi}$ is exponentially localised and hence that its total energy is finite.

What is the degree of the field (2.4)? A short calculation gives

$$\deg[\boldsymbol{\phi}] = \begin{cases} n & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even.} \end{cases}$$

Thus for a given degree $n \neq 0$ there are infinitely many solutions of the static field equations of the form (2.4), one for each odd m.

Using a shooting method we have solved the ordinary differential equation (2.8) numerically for a range of values of n and m and computed the energy of the corresponding hedgehog field from (2.7). We find that, amongst all solutions of the static field equations of the form (2.4) with degree 1 , the field with $n = m = 1$ has the lowest energy. We write $\boldsymbol{\phi}^{(1)}$ for that field in standard orientation ($\chi = 0$) and denote the profile function by $f^{(1)}$. We have also numerically tested the stability of the field $\boldsymbol{\phi}^{(1)}$ against perturbations which destroy the rotational symmetry and found it to be stable (we will describe our numerical method in more detail in section 5). Thus, in analogy to the Skyrme model, the minimal energy configuration amongst all fields of degree one is of the hedgehog form. The profile function $f^{(1)}$ is plotted in figure 1.a) and a plot of the energy density e of the field, which is given by the expression in round brackets in (2.7), is shown in figure 1.b). The actual field $\boldsymbol{\phi}^{(1)}$ is displayed in figure 2.a).

There is a whole manifold $\mathcal{M}_1$ of minima of $E_{|Q_1}$ obtained by acting with G on $\boldsymbol{\phi}^{(1)}$. All elements of $\mathcal{M}_1$ are baby Skyrmions as defined at the end of section 1. $\mathcal{M}_1$ is three-dimensional, so a baby Skyrmion is characterised by its two-dimensional position vector and an angle χ specifying its orientation. The value of E on $\mathcal{M}_1$ physically represents the mass E_1 of a baby Skyrmion. Numerically we find $E_1 = 1.564 \cdot 4\pi$.

3 Existence of Multisoliton Solutions

The basic ingredient for proving (1.14) is a superposition procedure for solitons. In our model such a procedure can be found using the stereographic projection

$$p : \mathbf{C} \cup \{\infty\} \mapsto S^2$$

$$u = u_1 + iu_2 \quad \mapsto \quad (\frac{2u_1}{1+|u|^2}, \frac{2u_2}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2}). \tag{3.1}$$

This allows us to translate a function

$$u : \mathbf{R}^2 \cup \{\infty\} \mapsto \mathbf{C} \cup \{\infty\}, \qquad u(\infty) = 0 \tag{3.2}$$

into a configuration $\boldsymbol{\phi}^u$ via

$$\boldsymbol{\phi}^u = p \circ u \tag{3.3}$$

and vice versa. If we think of u as a function of $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ then its degree, which equals the degree of $\boldsymbol{\phi}^u$, is the number of poles in z, counted with multiplicity, minus the number of poles in $\bar{z}$, also counted with multiplicity. For more details on the formulation of two-dimensional Skyrme models in terms of u instead of $\boldsymbol{\phi}$ we refer the reader to [10]. Here we use it to define: the superposition of the configurations $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ is the configuration $\boldsymbol{\phi}^w$ where $w = u + v$. Then clearly $\deg[\boldsymbol{\phi}^w] = \deg[\boldsymbol{\phi}^u] + \deg[\boldsymbol{\phi}^v]$.

Now let $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ be multisolitons of degrees k and l. We want to show that under certain circumstances $E[\boldsymbol{\phi}^w] < E[\boldsymbol{\phi}^u] + E[\boldsymbol{\phi}^v]$. The idea for our proof is taken from an unpublished paper by Kugler and Castillejo [7] where the authors claim to to prove the corresponding statement for the Skyrme model. However, their proof contains an unjustified assumption as we shall see further below.

Consider the situation where the multisolitons $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ are well separated. More precisely we assume that we can divide $\mathbf{R}^2$ into two regions such that u is small in region 2 and v small in region 1. The smallness of u means that $\boldsymbol{\phi}^u$ is close to the vacuum $\boldsymbol{n}$ in region 2 and hence of the form

$$\boldsymbol{\phi}^u = \sqrt{1 - \boldsymbol{\varphi}^u \cdot \boldsymbol{\varphi}^u}\boldsymbol{n} + \boldsymbol{\varphi}^u \approx \boldsymbol{n} + \boldsymbol{\varphi}^u + \mathcal{O}((\boldsymbol{\varphi}^u \cdot \boldsymbol{\varphi}^u), \tag{3.4}$$

where $\boldsymbol{\varphi}^u \cdot \boldsymbol{n} = 0$. Then, if $\boldsymbol{\phi}^u$ satisfies the Euler-Lagrange equation (1.11), $\boldsymbol{\varphi}^u$ satisfies the linearised equation

$$(\Delta - \mu^2)\boldsymbol{\varphi}^u = 0 \tag{3.5}$$

in region 2. Similarly, $\boldsymbol{\phi}^v$ is close to the vacuum in region 1 and one defines $\boldsymbol{\varphi}^v$ analogously to $\boldsymbol{\varphi}^u$. It satisfies

$$(\Delta - \mu^2)\boldsymbol{\varphi}^v = 0 \tag{3.6}$$

in region 1. One checks that $\boldsymbol{\phi}^w$ has the following expansion in powers of $\boldsymbol{\varphi}^v$ in region 1

$$\boldsymbol{\phi}^w \approx \boldsymbol{\phi}^u + \boldsymbol{\epsilon}^v \times \boldsymbol{\phi}^u + \frac{1}{2}\boldsymbol{\epsilon}^v \times (\boldsymbol{\epsilon}^v \times \boldsymbol{\phi}^u), \tag{3.7}$$

where $\boldsymbol{\epsilon}^v$ is linear in $\boldsymbol{\varphi}^v$ but also depends non-linearly on $\boldsymbol{\phi}^u$:

$$\boldsymbol{\epsilon}^v = \frac{1}{2}\boldsymbol{\phi}^u \times \left((1+\boldsymbol{\phi}^u\cdot\boldsymbol{n})\boldsymbol{\varphi}^v - (\boldsymbol{\phi}^u\cdot\boldsymbol{\varphi}^v)\boldsymbol{n}\right). \tag{3.8}$$

Similarly, in region 2

$$\boldsymbol{\phi}^w \approx \boldsymbol{\phi}^v + \boldsymbol{\epsilon}^u \times \boldsymbol{\phi}^v + \frac{1}{2}\boldsymbol{\epsilon}^u \times (\boldsymbol{\epsilon}^u \times \boldsymbol{\phi}^v), \tag{3.9}$$

where $\boldsymbol{\epsilon}^u$ depends linearly on $\boldsymbol{\varphi}^u$ but non-linearly on $\boldsymbol{\phi}^v$:

$$\boldsymbol{\epsilon}^u = \frac{1}{2}\boldsymbol{\phi}^v \times \left((1+\boldsymbol{\phi}^v\cdot\boldsymbol{n})\boldsymbol{\varphi}^u - (\boldsymbol{\phi}^v\cdot\boldsymbol{\varphi}^u)\boldsymbol{n}\right). \tag{3.10}$$

Using these formluae we evaluate $E[\boldsymbol{\phi}^w]$: in region 1 we keep terms which involve $\boldsymbol{\phi}^u$ only, terms which are linear in $\boldsymbol{\varphi}^v$ and those terms quadratic in $\boldsymbol{\varphi}^v$ which are independent of $\boldsymbol{\phi}^u$. Similarly, in region 2 we keep terms which involve $\boldsymbol{\phi}^v$ only, terms which are linear $\boldsymbol{\varphi}^u$ and those terms quadratic in $\boldsymbol{\varphi}^u$ which are independent of $\boldsymbol{\phi}^v$. Denoting integration over region 1 and 2 simply by the suffix 1 and 2 respectively we find

$$\begin{aligned}
E[\boldsymbol{\phi}^w] &\approx \int_1 d^2x\, e\left(\boldsymbol{\phi}^u + \boldsymbol{\epsilon}^v \times \boldsymbol{\phi}^u + \frac{1}{2}\boldsymbol{\epsilon}^v \times (\boldsymbol{\epsilon}^v \times \boldsymbol{\phi}^u)\right) \\
&+ \int_2 d^2x\, e\left(\boldsymbol{\phi}^v + \boldsymbol{\epsilon}^u \times \boldsymbol{\phi}^v + \frac{1}{2}\boldsymbol{\epsilon}^u \times (\boldsymbol{\epsilon}^u \times \boldsymbol{\phi}^v)\right) \\
&\approx \int_1 d^2x\, e(\boldsymbol{\phi}^u) + \int_2 d^2x \left(\frac{1}{2}\partial_i\boldsymbol{\varphi}^u\cdot\partial_i\boldsymbol{\varphi}^u + \frac{1}{2}\mu^2\boldsymbol{\varphi}^u\cdot\boldsymbol{\varphi}^u\right) \\
&+ \int_2 d^2x\, e(\boldsymbol{\phi}^v) + \int_1 d^2x \left(\frac{1}{2}\partial_i\boldsymbol{\varphi}^v\cdot\partial_i\boldsymbol{\varphi}^v + \frac{1}{2}\mu^2\boldsymbol{\varphi}^v\cdot\boldsymbol{\varphi}^v\right) \\
&+ \int_1 d^2x\, \boldsymbol{j}_i^u\cdot\partial_i\boldsymbol{\epsilon}^v + \mu^2\boldsymbol{\epsilon}^v\cdot\boldsymbol{n}\times\boldsymbol{\phi}^u + \int_2 d^2x\, \boldsymbol{j}_i^v\cdot\partial_i\boldsymbol{\epsilon}^u + \mu^2\boldsymbol{\epsilon}^u\cdot\boldsymbol{n}\times\boldsymbol{\phi}^v,
\end{aligned} \tag{3.11}$$

where $\boldsymbol{j}_i^u$ and $\boldsymbol{j}_i^v$ denote the current $\boldsymbol{j}_i$ (1.12) evaluated on the fields $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ respectively. The integrals in the third line represent the contribution to energy of the soliton $\boldsymbol{\phi}^u$ from region 1 and the leading contribution from region 2. The integrals in the fourth line are the contribution to the energy of $\boldsymbol{\phi}^v$ from region 2 and the leading contribution from region 1. The formulae for subleading contributions are complicated, but it is clear that the sum of all terms which only involve either $\boldsymbol{\phi}^u$ or $\boldsymbol{\phi}^v$ is just $E[\boldsymbol{\phi}^u]+E[\boldsymbol{\phi}^v]$. The cross terms are more interesting. Integrating by parts in the last line of the equation above, and using the fact that both $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ satisfy the Euler-Lagrange equations (1.11), we are only left with a boundary term

$$E[\boldsymbol{\phi}^w] \approx E[\boldsymbol{\phi}^u] + E[\boldsymbol{\phi}^v] + \int_\Gamma (\boldsymbol{j}_i^v\cdot\boldsymbol{\epsilon}^u - \boldsymbol{j}_i^u\cdot\boldsymbol{\epsilon}^v)dS_i. \tag{3.12}$$

Here Γ is a curve without self-intersections separating the region 1 from the region 2 and $dS_i = \varepsilon_{ij}\dot{\gamma}_j dt$ for any parametrisation of $\gamma(t)$ of Γ for which region 1 is on the left and region

2 on the right (ε_{ij} is the antisymmetric tensor in two dimensions normalised so that $\varepsilon_{12} = 1$). We now assume further that Γ lies in a region where both $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ are close to the vacuum and keep only terms which are linear in both $\boldsymbol{\varphi}^u$ and $\boldsymbol{\varphi}^v$. Then, interpreting the difference $E[\boldsymbol{\phi}^w] - E[\boldsymbol{\phi}^u] - E[\boldsymbol{\phi}^v]$ as the potential describing the interaction of the solitons $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$ we find that the leading term V of that potential is given by the simple formula

$$V = \int_\Gamma (\boldsymbol{\varphi}^v \cdot \partial_i \boldsymbol{\varphi}^u - \boldsymbol{\varphi}^u \cdot \partial_i \boldsymbol{\varphi}^v) dS_i. \tag{3.13}$$

To prove (1.14) one needs to show that we can always arrange for V to be negative. Suppose that this is not already the case, and consider the case where V is positive. Then it can be made negative by an iso-rotation by 180° of either $\boldsymbol{\phi}^u$ or $\boldsymbol{\phi}^v$ (which does not change the individual energies of $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^v$). However, the proof is incomplete unless we can rule out that V is zero. This possibility was not considered in [7]. In fact we shall see that this does indeed happen for all relative iso-orientations in a slightly modified version of our model. Thus we can only use the result (3.13) for constructing new multisolitons out of solitons for which the asymptotic field $\boldsymbol{\varphi}$ is known.

4 New Multisolitons from Old

Although we already have enough information about the asymptotic field of a baby Skyrmion to prove the existence of a 2-soliton in our model, it is useful to interpret the asymptotic field further before proceeding with the proof.

For large r the profile function $f^{(1)}$ approaches 0 exponentially fast and we can therefore approximate $\sin f^{(1)} \sim f^{(1)}$ and $\cos f^{(1)} \sim 1$. Using the asymptotic expression (2.11) we write the asymptotic form $\boldsymbol{\varphi}^{(1)}$ of $\boldsymbol{\phi}^{(1)}$ as

$$\boldsymbol{\varphi}^{(1)}(\boldsymbol{x}) = \frac{p\mu}{2\pi}(K_1(\mu r)\cos(\theta - \chi), K_1(\mu r)\sin(\theta - \chi), 0), \tag{4.1}$$

where we have written p for c_1. Alternatively, introducing the orthogonal vectors

$$\boldsymbol{p}_1 = p(\cos\chi, \sin\chi) \qquad \boldsymbol{p}_2 = p(-\sin\chi, \cos\chi) \tag{4.2}$$

and $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$ we can write

$$\varphi_a^{(1)}(\boldsymbol{x}) = \frac{\mu}{2\pi}\boldsymbol{p}_a \cdot \hat{\boldsymbol{x}} K_1(\mu r) = -\frac{1}{2\pi}\boldsymbol{p}_a \cdot \nabla K_0(\mu r) \qquad a = 1, 2. \tag{4.3}$$

Now, given that the Green function of the static Klein-Gordon equation in two dimensions is $K_0(\mu r)$:

$$(\Delta - \mu^2) K_0(\mu r) = -2\pi\delta^{(2)}(\boldsymbol{x}), \tag{4.4}$$

it follows that

$$(\Delta - \mu^2)\varphi_a^{(1)}(\boldsymbol{x}) = \boldsymbol{p}_a \cdot \nabla \delta^{(2)}(\boldsymbol{x}) \qquad a = 1, 2. \tag{4.5}$$

Thus the asymptotic field $\boldsymbol{\varphi}^{(1)}$ may be thought of as produced by a pair of orthogonal dipoles, one for each of the components $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$, in a linear field theory, namely Klein-Gordon theory. The strength of the dipole can be calculated from the asymptotic form of $f^{(1)}$. We find

$$p = 24.16. \tag{4.6}$$

Consider now the set up of section 3, with $\boldsymbol{\phi}^u$ a baby Skyrmion centred at the origin and iso-rotated relative to the standard hedgehog field $\boldsymbol{\phi}^{(1)}$ by χ_1 and $\boldsymbol{\phi}^v$ a second baby Skyrmion iso-rotated relative to the standard hedgehog by an angle χ_2 and centred at $\boldsymbol{R}$, where $R := |\boldsymbol{R}| << 1/\mu$. The asymptotic field of the first baby Skyrmion is

$$\varphi_a^u(\boldsymbol{x}) = -\frac{1}{2\pi}\boldsymbol{d}_a \cdot \nabla K_0(\mu r) \qquad a = 1, 2. \tag{4.7}$$

where

$$\boldsymbol{d}_1 = p(\cos\chi_1, \sin\chi_1) \qquad \boldsymbol{d}_2 = p(-\sin\chi_1, \cos\chi_1), \tag{4.8}$$

and the asymptotic field of the second is

$$\varphi_a^v(\boldsymbol{x}) = -\frac{1}{2\pi}\boldsymbol{p}_a \cdot \nabla K_0(\mu|\boldsymbol{x} - \boldsymbol{R}|) \qquad a = 1, 2, \tag{4.9}$$

where

$$\boldsymbol{p}_1 = p(\cos\chi_2, \sin\chi_2) \qquad \boldsymbol{p}_2 = p(-\sin\chi_2, \cos\chi_2). \tag{4.10}$$

Then, using that, in region 1,

$$(\Delta - \mu^2)\varphi_a^u = \boldsymbol{d}_a \cdot \nabla \delta^{(2)}(\boldsymbol{x}) \quad \text{and} \quad (\Delta - \mu^2)\varphi_a^v = 0 \qquad a = 1, 2 \tag{4.11}$$

and converting the line integral (3.13) into an area integral over region 1 we find the potential for the interaction of two well separated baby Skyrmions

$$\begin{aligned} V_{11} &= \int_1 d^2x \, \boldsymbol{\varphi}^v \cdot (\Delta - \mu^2)\boldsymbol{\varphi}^u \\ &= \sum_{a=1,2} \frac{1}{2\pi}(\boldsymbol{d}_a \cdot \nabla)(\boldsymbol{p}_a \cdot \nabla) K_0(\mu R) \\ &= \frac{p^2}{\pi}\cos\psi \Delta K_0(\mu R) \\ &= \frac{p^2\mu^2}{\pi}\cos\psi K_0(\mu R), \end{aligned} \tag{4.12}$$

where $\psi = \chi_1 - \chi_2$ describes the relative iso-orientation. In the last step we have used (4.4) and have omitted the $\delta^{(2)}$-function term because we are only interested in large separations $R > 1/\mu$. Thus we see that the potential for the interaction of two well separated baby Skyrmions is the same as that calculated in a linear field theory, namely Klein Gordon theory for a pair of scalar fields, for the interaction of two pairs of orthogonal dipoles. The important point is that baby Skyrmions not only act as sources of dipole fields but also react to an external field like a pair of orthogonal dipoles. This result deserves some comments. Firstly, it is instructive to compare it to a similar result in the Skyrme model. There a certain non-linear superposition procedure, the product ansatz, also leads to an interaction potential which has a "linear" interpretation: it describes the interaction between two triplets of mutually orthogonal scalar dipoles in a linear field theory for the pion fields, see [8] and also [9].

Secondly, we are now in a position to explain the caveat at the end of the previous section concerning the use of (3.13) for proving (1.14). In [10] a similar model to ours was studied with the potential term $\mu^2(1 - \boldsymbol{\phi}\cdot\boldsymbol{n})$ replaced by $(1 - \boldsymbol{\phi}\cdot\boldsymbol{n})^4$. In that model (where the mesons are massless) the solitons of degree 1 also have the hedgehog form (2.4) (with a particularly simple profile function) and their asymptotic field can still be interpreted in terms of a pair of dipoles. The Green function of the linearised theory, however, is $-\frac{1}{2\pi}\ln R$. Thus a calculation analogous to ours yields a potential proportional to $\cos\psi\Delta\ln R$ which is identically zero for all values of ψ. Numerically, two solitons of degree one are found to repel each other so that there are no 2-solitons in that model. The example shows that the "linear forces" between solitons, calculated via (3.13), may vanish for all relative iso-orientations. In that situation the inter-soliton forces are entirely due to non-linear effects and cannot be calculated with the methods of the previous section.

Returning to the formula (4.12) we see that V_{11} is negative if one baby Skyrmion is iso-rotated by 180° relative to the other, i.e. if $\psi = \pi$ in the above expression. Thus we conclude that $E_2 < 2E_1$. In fact one finds that already the minimum of the energy amongst hedgehog fields of degree 2 is less than $2E_1$. We write $\boldsymbol{\phi}^{(2)}$ for the hedgehog field (2.4) with $n = 2$ in standard orientation ($\chi = 0$) whose profile function satisfies (2.6),(2.8) and (2.5) with $m = 1$. For its energy, or mass, we find $E_2 = 2.936\cdot 4\pi$. We have again checked the stability of $\boldsymbol{\phi}^{(2)}$ against more general perturbations numerically and conclude that it minimises the energy amongst all fields of degree 2. Its profile function is plotted in figure 1.a) and its energy density in figure 1.b). Note that the maximum of the energy density is not at the origin but at $r \approx 1.8$. This is again reminiscent of the Skyrme model, where the energy of the static solution of degree 2 is concentrated in a toroidal region [11]. The field $\boldsymbol{\phi}^{(2)}$ is shown in figure 2.b).

Before we can use the 2-soliton as an input for the construction of higher multisolitons we need to understand the asymptotic form $\boldsymbol{\varphi}^{(2)}$ of the field $\boldsymbol{\phi}^{(2)}$. Since the 2-soliton is of the hedgehog form this is not difficult. The asymptotic field is now

$$\boldsymbol{\varphi}^{(2)} = \frac{q\mu^2}{2\pi}(K_2(\mu r)\cos(2\theta - \chi), K_2(\mu r)\sin(2\theta - \chi), 0) \tag{4.13}$$

and can be expressed in terms of the quadrupole moments

$$\boldsymbol{q}_1 = q(\cos\chi, \sin\chi) \qquad \boldsymbol{q}_2 = q(-\sin\chi, \cos\chi) \tag{4.14}$$

and the second order differential operator $\boldsymbol{D} = (\partial_1^2 - \partial_2^2, 2\partial_1\partial_2)$ via

$$\varphi_a^{(2)} = \frac{1}{2\pi}\boldsymbol{q}_a\cdot\boldsymbol{D}K_0(\mu r) \qquad a = 1, 2. \tag{4.15}$$

Then, using again (4.4), it follows that

$$(\Delta - \mu^2)\varphi_a^{(2)} = -\boldsymbol{q}_a\cdot\boldsymbol{D}\delta^{(2)}(\boldsymbol{x}). \tag{4.16}$$

Thus we may think of the asymptotic field $\boldsymbol{\varphi}^{(2)}$ as being due to a pair of orthogonal quadrupoles (in two dimensions all multipoles have two real components). For the strength of the quadrupole we find

$$q = 53.6. \tag{4.17}$$

Now consider the superposition of a baby Skyrmion at the origin and iso-rotated relative to the standard hedgehog by χ_1 and a 2-soliton centred at $\boldsymbol{R}$, $R >> 1/\mu$, and iso-rotated relative to the standard hedgehog $\boldsymbol{\phi}^{(2)}$ by χ_2. Thus the asymptotic field and the dipole moments of the baby Skyrmion are as in (4.7) and (4.8) and the asymptotic field of the second is

$$\varphi_a^v(\boldsymbol{x}) = \frac{1}{2\pi}\boldsymbol{q}_a\cdot\boldsymbol{D}K_0(\mu|\boldsymbol{x} - \boldsymbol{R}|) \qquad a = 1, 2, \tag{4.18}$$

where

$$\boldsymbol{q}_1 = q(\cos\chi_2, \sin\chi_2) \qquad \boldsymbol{q}_2 = q(-\sin\chi_2, \cos\chi_2). \tag{4.19}$$

Then, inserting these expressions into the general formula (3.13) and using (4.11) we find the potential V_{12} describing the interaction between a baby Skyrmion and a 2-soliton :

$$\begin{aligned} V_{12} &= \sum_{a=1,2}\frac{1}{2\pi}(\boldsymbol{d}_a\cdot\nabla)(\boldsymbol{q}_a\cdot\boldsymbol{D})K_0(\mu R) \\ &= \frac{pq}{2\pi}(\cos\psi\,\partial_1 + \sin\psi\,\partial_2)\Delta K_0(\mu R) \\ &= -\frac{pq\mu^3}{2\pi}\cos(\psi - \vartheta)K_1(\mu R), \end{aligned} \tag{4.20}$$

where (R, ϑ) are polar coordinates for the relative position vector $\boldsymbol{R}$ and ψ is defined as before. Thus the interaction between a baby Skyrmion and a 2-soliton depends both on their relative iso-orientation and on their relative position in physical space. By choosing $\psi - \vartheta = 0$ we can again arrange for the potential to be negative, so we expect there to be a 3-soliton solution in our model. Before describing that solution in the next section we calculate the potential V_{22} for the interaction between two 2-solitons.

Thus we look at the superposition of a 2-soliton at the origin and iso-rotated relative to the standard hedgehog field by χ_1 and a second 2-soliton centred at $\boldsymbol{R}$ and iso-rotated relative to the standard hedgehog field by χ_2. For the former the quadrupole moments are

$$\boldsymbol{e}_1 = q(\cos\chi_1, \sin\chi_1) \qquad \boldsymbol{e}_2 = q(-\sin\chi_1, \cos\chi_1) \tag{4.21}$$

and for the latter the quadrupole moments are $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ as defined in (4.19). Thus, the interaction potential is

$$\begin{aligned} V_{22} &= \sum_{a=1,2} \frac{1}{2\pi}(\boldsymbol{e}_a \cdot \boldsymbol{D})(\boldsymbol{q}_a \cdot \boldsymbol{D}) K_0(\mu R) \\ &= -\frac{q^2}{\pi}\cos\psi \Delta^2 K_0(\mu R) \\ &= -\frac{q^2\mu^4}{\pi}\cos\psi K_0(\mu R). \end{aligned} \tag{4.22}$$

Thus V_{22} has the opposite sign from V_{11}, but the same functional dependence on the separation R and the relative iso-orientation ψ. This deserves a comment, as it may seem surprising that the dipole-dipole potential V_{11} does not necessarily dominate over the quadrupole-quadrupole potential V_{22} at large R. However, it is characteristic of linear field theories with an exponentially decaying Green function that the leading term in a field produced by an n-pole is independent of n. This observation will be important for us later, when we study multisolitons whose fields are not of the simple hedgehog form; it means that all multipoles in the expansion of the asymptotic field are potentially equally important when studying the interaction of such multisolitons.

For now the most important feature of V_{22} is that it is negative when $\psi = 0$. Thus we conclude $E_4 < E_2 + E_2$. However, without further insights into the properties of the 3-soliton we cannot say anything about the relative size of E_4 and $E_1 + E_3$.

5 Numerical Results for Higher Multisolitons

There are a number of ways to search numerically for stationary points of energy functionals and to check whether a given stationary point is a local minimum. It is very hard, however, to ascertain whether local minima found in this way are global minima. We are similarly not able to prove that the configurations to be described in this section are multisolitons in the strict sense of the definition at the end of section 1. Instead we will consider a numerically found configuration to be a multisoliton if it satisfies certain numerical checks for a local minimum and the inequality (1.14).

Here we look for stationary points of the energy functional E by solving a suitable time-dependent equation which reduces to (1.11) in the static limit. The time evolution according to that equation should stop at stationary points of E. The equation we use is the Lorentz covariant equation

$$\partial^\alpha(\boldsymbol{\phi} \times \partial_\alpha \boldsymbol{\phi} - \partial^\beta \boldsymbol{\phi}(\partial_\beta \boldsymbol{\phi} \cdot \boldsymbol{\phi} \times \partial_\alpha \boldsymbol{\phi}) = \mu^2 \boldsymbol{\phi} \times \boldsymbol{n}, \tag{5.1}$$

where $\boldsymbol{\phi}$ is now a function of $x^\alpha = (t, \boldsymbol{x})$ and the indices $\alpha, \beta = 0, 1, 2$ are raised and lowered with the Lorentzian metric $\mathrm{diag}(1, -1, -1)$, with an added friction term to absorb the kinetic energy. To solve the resulting equation numerically we use a finite difference scheme to evaluate the space derivatives and integrate the time evolution using the 4th order Runge-Kutta method. We use a square grid of 200×200 points and set the time increment dt to half the length of a lattice site. Most of our simulations are performed on a grid extending in both the x^1 and x^2 direction from -20 to 20.

As the initial configuration we take a particular stationary point of E, namely the hedgehog field (2.4) of degree n with the profile function satisfying the boundary condition (2.5) for $m = 1$ and the ordinary differential equation (2.8). We then add a small perturbation which breaks the symmetry of the hedgehog. We have already noted that the hedgehog fields for $n = 1$ and $n = 2$ are stable against such perturbations. We have also investigated the cases $3 \leq n \leq 6$: in those cases the hedgehog fields are unstable and the time evolution ends at less symmetric configurations with lower energy. These final configurations are stable with respect to further perturbations, and their energies satisfy the inequality (1.14) so we take them to be multisolitons.

From the energies E_n of those multisolitons we calculate the "ionisation energies" Δ_{kl} defined via

$$\Delta_{kl} := E_k + E_l - E_n, \text{where} \quad 1 \leq k, l \leq n, \quad k + l = n. \tag{5.2}$$

In table 1 we summarise our results. All the energy values listed in the table are obtained by integrating the solitons' energy density over the grid. The resulting values for E_1 and E_2 are 1% smaller than the more accurate values given earlier in the paper, which were obtained by solving the ordinary differential equation (2.8) and integrating (2.7). For the calculation of the ionisation energies, however, it is important that all energies are calculated in the same way.

n	$E_n/4\pi$	$k+l$	$\Delta_{kl}/4\pi$
1	1.549	-	-
2	2.907	$1+1$	0.191
3	4.379	$2+1$	0.077
4	5.800	$3+1$ $2+2$	0.128 0.014
5	7.282	$4+1$ $3+2$	0.068 0.005
6	8.693	$5+1$ $4+2$ $3+3$	0.138 0.015 0.066

Table 1

Multisoliton energies and ionisation energies

The results for $n \geq 3$ deserve a more detailed discussion. For $n = 3$ our numerical procedure leads to the configuration displayed in figure 3.a). A plot of the energy density e (1.7) as a function of position is shown in figure 3.b). Unlike the solutions discussed so far the energy of the 3-soliton is not rotationally symmetric. Instead the configuration is like a linear molecule made up of three (distorted) baby Skyrmions aligned so that any two neighbours are in the most attractive relative orientation.

In principle it is still possible to analyse the asymptotic field of the 3-soliton in terms of multipole moments. In practice the absence of a continuous symmetry makes the analysis hard and we know from the previous section that we may have to consider many multipole moments. We have therefore made no attempt at deriving the potential for the interaction between 3-solitons and other multisolitons in our model and rely on numerical evidence for showing the existence of higher multisolitons.

The field and the energy density for the 4-soliton are plotted in figure 4. The 4-soliton is again like a linear molecule, but this time made up of two 2-solitons. The plot of the field shows that the two 2-solitons have the same iso-orientation, as expected from our discussion of the potential V_{22}. The picture suggests, and table 1 confirms, that it costs very little energy to break the 4-soliton into 2-solitons.

The field for the 5-soliton is the least symmetric of all the multisolitons we have studied. The field and its energy density are shown in figure 5. The 5-soliton consists of an almost undistorted 2-soliton and 3-soliton close together. Table 1 shows that the binding between those constituents is very weak. To check whether the binding is in fact a boundary effect we have therefore repeated the simulation on a grid of 250×250 points extending in x^1 and x^2 from -25 to 25. The result is identical to that of the first simulation.

Finally the 6-soliton is made up of three 2-solitons centred at the vertices of an equilateral triangle. The plot of the field in figure 6.a) reveals that the 2-solitons all have the same iso-orientation, so that the interaction energy between any two 2-solitons is minimised. Although the binding is again quite weak, the distortion of the individual 2-solitons as a result of their interaction is clearly visible in the plot of the energy density in figure 6.b).

The pictures of the energy density for $n \geq 4$ suggests that the 2-soliton (and to a lesser extent the 3-soliton) serves as a basic building block for higher multisolitons. Table 1 provides further evidence for this observation: the largest ionisation energy in table 1 is Δ_{11}, showing that the 2-soliton is most strongly bound, and the ionisation energy Δ_{kl} for $n = k + l > 2$ is least if $k = 2$ or $l = 2$, showing that it is easiest to break up multisolitons in a way that produces at least one 2-soliton.

6 Conclusions

We have studied multisoliton solutions in a non-linear field theory which may be interpreted physically as a two-dimensional version of the Skyrme model for nuclear physics. Our main analytical result is the expression (3.13) for the leading term in the potential describing the interaction of two well separated solitons in terms of the asymptotic fields of the solitons. In those cases where the asymptotic field of the solitons is known the leading term could be written down explicitly in terms of the multipoles associated with the solitons and the Green function of the linearised theory. The resulting formula can then be used to prove the existence of multisolitons of higher degree.

Using numerical methods we could explicitly display multisolitons of degree $1 \leq n \leq 6$. It turns out that in our model solitons of degree 1 and 2 are invariant under a $SO(2)$ action, but that higher multisolitons are only invariant under the action of finite groups. This is rather reminiscent of the Skyrme model where solitons of degree 1 and 2 have continuous symmetries [11], but higher multisolitons have only discrete symmetries, see [12] and also [13].

The relationship between solitons and the associated multipoles is intriguing and deserves further study. It is quite possible that one can prove general statements about the multipole expansion of the asymptotic field of an n-soliton for arbitrary n which would allow one to complete the proof of the inequality (1.14) for general k, l and n. This would be of wider interest since the method described in section 3 applies to any field theory with soliton solutions provided there is some sort of superposition procedure for well-separated solitons and the linearisation of the theory is of a suitable form.

Finally it would be interesting to see to what extent the multipole description can be used to understand the dynamical properties of the solitons in our model. The form of the potential V_{11} shows that the forces between two well separated baby Skyrmions, like those between Skyrmions, depend both on the relative separation and the relative orientation. This suggests that the interactive dynamics of two baby Skyrmions is more complicated than that of other topological solitons in two dimensions, such as $\mathbf{CP}^1$ lumps [14] or the solitons studied in [10], but also possibly more relevant to Skyrmion dynamics in three dimensions. We are presently investigating this point and will report on it elsewhere [15].

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Figure Captions

Figure 1

a) Profile functions for the baby Skyrmion ($n = 1$) and the 2-soliton ($n = 2$).

b) Energy densities e (1.7) as a function of r for the baby Skyrmion ($n = 1$) and the 2-soliton ($n = 2$).

Figure 2

a) Plot of the field $\boldsymbol{\phi}^{(1)}$. At every lattice site in physical space we plot an arrow whose direction and magnitude is that of $(\phi_1^{(1)}, \phi_2^{(1)})$ (we identify the axis in the target space S^2 with those in physical space). At the base of the arrow we put a '+' if $\phi_3^{(1)}$ is positive and a '×' if $\phi_3^{(1)}$ is negative. Thus the vacuum is represented simply by a '+'. The labels x and y refer to the first and second component of the vector $\boldsymbol{x}$.

b) Plot of the field $\boldsymbol{\phi}^{(2)}$ using the same conventions as in a).

Figure 3

a) Field of the 3-soliton; conventions as for figure 2.a).

b) Energy density of the 3-soliton in the range $-10 \leq x, y \leq 10$.

Figure 4

a) Field of the 4-soliton; conventions as for figure 2.a).

b) Energy density of the 4-soliton in the range $-10 \leq x, y \leq 10$.

Figure 5

a) Field of the 5-soliton; conventions as for figure 2.a).

b) Energy density of the 5-soliton in the range $-10 \leq x, y \leq 10$.

Figure 6

a) Field of the 6-soliton; conventions as for figure 2.a).

b) Energy density of the 6-soliton in the range $-10 \leq x, y \leq 10$.