Commun. Math. Phys. 245, 123–147 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-003-1006-2

Communications in
**Mathematical
Physics**

The Interaction Energy of Well-Separated Skyrme Solitons

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Received: 8 January 2003 / Accepted: 31 January 2003
Published online: 17 December 2003 –

Abstract: We prove that the asymptotic field of a Skyrme soliton of any degree has a non-trivial multipole expansion. It follows that every Skyrme soliton has a well-defined leading multipole moment. We derive an expression for the linear interaction energy of well-separated Skyrme solitons in terms of their leading multipole moments. This expression can always be made negative by suitable rotations of one of the Skyrme solitons in space and iso-space. We show that the linear interaction energy dominates for large separation if the orders of the Skyrme solitons' multipole moments differ by at most two. In that case there are therefore always attractive forces between the Skyrme solitons.

1. Skyrme Solitons

The fundamental field of Skyrme's theory [1] is a map

$$U : \mathbb{R}^3 \to SU(2). \tag{1.1}$$

We denote points in $\mathbb{R}^3$ by x with coordinates x_i, $i = 1, 2, 3$, and Euclidean length $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Sometimes we write $\hat{x}$ for the unit vector x/r. It is often useful to parametrise U in terms of the Pauli matrices τ_1, τ_2 and τ_3 and the triplet of pion fields π_1, π_2 and π_2 as

$$U(x) = \sigma(x) + i\pi_a(x)\tau_a, \tag{1.2}$$

where summation over the repeated index a is implied and the field σ is determined by the constraint $\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$. In this introductory section we do not specify the class of functions to which U belongs. It is assumed to be sufficiently smooth for all the following operations to make sense.

The Skyrme energy functional is best written in terms of the Lie-algebra valued currents

$$L_i = U^\dagger \partial_i U \tag{1.3}$$

or

$$R_i = U \partial_i U^\dagger, \tag{1.4}$$

where $\partial_i = \partial/\partial x_i$. It is

$$E[U] = -\int d^3x \left(\frac{1}{2}\mathrm{tr}(L_i L_i) + \frac{1}{16}\mathrm{tr}([L_j, L_i][L_j, L_i]) \right). \tag{1.5}$$

The Euler-Lagrange equation for stationary points of this functional is conveniently expressed in terms of the modified currents

$$\tilde{L}_i = L_i - \frac{1}{4}[L_j, [L_j, L_i]] \tag{1.6}$$

and

$$\tilde{R}_i = R_i - \frac{1}{4}[R_j, [R_j, R_i]], \tag{1.7}$$

where we again use the convention that repeated indices are summed over. It reads

$$\partial_i \tilde{L}_i = 0 \tag{1.8}$$

or, equivalently,

$$\partial_i \tilde{R}_i = 0. \tag{1.9}$$

Here we are interested in finite-energy solutions of the Euler-Lagrange equation. It is shown in [2] that the finite energy requirement implies that the map U tends to a constant value at infinity in a weak sense. We choose that constant to be the identity element $1 \in SU(2)$ and demand

$$\lim_{r\to\infty} U(x) = 1. \tag{1.10}$$

The boundary condition (1.10) means that the domain of U is effectively compactified to a three-sphere. Since the target space is also a three-sphere, maps satisfying (1.10) have an associated integer degree. The first rigorous proof that for a finite-energy Skyrme configuration in a very general class of functions the degree

$$\deg[U] = -\frac{1}{24\pi^2}\int d^3x\, \epsilon_{ijk}\, \mathrm{tr}\left(L_i L_j L_k\right) \tag{1.11}$$

is an integer was given in [3]. This result means that the space

$$\mathcal{C} = \{U : \mathbb{R}^3 \to SU(2) \mid E[U] < \infty\} \tag{1.12}$$

of finite-energy configurations is a disjoint union of sectors

$$\mathcal{C}_k = \{U : \mathbb{R}^3 \to SU(2) \mid E[U] < \infty \quad \text{and} \quad \deg[U] = k\} \tag{1.13}$$

labelled by the integers $k \in \mathbb{Z}$.

The symmetry group of Skyrme's theory will play an important role in our discussion. The energy functional (1.5), the boundary condition (1.10) and the degree (1.11) are invariant under the action of the Euclidean group of translations and rotations in $\mathbb{R}^3$ and under rotations of the pion fields

$$\pi_a \mapsto G_{ab}\pi_b, \qquad G \in SO(3), \tag{1.14}$$

which we call iso-rotations. Reflections in space $S : x \mapsto -x$ and in iso-space $\pi_a \mapsto -\pi_a$ both leave the energy invariant but each changes the sign of the degree. The pull-back of Skyrme configurations U via S provides a map

$$\tilde{S} : \mathcal{C}_k \to \mathcal{C}_{-k}, \quad U \mapsto U \circ S \tag{1.15}$$

which preserves the energy.

It was shown in [4] that the energy in each topological sector is bounded below by a multiple of the degree. It follows from the results of [5] that the bound cannot be attained for the standard version of the Skyrme model described here, so that we have the strict inequality

$$E[U] > 12\pi^2|k|. \tag{1.16}$$

The bound ensures that the infima

$$I_k = \inf\{E[U] \,|\, U \in \mathcal{C}_k\} \tag{1.17}$$

are well-defined. The question of whether the infima are attained was first addressed by Esteban in the paper [2]. Amongst other things Esteban proved that, for a suitable class of functions,

$$I_k \leq I_l + I_{k-l} \tag{1.18}$$

for all $k, l \in \mathbb{Z}$. She also showed that infima are attained provided one assumes the strict inequality

$$I_k < I_l + I_{k-l} \tag{1.19}$$

for all $k \in \mathbb{Z} - \{0, \pm 1\}$ and $l \in \mathbb{Z} - \{0, k\}$ in the range $|l| + |k - l| < \sqrt{2}|k|$. In [3] it was shown that the result still holds if one widens the class of allowed functions but the inequality (1.19) remains a necessary assumption in the proof. The strict inequality is also of interest in physics. As we shall see it is related to the question of attractive forces in the Skyrme model.

For low values of k the existence and nature of minima of the Skyrme energy functional is understood in more detail. For fields of degree one, the highly symmetric hedgehog ansatz

$$U_H(x) = \exp(if(r)\hat{x}_a\tau_a), \tag{1.20}$$

introduced already by Skyrme, leads to an ordinary differential equation for the profile function f. With the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$ the resulting Skyrme configuration has degree 1 and is called the Skyrmion. It was shown in [6] that the Skyrmion minimises the Skyrme energy functional amongst all degree one configurations of the hedgehog form. In [2] the existence of the minimum of the Skyrme energy functional in $\mathcal{C}_1$ was proved, but it has not been established rigorously that minimising configurations have the symmetry of the hedgehog field (1.20). Note that if U_1 is a minimal energy configuration in $\mathcal{C}_1$ then the reflected configuration $\tilde{S}(U_1)$ has the same energy and minimises the energy in $\mathcal{C}_{-1}$.

In the following we use the term Skyrme solitons for minimal energy solutions of the Skyrme equation with non-zero degree k. There is overwhelming numerical evidence that the minimum in $\mathcal{C}_2$ is attained by a configuration of toroidal symmetry [7]. For higher degree, too, much is known numerically about the minima of Skyrme's energy functional in the sector $\mathcal{C}_k$. Numerical searches, assisted by analytical ansätze and investigations of the possible symmetries of Skyrme solitons, suggest the existence of Skyrme solitons in all sectors $\mathcal{C}_k$ up to $k = 22$. The energies are sufficiently accurately computed that it appears that the inequality (1.19) is satisfied for all k in the range $2 \le k \le 20$ and all l in the range $0 < l < k$, see [8–10].

The existence of attractive forces between two Skyrmions was already shown by Skyrme, using the product ansatz. In this paper we investigate the existence of attractive forces between general Skyrme solitons. It is perhaps worth stressing that our arguments in fact apply to any finite-energy solution of the Skyrme equation, not just minimal ones. An earlier attempt at proving the existence of attractive forces between Skyrme solitons was made in the unpublished paper [11]. Our approach is partly inspired by ideas in [11] but also fills important gaps left there. Our main tool is an asymptotic expansion of Skyrme solitons. In Sect. 2 we show that Skyrme fields have an asymptotic expansion in powers of $1/r$ and that for non-trivial Skyrme solitons that expansion always has a non-trivial leading multipole. In Sect. 3 we study the interaction energy of two Skyrme solitons and show that it is dominated by a certain linear interaction energy provided the orders of the leading multipoles of the Skyrme solitons do not differ by more than 2. In Sect. 4 we derive an expression for the linear interaction energy of two multipoles, and show that it can always be made negative by suitable relative rotations in space and iso-space. At the end of this paper we briefly comment on the relationship between our results and Esteban's work, and on the implications for the existence of Skyrme solitons of arbitrary degree.

2. The Asymptotics of Skyrme Solitons

The aim of this section is to show that if U and the currents L_j are a little better than continuous, then U is smooth in $\mathbb{R}^3$, and has a non-trivial asymptotic expansion in powers of $1/r$ and $\log r$ as $r \to \infty$. By *non-trivial* we mean here that if U is non-constant, then it cannot happen that all terms in the asymptotic expansion vanish. Put another way, it is not possible for U to approach 1 at infinity faster than *every* power of $1/r$ unless U is identically equal to 1 in $\mathbb{R}^3$.

To make a precise statement, say that the (possibly matrix-valued) function f is in $C_b^{0,\alpha}(\mathbb{R}^3)$, with $0 < \alpha < 1$, if f satisfies

$$\sup_{x\in\mathbb{R}^3} |f| + \sup_{x,x'\in\mathbb{R}^3, x\neq x'} \frac{(1+r+r')^\alpha |f(x) - f(x')|}{|r-r'|^\alpha + (1+r+r')^\alpha d(\omega,\omega')^\alpha} < \infty. \tag{2.1}$$

Here $x = r\omega$, $x' = r'\omega'$, where ω is regarded as an angular variable (unit vector) living on the unit 2-sphere, and $d(\omega, \omega')$ is the 2-sphere distance between ω and ω'. If we consider the space $C_b^{0,\alpha}(K)$, where K is a bounded subset of $\mathbb{R}^3$, then this is precisely the same as the usual space of (bounded) Hölder-continuous functions, with Hölder exponent α. However, $C_b^{0,\alpha}(\mathbb{R}^3)$ is slightly different from the usual space of bounded Hölder-continuous functions on $\mathbb{R}^3$.

The main result of this section can be summarized as follows:

Theorem 2.1. *Suppose that for some $\delta > 0$,*

$$(1+r)^\delta(U-1) \in C_b^{0,\alpha}, \quad (1+r)^\delta L_j \in C_b^{0,\alpha}. \tag{2.2}$$

Suppose further that U satisfies the Skyrme differential Eqs. (1.8) in the sense of distributions.

Then U is smooth in $\mathbb{R}^3$ and has a complete asymptotic expansion in powers of $1/r$ *and* $\log r$*, for large r. If U is non-constant, then this expansion has a leading term which is harmonic, hence a multipole.*

Note that the hypotheses (2.2) force U to approach 1 and the L_j to approach 0 like $r^{-\delta}$ as $r \to \infty$. As a technical remark, we point out that the assumption of Hölder-continuous currents implies that U will have a Hölder-continuous first derivative. The second derivatives $\partial_j \partial_k U$ are then defined only in the sense of distributions, but in the Skyrme equation $\partial_j \partial_k U$ enters linearly, and is multiplied by continuous functions (smooth functions of the currents) see Sect. 2.1 below. In particular, the left-hand side of the Skyrme equation $\partial_j \tilde{L}_j = 0$ makes sense as a distribution if (2.2) holds.

The assumptions (2.2) of Theorem (2.1) do not follow immediately from the variational analysis used by Esteban in [2]. Her methods only give that the derivatives $\partial_j U$ are locally square-integrable (and that the components of U are locally bounded). On physical grounds, one expects minimizers of the Skyrme energy functional to satisfy (2.2), but it would be desirable to bridge the gap between the analysis given here and what was proved rigorously in [2 and 3]. This issue will not be pursued further here.

The proof of Theorem (2.1) proceeds in four steps, each of which takes up one of the following subsections. In the first subsection we rewrite the Skyrme equation in order to make explicit the form of the non-linearities. The equation is *quasilinear*, in the sense that the derivatives of highest order (2) enter linearly. The Skyrme equation can therefore be regarded as a second-order linear elliptic PDE, with Hölder-continuous coefficients. Then standard regularity theorems (Schauder estimates) yield that U is smooth.

In Sect. 2.2 we study the behaviour of the Skyrme equation at spatial infinity, by introducing coordinates (s, ω), where $s = r^{-1}$, so that the 2-sphere at infinity becomes a genuine boundary at $s = 0$. After a rescaling, the Laplacian of $\mathbb{R}^3$ is replaced by

$$\Delta_b = (s\partial_s)^2 - s\partial_s + \Delta_\omega, \tag{2.3}$$

where Δ_ω is the Laplacian of the unit 2-sphere. The analysis is now guided by the corresponding analysis of a system of ordinary differential equations with regular singular point at $s = 0$. In particular, one does not expect the solutions to be smooth near $s = 0$, but one does expect a non-trivial expansion in powers of s (and possibly $\log s$). The b-calculus of [12] enables us to make this kind of argument precise. Thus in Sect. 2.2 we show that if $U = 1 + u$, then u is *conormal* at $s = 0$, which is to say that u and all derivatives of the form $(s\partial_s)^m \nabla_\omega^n u$ are continuous as $s \to 0$. (This condition is strictly weaker than u being smooth near $s = 0$.) In Sect. 2.3, we show that it is not possible for U to approach 1 faster than $r^{-\mu}$ for *every* $\mu > 0$ unless $U = 1$ in $\mathbb{R}^3$. Finally in Sect. 2.4 we combine this fact with another application of the b-calculus to show the existence of a non-trivial asymptotic expansion in powers of r^{-1} and $\log r$ (or equivalently in powers of s and $\log s$).

2.1. Rewriting the Skyrme equation. To begin with we work near a fixed point of $\mathbb{R}^3$, which we may as well take to be the origin 0. By replacing $U(x)$ by $U(0)^{-1}U(x)$ we can assume that $U(0) = 1$. Write

$$U(x) = 1 + u(x), \tag{2.4}$$

so that $u(0) = 0$. Because U is unitary, the 2×2 complex matrix u will satisfy the algebraic constraints

$$u + u^\dagger + uu^\dagger = 0, \quad \operatorname{tr}(u) + \det u = 0. \tag{2.5}$$

In particular, u is neither exactly skew-hermitian nor trace-free. Then

$$L_i = \partial_i u + u^\dagger \partial_i u \tag{2.6}$$

and

$$\partial_i L_i = (1 + u^\dagger)\Delta u + \partial_i u^\dagger \partial_i u. \tag{2.7}$$

The cubic term in the currents can be written

$$\frac{1}{2} L_j [L_j, L_i] = \frac{1}{2}(\partial_j u + u^\dagger \partial_j u)[\partial_j u + u^\dagger \partial_j u, \partial_i u + u^\dagger \partial_i u]. \tag{2.8}$$

Taking the divergence, and using the notation

$$v_{ij} = \partial_i u^\dagger \partial_j u, \tag{2.9}$$

we obtain

$$\frac{1}{2} L_j [L_j, L_i] = (1 + u^\dagger)(T + F), \tag{2.10}$$

where

$$\begin{aligned} T &= T(u, \partial u, \partial^2 u) \\ &= \frac{1+u}{2}\left(L_j[L_j, (1+u^\dagger)\Delta u] + L_j[(1+u^\dagger)\partial_i u \partial_j u, L_i]\right) \end{aligned} \tag{2.11}$$

and

$$F = F(u, \partial u) = \frac{1+u}{2}\left(v_{ij}[L_i, L_j] + L_j[L_j, v_{ii}] + L_j[(1+u^\dagger)v_{ij}, L_i]\right). \tag{2.12}$$

The nonlinear terms have been divided here so that T is polynomial in the first derivatives of u and linear in its second derivatives, while F only contains u and its first derivatives.

With this notation, the full Skyrme equation can be written

$$P(u, \partial u, \partial^2 u) = Q(u, \partial u) + F(u, \partial u), \tag{2.13}$$

where

$$P(u, \partial u, \partial^2 u) = \Delta u + T(u, \partial u, \partial^2 u) \text{ and } Q(u, \partial u) = -(1+u)v_{ii}. \tag{2.14}$$

Notice that Q is (approximately) quadratic in u, T is of degree three and F is of degree four. Note also that T is linear in $\partial_i \partial_j u$ (and quadratic in $\partial_i u$).

Because of the assumed Hölder continuity of the currents, the coefficients of the differential operator

$$f \longmapsto P(u, \partial u, \partial^2 f) \tag{2.15}$$

are Hölder continuous, and this operator is linear and elliptic in a small neighbourhood K of 0. Moreover, the RHS $Q + F$ of (2.13) is also in $C_b^{0,\alpha}(K)$, so by elliptic regularity, it follows that $u \in C_b^{2,\alpha}(K)$. Then the currents are in $C_b^{1,\alpha}(K)$ and the process continues to show that u is in $C_b^{k,\alpha}(K)$ for every k. Thus u is smooth near 0. Since the point 0 was arbitrary, the argument shows that u is smooth in $\mathbb{R}^3$.

2.2. Boundary regularity. In order to analyze the asymptotic behaviour of the field U, we shall make a transformation of the problem which involves passing from $\mathbb{R}^3$ to a "compactification" $\overline{\mathbb{R}^3}$ in which the sphere at infinity becomes a genuine boundary. This is easily achieved by introducing the coordinate $s = 1/r$ along with angular coordinates θ and φ in $\mathbb{R}^3$. Then in the space $[0, \infty) \times S^2$ with coordinates (s, θ, φ), the boundary $s = 0$ corresponds to $r = \infty$ and θ and φ give coordinates on the boundary, which is the 2-sphere "at infinity". $\overline{\mathbb{R}^3}$ is defined to be the union of $\mathbb{R}^3$ with the 2-sphere at infinity attached in this way.

It can be cumbersome to work with explicit coordinates on the 2-sphere, so we again use ω for points on S^2 and represent any point other than the origin of $\overline{\mathbb{R}^3}$ in the form (s, ω).

Next, introduce rescaled derivatives,

$$D_i = r\partial_i = \frac{1}{s}\partial_i. \tag{2.16}$$

These vector fields have the property that they are linear combinations, with coefficients that are smooth, down to $s = 0$, of the basic vector fields $s\partial_s$, ∂_θ and ∂_φ. The Euclidean Laplacian takes the form

$$\Delta = s^2 \Delta_b \tag{2.17}$$

with Δ_b defined in (2.3).

Now we write $U = 1 + u$ for large r (that is, for small positive s) and make the rescalings of (2.13) suggested by (2.16) and (2.17). The result is a 'b' version of the Skyrme equation,

$$P_b(u, Du, D^2u) = Q_b(u, Du) + s^2 F_b(u, Du), \tag{2.18}$$

where

$$P_b(u, Du, D^2u) = \Delta_b u + s^2 T_b(u, Du, D^2u), \tag{2.19}$$
$$Q_b(u, Du) = -(1 + u)D_i u^\dagger D_i u, \tag{2.20}$$

these terms being obtained by replacing ∂_i by the rescaled derivative D_i wherever they occur.

The reason for reformulating the equation in this way is that there is a well-established theory, called the b-calculus, which can be used to analyze equations of this kind

[12]. The b-calculus is concerned with b-differential operators. In the present situation, a b-differential operator is just a differential operator of the form

$$P = \sum_{a+b+c\leq m} C_{abc}(s,\omega)(s\partial_s)^a \partial_\theta^b \partial_\varphi^c, \tag{2.21}$$

where the coefficients C_{abc} are smooth up to the boundary $s = 0$. From (2.3) it is clear that the rescaled Laplacian Δ_b is an example of such an operator. The set of all such operators will be denoted by Diff_b, those of order k by Diff_b^k.

The first aspect of this theory that is needed is the counterpart of the elliptic regularity for Hölder spaces that we used in the previous subsection. For this we need b-Hölder spaces, already introduced in (2.1). With the new variable s, we have $f(s,\omega) \in C_b^{0,\alpha}$ if

$$\sup_{x\in\mathbb{R}^3} |f| + \sup_{(s,\omega)\neq(s',\omega')} \frac{(s+s')^\alpha |f(s,\omega) - f(s',\omega')|}{|s-s'|^\alpha + (s+s')^\alpha d(\omega,\omega')^\alpha} < \infty. \tag{2.22}$$

Now put

$$C_b^{k,\alpha} = \{f : Lf \in C_b^{0,\alpha} \text{ for all } b\text{-differential operators } L \in \mathrm{Diff}_b^k\}$$

and

$$\mathcal{A} = \{u : Lu \in C_b^{0,\alpha} \text{ for all } L \in \mathrm{Diff}_b\}.$$

In order to force functions to decay as $s \to 0$, we introduce *weighted* versions of these spaces,

$$s^\delta C_b^{k,\alpha} = \{u = s^\delta v : v \in C_b^{k,\alpha}\}, \quad s^\delta \mathcal{A} = \{u = s^\delta v : v \in \mathcal{A}\}.$$

The following is a very special case of elliptic regularity for b-differential operators [13].

Theorem 2.2. *Consider the differential operator $P = \Delta_b + s^2 E$, where E is a second-order differential operator with coefficients smooth up to the boundary $s = 0$. Suppose that $Pu = f$ near $s = 0$, with u and f in $s^\delta C_b^{0,\alpha}$. Then if δ is not an integer, it follows that $u \in s^\delta C_b^{2,\alpha}$.*

We want to apply this to the Skyrme equation, written in the form (2.19). After the last section, we know that the coefficients of the perturbing term E are smooth for $s > 0$, but all we know at $s = 0$ is the original assumption that the currents are in $s^\delta C_b^{0,\alpha}$. The elliptic regularity statement still holds in this case (though this statement does not seem to be available in the literature). Applying this result, for δ a small positive number, we obtain first that $u \in s^\delta C_b^{2,\alpha}$, which gives that the coefficients are in $s^\delta C_b^{1,\alpha}$, so $u \in s^\delta C_b^{3,\alpha}$ and so on. Iterating, we find $u \in s^\delta \mathcal{A}$.

This is a major step forward: in particular, the angular variation of u is now very well controlled. However, u may still be very far from being smooth up to the boundary or having an asymptotic expansion there. Indeed, horrors like $s^\delta \sin\log s$ lie in $s^\delta \mathcal{A}$.

2.3. Power-law decay of u. In this section and the next we establish that a topologically non-trivial solution of the Skyrme equation must have a non-trivial asymptotic expansion in powers of $1/r$. We show first that it is not possible for a topologically non-trivial solution to approach 1 faster than every power of $1/r$. This result is then fed into an iterative analysis of the equation in the next section. These two sections can, however, be read in either order. The main result of this section is as follows.

Theorem 2.3. *If the topological charge of the Skyrme field U is non-zero and if U satisfies the hypotheses of Theorem 2.1, then there exists some $\mu \in \mathbb{R}^+$ such that $r^\mu u(r,\omega)$ does* not *tend to zero as $r \to \infty$.*

To set this result in a more general context, recall that a partial differential equation $Lu = 0$ is said to have the unique continuation property at a point 0, say, if the following is true:

$$\text{If all derivatives of } u \text{ vanish at } 0, \text{ then } u = 0 \text{ in some neighbourhood of } 0. \tag{2.23}$$

The methods used to establish that a second-order PDE has the unique continuation property establish analogous statements with somewhat weaker hypotheses. For example, a simplified version of Theorem 17.2.6 of [16] is as follows.

Theorem 2.4. *Let $a_{jk}(x)$ be smooth and positive-definite in a neighbourhood X of 0, and suppose that $a_{jk}(0) = \delta_{jk}$. Suppose that for $x \in X$, the smooth function u satisfies*

$$|a_{jk}(x)\partial_j\partial_k u| \le A\,(|u(x)| + |\nabla u(x)|) \tag{2.24}$$

for some constant A, and

$$|x|^{-\mu}|u(x)| \to 0 \ \textit{as} \quad |x| \to 0 \quad \textit{for} \text{ every } \mu \in \mathbb{R}^+. \tag{2.25}$$

Then $u = 0$ identically in a neighbourhood of 0.

Using this result we can prove the following:

Theorem 2.5. *Let U satisfy the hypotheses of Theorem (2.1). Then, if the topological charge of U is non-zero, U cannot be constant in any open subset of $\mathbb{R}^3$.*

Proof. Define

$$W = \{x \in \mathbb{R}^3 : U(y) = U(x) \text{ for all } y \text{ in some neighbourhood of } x\}. \tag{2.26}$$

Then W is open by definition. By Theorem 2.4, W is also closed. To see this, suppose that $x_n \in W$, $x_n \to x_0 \in \mathbb{R}^3$. Then all derivatives of U are zero at x_0, so (2.25) holds (with 0 replaced by x_0). The Skyrme equation, in the form (2.13), implies a differential inequality of the form (2.24) in a neighbourhood of x_0. It follows[1] that U is identically constant in a neighbourhood of x_0, so that $x_0 \in W$. Since $\mathbb{R}^3$ is connected, $W = \emptyset$ or $W = \mathbb{R}^3$. $\square$

[1] Unique continuation theorems do not extend wholesale to systems. However, in our case, the leading term is the Laplacian and the other second-order terms $C(u, \partial u, \partial^2 u)$ are non-scalar but small near x_0. The proof of Theorem 2.4 goes through in this case.

We will use this theorem to give an indirect proof of Theorem 2.3. Suppose $r^{\mu}u(x) \to 0$ as $r \to \infty$ for every μ. Adapting Theorem 2.4 we shall show that then $u = 0$ for *all* sufficiently large r. Thus U is constant in an open set, hence by the previous result constant everywhere.

So consider the b-differential operator

$$P_b = \Delta_b + s^2 T_b \tag{2.27}$$

on $\overline{\mathbb{R}^3}$, where the coefficients of T_b are in $C_b^{0,\alpha}$. By rescaling Theorem 17.2.6 in [16], we obtain

Theorem 2.6. *Suppose that*

$$|P_b u(s,\omega)| \leq A s^{\delta}(|u(s,\omega)| + |Du(s,\omega)|) \textit{ for all } 0 < s < s_0, \omega \in S^2 \tag{2.28}$$

and that

$$s^{-\mu}|u(s,\omega)| \to 0 \textit{ as } s \to 0 \textit{ for all } \mu. \tag{2.29}$$

Then $u(s,\omega) = 0$ for $0 \leq s < s_1$, where s_1 is some small positive number.

The “b” version (2.18) of the Skyrme equation implies a differential inequality of the form (2.28), just as before. It follows that if u decays faster than any power of r, then $u = 0$ for all sufficiently large r. By the remarks before the statement of Theorem 2.6, the proof of Theorem 2.3 is now complete.

2.4. Refined regularity, asymptotic expansions. The main result of this section can be stated as follows:

Theorem 2.7. *Let $U = 1 + u$ satisfy the hypotheses of Theorem 2.1. Then there is some integer $M \geq 1$ and an asymptotic expansion*

$$u \sim \sum_{j=M}^{2M} Y_j(\omega) r^{-(j+1)} + \sum_{j=2M+1}^{\infty} r^{-(j+1)} w_j(\omega, \log r) \quad \textit{for large } r. \tag{2.30}$$

Here the Y_j are Lie-algebra valued spherical harmonics,

$$\Delta_\omega Y_j = -j(j+1) Y_j, \tag{2.31}$$

and $Y_M \neq 0$, so that the piece $\sum_{j=M}^{2M} Y_j(\omega) r^{-(j+1)}$ is a non-zero harmonic function. The functions w_j are smooth in ω and polynomial in $\log r$.

It will follow from the proof that the terms w_{2M+1} to w_{3M+1} are of at most degree 1 in $\log r$, the terms w_{3M+2} to w_{4M+2} are of at most degree 2 in $\log r$ and so on. We remark also that the asymptotic expansion can safely be differentiated term by term to give asymptotic expansions of all derivatives of u.

In order to motivate the proof, consider the equation

$$\Delta_b u = f, \tag{2.32}$$

where u and f are defined for small s. If f has the form $f = s^\lambda g(\omega)$, then a solution can be found as follows. Expand g as a sum of spherical harmonics,

$$g = \sum_{j=0}^{\infty} g_j, \text{ where } \Delta_\omega g_j = -j(j+1)g_j$$

and seek a solution $u = \sum u_j(s)g_j$. Then u_j must satisfy

$$[(s\partial_s)^2 - (s\partial_s) - j(j+1)]u_j(s) = s^\lambda.$$

This is solved by

$$u_j(s) = \frac{s^\lambda}{\lambda(\lambda-1) - j(j+1)}$$

provided there is no resonance, that is to say

$$\lambda \neq -j, j+1.$$

In the resonant case, $u_j(s)$ has the form $s^\lambda(A + B\log s)$. The general solution is obtained by combining this with an arbitrary solution of the homogeneous equation $\Delta_b v = 0$. If we require $u \to 0$ as $s \to 0$, then v must itself go to zero and hence will be a sum of multipoles, $v = \sum_{j=0}^{\infty} Y_j(\omega)s^{j+1}$, where Y_j satisfies (2.31).

The b-calculus extends results of this kind to functions in $s^\delta\mathcal{A}$ (which, as we have seen, can be far from having expansions in powers of s). In order to summarize the needed results, write

$$f = \mathcal{O}(s^\delta) \text{ instead of } f \in s^\delta\mathcal{A}. \tag{2.33}$$

This conforms to the use of the "$\mathcal{O}$"-notation in the rest of the paper, but has the additional property

$$\text{If } f = \mathcal{O}(s^\delta), \text{ then } Lf = \mathcal{O}(s^\delta) \text{ for all } L \in \mathrm{Diff}_b. \tag{2.34}$$

Lemma 2.8. *Suppose that u and f defined near $s = 0$ satisfy (2.32). If $u = \mathcal{O}(s^\delta)$ and $f = \mathcal{O}(s^{n+\delta})$, where $\delta > 0$ and n is a positive integer, then*

$$u = \sum_{j=0}^{n-1} Y_j(\omega)s^{j+1} + \mathcal{O}(s^{n+\delta}), \tag{2.35}$$

where Y_j satisfies (2.31). In particular the sum on the RHS is harmonic

$$\Delta_b \sum_{j=0}^{n-1} Y_j(\omega)s^{j+1} = 0. \tag{2.36}$$

Lemma 2.9. *Suppose that u and f defined near $s = 0$ satisfy (2.32). If $u = \mathcal{O}(s^\delta)$ and*

$$f = \sum_{j=0}^{n-1} f_j(\omega, \log s)s^{j+1} + \mathcal{O}(s^{n+\delta}), \tag{2.37}$$

where $\delta > 0$, n is a positive integer and f_j is polynomial of degree m_j in $\log s$*, then*

$$u = \sum_{j=0}^{n-1} w_j(\omega, \log s)s^{j+1} + \mathcal{O}(s^{n+\delta}), \tag{2.38}$$

where w_j is a polynomial of degree $m_j + 1$ in $\log s$.

These results will be applied to the Skyrme equation, now rewritten as

$$\Delta_b u = Z_b(u) := Q_b(u) - s^2 T_b(u) + s^2 F_b(u). \tag{2.39}$$

Lemma 2.10. *Suppose that $U = 1 + u$ satisfies the Skyrme equation and $u = \mathcal{O}(s^\delta)$. Then $u = \mathcal{O}(s^2)$.*

Proof. On the RHS of (2.39), Q_b is quadratic in Du and the other terms in Z_b are of even higher degree. Hence $Z_b = \mathcal{O}(s^{2\delta})$. Applying Lemma 2.8 we obtain that $u = \mathcal{O}(s^{2\delta}) + \mathcal{O}(s)$. If $2\delta < 1$, we can iterate this argument to obtain eventually $u \in \mathcal{O}(s)$. In [14], it is shown that a solution of the Skyrme equation cannot have leading term $1/r = s$ in its asymptotic expansion. It follows that $u = \mathcal{O}(s^{1+\delta})$ (for a possibly smaller $\delta > 0$). Hence $Z_b(u) = \mathcal{O}(s^{2+2\delta})$ and so, applying Lemma 2.8 again, $u = Y_1 s^2 + \mathcal{O}(s^{2+\delta})$. □

Combining Theorem 2.3 with Lemma 2.8, we see that there exists an integer $M \geq 1$ with the property that

$$u = Y_M(\omega)s^{M+1} + \mathcal{O}(s^{M+1+\delta}), \tag{2.40}$$

where Y_M is a *non-vanishing* spherical harmonic. We can now complete the proof of Theorem 2.7 in the following iterative fashion. From the structure of $Z_b(u)$ it follows from (2.40) that

$$Z_b(u) = f_{2M+1}s^{2M+2} + \mathcal{O}(s^{2M+2+\delta}). \tag{2.41}$$

Applying Lemma 2.9,

$$u = \sum_{j=M}^{2M} Y_j(\omega)s^{j+1} + w_{2M+1}(\omega, \log s)s^{2M+2} + \mathcal{O}(s^{2M+2+\delta}), \tag{2.42}$$

where $w_{2M+1}(\omega, \log s)$ is of degree at most 1 in $\log s$. Now this expression for u is substituted into Z_b, to give

$$Z_b = \sum_{2M+1}^{3M+1} f_j(\omega)s^{j+1} + f_{3M+2}(\omega, \log s)s^{3M+3} + +\mathcal{O}(s^{3M+3+\delta}), \tag{2.43}$$

where f_{3M+2} is of degree at most 1 in $\log s$. Now apply Lemma 2.9 to get

$$u = \sum_{j=M}^{2M} Y_j(\omega)s^{j+1} + \sum_{j=2M+1}^{3M+1} w_j(\omega, \log s)s^{j+1} \\ +w_{3M+2}(\omega, \log s)s^{3M+3} + \mathcal{O}(s^{3M+3+\delta}), \tag{2.44}$$

where the functions $w_{2M+1}, ..., w_{3M+1}$ are of degree at most 1 in $\log s$ and w_{3M+2} is of degree at most 2 in $\log s$. Carrying on in this way we obtain a complete asymptotic expansion.

To complete the proof of Theorem 2.7 note finally that the first $M+1$ terms in the expansion are actually the pion field. Indeed, if the expansion (2.30) is substituted into the algebraic constraints (2.5) we see that the harmonic piece $\sum_M^{2M} Y_j s^{j+1}$ is skew-adjoint and trace-free – the quadratic corrections enter at order s^{2M+2}. □

The upshot of this section is that every Skyrme soliton has a leading Lie-algebra valued multipole field (called a 2^M-pole)

$$u_M(x) = i\tau_a \sum_{m=-M}^{M} \frac{4\pi}{2M+1} Q^a_{Mm} \frac{Y_{Mm}(\theta, \varphi)}{r^{M+1}}, \tag{2.45}$$

where Y_{Mm} are the usual spherical harmonics on S^2, see Appendix A. The leading multipole moment Q^a_{Mm} is independent of the location of the Skyrme soliton, and is acted on naturally by rotations and iso-rotations. It is a key ingredient in the calculations of the following sections. As already mentioned one can show that Skyrme solitons never have asymptotic monopole fields [14]. The leading multipole field of the $B = 1$ hedgehog (1.20) is a triplet of dipoles, and dipoles are known to occur as leading multipoles in a number of Skyrme solitons. The highest leading multipole known from numerical work is an octupole, which occurs in a $B = 7$ configuration with icosahedral symmetry [9].

3. The Interaction Energy of Two Skyrme Solitons

Suppose we have Skyrme solitons $U^{(1)}$ and $U^{(2)}$ of degrees k and l which minimise the energy in the sectors $\mathcal{C}_k$ and $\mathcal{C}_l$. Since the total energies of $U^{(1)}$ and $U^{(2)}$ are finite there must be balls B_1 and B_2 in $\mathbb{R}^3$ so that most of the energy of $U^{(1)}$ and $U^{(2)}$ is concentrated in, respectively, B_1 and B_2. Outside the balls B_1 and B_2 the asymptotic analysis of the previous section applies. Suppose that the leading multipole of $U^{(1)}$ is a 2^M-pole and the leading multipole of $U^{(2)}$ is a 2^N-pole. Denoting the radii of B_1 and B_2 by D_1 and D_2, and with the abbreviation (2.45) for a generic Lie-algebra valued multipole field we have

$$U^{(1)}(x) \sim 1 + u_M(x) \quad \text{for} \quad x \notin B_1 \tag{3.1}$$

and

$$U^{(2)}(x) \sim 1 + v_N(x) \quad \text{for} \quad x \notin B_2. \tag{3.2}$$

Using the translational invariance of the Skyrme energy functional we can assume without loss of generality that B_1 is centred at $X_+ = (0, 0, R/2)$ and that B_2 is centred at $X_- = (0, 0, -R/2)$, where R is so large that B_1 and B_2 do not overlap, i.e. $R > D_1 + D_2$. The parameter R will be interpreted as the separation of the Skyrme solitons. There is clearly an ambiguity in the definition of such a separation parameter, but this does not affect our calculation of leading terms in the limit where R becomes large. Then we define the following product configuration:

$$U_R(x) = U^{(1)}(x)U^{(2)}(x). \tag{3.3}$$

This configuration has degree $k+l$ and we shall see shortly that its energy is finite, so that $U_R \in \mathcal{C}_{k+l}$.

Our goal is to study the energy of the product configuration U_R perturbatively in the limit of large R and to compute the leading terms in powers of $1/R$. A similar calculation for moving and spinning Skyrmions was performed in [15], where some further details are given. Let $L_i^{(1)}$ and $\tilde{L}_i^{(1)}$ be the currents (1.3) and (1.6) constructed out of $U^{(1)}$, and $R_i^{(2)}$ and $\tilde{R}_i^{(2)}$ be the currents (1.4) and (1.7) constructed out of $U^{(2)}$. Then one computes

$$E[U_R] = E[U^{(1)}] + E[U^{(2)}] + W_2 + W_4. \tag{3.4}$$

The energies $E[U^{(1)}]$ and $E[U^{(2)}]$ are simply the energies of the Skyrme solitons $U^{(1)}$ and $U^{(2)}$ and therefore independent of R. The interaction terms W_2 and W_4 are given by integrals over $\mathbb{R}^3$,

$$W_2 = \int d^3x\, w_2 \quad \text{and} \quad W_4 = \int d^3x\, w_4, \tag{3.5}$$

with integrands

$$w_2 = \operatorname{tr}\left(L_i^{(1)}\tilde{R}_i^{(2)} + \tilde{L}_i^{(1)}R_i^{(2)} - L_i^{(1)}R_i^{(2)}\right) \tag{3.6}$$

and

$$\begin{aligned} w_4 = -\frac{1}{8}\operatorname{tr}\Big([L_i^{(1)}, R_j^{(2)}][L_i^{(1)}, R_j^{(2)}] \\ +[L_i^{(1)}, R_j^{(2)}][R_i^{(2)}, L_j^{(1)}] + [L_i^{(1)}, L_j^{(1)}][R_i^{(2)}, R_j^{(2)}]\Big). \end{aligned} \tag{3.7}$$

We shall see shortly that the term W_2 contains the leading contribution to the interaction energy. However, for the presentation of our method of computation it is more convenient to begin with the quartic term W_4. We split the integration region $\mathbb{R}^3$ into the balls B_1 and B_2 and the complement $C = \mathbb{R}^3 - (B_1 \cup B_2)$. To illustrate our method, consider the integral

$$\begin{aligned} I &= -\frac{1}{8}\int_{B_1} d^3x \operatorname{tr}\left([L_i^{(1)}, R_j^{(2)}][L_i^{(1)}, R_j^{(2)}]\right) \\ &\leq \frac{1}{4}\int_{B_1} d^3x \operatorname{tr}\left(L_i^{(1)}L_i^{(1)}\right)\operatorname{tr}\left(R_j^{(2)}R_j^{(2)}\right). \end{aligned} \tag{3.8}$$

The currents $L_i^{(1)}$ are smooth functions and hence bounded on the compact domain B_1. Therefore

$$|I| \leq -K\int_{B_1} d^3x \operatorname{tr}\left(R_j^{(2)}R_j^{(2)}\right) \tag{3.9}$$

for some positive constant K. Since B_1 is far away from the centre of soliton $U^{(2)}$, the leading contribution to the integral (3.9) is obtained by replacing $R_j^{(2)}$ by the leading multipole component $-i\partial_j v_N$. Using

$$|\partial_j v_N|(x) \leq \frac{K'}{|x - X_-|^{N+2}} \tag{3.10}$$

for a further positive constant K' we conclude that

$$I = \mathcal{O}\left(\frac{1}{R^{2N+4}}\right). \tag{3.11}$$

A similar calculation for the other terms in W_4 shows that

$$\int_{B_1} d^3x\, w_4 = \mathcal{O}\left(\frac{1}{R^{2N+4}}\right). \tag{3.12}$$

Considering the contribution from B_2 we find by the same argument

$$\int_{B_2} d^3x\, w_4 = \mathcal{O}\left(\frac{1}{R^{2M+4}}\right). \tag{3.13}$$

Finally, the remaining integral over C can be estimated as follows. Define

$$F(R) = \int_C d^3x\, \frac{1}{|x - X_+|^{2M+4}} \frac{1}{|x - X_-|^{2N+4}}, \tag{3.14}$$

noting that the integral converges for all values of $R > D_1 + D_2$. Then there is a positive constant K'' such that

$$\int_C d^3x\, w_4 < K''\, F(R). \tag{3.15}$$

The large R behaviour of F can be estimated by a scaling analysis. Fix $R_0 > D_1 + D_2$ and consider $R > R_0$. Changing integration variables $x \mapsto (R_0/R)x$ one computes

$$\begin{aligned} F(R) = \left(\frac{R_0}{R}\right)^{2M+2N+5} \Bigg[& F(R_0) \\ & + \int_{S_1(R)\cup S_2(R)} d^3x\, \frac{1}{|x - X_+|^{2M+4}} \frac{1}{|x - X_-|^{2N+4}} \Bigg], \end{aligned} \tag{3.16}$$

where $S_1(R)$ and $S_2(R)$ are the thick shells,

$$\begin{aligned} S_1(R) &= \{x \in \mathbb{R}^3 | (R_0/R)D_1 \leq |(x_1, x_2, x_3 - (R_0/2)| < D_1\}, \\ S_2(R) &= \{x \in \mathbb{R}^3 | (R_0/R)D_2 \leq |(x_1, x_2, x_3 + (R_0/2)| < D_2\}, \end{aligned} \tag{3.17}$$

which converge to punctured balls

$$\begin{aligned} B_1^0 &= \{x \in \mathbb{R}^3 - \{(0, 0, R_0/2)\} \,|\, |(x_1, x_2, x_3 - (R_0/2)| < D_1\}, \\ B_2^0 &= \{x \in \mathbb{R}^3 - \{(0, 0, -R_0/2)\} \,|\, |(x_1, x_2, x_3 + (R_0/2)| < D_2\} \end{aligned} \tag{3.18}$$

in the limit $R \to \infty$. In that limit the integral over $S_1(R)$ diverges like R^{2N+1} and that over $S_2(R)$ like R^{2M+1}. Combining this with the factor $R^{-(2M+2N+5)}$ we deduce that $F(R)$ decays for large R like $R^{-(2N+4)}$ and $R^{-(2M+4)}$, just like the contributions (3.12) and (3.13). Combining all the terms, we conclude that the leading terms in W_4 decay for large R according to $R^{-(2N+4)}$ and $R^{-(2M+4)}$.

In order to evaluate W_2 we first divide the region of integration into the half-spaces

$$H^+ = \{x \in \mathbb{R}^3 \,|\, x_3 > 0\} \qquad \text{and} \qquad H^- = \{x \in \mathbb{R}^3 \,|\, x_3 < 0\}. \tag{3.19}$$

In H^+ we replace $U^{(2)}$ by the leading term $1 + v_N$ and in H^- we replace $U^{(1)}$ by the leading contribution $1 + u_M$. The result is

$$W_2 \approx \int_{H^+} d^3x\, \frac{1}{4}\mathrm{tr}\left(L_i^{(1)}[\partial_j v_N, [\partial_j v_N, \partial_i v_N]]\right) - \mathrm{tr}\left(\partial_i v_N \tilde{L}_i^{(1)}\right)$$
$$- \int_{H^-} d^3x\, \frac{1}{4}\mathrm{tr}\left(R_i^{(2)}[\partial_j u_M, [\partial_j u_M, \partial_i u_M]]\right) + \mathrm{tr}\left(\partial_i u_M \tilde{R}_i^{(2)}\right). \quad (3.20)$$

Now integrating by parts and using the Euler-Lagrange equations $\partial_i \tilde{L}_i^{(1)} = \partial_i \tilde{R}_i^{(2)} = 0$ for the individual Skyrme solitons we convert two of the terms into an area integral

$$- \int_{H^+} d^3x\, \mathrm{tr}\left(\partial_i v_N \tilde{L}_i^{(1)}\right) + \int_{H^-} d^3x \mathrm{tr}\left(\partial_i u_M \tilde{R}_i^{(2)}\right)$$
$$= \int_{x_3=0} dx_1 dx_2\, \mathrm{tr}\left(v_N \tilde{L}_3^{(1)} + u_M \tilde{R}_3^{(2)}\right). \quad (3.21)$$

Since the x_1x_2-plane is far away from both Skyrme solitons the leading contribution to this area integral can be expressed entirely in terms of the asymptotic fields:

$$\Delta E = 2\sum_{a=1}^{3} \int_{x_3=0} dx_1 dx_2\, (u_M^a \partial_3 v_N^a - v_N^a \partial_3 u_M^a). \quad (3.22)$$

A simple scaling analysis shows that ΔE falls off like $R^{-(N+M+1)}$ for large R. We skip the details here because we shall show how to evaluate ΔE *exactly* in the next section. The remaining terms in (3.20) can be estimated with the techniques used in estimating W_2. The result is

$$W_2 = \Delta E + \mathcal{O}\left(\frac{1}{R^{3N+6}}\right) + \mathcal{O}\left(\frac{1}{R^{3M+6}}\right). \quad (3.23)$$

Combining all terms in (3.4) we conclude that

$$E[U_R] = E[U^{(1)}] + E[U^{(2)}] + \Delta E + \mathcal{O}\left(\frac{1}{R^{2N+4}}\right) + \mathcal{O}\left(\frac{1}{R^{2M+4}}\right). \quad (3.24)$$

Note that ΔE is the leading contribution to the interaction energy if $|N - M| \leq 2$, i.e. if the orders of the leading multipoles of the two Skyrme solitons differ by at most two. We will comment on the validity of this assumption at the end of this paper.

4. Harmonic Functions and Their Interaction Energy

In order to compute the interaction energy ΔE we need to derive some general results about harmonic functions. We define the regions

$$H_\delta^- = \{x \in \mathbb{R}^3 \,|\, x_3 < \delta\} \qquad \text{and} \qquad H_\delta^+ = \{x \in \mathbb{R}^3 \,|\, x_3 > -\delta\}, \quad (4.1)$$

where the positive parameter δ is introduced for technical reasons. Then we introduce the spaces

$$\mathcal{H}^- = \{f : H_\delta^- \to \mathbb{R} \,|\, \Delta f = 0, \quad \lim_{r\to\infty} f(x) = 0\} \quad (4.2)$$

$$\mathcal{H}^+ = \{g : H_\delta^+ \to \mathbb{R} \mid \Delta g = 0, \quad \lim_{r\to\infty} g(x) = 0\}. \tag{4.3}$$

Elements of $\mathcal{H}^-$ tend to zero at the boundary "at infinity" of H_δ^-, elements of $\mathcal{H}^+$ tend to zero at the boundary "at infinity" of H_δ^+. No additional restriction is placed on the behaviour at the boundaries $x_3 = \pm\delta$.

For the calculations in this section it is convenient to split $\mathbb{R}^3$ into $\mathbb{R}^2 \times \mathbb{R}$ and denote vectors in $\mathbb{R}^2$ by bold letters, e.g. $\boldsymbol{x} = (x_1, x_2)$. We then write $x = (\boldsymbol{x}, x_3)$. The most general element of $\mathcal{H}^-$ can be written as

$$f(x) = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2 2k}\ p(\boldsymbol{k})\ \exp(i\boldsymbol{k}\cdot\boldsymbol{x} + kx_3), \tag{4.4}$$

where $k = \sqrt{\boldsymbol{k}^2}$ and the volume element $d^2\boldsymbol{k}/((2\pi)^2 2k)$ arises from the combination of d^3k with the delta-function $\delta(\boldsymbol{k}^2 - k^2)$ which ensures that f satisfies the Laplace equation. Since f is real the Fourier transform p satisfies

$$\bar{p}(\boldsymbol{k}) = p(-\boldsymbol{k}). \tag{4.5}$$

Similarly, the most general element of $\mathcal{H}^+$ can be written as

$$g(x) = \int \frac{d^2\boldsymbol{l}}{(2\pi)^2 2l}\ q(\boldsymbol{l})\ \exp(i\boldsymbol{l}\cdot\boldsymbol{x} - lx_3), \tag{4.6}$$

with $l = \sqrt{\boldsymbol{l}^2}$ and

$$\bar{q}(\boldsymbol{k}) = q(-\boldsymbol{k}). \tag{4.7}$$

There is a natural pairing between elements of $\mathcal{H}^-$ and those of $\mathcal{H}^+$,

$$\langle f, g\rangle = \int_{x_3=0} dx_1 dx_2\ (g\partial_3 f - f\partial_3 g) \qquad \text{for} \quad f \in \mathcal{H}^-, g \in \mathcal{H}^+. \tag{4.8}$$

Using the expansion (4.4) and (4.6) we find, in terms of the Fourier modes,

$$\langle f, g\rangle = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2 2k}\ p(\boldsymbol{k})q(-\boldsymbol{k}). \tag{4.9}$$

This pairing is of interest to us since the interaction energy (3.22) is proportional to the sum over the pairings $\langle u_M^a, v_N^a\rangle$. Therefore, we also refer to the expressions (4.8) and (4.9) as the interaction energy of the harmonic functions f and g.

It is clear that the pairing (4.8) may vanish for some pairs of harmonic functions $f \in \mathcal{H}^-$, $g \in \mathcal{H}^+$. This happens for example if the support of the Fourier transform p is complementary to that of the Fourier transform q. However, we shall now show that the interaction energy of multipoles can always be made non-zero by rotating one of the functions.

4.1. Multipole fields. For $f \in \mathcal{H}^-$ we have an alternative expansion in spherical harmonics Y_{Mm},

$$f(x) = \sum_{M \geq 0} \sum_{m=-M}^{M} \frac{4\pi}{(2M+1)} Q_{Mm} \frac{Y_{Mm}(\theta_+, \varphi_+)}{r_+^{M+1}}, \tag{4.10}$$

where $r_+ = |x - X_+|$ and (θ_+, φ_+) are spherical coordinates centred at $X_+ = (0, 0, R/2)$. In this section we only need to assume $R > 0$, but in our applications we will be interested in the large R limit. The coefficients Q_{Mm} are called the multipole moments of the function f. Assume that f has non-vanishing multipole moments and suppose M is the smallest integer such that $Q_{Mm} \neq 0$ for some $m = -M, ..., M$. The function

$$f_M(x) = \frac{4\pi}{(2M+1)} \sum_{m=-M}^{M} Q_{Mm} \frac{Y_{Mm}(\theta_+, \varphi_+)}{r_+^{M+1}} \tag{4.11}$$

is a 2^M-pole field and Q_{Mm} are the leading multipole moments of f.

It is often convenient to write multipole fields in terms of partial derivatives of the Coulomb potential centred at X_+:

$$\phi_+(x) = \frac{1}{r_+} = \frac{1}{\sqrt{\rho^2 + (x_3 - R/2)^2}}, \tag{4.12}$$

where $\rho^2 = \boldsymbol{x}^2$. The function $\partial_3^{m_3} \partial_2^{m_2} \partial_1^{m_1}\ \phi_+(x)$ is a 2^M-pole field if $m_1 + m_2 + m_3 = M$. However, not all of the fields obtained in this way are independent. We introduce the complex derivatives

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{4.13}$$

and note that $\Delta = \partial_3^2 + 4\partial\bar{\partial}$. Then, since $\Delta\phi_+(x) = 0$ we have

$$\partial\bar{\partial}\,\phi_+(x) = -\frac{1}{4}\partial_3^2\,\phi_+(x). \tag{4.14}$$

Thus a basis for 2^M-pole fields is given by $\partial_3^{m_3} \partial^n \bar{\partial}^{\bar{n}} \phi_+(x)$, where $M = m_3 + n + \bar{n}$ and either n or $\bar{n}$ can be taken to be zero. In Appendix A we derive the exact relation between the functions $\partial_3^{M+m} \partial^{-m} \phi_+(x)$, $-M \leq m < 0$, and $\partial_3^{M-m} \bar{\partial}^m \phi_+(x)$, $0 \leq m \leq M$, on the one hand and the spherical harmonics Y_{Mm} centred at X_+ on the other. The result is that we have the alternative expansion of an 2^M-pole field

$$f_M(x) = \sum_{-M \leq m \leq 0} A_{Mm}\ \partial_3^{M+m} \partial^{-m} \phi_+(x) + \sum_{1 \leq m \leq M} A_{Mm}\ \partial_3^{M-m} \bar{\partial}^m \phi_+(x), \tag{4.15}$$

where the coefficients A_{Mm}, $m = -M, ..., M$ are directly proportional to the multipole moments Q_{Mm}. It follows from the results in Appendix A that

$$A_{Mm} = \sqrt{\frac{4\pi}{2M+1}} \frac{(-1)^{M+m} 2^m}{\sqrt{(M-m)!(M+m)!}} Q_{Mm} \tag{4.16}$$

for $m \geq 0$ and

$$A_{Mm} = \sqrt{\frac{4\pi}{2M+1}} \frac{(-1)^M 2^{|m|}}{\sqrt{(M-m)!(M+m)!}} Q_{Mm} \tag{4.17}$$

for $m < 0$. Note in particular that the reality of f_M is equivalent to

$$A_{M(-m)} = \bar{A}_{Mm}. \tag{4.18}$$

Multipole fields have a remarkably simple Fourier transform, which will be important for us. We use the representation

$$\phi_+(x) = \int \frac{d^2\boldsymbol{k}}{2\pi k} \; e^{-k|x_3 - \frac{R}{2}|} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{4.19}$$

which can be verified as follows. Exploiting the invariance of ϕ_+ under rotations in the x_1x_2-plane we may assume that $\boldsymbol{x} = (\rho, 0)$. Using polar coordinates (k, ψ) for $\boldsymbol{k}$ we first carry out the dk integration and then the angular integration:

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\psi}{2\pi} \int_0^\infty dk \, e^{-k|x_3 - \frac{R}{2}|} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \\
&= \int_0^{2\pi} \frac{d\psi}{2\pi} \frac{-1}{(i\rho\cos\psi - |x_3 - R/2|)} && (4.20) \\
&= \oint_{S^1} \frac{dw}{2\pi i} \frac{-2}{(i\rho w^2 - 2|x_3 - R/2|w + i\rho)}, && (4.21)
\end{aligned}$$

where we changed variables to $w = e^{i\psi}$ in the last line. Expanding the integrand in partial fractions and using the residue theorem then yields the expression (4.12). We compute the Fourier transform of the multipole field (4.15) by differentiating (4.19) under the integral sign. Note that

$$\begin{aligned}
\partial \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) &= \frac{i}{2} k e^{-i\psi} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \quad \text{and} \\
\bar{\partial} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) &= \frac{i}{2} k e^{i\psi} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}),
\end{aligned} \tag{4.22}$$

so we find

$$f_M(x) = \frac{1}{2\pi} \int d^2\boldsymbol{k} \; e^{-k(\frac{R}{2} - x_3)} \, k^{M-1} \sum_{-M \leq m \leq M} \left(\frac{i}{2}\right)^{|m|} A_{Mm} e^{im\psi} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}). \tag{4.23}$$

Here we have used that $x_3 < \delta$ so that in particular $x_3 < \frac{R}{2}$. Thus with the normalisation (4.4) we arrive at the following simple expression for the Fourier transform

$$p_M(\boldsymbol{k}) = 4\pi e^{-k\frac{R}{2}} k^M \sum_{-M \leq m \leq M} \left(\frac{i}{2}\right)^{|m|} A_{Mm} e^{im\psi}. \tag{4.24}$$

This function factorises into a k-dependent part and the function

$$\Theta(\psi) = \sum_{-M \leq m \leq M} \left(\frac{i}{2}\right)^{|m|} A_{Mm} e^{im\psi} \tag{4.25}$$

of the angle ψ. The k-dependent part $e^{-k\frac{R}{2}}k^M$ is non-zero for $k \neq 0$ and the function Θ only vanishes identically if $A_{Mm} = 0$ for all $m = -M, ..., M$, i.e. if the 2^M-pole field is trivial.

4.2. The interaction energy of two scalar multipoles. The interaction energy of two multipoles can be expressed in a remarkably compact way. Let

$$\phi_-(x) = \frac{1}{|x - X_-|} \tag{4.26}$$

be the Coulomb potential centred at $X_- = (0, 0, -R/2)$ and consider the multipole field

$$g_N(x) = \sum_{-N \leq n \leq 0} B_{Nn}\ \partial_3^{N+n}\partial^{-n}\phi_-(x) + \sum_{1 \leq n \leq N} B_{Nn}\ \partial_3^{N-n}\bar{\partial}^n\phi_-(x) \tag{4.27}$$

with $B_{N(-n)} = \bar{B}_{Nn}$. By the same calculation as for f_M above we find the Fourier transform of g_N in the x_1x_2-plane:

$$q_N(\boldsymbol{k}) = 4\pi e^{-k\frac{R}{2}}k^M \sum_{-N \leq n \leq N} \left(\frac{i}{2}\right)^{|n|} B_{Nn}e^{in\psi}. \tag{4.28}$$

The interaction energy of the two multipole fields f_M and g_N

$$V_{MN} = \langle f_M, g_N \rangle \tag{4.29}$$

can now be computed using the formula (4.9). Using the factorisation property of the Fourier transforms p_N and q_N, it is easy to perform the integration over $\boldsymbol{k}$. Assuming without loss of generality that $M \leq N$ we first carry out the integration over the angle ψ to find

$$V_{MN} = 4\pi \int_0^\infty dke^{-kR}k^{N+M} \sum_{m=-M}^{M} 2^{-2|m|}\bar{A}_{Mm}B_{Nm}, \tag{4.30}$$

where we have used the reality condition for the coefficients A_{Mm} and B_{Nm}. Computing the remaining integral we obtain the final result

$$V_{MN} = 4\pi \frac{(M+N)!}{R^{M+N+1}} \sum_{m=-M}^{M} 2^{-2|m|}\bar{A}_{Mm}B_{Nm}. \tag{4.31}$$

This formula has a number of interesting features. The interaction energy depends only on the separation of the multipoles and on the combination $\sum_{m=-M}^{M} 2^{-2|m|}\bar{A}_{Mm}B_{Nm}$ of the multipole components. As explained in Appendix A, the multipole moments Q_{Nn} of a 2^N-pole can be thought of as elements of the $(2N+1)$-dimensional unitary irreducible representation W_N of $SO(3)$. The vector B with $2N+1$ components $B_{Nn}, -N \leq n \leq N$ is naturally an element of W_N. Rotations $G \in SO(3)$ about the centre X_- of the multipole field g_N act on the multipole components via $B_{Nn} \mapsto \sum_{n'=-N}^{N} U_{nn'}^N(G)B_{Nn'}$, where U^N is a $(2N+1)$-dimensional irreducible representation of $SO(3)$ (because of the rescaling (4.16) and (4.17) this is not the standard unitary representation). With our assumption that $M \leq N$ we can use the multipole components A_{Mm}, $-M \leq m \leq M$, to define the linear form

$$F_A : W_N \to \mathbb{R}, \qquad B \mapsto \sum_{m=-M}^{M} 2^{-2|m|} \bar{A}_{Mm} B_{Nm}. \tag{4.32}$$

By assumption, the components A_{Mm} are not all zero, and therefore the map F_A is non-degenerate. Writing the formula (4.31) in terms of this map as

$$V_{MN} = 4\pi \frac{(M+N)!}{R^{M+N+1}} F_A(B), \tag{4.33}$$

we immediately deduce the following result.

Theorem 4.1. *The interaction energy of an 2^M-pole and a 2^N-pole separated by a distance R is always non-vanishing for some relative orientation of the two multipoles. When such an orientation is chosen, the modulus of the interaction energy decreases with the separation as $R^{-(M+N+1)}$.*

Proof. Assuming without loss of generality that the multipoles are separated along the x_3-axis and that $M \leq N$, we have the formula (4.33) for the interaction energy. Since the map (4.32) defined in terms of the (non-vanishing) multipole components A_{Mm} of the 2^M pole at X_+ is non-degenerate it has a $2N$-dimensional kernel. It then follows from the irreducibility of the $(2N+1)$-dimensional representation W_N that there exists a rotation $G \in SO(3)$ such that $U^N(G)B$ is not in the kernel of F_A for some G. For that G we thus have $F_A(U^N(G)B) = \kappa \neq 0$ and $V_{MN} = 4\pi\kappa(M+N)!R^{-(M+N+1)}$. □

5. Attractive Forces and Existence of Minima

The arguments of the previous section apply to the asymptotic pion fields of the Skyrme solitons $U^{(1)}$ and $U^{(2)}$ discussed in Sect. 3. In particular we note that the interaction energy ΔE (3.22) for the leading multipole fields u_M and v_N is just the sum over iso-components of pairings of the form (4.29)

$$\Delta E_a = -2\langle u_M^a, v_N^a \rangle = 2 \int_{x_3=0} dx_1 dx_2 \ (u_M^a \partial_3 v_N^a - v_N^a \partial_3 u_M^a). \tag{5.1}$$

Now pick one of the iso-indices, say $a = 1$, and use iso-rotations of the Skyrme solitons to make sure that the first iso-components u_M^1 and v_N^1 are non-vanishing. It then follows from Theorem 4.1 that we can make the multipole interaction energy ΔE_1 non-zero by spatial rotations of one of the Skyrme solitons. This fact is the main upshot of the calculations in the previous section and a crucial input for the following argument which was missing in [11].

Now consider the sum

$$\Delta E = \Delta E_1 + \Delta E_2 + \Delta E_3. \tag{5.2}$$

We would like to show that we can always arrange for ΔE to be negative by a suitable iso-rotation of one of the Skyrme solitons. We may assume that, possibly after re-labelling the pion fields,

$$\Delta E_1 \geq \Delta E_2 \geq \Delta E_3. \tag{5.3}$$

If $\Delta E < 0$ we are done, so suppose that $\Delta E \geq 0$. Since we know that not all ΔE_a vanish we can conclude that $\Delta E_1 > 0$. Now perform an iso-rotation of Skyrme soliton 1 by 180 degrees around the third iso-spin axis. This reverses the sign of $\pi_1^{(1)}$ and $\pi_2^{(1)}$ and hence also the sign of ΔE_1 and ΔE_2. The new value of ΔE is

$$\begin{aligned} \Delta E &= -\Delta E_1 - \Delta E_2 + \Delta E_3 \\ &= -\Delta E_1 - (\Delta E_2 - \Delta E_3) < 0 \end{aligned} \tag{5.4}$$

since $-\Delta E_1 < 0$ and, with our ordering, $-(\Delta E_2 - \Delta E_3) \leq 0$.

Thus, the contribution ΔE to the interaction energy of two Skyrme solitons $U^{(1)}$ and $U^{(2)}$ can always be made less than zero by suitable rotations and iso-rotations of Skyrme soliton 1. It follows from the discussion at the end of Sect. 3 that for $|N - M| \leq 2$ and sufficiently large separation parameter R,

$$E[U_R] < E[U^{(1)}] + E[U^{(1)}]. \tag{5.5}$$

We conclude with a few comments on the implications of our result for the question of existence of general Skyrme solitons. As explained in Sect. 1, Esteban proved the existence of Skyrme solitons of arbitrary degree provided the strict inequality (1.19) holds. Our result (5.5) implies the inequality in those cases where minima exist in the sectors l and $k - l$, and where the associated multipoles have orders which do not differ by more than two. Since monopole fields do not arise in Skyrme solitons, the interaction energy ΔE dominates at large separation if the leading multipole moments in Skyrme solitons are at most octupoles. As explained at the end of Sect. 2, the $B = 7$ Skyrme soliton is believed to have octupoles as leading multipoles, but there is no numerical evidence for leading multipoles of higher order. Unfortunately, it seems very difficult to rule out this possibility in general.

Even if one could prove (or circumvent) the assumption concerning multipoles, the existence of attractive forces between Skyrme solitons is not sufficient to establish the inequality (1.19) for infima. Physically it seems reasonable that the existence of attractive forces should imply the existence of minima in every sector. However, we have not been able to develop this observation into a mathematical proof. Further thoughts and speculations in this direction can be found in [17].

Acknowledgement. BJS acknowledges financial support through an Advanced Research Fellowship of the Engineering and Physical Sciences Research Council. MAS thanks Rafe Mazzeo for directing him to the unique continuation theorem used in Sect. 2. NSM is grateful to Walter Kohn for drawing his attention to the formula (4.4).

A. Spherical Harmonics

In this appendix we derive the relation between the standard spherical harmonics and the following functions on $\mathbb{R}^3 - \{0\}$ used in the multipole expansion in Sect. 4.1:

$$F_{Nn}(x) = \begin{cases} \bar{\partial}^n \partial_3^{N-n} \left(\frac{1}{r}\right) & \text{if} \quad n \geq 0 \\ \partial^{-n} \partial_3^{N+n} \left(\frac{1}{r}\right) & \text{if} \quad n < 0 \end{cases}. \tag{A.1}$$

Here $N \geq 0$ and $-N \leq n \leq N$ and $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$. These functions are harmonic in their domain:

$$\Delta F_{Nn} = 0. \tag{A.2}$$

They are also homogeneous of degree $-(N+1)$ so that they can be written as

$$F_{Nn} = \frac{1}{r^{N+1}} \Phi_{Nn}(\theta, \varphi), \tag{A.3}$$

where (θ, φ) are the usual spherical coordinates on the two-sphere centred at the origin. Since the Laplace operator takes the following form in spherical coordinates:

$$\Delta = \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{1}{r^2} \Delta_\omega, \tag{A.4}$$

where Δ_ω is the Laplace operator on the 2-sphere of unit radius, it follows from (A.2) that

$$\Delta_\omega \Phi_{Nn} = -N(N+1)\Phi_{Nn}. \tag{A.5}$$

Define the generator of rotations about the 3-axis

$$J_3 = -i \frac{\partial}{\partial \varphi} \tag{A.6}$$

and express it in terms of complex coordinates $z = x_1 + i x_2$ and complex derivatives in the $x_1 x_2$ plane:

$$J_3 = z\partial - \bar{z}\bar{\partial}. \tag{A.7}$$

The operator ∂ acts as a raising operator and the operator $\bar{\partial}$ acts as a lowering operator for J_3:

$$[J_3, \bar{\partial}] = \bar{\partial} \quad \text{and} \quad [J_3, \partial] = -\partial. \tag{A.8}$$

Thus if ϕ_n is a function on $\mathbb{R}^3$ which is an eigenfunction of J_3 with eigenvalue n then $\bar{\partial}\phi_n$ is an eigenfunction with eigenvalue $n+1$ provided it is not zero. Similarly $\partial\phi_n$ is an eigenfunction of J_3 with eigenvalue $n-1$ provided it is not zero.

It follows from (A.1) that

$$F_{Nn} = \begin{cases} \bar{\partial}^n F_{N-n,0} & \text{if} \quad n \geq 0 \\ \partial^{-n} F_{N+n,0} & \text{if} \quad n < 0. \end{cases} \tag{A.9}$$

Noting that, by rotational symmetry about the 3-axis,

$$J_3 \Phi_{N0} = 0 \quad \text{for all } N, \tag{A.10}$$

we conclude that

$$J_3 F_{Nn} = n F_{Nn}. \tag{A.11}$$

It follows that

$$\Phi_{Nn} = r^{N+1} F_{Nn} \tag{A.12}$$

also satisfies

$$J_3 \Phi_{Nn} = n \Phi_{Nn}. \tag{A.13}$$

Thus, to sum up, the Φ_{Nn} are functions on S^2, which are eigenfunctions of both the Laplace operator (A.5) and the operator J_3 with eigenvalues respectively $-N(N+1)$ and n. It follows from standard harmonic analysis on S^2 that they must be proportional to the spherical harmonics Y_{Nn}.

In the case $n = 0$ we can determine the proportionality constant by evaluating both Φ_{N0} and Y_{N0} on the positive 3-axis. With the usual normalisation [18] we find

$$Y_{N0} = \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^N}{N!} \Phi_{N0}. \tag{A.14}$$

The relation between Y_{Nn} and Φ_{Nn} for $n \neq 0$ is harder to compute. Let us assume initially that $n > 0$. Starting with the standard expression for the associated Legendre function in terms of Legendre polynomials

$$P_N^n(\cos\theta) = (-1)^n \sin^n\theta \left(\frac{d}{d\cos\theta}\right)^n P_N(\cos\theta) \tag{A.15}$$

and the expression of the spherical harmonic in terms of the associated Legendre function (see e.g. [18] p. 99) we have the relation

$$Y_{Nn}(\theta,\varphi) = (-1)^n \sqrt{\frac{(N-n)!}{(N+n)!}} \left(\sin\theta e^{i\varphi}\right)^n \left(\frac{\partial}{\partial\cos\theta}\right)^n Y_{N0}(\theta,\varphi). \tag{A.16}$$

Then using (A.14) and the definition of Φ_{N0} we deduce

$$Y_{Nn}(\theta,\varphi) = r^{N+1} \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N+n}}{N!} \sqrt{\frac{(N-n)!}{(N+n)!}} z^n D_\theta^n \partial_3^N \left(\frac{1}{r}\right), \tag{A.17}$$

where

$$D_\theta = \frac{1}{r}\frac{\partial}{\partial\cos\theta} = -\frac{x_3}{\rho}\frac{\partial}{\partial\rho} + \partial_3 \tag{A.18}$$

and $z = \rho e^{i\varphi}$ with $\rho = r\sin\theta$ as in the main text of the paper. Then we use the commutation relation

$$[D_\theta, \partial_3] = \frac{1}{\rho}\frac{\partial}{\partial\rho} \tag{A.19}$$

to move D_θ past ∂_3 in (A.17). Noting that $D_\theta\, r = 0$ we find

$$Y_{Nn}(\theta,\varphi) = r^{N+1} \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N+n}}{\sqrt{(N-n)!(N+n)!}} z^n \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \partial_3^{N-n} \left(\frac{1}{r}\right). \tag{A.20}$$

Now exploit that on any function f which only depends on ρ and x_3,

$$\frac{1}{\rho}\frac{\partial f}{\partial\rho}(\rho, x_3) = \frac{2}{z}\bar{\partial} f(\rho, x_3) \tag{A.21}$$

to conclude

$$Y_{Nn}(\theta,\varphi) = r^{N+1} \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N+n}\, 2^n}{\sqrt{(N-n)!(N+n)!}} \bar{\partial}^n \partial_3^{N-n} \left(\frac{1}{r}\right). \tag{A.22}$$

Thus we finally arrive at the promised relationship between Y_{Nn} and Φ_{Nn}, valid for $n \geq 0$:

$$Y_{Nn} = \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N+n} 2^n}{\sqrt{(N-n)!(N+n)!}} \Phi_{Nn}. \tag{A.23}$$

To deduce the corresponding result for $n < 0$ we note that $\Phi_{Nn} = \bar{\Phi}_{N(-n)}$ and $Y_{Nn} = (-1)^n \bar{Y}_{N(-n)}$ for $n < 0$. Thus for $n < 0$:

$$Y_{Nn} = \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N} 2^{|n|}}{\sqrt{(N-n)!(N+n)!}} \Phi_{Nn}. \tag{A.24}$$

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Communicated by P. Constantin