hep-th/9603101
ITF-96-05

The spectrum of Bogomol'nyi solitons in gauged linear sigma models

B.J. Schroers[†]

Institute for Theoretical Physics
University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam
The Netherlands

Abstract

Gauged linear sigma models with $\mathbf{C}^m$-valued scalar fields and gauge group $U(1)^d$, $d \leq m$, have soliton solutions of Bogomol'nyi type if a suitably chosen potential for the scalar fields is also included in the Lagrangian. Here such models are studied on $(2+1)$-dimensional Minkowski space. If the dynamics of the gauge fields is governed by a Maxwell term the appropriate potential is a sum of generalised Higgs potentials known as Fayet-Iliopoulos D-terms. Many interesting topological solitons of Bogomol'nyi type arise in models of this kind, including various types of vortices (e.g. Nielsen-Olesen, semilocal and superconducting vortices) as well as, in certain limits, textures (e.g. $\mathbf{CP}^{m-1}$ textures and gauged $\mathbf{CP}^{m-1}$ textures). This is explained and general results about the spectrum of topological defects both for broken and partially broken gauge symmetry are proven. When the dynamics of the gauge fields is governed by a Chern-Simons term instead of a Maxwell term a different scalar potential is required for the theory to be of Bogomol'nyi type. The general form of that potential is given and a particular example is discussed.

May 1996

[†] e-mail: `schroers@phys.uva.nl`

1. Introduction

In the study of topological solitons, field theories of Bogomol'nyi type occupy a special place. Mathematically such field theories are characterised by the fact that their soliton solutions may be obtained by solving first order differential equations (called the Bogomol'nyi equations) instead of the second order Euler-Lagrange equations. A further characteristic property is the existence of static multisoliton solutions made up of arbitrarily placed single solitons. Physically one may think of this property as reflecting the absence of static forces between well-separated single solitons. It follows that, in theories of Bogomol'nyi type, the space of static multisoliton solutions is larger and more interesting than in generic field theories with topological solitons. Defining the moduli spaces M_N to be the spaces of static soliton solutions of soliton number N, one finds for example that in theories of Bogomol'nyi type the dimension of M_N increases linearly with N. Starting with the proposal by Manton [1] in the context of magnetic monopoles, moduli spaces have become a powerful tool for investigating the physics of several interacting solitons in theories of Bogomol'nyi type . This program is particularly advanced in the case of vortices in the abelian Higgs model. Here moduli space techniques have been used successfully to study the interacting dynamics of several vortices [2] and, more recently, the statistical mechanics of vortices [3].

The purpose of this paper is to explore a certain class of (2+1)-dimensional field theories with topological solitons of Bogomol'nyi type which provide a natural generalisation of the abelian Higgs model. In two spatial dimensions, it is useful to distinguish two types of topological solitons: vortices and textures (depending on the context the latter are also often called lumps or baby-Skyrmions). Generally we may distinguish the two by the origin of their topological stability. If the stability is due to a non-trivial first homotopy group of the vacuum manifold we speak of vortices, if it is due to a non-trivial second homotopy group of the vacuum manifold we speak of textures (though care has to be taken in interpreting these statements, as first pointed out in ref. [4] and as we shall also see later). More specifically, vortices arise in field theories with charged scalar fields when a $U(1)$ gauge symmetry is spontaneously broken, whereas textures typically arise in non-linear sigma models. Typical and much studied examples of such textures are the lumps in $\mathbf{CP}^m$ sigma models.

Remarkably, both vortices and textures can be studied in a unified manner in the framework of gauged linear sigma models (GLSM's) with a judiciously chosen Higgs potential. The special potentials which are required here are known to supersymmetry theorists as Fayet-Iliopoulos D-terms, and GLSM's with these potentials have been studied intensively in recent years in the context of topological sigma models and string theory, see for example refs. [5][6]. It has also been emphasised in that context that, in suitable limits, GLSM's with Fayet-Iliopoulos D-terms become non-linear sigma models whose target spaces are certain special manifolds called toric varieties (of which the projective spaces $\mathbf{CP}^m$ are particular examples). Thus there exists a unifying framework for studying textures and vortices of Bogomol'nyi type, with a host of beautiful mathematical results, whose implications for the study of topological defects in (2+1) dimensional field theories do not appear to have been fully exploited.

However, while there is significant overlap between the questions studied by string theorists and those which one might ask in the context of (2+1)-dimensional field theories, there are also important differences. While in the context of topological sigma models and string theory GLSM's have typically been studied on compact Riemann surfaces, the field theorist would naturally study the models on flat Minkowski space. Thus, in the (2+1)-dimensional case static fields are defined on (non-compact) $\mathbf{R}^2$, and it is natural also to consider time-dependent fields. This leads to additional questions. On a non-compact domain the convergence of integrals needs to be checked carefully. This results in extra constraints which affect the spectrum of Bogomol'nyi solitons. The inclusion of the time coordinate allows one to study time-dependent fields and to consider gauge fields whose dynamics is governed by a Chern-Simons term. The latter is physically interesting because topological solitons in such theories display anyonic statistics.

This paper begins with two general sections in which GLSM's are introduced and general results about the spectrum of Bogomol'nyi solitons are derived. Most of the algebraic manipulations in these sections have appeared at various points in the literature, but the parts of the analysis which deal with the non-compactness of $\mathbf{R}^2$ appear to be new. In sect. 4 we show how the non-linear $\mathbf{CP}^1$ sigma model can be understood in a very precise way as a limiting case of the semilocal vortex model in the context of GLSM's, and we explain how this limiting procedure can be generalised. Our contribution in this section is mainly to link work done in the study of topological defects with the mathematical framework of toric varieties. In sect. 5 we study in detail a family of models with two complex scalar fields and gauge group $U(1) \times U(1)$. We show that the topological solitons in this family include, for various parameter values and in various limits, superconducting vortices and gauged $\mathbf{CP}^1$ lumps. The latter are solitons in the gauged $O(3)$ sigma model introduced in [7]. The present paper developed out of an attempt better to understand the mathematical structure and physical interpretation of that model, and we will see that the framework of GLSM's with Fayet-Iliopoulos D-terms provides a satisfactory understanding of both. In sect. 6 we write down a general Lagrangian for GLSM's of Bogomol'nyi type with Chern-Simons terms for the gauge fields. Again we illustrate the general results in a particular model with two complex scalar fields. Finally, the purpose of sect. 7 is to draw together the rather diverse viewpoints which enter this paper and to highlight some open questions.

2. Gauged linear sigma models

Perhaps the most natural way to introduce GLSM's with Fayet-Iliopoulos D terms without invoking supersymmetry is to recall the manipulations that lead to the establishment of the Bogomol'nyi equations for the abelian Higgs model. The basic fields in the abelian Higgs model are a complex scalar and a $U(1)$ gauge field, but as we shall see presently Bogomol'nyi equations can still be established, *mutatis mutandis*, if we consider instead a $\mathbf{C}^m$-valued scalar field w and gauge any subgroup of the maximal torus of the unitary group $U(m)$ acting on w in the fundamental representation. The maximal torus of $U(m)$ is m-dimensional and a choice of generators t_a, $a = 1, ...m$, defines an isomorphism between it and $U(1)^m$. Concretely we take the generators to be diagonal $m \times m$ matrices with integer entries, so we can write in components

$$t^a_{\alpha\beta} = Q^a_\alpha \delta_{\alpha\beta} \qquad (\alpha, \beta = 1, ..., m, \text{ no sum over } \alpha). \tag{2.1}$$

Introducing a $U(1)$ gauge fields A^a_μ for each of the generators we define the covariant derivative

$$D_\mu w = \partial_\mu w + i \sum_{a=1}^{m} t_a A^a_\mu \, w, \tag{2.2}$$

and the curvatures

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu. \tag{2.3}$$

Here the space-time indices $\mu, \nu, ..$ run over $\{0, 1, 2\}$. In the following we shall exclusively work on (2+1)-dimensional Minkowski space with signature $(-, +, +)$, whose points we denote by x^μ. Occasionally we also use latin indices $i,j,k... \in \{1,2\}$ to label the spatial components of x^μ, or polar coordinates (ρ,θ) for the spatial vector (x_1,x_2). Finally we should define a suitable "Higgs potential" for each generator t_a of the gauge group $U(1)^m$. The appropriate potential turns out to be the Fayet-Iliopoulos D-term $(R_a - w^\dagger t_a w)^2$ [8], where R_a is a parameter of dimension mass. There are a number of ways to think about this term, but we only point out as an aside that in terms of the symplectic geometry of $\mathbf{C}^m$, with symplectic form $dw^\dagger \wedge dw$, $H_a = R_a - w^\dagger t_a w$ is a Hamiltonian for the $U(1)$ action on $\mathbf{C}^m$ generated by t_a. The interested reader is referred to the book [9] for more details on this point of view.

Thus we can write down the general Lagrangian density which is the main subject of this paper. In natural units $c = \hbar = 1$ it reads

$$\mathcal{L} = -\frac{1}{2}(D_\mu w)^\dagger D^\mu w - \sum_{a=1}^m \frac{1}{4e_a^2}(F^a_{\mu\nu})^2 - \sum_{a=1}^m \frac{e_a^2}{8}(R_a - w^\dagger t_a w)^2. \tag{2.4}$$

It depends on m coupling constants e_a of dimension $(\text{mass})^{\frac{1}{2}}$, and on the m parameters R_a. The dependence of the theory on these parameters is one of our main interests. Note in particular that we can trivially eliminate any of the gauge fields A^a_μ by setting them and the corresponding coupling constants e_a to zero.

We are mostly interested in static fields, for which the energy functional has the form

$$E = \frac{1}{2}\int d^2x\, (D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + \sum_{a=1}^m \frac{1}{e_a^2}(F^a_{12})^2 + \sum_{a=1}^m \frac{e_a^2}{4}(R_a - w^\dagger t_a w)^2. \tag{2.5}$$

To ensure that the energy of a configuration is finite we also impose the boundary conditions

$$\lim_{\rho\to\infty}(R_a - w^\dagger t_a w) = 0 \tag{2.6}$$

$$\lim_{\rho\to\infty} D_i w = 0. \tag{2.7}$$

Below we will state more precisely how quickly these limits should be attained. For now we proceed, assuming that the decay of the fields at infinity is fast enough to justify the following manipulations. Thus using the algebraic identity

$$\begin{aligned} E = &\frac{1}{2}\int d^2x |(D_1 \pm iD_2)w|^2 + \sum_{a=1}^m |\frac{1}{e_a}F^a_{12} \pm \frac{e_a}{2}(R_a - w^\dagger t_a w)|^2 \\ &\mp \frac{1}{2}\int d^2x\, i(D_1 w)^\dagger D_2 w - i(D_2 w)^\dagger D_1 w + \sum_{a=1}^m F^a_{12}(R_a - w^\dagger t_a w) \end{aligned} \tag{2.8}$$

and integrating by parts we deduce

$$E = \frac{1}{2} \int d^2x \, |(D_1 \pm iD_2)w|^2 + \sum_{a=1}^{m} |\frac{1}{e_a} F_{12}^a \pm \frac{e_a}{2}(R_a - w^\dagger t_a w)|^2 \\ \mp \frac{1}{2} \sum_{a=1}^{m} R^a \int d^2x \, F_{12}^a, \tag{2.9}$$

where we have also used that

$$w^\dagger (D_1 D_2 - D_2 D_1) w = i \sum_{a=1}^{m} w^\dagger t_a w \, F_{12}^a. \tag{2.10}$$

Finally defining suitably normalised magnetic fluxes

$$\Phi_a = -\frac{1}{2\pi} \int d^2x \, F_{12}^a \tag{2.11}$$

and

$$T = \sum_{a=1}^{m} R_a \Phi_a \tag{2.12}$$

we deduce the inequality

$$E \geq \pi |T|. \tag{2.13}$$

More precisely we have

$$E = \pi T \tag{2.14}$$

if and only if the Bogomol'nyi equations hold

$$(D_1 + iD_2)w = 0 \tag{2.15a}$$

$$F_{12}^a + \frac{e_a^2}{2}(R_a - w^\dagger t_a w) = 0, \tag{2.15b}$$

and

$$E = -\pi T \tag{2.16}$$

if and only if the (anti)-Bogomol'nyi equations hold

$$(D_1 - iD_2)w = 0 \tag{2.17a}$$

$$F_{12}^a - \frac{e_a^2}{2}(R_a - w^\dagger t_a w) = 0. \tag{2.17b}$$

Since we are working on non-compact $\mathbf{R}^2$ the fluxes Φ_a are not necessarily integers and do not, in general, have a clear topological meaning. They are still useful labels for solutions of the Bogomol'nyi equations because they are conserved if one rules out infinite energy configurations. This follows from Faraday's law of induction

$$\frac{d\Phi_a}{dt} = -\frac{1}{2\pi} \oint_C E_i^a dx_i, \tag{2.18}$$

where the contour C is the circle at spatial infinity and $E_i^a = F_{0i}^a$ is the electric field of the a-th gauge field. The integral on the right hand side is only non-zero if the electric field falls off for large ρ no faster than $1/\rho$, which is precisely the condition for the electric field to have infinite energy. Since the quantity T is defined in terms of the fluxes Φ_a it thus appears that T, while conserved, also does not have a clear topological significance. In fact T is a linear combination of topological quantities, as we shall now show after introducing some notation.

Thus we define the gauge potentials and curvatures

$$a_\alpha = \sum_{a=1}^{m} Q_{a\alpha} A_a \tag{2.19a}$$

$$f_{12}^\alpha = \sum_{a=1}^{m} Q_{a\alpha} F_{12}^a, \tag{2.19b}$$

and we introduce the set of parameters r_α through

$$R_a = \sum_{\alpha=1}^{m} Q_{a\alpha} r_\alpha \tag{2.20}$$

(note that the matrix $\mathbf{Q}$ with entries $Q_{a\alpha}$ is invertible by virtue of the generators t_a being independent). The parameters r_α are convenient for discussing the spontaneous symmetry breaking in this model. The potential

$$\sum_{a=1}^{m} \frac{e_a^2}{8} (R_a - w^\dagger t_a w)^2 \tag{2.21}$$

vanishes if and only if $w^\dagger t_a w = R_a$ for all a, or equivalently if

$$|w_\alpha|^2 = r_\alpha, \qquad \alpha = 1, ..., m. \tag{2.22}$$

Thus, in order to have any chance of finding finite energy solutions we must require

$$r_\alpha \geq 0, \qquad \alpha = 1, ..., m. \tag{2.23}$$

Further we see that the $U(1)$ gauge group which rotates the phase of w_α for some given $\alpha \in \{1,...,m\}$ is spontaneously broken if $r_\alpha > 0$ but unbroken if $r_\alpha = 0$. By the mechanism familiar from the abelian Higgs model, the corresponding fluxes

$$\varphi_\alpha = -\frac{1}{2\pi}\int d^2x\, f_{12}^\alpha \tag{2.24}$$

are quantised in the case of broken gauge symmetry, but can take any value in the unbroken case. More specifically, if $r_\alpha > 0$ we deduce from the boundary condition (2.7) that

$$\varphi_\alpha = \frac{1}{2\pi}\oint_C dx_i\, a_i^\alpha = \frac{1}{2\pi i}\oint_C d\theta\, \frac{\partial}{\partial\theta}\ln w_\alpha = n_\alpha, \tag{2.25}$$

where n_α is the winding number of $\lim_{\rho\to\infty} w_\alpha$ viewed as a map from the circle C at spatial infinity to the circle defined by $|w_\alpha|^2 = r_\alpha$. By a standard topological argument [10] the winding number n_α also equals the number of zeros of the scalar field w_α counted with multiplicity (which may be positive or negative).

Finally writing T in terms of the fluxes φ_α yields the promised topological interpretation:

$$T = \sum_{\alpha=1}^{m} r_\alpha \varphi_\alpha. \tag{2.26}$$

Thus T only depends on those fluxes φ_α for which the corresponding vacuum expectation value r_α is non-vanishing. It is therefore a (positive, real) linear combination of winding numbers. A corollary of the topological interpretation of T is that solutions of the Bogomol'nyi equations satisfy the static Euler-Lagrange equations derived from (2.4), as one can also check by direct computation.

To end this section we note some similarities between the discussion in the previous paragraph and the mass formula for BPS monopoles in Yang-Mills-Higgs theory with gauge group $SU(m+1)$, see [11] and references therein. Like the expression for T the mass of a BPS monopole is given by a linear combination of m (= rank $SU(m+1)$) magnetic fluxes. The coefficients in this linear combination depend on the the asymptotic value of the Higgs field and therefore on the symmetry breaking pattern. If the gauge symmetry is maximally broken to $U(1)^m$ all magnetic fluxes have a topological interpretation in terms of elements of $\Pi_2\left(SU(m+1)/U(1)^m\right) = \mathbf{Z}^m$ and the BPS mass formula depends on all of them. For less then maximal symmetry breaking only some (but at least one) of the fluxes have a topological interpretation, but the others remain useful labels for BPS monopoles and are usually called magnetic weights. The mass of a BPS monopole, like the mass of the vortices discussed in this paper, depends only on those magnetic fluxes which have a topological significance.

3. The Bogomol'nyi equations on $\mathbf{R}^2$

For the rest of this paper we focus on the Bogomol'nyi equations (2.15a) and (2.15b). We are interested in finite energy solutions of these equations, and we need to to specify more carefully the boundary condition which ensure that the energy and flux integrals written down in the previous section converge. For this purpose it is convenient to introduce the complex notation $z = x_1 + ix_2$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $a_\alpha = \frac{1}{2}(a_1^\alpha - ia_2^\alpha)$. Then defining the matrix $M_{\alpha\beta} = \sum_{a=1}^m e_a^2 Q_{a\alpha} Q_{a\beta}$ the Bogomol'nyi equations (2.15a) and (2.15b) become

$$(\partial_{\bar z} + i\bar a_\alpha) w_\alpha = 0 \tag{3.1a}$$

$$f_{12}^\alpha + \frac{1}{2}\sum_{\beta=1}^m M_{\alpha\beta}(r_\beta - |w_\beta|^2) = 0. \tag{3.1b}$$

These two first order equations imply one second order equation for w_α as follows. At points where $w_\alpha \neq 0$ the first of the above equations is equivalent to

$$\bar a_\alpha = i\partial_{\bar z} \ln w_\alpha, \tag{3.2}$$

which in turn implies

$$f_{12}^\alpha = \frac{1}{2}\Delta \ln |w_\alpha|^2. \tag{3.3}$$

To extend this equation to the whole plane one uses the $\bar\partial$-Poincaré lemma in a standard fashion, see e.g. [10]. Supposing that w_α has zeros at $\{z_{\alpha,s} | s = 1, ..., n_\alpha\}$ and using that in two dimensions

$$\Delta \ln |z - z_s|^2 = 4\pi\delta(z - z_s) \tag{3.4}$$

one concludes

$$f_{12}^\alpha = \frac{1}{2}\Delta \ln |w_\alpha|^2 - 2\pi \sum_{s=1}^{n_\alpha} \delta(z - z_{\alpha,s}). \tag{3.5}$$

Combining this with (3.1b) we arrive at the promised second order equation for w_α:

$$\Delta \ln |w_\alpha|^2 + \sum_{\beta=1}^m M_{\alpha\beta}(r_\beta - |w_\beta|^2) = 4\pi \sum_{s=1}^{n_\alpha} \delta(z - z_{\alpha,s}). \tag{3.6}$$

This equation should be thought of as a generalised vortex equation, in the sense that in the case $m = 1$ and $t_1 = 1$ it reduces to the equation

$$\Delta \ln |w|^2 + e^2(r - |w|^2) = 4\pi \sum_{s=1}^N \delta(z - z_s) \tag{3.7}$$

(we have omitted the label $\alpha = 1$ and written $N = n_1$), which is the key to the study of Nielsen-Olesen vortices in the abelian Higgs model at critical coupling. As mentioned in the introduction many of the questions we will address in this paper are motivated by what is known about the abelian Higgs model; it may thus be useful briefly to summarise some salient results.

The first question that one needs to settle is that of existence and uniqueness of solutions. In the case of (3.7) this question is completely answered in [10]. It is shown there that no finite-energy solutions exist if $r \leq 0$, but that for $r > 0$, (3.7) has a unique finite energy solution for each integer $N \geq 0$ and given zeros z_s, $s = 1, ..., N$. It follows that for given $N \geq 0$ (and thus fixed energy) the Bogomol'nyi equations in the abelian Higgs model have an N complex parameter family of solutions, the N complex parameters determining the positions of the zeros of the w-field up to permutations. This parameter space is called the moduli space of N-vortices. The moduli space is in fact a differentiable manifold and has a natural Riemannian metric, inherited from the field theory kinetic energy functional. This metric is crucial in the so-called moduli space to vortex dynamics. In that scheme the interacting dynamics of N vortices is approximated by geodesic motion on the moduli space of N-vortices. In [2] it is shown that the Bogomol'nyi property of the field theory implies that the metric on the moduli space is Kähler. The Kähler property together with symmetries is then used to compute the metric explicitly in the case $N = 2$; by studying the geodesics of this metric much can be learnt about the interaction of two vortices. The Kähler property is also crucial in the study of statistical mechanics of vortices on Riemann surfaces in ref. [3].

The example of vortices in the abelian Higgs model shows paradigmatically how the Bogomol'nyi property is the key to understanding difficult and interesting aspects of vortex physics, such as vortex interactions and statistical mechanics of vortices. The family of theories defined in this and the previous section also have soliton solutions of Bogomol'nyi type whose physical properties can be studied with similar methods. In this paper we will not attempt to do this in full generality but instead go through some of the steps outlined above for a few specific models. However, to end this general section we show how one can deduce some general information about the spectrum of Bogomol'nyi solitons directly from (3.6).

In particular we can now state and partly prove necessary conditions for finite energy solutions of the Bogomol'nyi equations. The energy of solutions of the equations (3.1a)

and (3.1b) attains the Bogomol'nyi bound πT, with T given by (2.26). It thus follows from (3.1b) that all fluxes and hence the energy are well-defined if $(r_\alpha - |w_\alpha|^2)$ is integrable over $\mathbf{R}^2$ for all α.

If $r_\alpha > 0$ for *all* α the gauge symmetry is completely broken and the analysis of the finite energy condition is straightforward. In this case all scalar and gauge fields approach their vacuum values exponentially as can be seen by linearising the equations (3.6) for large ρ and small $\epsilon_\alpha := |w_\alpha|^2 - r_\alpha$. Then we may make the approximation $\ln |w_\alpha|^2 \approx \ln r_\alpha + \epsilon_\alpha / r_\alpha$ and (3.6) becomes

$$\frac{1}{r_\alpha} \Delta \epsilon_\alpha - \sum_{\beta=1}^{m} M_{\alpha\beta}\ \epsilon_\beta = 0. \tag{3.8}$$

Since the matrix $M_{\alpha\beta}$ is positive definite it has positive eigenvalues, and therefore solutions of (3.8) are a linear superposition of functions which decay exponentially for large ρ, as claimed. Thus in the case of completely broken gauge symmetry the boundary condition (2.6) and (2.7) are sufficient to ensure finite energy of the solution.

If on the other hand some of the gauge symmetry remains unbroken i.e. if $r_\beta = 0$ for some (at least one) $\beta \in \{1, ..., m\}$ then we expect on physical grounds that the corresponding scalar fields w_β and (because of the coupling) possibly some other fields have long range components. However, since the differential equation (3.6) is singular at $|w_\beta|^2 = 0$, linearisation around this value can no longer be used to establish the asymptotic form of the fields. In [12] the asymptotics of solutions of the Bogomol'nyi equations in the gauged $O(3)$ sigma model was derived rigorously by suitably transforming the equations and using tools from non-linear functional analysis. Since, as we shall explain in sect. 5.1, that model is a special case of the GLSM's studied here, this treatment can presumably be extended to deal with the general situation (see also [13]). We will not attempt to give such a rigorous treatment here but guided by the example of the gauged $O(3)$ model we conjecture that for those $\beta \in \{1, ..., m\}$ for which $r_\beta = 0$ the components w_β of a solution of (3.6) approach the vacuum according to a power law

$$|w_\beta|^2 = O(\rho^{-2\tilde{\eta}_\beta}) \qquad \text{for large} \quad \rho, \tag{3.9}$$

or possibly according to a power law modified by logarithmic terms [1]

$$|w_\beta|^2 = O\left(\frac{1}{\rho^2 \ln^2 \rho}\right) \qquad \text{for large} \quad \rho. \tag{3.10}$$

[1] This was pointed out to me by Trevor Samols

The specific form of the logarithmic correction is suggested by the observation that $|w|^2 = 2/(e^2\rho^2 \ln^2 \rho)$ solves (3.7) with $r = 0$ provided $z \neq z_r$, in particular therefore in the asymptotic region. Smooth solutions of (3.6) with the asymptotic from (3.10) will automatically have finite energy but to ensure finite energy in the case of (3.9) we need to impose

$$\tilde{\eta}_\beta > 1. \tag{3.11}$$

Note that we have not made any claim about the asymptotic form of those components w_α for which $r_\alpha > 0$. Depending on the coupling via the matrix $M_{\alpha\beta}$ they may approach the vacuum exponentially fast or also according to (3.9) or (3.10).

Given the asymptotic form of a solution, its fluxes and hence energy can be computed explicitly from (3.5) as follows. Using Stokes's theorem we first obtain

$$\int d^2x\, \Delta \ln |w_\alpha|^2 = \oint_C dx_j \varepsilon_{ij} \partial_i \ln |w_\alpha|^2 = \oint_C d\theta\ \rho \frac{\partial}{\partial\rho} \ln |w_\alpha|^2, \tag{3.12}$$

where the contour C is again the circle at spatial infinity and $\varepsilon_{12} = -\varepsilon_{21} = 1$. Then using again the expansion $\ln(r_\alpha + \epsilon_\alpha) \approx \ln r_\alpha + \epsilon_\alpha / r_\alpha$, valid when $r_\alpha \neq 0$ and ϵ_α small, and integrating (3.5) we deduce

$$r_\alpha > 0 \ \Rightarrow\ \varphi_\alpha = n_\alpha \in \mathbf{Z}^{\geq 0} \tag{3.13}$$

regardless of how $|w_\alpha|^2$ approaches r_α. If on the other hand $r_\alpha = 0$, we expect that $|w_\alpha|^2$ tends to zero according to (3.9) or (3.10). Then the result of the integration can be summarised in

$$r_\alpha = 0 \ \Rightarrow\ \varphi_\alpha = n_\alpha + \eta_\alpha, \quad n_\alpha \in \mathbf{Z}^{\geq 0} \quad \text{and} \quad \eta_\alpha \in \mathbf{R}^{\geq 1}. \tag{3.14}$$

The real number η_α equals $\tilde{\eta}_\alpha$ if $|w_\alpha|^2$ decays according to the power law (3.9) and is 1 if $|w_\alpha|^2$ approaches zero according to (3.10).

The above formulae may be interpreted by saying that the flux φ_α counts the zeros of w_α with multiplicity. If $r_\alpha > 0$, all zeros are at finite ρ and have positive integer multiplicity. If $r_\alpha = 0$ then w_α has a zero at infinity and we may interpret the number η_α as its multiplicity; it is ≥ 1 but not necessarily integer.(One shows similarly that for square-integrable solution of the (anti) Bogomol'nyi equations (2.17a) and (2.17b), $\varphi_\alpha \in \mathbf{Z}^{\leq 0}$ if $r_\alpha > 0$ and $\varphi_\alpha \in \mathbf{R}^{\leq -1}$ if $r_\alpha = 0$.)

Finally we observe a useful restriction on the fluxes φ_α which can be deduced from (3.1b) if $r_\beta = 0$ for some $\beta \in \{1, ..., m\}$. Solving for $|w_\beta|^2$ and using the definition (2.19b) yields

$$\sum_{a=1}^{m} (\mathbf{Q}^{-1})_{\beta a} \frac{F_{12}^a}{e_a^2} = \frac{1}{2}|w_\beta|^2 \geq 0 \tag{3.15}$$

so that

$$\sum_{a=1}^{m} (\mathbf{Q}^{-1})_{\beta a} \frac{\Phi_a}{e_a^2} \leq 0. \tag{3.16}$$

Here the equality holds only if w_β vanishes everywhere.

4. Semilocal vortices, $\mathbf{CP}^1$ lumps and toric varieties

For specific calculations we will mostly concentrate on the GLSM with two scalar fields w_1 and w_2 in this paper. Our favourite choice of generators of $U(1)$ subgroups of the maximal torus in this case is

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.1}$$

(we will mostly omit writing the identity matrix t_1 in the following formulae). Then, to avoid crowded notation, we write A_μ and B_μ for the gauge fields A_μ^1 and A_μ^2, A and B for the corresponding complex fields A^1 and A^2, and $F_{\mu\nu}$ and $G_{\mu\nu}$ for the corresponding field strengths.

In this section we restrict attention to the case where one of the gauge fields is set to zero. Specifically consider the case $e_2 = 0$ and $B_\mu = 0$. The resulting model has been much studied in the recent literature and has vortex solutions known as semilocal vortices, see refs. [4], [14] and [15]. One interesting aspect of this model is that it contains stable vortices although the vacuum manifold, defined as the submanifold of $\mathbf{C}^2$ where the potential $(R_1 - w^\dagger w)$ vanishes, is a three-sphere (provided $R_1 > 0$) and therefore has trivial first homotopy group. The reason why there are nonetheless stable vortex solution, first explained in [4], is that in addition to condition (2.6) we have the condition (2.7) and this forces w to lie both on the vacuum manifold and on a gauge orbit of the gauged $U(1)$ at spatial infinity. Thus, in terms of spatial polar coordinates (ρ, θ) w has to be of the form $(w_1, w_2) = (c_1 e^{iN_1\theta}, c_2 e^{iN_1\theta})$ for large ρ with N_1 some integer and c_1 and c_2 complex constants satisfying $c_1^2 + c_2^2 = R_1$. The point is that the gauge group acts without fixed points on the vacuum manifold, so that gauge orbits are necessarily loops and never just a point. The integer N_1 is the degree of the map $w_{|\rho=\infty}$ from the circle at spatial infinity into one of these gauge orbits. Evaluating the formulae (2.25) and (2.26) for the topological lower bound in this case we find $T = R_1 N_1$, showing that vortex solutions of the Bogomol'nyi equations with $N_1 \neq 0$ cannot decay into the vacuum.

It has also been observed in the literature [16] that there is a close connection between semilocal vortices and topological solitons in the $\mathbf{CP}^1$ (or $O(3)$) sigma model. This connection is usually discussed in terms of energy scales, with the sigma model being thought of as a low-energy effective theory of the vortex model. However, from the present point of view it is more convenient to keep the energy under consideration fixed and vary the parameter e_1 (recall that this has dimensions $(\text{mass})^{\frac{1}{2}}$). In fact it is easy to see that in limit $e_1 \to \infty$ the semilocal vortex model reduces to the $\mathbf{CP}^1$ model. To keep the energy finite we simultaneously impose the constraint

$$R_1 - w^\dagger w = 0. \tag{4.2}$$

Then, since the kinetic term of the gauge field A_μ disappears form the Lagrangian in the limit $e_1 \to \infty$ we can eliminate the gauge field altogether form the energy functional via the equation

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{2i}((D_\mu w)^\dagger w - w^\dagger D_\mu w) = 0, \tag{4.3}$$

which implies

$$R_1 A_\mu = i w^\dagger \partial_\mu w. \tag{4.4}$$

It is a standard result, see e.g. [17], that the resulting model is equivalent to the $\mathbf{CP}^1$ sigma model. Geometrically, the condition (4.2) forces w to lie on a three-sphere S^3, and the appearance of the covariant derivative with the gauge potential given by (4.4) means that the Lagrangian depends on w only up to an overall phase. Thus, defining the equivalence relation $\sim$ via $(w_1, w_2) \sim e^{i\chi}(w_1, w_2)$, $\chi \in [0, 2\pi)$, the limit $e_1 \to \infty$ leads to the non-linear sigma model with target space $(S^3/\sim) \cong \mathbf{CP}^1 \cong S^2$. For later use, we note that this can be made explicit by introducing the $\mathbf{CP}^1$-valued field

$$u = \frac{w_2}{w_1} \tag{4.5}$$

or the S^2-valued field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, defined via

$$\phi_l = w^\dagger \tau_l w, \qquad l = 1, 2, 3, \tag{4.6}$$

where τ_1, τ_2 and τ_3 are the three Pauli matrices. The Lagrangian density can then be written in terms of u or $\vec{\phi}$, see ref. [17].

Note also that the topological character of the topological defect changes as we take the limit $e_1 \to \infty$. For the semilocal vortex we have already interpreted the topological bound T as (R_1 times) the degree of the w viewed as a mapping from the circle at spatial infinity to a gauge orbit on the vacuum manifold. However, in the limit $e_1 \to \infty$, points on gauge orbits should be identified, so that now w maps the entire circle at spatial infinity into on point on the target space $\mathbf{CP}^1$. This allows us to regard $u = w_2/w_1$ as a map from compactified space $\mathbf{R}^2 \cup \{\infty\} \cong S^2$ to $\mathbf{CP}^1$. The flux $\Phi_1 = -1/2\pi \int d^2x F_{12}$, with F_{12} computed from (4.4), equals the degree of that map. Thus, in the general terminology of the introduction topological solitons in the $\mathbf{CP}^1$ model are textures. However, in this particular context they are more usually called lumps.

The moduli spaces of both semilocal vortices and $\mathbf{CP}^1$ lumps have been studied in some detail. In [14] it is shown that the moduli space of semilocal vortices with magnetic flux number N is diffeomorphic to the space of polynomials of the form $P_N(z) = z^N + \sum_{s=0}^{N-1} a_s z^s$ and $Q_N(z) = \sum_{s=0}^{N-1} b_s z^s$. The coordinatisation of that space in terms of the complex coefficients a_s and b_s, $s = 0, ..., N-1$ shows that it can be identified with $\mathbf{C}^{2N}$. The translation from a point in the moduli space to an actual field configuration, however, is best done in terms of the zeros of P_N and Q_N. They are also the zeros of the scalar fields w_1 and w_2.

In the $\mathbf{CP}^1$ model, a lump of degree N is a rational function on $\mathbf{R}^2$ of degree N which tend to zero at infinity. Explicitly such a function can again be written in terms of the polynomials P_N and Q_N

$$u(z) = \frac{Q_N(z)}{P_N(z)}, \tag{4.7}$$

but now we have to require in addition that P_N and Q_N have no common zeros, or equivalently that the resultant of P_N and Q_N is non-vanishing. Writing R_N for the set where the resultant does vanish we conclude that the moduli space of degree N lumps is $\mathbf{C}^{2N} - R_N$. The interpretation of the moduli is now quite different from the vortex case. Writing for example a single lump configuration as $b/(z-a)$, the complex number a is the lump's position in $\mathbf{R}^2$ and the modulus and phase of the complex number b are its size and orientation respectively. Note that b has to be non-vanishing and that in the limit $|b| \to 0$ the lump becomes infinitely spiky. These features generalise to lumps of degree N.

The moduli encode information about internal as well as position degrees of freedom. The condition of the non-vanishing resultant removes precisely those points from the moduli space which correspond to infinitely spiky configurations.

As mentioned earlier, the Riemannian metric which the moduli spaces inherit from the field theory kinetic energy is crucial in the moduli space approximation to soliton dynamics. In ref. [18] this approximation is applied to lumps in the $\mathbf{CP}^1$ model and it turns out that the moduli space metric has two problematic features in this case (see also [19] for further discussion of this point). The metric is not finite because changes in the coefficient q_{N-1} require infinite kinetic energy. Furthermore it is not complete: there are geodesics which reach infinitely spiky configurations in finite time. By contrast the metric on the moduli space for semilocal vortices , studied in ref. [15] in the case $e_1 = 1$, is finite and complete. Thus the moduli space of semilocal vortices can be thought of in a very precise sense as a regularised version of the moduli space of $\mathbf{CP}^1$ lumps, to which it tends in the limit $e_1 \to \infty$.

Let us briefly consider the case where only the $U(1)$-factor generated by t_2 is gauged. Thus we set $e_1 = 0$ and $A_\mu = 0$. The Bogomol'nyi equations (2.15a) and (2.15b) are this case:

$$
\begin{aligned}
(\partial_{\bar z} + i\bar B)w_1 &= 0 \\
(\partial_{\bar z} - i\bar B)w_2 &= 0 && (4.8a) \\
G_{12} + \frac{1}{2}(R_2 - |w_1|^2 + |w_2|^2) &= 0. && (4.8b)
\end{aligned}
$$

The energy of a solution of these equations is $E = R_2\Phi_2$. Thus, if $R_2 < 0$ and $E \neq 0$ we necessarily have $\Phi_2 < 0$. However, it then follows from a standard vanishing theorem (a line bundle of negative degree cannot have a non-zero holomorphic section) that $w_1 = 0$. Similarly if $R_2 > 0$ we deduce that for any solution with non-vanishing energy $w_2 = 0$ In either case the vortex solutions of this model are just embedded Nielsen-Olesen vortices. In the limit $e_2 \to \infty$ we again obtain a non-linear sigma model, but this time the target space is $\{(w_1, w_2) \in \mathbf{C}^2 | R_2 = |w_1|^2 - |w_2|^2\}/\sim$, where $\sim$ is the equivalence relation $(w_1, w_2) \sim (e^{i\chi}w_1, e^{-i\chi}w_2)$. This space is known in mathematics as the weighted projective space $\mathbf{CP}^1_{(1,-1)}$ and is isomorphic to $\mathbf{C}$. Bogomol'nyi solitons in the corresponding non-linear sigma model are holomorphic maps from S^2 to $\mathbf{CP}^1_{(1,-1)}$. However, such maps are necessarily constant maps. Thus there are no non-trivial textures of Bogomol'nyi type in this model.

The phenomena encountered in this section form part of a very general story, much discussed in mathematics and string theory, see refs. [5] and [6]. Briefly, it goes as follows. Returning to the general notation of sect. 2, consider taking the limit

$$e_a \to \infty, \qquad a \in I \tag{4.9}$$

and simultaneously imposing

$$(R_a - w^\dagger t_a w) = 0, \qquad a \in I \tag{4.10}$$

for some subset of indices $I \subset \{1, ..., m\}$ which we can without loss of generality take to be $I = \{1, ..., d\}$, $d < m$. Assume that the R_a are such that the entire gauge symmetry is spontaneously broken. Also let us at first restrict our attention to the situation where the couplings and gauge fields labelled by the complementary indices are set to zero

$$e_a = 0, \qquad A^a_\mu = 0 \qquad \text{for} \quad a = d+1, ..., m. \tag{4.11}$$

Defining the currents

$$j^a_\mu = \frac{1}{2i}\left((D_\mu w)^\dagger t^a w - w^\dagger t^a D_\mu w\right), \tag{4.12}$$

the equations of motion for the gauge fields A^a_μ, $a \in I$ are then

$$j^a_\mu = 0 \quad \text{for} \quad a \in I \tag{4.13}$$

which one can solve explicitly for A^a_μ, $a \in I$:

$$\sum_{b=1}^{d} w^\dagger t_a t_b w \, A^b_\mu = i w^\dagger t_a \partial_\mu w. \tag{4.14}$$

The gauge fields A^a_μ, $a \in I$, can thus be eliminated, but the Lagrangian (2.4) is still invariant under transformations $w(x) \to e^{i\chi^a(x) t_a} w(x)$, $a \in I$, $\chi_a \in [0, 2\pi)$. Then defining the equivalence relation

$$w \sim e^{i\chi^a t_a} w, \qquad a \in I, \qquad \chi_a \in [0, 2\pi) \tag{4.15}$$

the fields w may be thought of as taking values in the non-linear space

$$Z = \{w \in \mathbf{C}^m | w^\dagger t_a w = R_a, a \in I\}/\sim . \tag{4.16}$$

The combined operation of imposing (4.10) and dividing by the action (4.15) of the torus $U(1)^d$ is called the symplectic quotient of $\mathbf{C}^m$ by $U(1)^d$ and is often written $\mathbf{C}^m//U(1)^d$. In fact what we are looking at here is a very special sort of symplectic quotient. With the original space being $\mathbf{C}^m$ and the group action being a $U(1)^d$-action the resulting quotient is a so-called toric variety of complex dimension $m-d$. The complex projective space and the weighted complex projective space which we encountered above are special examples of toric varieties. Toric varieties are naturally Kähler manifolds, and their topology (which depends on the values of the R_a) is well-studied. In particular it is known that under some restriction on the Q^a_α (2.1) the quotient space Z is compact. This holds for example if for some $a \in I$ all the Q^a_α are positive. (For more general conditions see [9].) Moreover, the second homology group $H^2(Z,\mathbf{R})$ is d-dimensional for generic values of R_a (note that as long as $d<m$, the real dimension of Z is ≥ 2).

In analogy to our discussion of the $\mathbf{CP}^1$ sigma model static fields which obey the condition (2.7) can here be regarded as maps from $\mathbf{R}^2 \cup \{\infty\} \cong S^2$ into Z. Such maps can be classified topologically by their multi-degree, the integral of the pull-back of the d generators of $H^2(Z,\mathbf{R})$. This generalises the observation made in the simple case of the $\mathbf{CP}^1$ sigma model. For every gauge field which we eliminate by taking the corresponding coupling constant to infinity the topological meaning of the magnetic flux number changes. If before elimination this number counts the number of times the circle at spatial infinity is wrapped round a certain loop in the vacuum manifold then, after elimination, it counts the number of times compactified space get wrapped around a certain generator of the second homology of the (now non-linear) target space.

Note that the argument in the last paragraph depends crucially on the fact that all gauge fields are either eliminated through the limit (4.9) or set to zero (4.11). Only in this case does the boundary condition (2.7) allow us to identify points at infinity to one point. However, it is also interesting to consider the mixed situation, where some gauge fields are eliminated and others remain as dynamical fields. This leads to gauged non-linear sigma models, where the geometric interpretation of the magnetic fluxes φ_α is more subtle. We shall see this in a particular example in the next section.

5. Topological solitons in $U(1) \times U(1)$ gauge theory

We now have all the ingredients necessary for the study of the case where the entire maximal torus of $U(2)$ is gauged. There are eight independent parameters in this model: the real numbers e_1, e_2, R_1, R_2 and the integer entries in the generators t_a :

$$t_1 = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{12} \end{pmatrix} \qquad t_2 = \begin{pmatrix} Q_{21} & 0 \\ 0 & Q_{22} \end{pmatrix}. \tag{5.1}$$

It is instructive to write out the Lagrangian density in components for this model

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}|D_\mu w_1|^2 - \frac{1}{2}|D_\mu w_2|^2 - \frac{1}{4e_1^2}F_{\mu\nu}^2 - \frac{1}{4e_2^2}G_{\mu\nu}^2 \\ &- \frac{e_1^2}{8}(R_1 - Q_{11}|w_1|^2 - Q_{12}|w_2|^2)^2 - \frac{e_2^2}{8}(R_2 - Q_{21}|w_1|^2 - Q_{22}|w_2|^2)^2. \end{aligned} \tag{5.2}$$

This Lagrangian density is of the type studied first by Witten in the seminal paper [20] and later by many others (see the book [21] and references therein) as a model for bosonic superconducting vortices. In that context the discussion is usually conducted in terms of the gauge fields $a_\mu = Q_{11}A_\mu + Q_{21}B_\mu$ and $b_\mu = Q_{12}A_\mu + Q_{22}B_\mu$ which couple only to the scalar fields w_1 and w_2 respectively. Writing $U(1)_a$ and $U(1)_b$ for the gauge groups which rotate the phases of w_1 and w_2 respectively the basic observation can be stated as follows. By adjusting parameters suitably one can arrange for one of the two gauge groups, say $U(1)_a$, to be spontaneously broken while $U(1)_b$ remains unbroken. Thus $U(1)_b$ may be interpreted as the gauge group of electromagnetism (there is a *caveat* which we explain below). Then there is a range of parameters for which the model has $U(1)_a$-vortex solutions in whose core the Higgs field w_2 of $U(1)_b$ is non-zero, thereby breaking the electromagnetic gauge group there. Witten showed that under these circumstances the core of the vortex becomes superconducting.

The Lagrangian density (5.2) is more general than the Lagrangians usually studied in the context of superconducting vortices because it allows for interactions between the gauge fields a_μ and b_μ (through the curvature terms) but it also has many special properties because it of Bogomol'nyi type. One general feature of Bogomol'nyi solitons in gauge theories with scalar fields is the cancellation between the static forces mediated by the gauge fields and those mediated by the scalar fields. Such a cancellation is of course only possible if the scalar and gauge fields are either both massless or both massive. Thus we should expect on general grounds that in our model, too, the scalar field w_2 should acquire a long range component when $U(1)_b$ is unbroken. In fact it is easy to check this explicitly. By the general condition stated after eq. (2.23), $U(1)_b$ is unbroken if and only if $r_2 = 0$, which is equivalent to $Q_{21}R_1 = Q_{11}R_2$. The vacuum expectation value of

$|w_1|^2$ is then $R_1/Q_{11} = R_2/Q_{21}$ and, by collecting the terms quadratic in $|w_2|$, the mass of w_2 is found to be zero. (A similar calculation shows that if we fix the parameters so that w_1 vanishes in the vacuum then w_1 becomes massless.) From the point of view of superconducting vortex physics this result is disappointing: one requires an exponentially localised w_2 condensate in order to interpret the unbroken gauge group $U(1)_b$ as the gauge group of electromagnetism. If the condensate only decays according to some power law the $U(1)_b$ gauge invariance is never properly restored outside the core of the vortex and the electromagnetic interpretation is not appropriate.

By adding a suitable perturbation to the Lagrangian (5.2) one could ensure an exponentially localised condensate, of course at the expense of destroying the Bogomol'nyi property. However, properties such as interactive dynamics and thermodynamics of such 'almost Bogomol'nyi' models can also be studied with relative ease by perturbation methods [22]. This may be interesting from the point of view of the phenomenology of superconducting vortices but we will not pursue it here. Instead we want to exhibit some detailed properties of the model (5.2) by picking a particular set of generators t_a.

Thus we return to the generators (4.1), i.e. we set $Q_{11} = Q_{12} = 1$ and $Q_{21} = -Q_{22} = 1$ so that $r_1 = (R_1 + R_2)/2$, $r_2 = (R_1 - R_2)/2$. We also continue with the index saving terminology and write $a_i = A_i + B_i$ and $b_i = A_i - B_i$, $a = (a_1 - ia_2)/2$ and $b = (b_1 - ib_2)/2$ as well as $f_{12} = F_{12} + G_{12}$ and $g_{12} = F_{12} - G_{12}$. Then the Bogomol'nyi equations (3.1)read

$$(\partial_{\bar{z}} + i\bar{a})w_1 = 0 \tag{5.3a}$$

$$(\partial_{\bar{z}} + i\bar{b})w_2 = 0 \tag{5.3b}$$

$$f_{12} + \frac{1}{2}(e_1^2 + e_2^2)(r_1 - |w_1|^2) + \frac{1}{2}(e_1^2 - e_2^2)(r_2 - |w_2|^2) = 0 \tag{5.3c}$$

$$g_{12} + \frac{1}{2}(e_1^2 - e_2^2)(r_1 - |w_1|^2) + \frac{1}{2}(e_1^2 + e_2^2)(r_2 - |w_2|^2) = 0, \tag{5.3d}$$

and the generalised vortex equations (3.6) are

$$\Delta \ln |w_1|^2 + (e_1^2 + e_2^2)(r_1 - |w_1|^2) + (e_1^2 - e_2^2)(r_2 - |w_2|^2) = 4\pi \sum_{s=1}^{n_1} \delta(z - z_{1,s}) \tag{5.4a}$$

$$\Delta \ln |w_2|^2 + (e_1^2 - e_2^2)(r_1 - |w_1|^2) + (e_1^2 + e_2^2)(r_2 - |w_2|^2) = 4\pi \sum_{t=1}^{n_2} \delta(z - z_{2,t}). \tag{5.4b}$$

As an aside we point out that essentially the same equations arise in the study of vortex solutions in the two-Higgs-doublet system [23]. Here we are interested in the way the properties of solutions, such as energies and magnetic fluxes, depend on the coupling constants e_1 and e_2 and on the values of the parameters r_1 and r_2. The latter determine the symmetry pattern. If $r_1 > 0$ and $r_2 > 0$ the gauge symmetry is completely broken. Then the fluxes φ_1 and φ_2 equal the integers n_1 and n_2 respectively. Although we do not prove this rigorously here we expect that given these integers there is a $(n_1 + n_2)$ complex parameter family of solutions of (5.4a) and (5.4b). Each solution is fully characterised by the unordered set of parameters $\{z_{1,1}, ..., z_{1,n_1}\}$ and $\{z_{2,1}, ..., z_{2,n_2}\}$ which label the zeros of the scalar fields w_1 and w_2. Note that, contrary to the situation for a single vortex, the magnitudes of the magnetic fields are not necessarily maximal at these zeros.

If $r_1 > 0$ and $r_2 = 0$, $U(1)_a$ is broken but $U(1)_b$ remains unbroken (and vice-versa for $r_1 = 0$ and $r_2 > 0$). Thus φ_1 equals again the integer n_1 which counts the zeros of w_1, but φ_2 may have a non-integral part. The interpretation, explained in sect. 3, is that w_2 has a zero at infinity whose order is at least one but may be non-integral. Thus only $[\varphi_2 - 1]$ zeros (where $[\varphi_2]$ denotes the integral part of φ_2) can be placed arbitrarily in $\mathbf{R}^2 \cup \{\infty\}$. For given $\varphi_1 = n_1$ and φ_2 we therefore expect there to be a $n_1 + [\varphi_2] - 1$ complex parameter family of solutions of (5.4a) and (5.4b).

Finally in the special case $r_1 = r_2 = 0$ the full gauge symmetry survives, but we shall see that the model has no finite energy solutions in this case. Before we discuss the spectrum of solitons in more detail we will now show that our model contains, in the limit $e_1 \to \infty$, the gauged $O(3)$ sigma model introduced in ref. [7] and analysed further in ref. [12].

5.1. The gauged $O(3)$ sigma model revisited

In taking the limit $e_1 \to \infty$ we can now follow the general recipe of sect. 4. Thus we impose simultaneously the constraint $|w_1|^2 + |w_2|^2 = R_1 > 0$. Then we solve for A_μ according to (4.14):

$$R_1 A_\mu = i w^\dagger \partial_\mu w - w^\dagger t_2 w B_\mu. \tag{5.5}$$

Since the field (w_1, w_2) is now only defined up to an overall phase we can again discuss the model in terms of the ratio $u = w_2/w_1$ (4.5). In particular the energy functional for static fields takes the form

$$E = \frac{1}{2} \int d^2x \, (r_1 + r_2) \frac{|D_1 u|^2 + |D_2 u|^2}{(1 + |u|^2)^2} + \frac{1}{e_2^2} G_{12}^2 + e_2^2 \left(\frac{r_1 |u|^2 - r_2}{1 + |u|^2} \right)^2, \tag{5.6}$$

where the covariant derivative on u is $D_i u = (\partial_i - 2iB_i)u$. This functional is a generalised form of the energy functional of the gauged $O(3)$ sigma model introduced in ref. [7] and formulated there in terms of the field $\vec{\phi}$ (4.6). To recover the formulae in ref. [7] one should set $r_1 = 1$, $r_2 = 0$ and $e_2 = 1$, and identify $-2B_i$ and $-2G_{12}$ with what is called A_i and F_{12} there. Then the energy functional (5.6) is 4 times the energy functional studied in ref. [7] and the magnetic flux

$$\Phi_2 = -\frac{1}{2\pi}\int d^2x\, G_{12} \tag{5.7}$$

is $\frac{1}{4\pi}\times$ the magnetic flux defined in ref. [7], which is denoted Φ there.

The inequality

$$E \geq \pi(r_1\varphi_1 + r_2\varphi_2), \tag{5.8}$$

is independent of the values of e_1 and e_2 and therefore also valid in the limit $e_1 \to \infty$. However, since we are now dealing with the 'mixed situation' anticipated at the end of sect. 4, the fluxes φ_1 and φ_2 have to be interpreted with care. First we note the following formulae which follow from the definitions and from eq. (5.5) through direct computation:

$$\varphi_1 = \frac{1}{2\pi i}\int d^2x\, \frac{\overline{D_1 u}D_2 u - \overline{D_2 u}D_1 u}{(1+|u|^2)^2} - \frac{1}{2\pi}\int d^2x\, \frac{2G_{12}|u|^2}{1+|u|^2}, \tag{5.9}$$

and

$$\varphi_2 = \frac{1}{2\pi i}\int d^2x\, \frac{\overline{D_1 u}D_2 u - \overline{D_2 u}D_1 u}{(1+|u|^2)^2} + \frac{1}{2\pi}\int d^2x\, \frac{2G_{12}}{1+|u|^2}. \tag{5.10}$$

Integrating by parts and using the boundary condition $|u|^2 = r_2/r_1$ at spatial infinity one finds that equivalently

$$\varphi_1 = \frac{1}{2\pi i}\int d^2x\, \frac{\partial_1\bar{u}\partial_2 u - \partial_2\bar{u}\partial_1 u}{(1+|u|^2)^2} - \frac{r_2}{r_1+r_2}\frac{1}{2\pi}\int d^2x\; 2G_{12}, \tag{5.11}$$

and

$$\varphi_2 = \frac{1}{2\pi i}\int d^2x\, \frac{\partial_1\bar{u}\partial_2 u - \partial_2\bar{u}\partial_1 u}{(1+|u|^2)^2} + \frac{r_1}{r_1+r_2}\frac{1}{2\pi}\int d^2x\; 2G_{12}. \tag{5.12}$$

These formulae are invariant under any change in the fields u and B_i which can be built up from a sequence of infinitesimal changes of the form $u \to u + \delta u$ and $B_i \to B_i + \delta B_i$ with δu and δB_i vanishing at spatial infinity, thus showing that φ_1 and φ_2 are homotopy invariant. However, to see if and when the magnetic fluxes φ_1 and φ_2 are integers it is best to recall how the magnetic fluxes are related to the numbers of zeros of the scalar fields in the corresponding vortex model. Since r_1 and r_2 should not both be zero let us for definiteness assume that $r_1 > 0$. Then φ_1 is an integer which counts with multiplicity the number of zeros of w_1 or equivalently the number of poles of u. Similarly the flux φ_2 counts the zeros of w_2 which are precisely those of u. If $r_2 > 0$, i.e. if the only remaining $U(1)$ gauge symmetry is spontaneously broken, then $|u|^2 = r_2/r_1$ at spatial infinity and all zeros of u are at finite ρ. Then φ_2 is also an integer which counts these zeros with multiplicity. If $r_2 = 0$, i.e. in the case of unbroken gauge symmetry, the condition $\varphi_2 \in \mathbf{R}^{\geq 1}$ tells us that u must have a zero at infinity of order at least 1.

Although the above interpretation of the quantities φ_1 and φ_2 seems the most useful in the present context it is also instructive to indicate a bundle theoretic description of the gauged $O(3)$ sigma model which does not refer to the vortex situation. In the language of fibre bundles u is a section of a bundle with structure group $U(1)$ and typical fibre $\mathbf{CP}^1$. When $r_2 > 0$ sections may be extended from $\mathbf{R}^2$ to $\mathbf{R}^2 \cup \{\infty\}$. Then the flux $2\Phi_2$ (5.7) is the first Chern class of a $U(1)$ bundle over a compact base space and therefore an integer; according to (5.11) and (5.12) it equals the difference between the number of poles and the number of zeros of u, both counted with multiplicity. When $r_2 = 0$, $2\Phi_2$ loses its topological meaning but, as long as $r_1 > 0$, there remains another topological attribute of the gauged $O(3)$ sigma model, which is a generalisation of the degree of fields u in the ungauged $\mathbf{CP}^1$ model. Such a generalised degree is defined and discussed in the context of the gauged (three-dimensional) Skyrme model in [24]. We do not want to review this discussion here, but simply note that the analogous generalised degree in the gauged $O(3)$ sigma model is φ_1. Thus the most natural labels for configurations in this model are the magnetic flux $2\Phi_2$ and the 'degree' φ_2, which counts with multiplicity the number of poles of u.

The Bogomol'nyi equations in this model may be derived either by determining the condition for the equality in (5.8) to hold, or by subtracting (5.3a) from (5.3b) and (5.3c) from (5.3d). In either case the result is

$$(\partial_{\bar{z}} - 2i\bar{B})u = 0 \tag{5.13a}$$

$$G_{12} + e_2^2 \left(\frac{r_1|u|^2 - r_2}{1 + |u|^2} \right) = 0. \tag{5.13b}$$

These equations imply a second order elliptic equation for u which can again be derived in two ways. Either, as first shown in ref. [12], from (5.13a) and (5.13b) via the $\bar{\partial}$-Poincaré lemma or by subtracting eq. (5.4a) from eq. (5.4b). In either case the result is

$$\Delta \ln |u|^2 - 4e_2^2 \left(\frac{r_1|u|^2 - r_2}{1+|u|^2} \right) = -4\pi \sum_{s=1}^{n_1} \delta(z - z_{1,s}) + 4\pi \sum_{t=1}^{n_2} \delta(z - z_{2,t}). \tag{5.14}$$

This equation with $r_2 = 0$ is analysed carefully by Yang in ref. [12]. It is shown there that in that situation and for given φ_1 and φ_2 it has a $\varphi_1 + [\varphi_2] - 1$ complex parameter family of solutions, thus proving the general counting argument given in the discussion of (5.4a) and (5.4b) in this particular case. Yang also rigorously established the relation (3.14) between the flux φ_2 and the asymptotic form of the scalar fields in this case.

Physically one may think of a solution of the Bogomol'nyi equations (5.13a) and (5.13b) as a texture carrying magnetic flux $2\Phi_2 = \varphi_1 - \varphi_2$ which counts the difference between the number (with multiplicity) of poles and the number of zeros of u. In the case of spontaneously broken gauge symmetry the flux is quantised and can be positive, negative or zero. In the unbroken case, however, it is clear from (5.7) and (5.13b) with $r_2 = 0$ that the magnetic flux $2\Phi_2$ is positive for non-trivial solutions. Thus we deduce that Φ_2 takes values in a finite interval

$$0 < 2\Phi_2 = \varphi_1 - \varphi_2 \leq \varphi_1 - 1. \tag{5.15}$$

This formula was derived independently by Samols (as referred to in ref. [7]) and by Yang in ref. [12]. In the next section we shall see that it is a consequence of the general inequality (3.16).

At the end of the introduction I mentioned that the wish better to understand the mathematical structure and physical significance of the gauged $O(3)$ sigma model was the starting point of this paper. In this section we have seen that mathematically this model is a limiting case of a certain GLSM with Fayet-Iliopoulos potential terms. Since the relevant GLSM has solutions which describe superconducting vortices of Bogomol'nyi type this observation also sheds light on the physics. When $r_2 = 0$ the gauged $O(3)$ sigma model has an unbroken $U(1)$ gauge group which we may attempt to identify with the gauge group of electromagnetism. In the core of a soliton solution the gauge symmetry is broken and in this sense we may think of the solitons in the gauged $O(3)$ sigma model as superconducting textures. As in our discussion of superconducting vortices in the previous section the Bogomol'nyi property of the model leads to a power-law localisation of the scalar condensate, which is phenomenologically disastrous . However, this problem can again be solved by moving away from the Bogomol'nyi limit. In fact a model of the required type is studied in [25]. The soliton solutions, called Skyrme-Maxwell solitons there, are exponentially localised. Asymptotically there is an unbroken $U(1)$ gauge group which is broken in the centre of a soliton. Such a topological soliton may thus properly be called a superconducting texture. These textures only involve one scalar field and one gauge field; from a mathematical point of view they therefore appear to provide a more economical model for superconducting topological defects than superconducting vortices.

5.2. Symmetry breaking patterns and the soliton spectrum

The simple energy formula

$$E_{\mathrm{BPS}} = \pi(r_1\varphi_1 + r_2\varphi_2) \tag{5.16}$$

for Bogomol'nyi solitons in the $m = 2$ GLSM and the inequality (3.16) have some remarkable consequences for the soliton spectrum, which we now want to explain. We stick to the choice of generators (4.1) but consider general values of the coupling constants e_1 and e_2 (including the non-linear sigma model limit $e_1 \to \infty$). Again we organise the discussion according to the symmetry breaking pattern.

First consider the case of completely broken symmetry, i.e. $r_1 > 0$ and $r_2 > 0$. Then fluxes φ_1 and φ_2 take arbitrary non-negative integer values n_1 and n_2. The resulting spectrum depends in an interesting way on whether r_1/r_2 is irrational or rational. In the former case there is no degeneracy in the energy level for given φ_1 and φ_2 other than that coming from the arbitrary positions of the n_1 and n_2 zeros of w_1 and w_2 discussed earlier. When r_1/r_2 is rational, however, solutions with different flux quantum numbers may be degenerate in energy. More precisely, writing $r_1/r_2 = p/q$, where p and q are two integers which are relatively prime, we deduce from formula (5.16) that the solution with $(\varphi_1, \varphi_2) = (n_1, n_2)$ has the same energy as the solution with $(\varphi_1, \varphi_2) = (n_1 + q, n_2 - p)$ (provided these integers are still positive). In the case $e_1 = e_2$, where the equations (5.4a) and (5.4b) decouple this degeneracy is easily understood: if the masses of a single w_1-vortex and a single w_2-vortex have a rational ratio, then we expect degeneracies in the energy levels of superpositions of w_1- and w_2-vortices. However, for $e_1 \neq e_2$, and particularly in the limit $e_1 \to \infty$, this degeneracy is more remarkable. In this limit we interpret φ_1 as the generalised degree of u and $2\Phi_2$ as the magnetic flux carried by the texture; in terms of these eq. (5.16) reads

$$E_{\mathrm{BPS}} = \pi \left((r_1 + r_2)\text{`degree'}[u] - 2r_2\Phi_2 \right). \tag{5.17}$$

Thus the degeneracy for rational r_1/r_2 means physically that in the gauged $O(3)$ sigma model changing the (generalised) degree of a configuration is energetically equivalent to changing, by a suitable number of units, the magnetic flux it carries.

If $r_1 > 0$ and $r_2 = 0$ then $U(1)_a$ is broken and $U(1)_b$ remains unbroken; in that case φ_1 is still quantised as a non-negative integer, but φ_2 may now have a non-integer part. The energy (5.16) depends only on φ_1 and is degenerate with respect to changes in φ_2. Note, however, that the range of φ_2 is restricted by the finite-energy condition $\varphi_2 \geq 1$ and the further constraint coming from (3.16) for non-vanishing solutions. In this case it reads

$$\varphi_2 < \frac{e_1^2 - e_2^2}{e_1^2 + e_2^2}\varphi_1. \tag{5.18}$$

Together these conditions force φ_2 to lie in a finite interval. In particular they may force φ_2 to vanish for certain values of e_1 and e_2. This holds in particular for $e_1 = e_2$ where this result, too, is easily understood. The equation (5.4b) for w_2 is then just the abelian Higgs vortex equation (3.7). As mentioned in sect. 3 this equation has no finite energy solution other than the trivial solution when $r = 0$. In the non-linear sigma model limit $e_1 \to \infty$ the formula (5.18) yields the promised re-derivation of (5.15). It now reads $\varphi_2 < \varphi_1$, which we combine with the earlier condition $\varphi_2 \in \mathbf{R}^{\geq 1}$ to

$$1 \leq \varphi_2 < \varphi_1. \tag{5.19}$$

This is equivalent to the inequality (5.15). Note in particular that this condition cannot be satisfied if $\varphi_1 = 1$ and that there is therefore no solution with precisely one pole in the gauged $O(3)$ sigma model. This result was first derived for the spherically symmetric case in ref. [7] and proved in general in ref. [12].

The reverse situation, with $r_1 = 0$ and $r_2 > 0$ can be discussed in an analogous fashion, and the case $r_1 = r_2 = 0$ does not lead to any interesting solution: now the condition (3.16) implies, for non-trivial solutions, (5.18) as well as

$$\varphi_1 < \frac{e_1^2 - e_2^2}{e_1^2 + e_2^2}\varphi_2 \tag{5.20}$$

which is incompatible with (5.18). Thus the trivial solutions $w_1 = w_2 = 0$ is the only finite energy solution of the equations (5.4a) and (5.4b) in this case.

6. Abelian Chern-Simons models of Bogomol'nyi type

Consider the following Lagrangian density for a theory where the dynamics of all the gauge fields are governed by Chern-Simons terms:

$$\mathcal{L}_{CS} = -\frac{1}{2}(D_\mu w)^\dagger D^\mu w - \sum_{a=1}^{m} \frac{1}{2\kappa_a} A^a_\mu \partial_\nu A^a_\lambda \epsilon^{\mu\nu\lambda} - V_{CS}(\kappa_a, w, R_a), \tag{6.1}$$

with m dimensionless coupling constants κ_a. In this section we want to determine the potentials V_{CS} such that the model has static soliton solutions of Bogomol'nyi type.

The Chern-Simons term is independent of the metric, so it does not contribute to the energy-momentum tensor. As a result the energy functional reads

$$E_{CS} = \frac{1}{2}\int d^2x (D_0 w)^\dagger D_0 w + (D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + V_{CS}. \tag{6.2}$$

However, the energy does depend on the gauge fields through the covariant derivatives. In particular the time-component of the gauge fields is determined as a function of the scalar fields and the spatial components of the field strength by the Chern-Simons version of Gauss's law:

$$j^a_0 = \frac{1}{\kappa_a} F^a_{12}, \tag{6.3}$$

where j^a_μ is defined as in (4.12). This constraint means that, even for static fields we cannot set $A^a_0 = 0$. Instead we find, for time-independent w_a,

$$\sum_{b=1}^{m} w^\dagger t_a t_b w \, A^b_0 = \frac{1}{\kappa_a} F^a_{12}. \tag{6.4}$$

Thus the energy functional E_{CS} can be expressed entirely in terms of spatial components of the gauge fields and the spatial part of the field strengths. To do this it is convenient to introduce the notation $\mathbf{F}$ for the column vector with components F^a_{12}/κ_a, $\mathbf{V}$ for the column vector with components $\kappa_a(R_a - w^\dagger t_a w)$ and $\mathbf{W}$ for the $m \times m$ matrix with components $W_{ab} = w^\dagger t_a t_b w$, $a, b = 1, .., m$. We claim that by setting

$$V_{CS} = \frac{1}{8}\mathbf{V}^t \mathbf{W} \mathbf{V} = \frac{1}{8} \sum_{a,b=1}^{m} \kappa_a (R_a - w^\dagger t_a w) \, w^\dagger t_a t_b w \; \kappa_b (R_b - w^\dagger t_b w) \tag{6.5}$$

we obtain an energy functional of Bogomol'nyi type:

$$\begin{aligned} E_{CS} &= \frac{1}{2}\int d^2x (D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + \mathbf{F}^t \mathbf{W}^{-1} \mathbf{F} + \frac{1}{4}\mathbf{V}^t \mathbf{W} \mathbf{V} \\ &= \frac{1}{2}\int d^2x |(D_1 \pm i D_2) w|^2 + (\mathbf{F} \pm \frac{1}{2}\mathbf{W}\mathbf{V})^t \mathbf{W}^{-1} (\mathbf{F} \pm \frac{1}{2}\mathbf{W}\mathbf{V}) \\ &\mp \frac{1}{2}\sum_{a=1}^{m} R^a \int d^2x F^a_{12}. \end{aligned} \tag{6.6}$$

Here the superscript t means transposition and we have again used the boundary condition (2.7) to integrate by parts and exploited the identity (2.10). As in sect. 2 we therefore deduce the inequality

$$E_{CS} \geq \pi |T| \tag{6.7}$$

with equality if and only if one of the Bogomol'nyi equations holds

$$(D_1 \pm iD_2)w = 0 \tag{6.8a}$$

$$\frac{F^a_{12}}{\kappa_a} \pm \frac{1}{2}\sum_{b=1}^{m} w^\dagger t_a t_b w \, \kappa_b (R_b - w^\dagger t_b w) = 0. \tag{6.8b}$$

In the case $m = 1$ the general prescription we have discussed leads to the abelian Chern-Simons-Higgs model with a sextic potential, which has been much discussed in the literature. One reason why this model has attracted attention is that it contains both topological and non-topological solitons [26]. The generalised models described here include, for $m = 2$, the choice of generators (4.1) and $\kappa_2 = 0$, the semilocal Chern-Simons model discussed in ref. [27], and have in general a variety of both topological and non-topological soliton solutions whose properties deserve to be studied further. However, here we now turn to another point of principal interest.

Like in the case of GLSM's with the gauge field governed by a Maxwell term it is interesting to consider limits in which some of the gauge fields are eliminated, thus leading to a (possibly gauged) non-linear sigma model. Specifically we can eliminate the Chern-Simons terms for the gauge fields A^a_μ, where a runs over the subset of indices $I = \{1, ..., d\}$, $d < m$, by taking the limit

$$\kappa_a \to \infty, \qquad a \in I \tag{6.9}$$

and simultaneously imposing

$$(R_a - w^\dagger t_a w) = 0, \qquad a \in I. \tag{6.10}$$

Then the equations of motion for the gauge fields A^a_μ, $a \in I$, are again (4.13), allowing us to express these gauge fields in terms of the scalar fields and the complementary set of gauge fields:

$$\sum_{b=1}^{d} w^\dagger t_a t_b w \, A^b_\mu = i w^\dagger t_a (\partial_\mu + i \sum_{b=d+1}^{m} t_b A^b_\mu) w. \tag{6.11}$$

However, there is a subtlety here: in the Chern-Simons case, unlike in the Maxwell case, simply taking the value of the potential in the limit (6.9) and (6.10) as the potential for the remaining degrees of freedom does not appear to lead to a model with interesting Bogomol'nyi solitons. The quickest way to see this is naively to apply the limit (6.9) and (6.10) to the Bogomol'nyi equation (6.8b). The equations labelled by $a \in I$ then lead to purely algebraic constraints on the scalar fields which eliminate many solutions. Here we therefore follow a different path and define a new potential as follows. Let $\tilde{I} = \{d+1, ..., m\}$ be the complementary index set to I and let $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{V}}$ be the column vectors with components $F_{12}^{\tilde{a}}/\kappa_{\tilde{a}}$ and $\kappa_{\tilde{a}}(R_{\tilde{a}} - w^\dagger t_{\tilde{a}} w)$ respectively, where $\tilde{a} \in \tilde{I}$. Further write $\widetilde{\mathbf{W}^{-1}}$ for the matrix with elements $(\mathbf{W}^{-1})_{\tilde{a}\tilde{b}}$, $\tilde{a}, \tilde{b} \in \tilde{I}$. Then the kinetic term for the gauge field in the Chern-Simons energy functional in the limit (6.9) is

$$\tilde{\mathbf{F}}^t \widetilde{\mathbf{W}^{-1}} \tilde{\mathbf{F}} \tag{6.12}$$

and thus the appropriate potential is

$$V_{\widetilde{CS}} = \frac{1}{8} \tilde{\mathbf{V}}^t \left(\widetilde{\mathbf{W}^{-1}}\right)^{-1} \tilde{\mathbf{V}}. \tag{6.13}$$

With this choice the energy functional

$$E_{\widetilde{CS}} = \frac{1}{2} \int d^2x \, (D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + \tilde{\mathbf{F}}^t \widetilde{\mathbf{W}^{-1}} \tilde{\mathbf{F}} + \frac{1}{4} \tilde{\mathbf{V}}^t \left(\widetilde{\mathbf{W}^{-1}}\right)^{-1} \tilde{\mathbf{V}} \tag{6.14}$$

satisfies

$$E_{\widetilde{CS}} \geq \pi |T| \tag{6.15}$$

with equality if and only if one of the Bogomol'nyi equations holds

$$(D_1 \pm i D_2) w = 0 \tag{6.16a}$$

$$\frac{F_{12}^{\tilde{a}}}{\kappa_{\tilde{a}}} \pm \frac{1}{2} \sum_{\tilde{b}=d+1}^{m} (\widetilde{\mathbf{W}^{-1}})^{-1}_{\tilde{a}\tilde{b}} V_{\tilde{b}} = 0, \qquad \tilde{a} \in \tilde{I}. \tag{6.16b}$$

In the first of these equations the gauge potentials A_i^a, $a \in I$ are again determined by (6.11).

The potential (6.13) is, in general, of a more complicated form than the sextic polynomial that is familiar in Chern-Simons theories of Bogomol'nyi type. Thus it may be useful to illustrate the general discussion by considering again the case $m = 2$, with entire maximal torus of $U(2)$ gauged. We will again use the index saving notation of sect. 5.

With t_1 and t_2 as in (4.1), the matrix $\mathbf{W}$ is

$$\mathbf{W} = \begin{pmatrix} w^\dagger w & w^\dagger t_2 w \\ w^\dagger t_2 w & w^\dagger w \end{pmatrix} \tag{6.17}$$

so that in the limit $\kappa_1 \to \infty$ and with $R_1 - w^\dagger w = 0$, the matrix $\widetilde{\mathbf{W}^{-1}}$ is just a number:

$$\widetilde{\mathbf{W}^{-1}} = \frac{R_1}{R_1^2 - (w^\dagger t_2 w)^2}. \tag{6.18}$$

Thus the Chern-Simons potential is

$$V_{\widetilde{CS}} = \frac{\kappa_2^2}{8R_1}(R_1^2 - (w^\dagger t_2 w)^2)(R_2 - w^\dagger t_2 w)^2 \tag{6.19}$$

and the Bogomol'nyi equations are

$$(D_1 \pm i D_2) w = 0 \tag{6.20a}$$

$$G_{12} \pm \frac{\kappa_2^2}{2R_1}(R_1^2 - (w^\dagger t_2 w)^2)(R_2 - w^\dagger t_2 w) = 0. \tag{6.20b}$$

Formulating this model in terms of the field u (4.5) is again instructive. Using also again the parameters r_1 and r_2 defined in sect. 5, the potential (6.19) takes the form

$$V_{\widetilde{CS}} = 2\kappa_2^2(r_1 + r_2)\frac{|u|^2(r_1|u|^2 - r_2)^2}{(1+|u|^2)^4}. \tag{6.21}$$

Choosing the upper sign in the Bogomol'nyi equations they now read

$$(\partial_{\bar z} - 2i\bar B)u = 0 \tag{6.22a}$$

$$G_{12} + 4\kappa_2^2(r_1 + r_2)\frac{|u|^2(r_1|u|^2 - r_2)}{(1+|u|^2)^3} = 0. \tag{6.22b}$$

In fact these equations and their soliton solutions are studied with slightly different notation in ref. [28] and ref. [29] for the case of unbroken $U(1)$ gauge symmetry $r_2 = 0$. In ref. [30] the case of broken gauge symmetry, $r_2 > 0$ is also considered. As often in Chern-Simons theories there are both topological and non-topological solitons in this model. All soliton solutions carry magnetic flux, and in the case of broken gauge symmetry the topological solitons may derive their topological stability from the (quantised) magnetic flux, from the number of poles of u (the "degree") or from a combination of both. For more details we refer the reader to the papers [28], [29] and [30] and also to ref. [31], where a related model is discussed. However, it should be clear from this brief case study that the family of gauged linear sigma models with Chern-Simons terms defined in this section have an even more diverse spectrum of solitons than their Maxwell cousins.

7. Discussion and Outlook

In (2+1) dimensions, many interesting topological solitons of Bogomol'nyi type can be studied in a unified way in terms of the gauged linear sigma models discussed in this paper. Thinking of the GLSM's with Maxwell term and Fayet-Iliopoulos potential terms as a generalisation of the abelian Higgs model and keeping in mind the phenomena encountered here, it would be interesting to address the following points in full generality.

Can one extend the analysis of ref. [10] and ref. [12] to establish rigorous general existence and uniqueness theorems for the generalised vortex equation (3.6)? If in the general model discussed in sect. 3 the gauge symmetry is completely broken (i.e. if all $r_\alpha > 0$) then all the fluxes φ_α are non-negative integers and the examples studied above suggest that for given fluxes there is a $\sum_{\alpha=1}^{m} \varphi_\alpha$ complex parameter family of solutions. If the gauge symmetry is only partially broken, say $r_\alpha > 0$ for $1 \leq \alpha \leq d$ and $r_\alpha = 0$ for $d < \alpha \leq m$, then I conjecture that there is generically a $\left(\sum_{\alpha=1}^{d} \varphi_\alpha + \sum_{\alpha=d+1}^{m} [\varphi_\alpha - 1]\right)$ complex parameter family of solutions for given fluxes. However, in this case the fluxes also have to satisfy the constraints (3.16). Here we have only shown that these constraints are necessary conditions for finite-energy solutions. It would be interesting to know whether they (or possibly some stricter version of them) are also sufficient for the existence of finite energy solutions.

Having counted the solutions it is then interesting to look at the structure - differentiable and metric - of the moduli spaces of solutions. The examples studied here suggest that the moduli spaces of vortices on $\mathbf{R}^2$ are typically diffeomorphic to $\mathbf{C}^D$ for some D (determined by the above counting arguments) and have smooth metrics. However, in the limit $e_a \to \infty$ for some $a \in \{1, ..., m\}$, where vortices turn into (possibly gauged) textures a number of interesting things happen. The interpretation of the moduli changes. Whereas vortex moduli simply characterise the (unordered) zeros of the scalar fields, the moduli for textures include internal as well as position parameters. Taking certain limits of the internal parameters corresponds to making the texture infinitely spiky, and these limiting values are therefore forbidden as moduli for textures. Thus the moduli spaces for textures are typically obtained from those of vortices by removing certain lower dimensional algebraic submanifolds. Metrically, the moduli spaces of textures are also worse behaved than those of vortices. The example of the $\mathbf{CP}^1$ lumps suggest that non-complete and non-finite metrics arise. In studying moduli spaces of textures it is therefore useful to bear in mind that they may be thought of as singular limits of smooth vortex moduli spaces. In that way, singularities may be understood and, if desired, circumvented.

Some of the questions raised so far are completely answered in the case where GLSM's are defined on compact Riemann surfaces. There the counting of solutions can be done using standard theorems in algebraic geometry, and much can be deduced about the topological and (complex) differentiable structure of the moduli spaces from the key observation that they (like the target spaces of non-linear sigma models encountered in sect. 4, and for similar reasons) are toric varieties; see ref. [6] for a pedagogical explanation of this statement and for further references. In particular the intersection ring of the moduli spaces is described in the literature, see again [6]. This is of interest from the point of view of vortex physics because knowing the intersection ring of the moduli spaces is the crucial ingredient in the study of the statistical mechanics of vortices in ref. [3]. There the statistical mechanics of Nielsen-Olesen vortices is studied on (amongst other Riemann surfaces) the torus, which is equivalent to studying vortices on the plane with periodic boundary conditions. Thus it appears that much of the information needed similarly to study the statistical mechanics of the more general types of vortices described here can be found in the mathematical and string theory literature. However, one important ingredient in the analysis of ref. [3], a certain normalisation factor, is lacking. Computing it requires some knowledge of the Riemannian structure that the moduli space inherits from the kinetic energy of the field theory. Since that metric is only natural form a (2+1)-dimensional point of view it has not been considered in the string theory context.

It is also interesting to ask some of the above questions for the Chern-Simons version of the GLSM's introduced in this paper. Already the first step, the general analysis of the soliton spectrum, is more complicated in this case. There are now both topological and non-topological solitons and since the scalar potential no longer has a universal polynomial structure the finite energy condition needs to be analysed for each model separately. It is also natural to contemplate further generalisations, such as mixed models, where some of the $U(1)$ gauge fields are governed by Maxwell terms and others by Chern-Simons terms. It should even be possible to construct general abelian GLSM's of Bogomol'nyi type where both a Maxwell and a Chern-Simons term is included for each gauge field. As shown first for the abelian Higgs model in ref. [32], later extended in ref. [33] and recently demonstrated in the context of the gauged $O(3)$ sigma model in refs. [28] and [30], this requires the introduction of additional neutral scalar fields. In such models the gauge fields can get their masses from the Chern-Simons term ("topological mass") or through the Higgs mechanism. The analysis in ref. [33] suggests that generalising this construction for abelian GLSM's will lead to models which can accommodate a truly bewildering array of massless, massive, topological and non-topological particles. Thus it seems that the framework of abelian GLSM's provides an extremely versatile kit for the construction of Bogomol'nyi type field theories in (2+1) dimensions. The Maxwell version mainly studied here also shows that this kit comes equipped with powerful tools for exploring the physics of these field theories.

Acknowledgements

I thank Robbert Dijkgraaf and Jae-Suk Park for discussions about the connection between GLSM's and toric varieties and Yisong Yang for communicating his results [13] prior to publication. I acknowledge financial support through a Pioneer Fund of the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).

References

[1] N.S. Manton, "A remark on the scattering of BPS monopoles", Phys. Lett. **110B** (1982) 54

[2] T.M. Samols, "Vortex scattering", Commun. Math. Phys. **145** (1992) 149

[3] N.S. Manton, "Statitstical mechanics of vortices", Nucl. Phys. **B400** (1993) 624 ;
P.A. Shah and N.S. Manton, "Thermodynamics of vortices in the plane", J. Math. Phys. **35** (1994) 1171 ;
P.A. Shah, "Statistical mechanics of vortices on Riemann surfaces", DAMTP preprint HEP 95-45

[4] T. Vachaspati and A. Achucarro, "Semilocal cosmic strings", Phys. Rev. **D44** (1991) 3067

[5] E. Witten, "Phases of N=2 theories in two dimensions", Nucl. Phys. **B403** (1993) 159

[6] D.R. Morrison and M.R. Plessner, "Summing the instantons: quantum cohomology and mirror symmetry in toric varieties", Nucl. Phys. **B440** (1995) 279

[7] B.J. Schroers, "Bogomol'nyi solitons in a gauged $O(3)$ sigma model", Phys. Lett. **356B** (1995) 291

[8] P. Fayet and J. Iliopoulos, "Spontaneously broken supergauge symmetries and Goldstone spinors", Phys. Lett. **51B** (1974) 461

[9] M. Audin, *The topology of torus actions on symplectic manifolds* (Birkhäuser, Basel, 1991)

[10] A. Jaffe and C. Taubes, *Vortices and monopoles* (Birkhäuser, Boston, 1980)

[11] E.J. Weinberg, "Fundamental monopole and multimonopole solutions for arbitrary simple gauge groups", Nucl. Phys. **B167** (1980) 500 ;
C.H. Taubes, "Surface integrals and monopole charges in non-abelian gauge theories", Commun. Math. Phys. **81** (1981) 299

[12] Y. Yang, "A necessary and sufficient condition for the existence of multisolitons in a self-dual gauged sigma model", preprint 1995, to appear in Commun. Math. Phys.

[13] Y. Yang, "On a system of nonlinear elliptic equations arising in theoretical physics", to appear

[14] G.W. Gibbons, M.E. Ortiz, F. Ruiz Ruiz and T.M. Samols, "Semilocal strings and monopoles", Nucl. Phys. **B385** (1992) 127

[15] R.A. Leese and T.M. Samols, "Interaction of semilocal vortices", Nucl. Phys. **B396** (1993) 639

[16] M. Hindmarsh, "Existence and stability of semilocal strings", Phys.Rev.Lett. **68** (1992) 1263; "Semilocal topological defects", Nucl. Phys. **B392** (1993) 461

[17] R. Rajaraman, *Solitons and instantons* (North-Holland, Amsterdam, 1982)

[18] R.S. Ward, "Slowly moving lumps in the $\mathbf{CP}^1$ model in (2+1) dimensions", Phys. Lett. **158B** (1985) 424 ;
R.A. Leese, "Low-energy scattering of solitons in the $\mathbf{CP}^1$ model", Nucl. Phys. **B344** (1990) 33

[19] J.M. Speight, "Low-energy dynamics of a $\mathbf{CP}^1$ lump on a sphere", J. Math. Phys. **36** (1995) 796

[20] E. Witten, "Superconducting strings", Nucl. Phys. **B249** (1985) 557

[21] A. Vilenkin and E.P.S. Shellard, *Cosmic strings and other topological defects* (Cambridge University Press, Cambridge, 1994)

[22] P.A. Shah, "Vortex scattering at near-critical coupling", Nucl. Phys. **B429** (1994) 259; P.A. Shah, "Phase transitions in a vortex gas", Nucl. Phys. **B438** (1995) 589

[23] G. Bimonte and G. Lozano, "Vortex solutions in two-Higgs-doublet systems", Phys. Lett. **326 B** (1994) 270

[24] C. Gross, "Topology of the Skyrmion bundle", J. Math. Phys. **36** (1995) 4406

[25] J. Gladikowski, B.M.A.G. Piette and B.J. Schroers, "Skyrme-Maxwell solitons in (2+1) dimensions", Phys. Rev. **D53** (1996) 844

[26] J. Hong, Y. Kim and P.Y. Pac, "On the multivortex solutions of the abelian Chern-Simons-Higgs theory", Phys. Rev. Lett. **64** (1990) 2230 ;
R. Jackiw and E. Weinberg, "Self-Dual Chern-Simons vortices", Phys. Rev. Lett **64** (1990) 2234 ;
R. Jackiw, K. Lee and E.J. Weinberg, "Self-Dual Chern-Simons Solitons", Phys. Rev. **D42** (1990) 3488.

[27] A. Khare, "Semilocal Chern-Simons vortices", Phys. Rev. **D46** (1992) R2287 ;
W.G. Fuertes and J.M. Guilarte, "Semilocal Chern-Simons defects", J. Math. Phys. **37** (1996) 554

[28] P.K. Ghosh and S.K. Ghosh, "Topological and non-topological solitons in a gauged $O(3)$ sigma model with Chern-Simons term" Phys. Lett. **366B** (1996) 199

[29] K. Arthur and D.H. Tchrakian, "Topological and non-topological self-dual Chern-Simons solitons in a gauged $O(3)$ sigma model", DIAS preprint STP-96-07

[30] K. Kimm, K. Lee and T. Lee, "Anyonic Bogomol'nyi soliton in a gauged $O(3)$ sigma model", Columbia preprint CU-TP-712, hep-th/9510141

[31] J. Gladikowski, "Topological Chern-Simons vortices in the gauged $O(3)$ sigma model", Durham preprint DTP 95-75, hep-th/9603056, to appear in Zeitschrift für Physik C

[32] C. Lee, K. Lee and H. Min, "Self-Dual Maxwell-Chern-Simons Solitons", Phys. Lett. **252B** (1990) 79

[33] C. Kim, "Self-dual vortices in the generalised abelian Higgs model with independent Chern-Simons interaction", Phys. Rev. **D47** (1993) 673