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Quaternionic and Poisson-Lie structures in 3d gravity: the cosmological constant as deformation parameter

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Abstract

Each of the local isometry groups arising in 3d gravity can be viewed as the group of unit (split) quaternions over a ring which depends on the cosmological constant. In this paper we explain and prove this statement, and use it as a unifying framework for studying Poisson structures associated with the local isometry groups. We show that, in all cases except for Euclidean signature with positive cosmological constant, the local isometry groups are equipped with the Poisson-Lie structure of a classical double. We calculate the dressing action of the factor groups on each other and find, amongst others, a simple and unified description of the symplectic leaves of $SU(2)$ and $SL(2,\mathbb{R})$. We also compute the Poisson structure on the dual Poisson-Lie groups of the local isometry groups and on their Heisenberg doubles; together, they determine the Poisson structure of the phase space of 3d gravity in the so-called combinatorial description.

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1 Introduction

In 3d gravity, solutions of the Einstein equations are locally isometric to a model spacetime which is determined by the signature of spacetime (Euclidean or Lorentzian) and the cosmological constant [1]. The isometry groups of these model spacetimes are therefore local

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isometry groups in 3d gravity. In the formulation of 3d gravity as a Chern-Simons gauge theory [2, 3], the local isometry groups play the role of gauge groups. For Euclidean gravity, the relevant groups are $SU(2) \times SU(2)$ for positive cosmological constant, $SL(2, \mathbb{C})$ for negative cosmological constant and the (double cover of the) Euclidean group $SU(2) \ltimes \mathbb{R}^3$ for vanishing cosmological constant. With Lorentzian signature the relevant local isometry groups are $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ for negative cosmological constant, $SL(2, \mathbb{C})$ for positive cosmological constant and the (double cover of the) Poincaré group $SL(2, \mathbb{R}) \ltimes \mathbb{R}^3$ for vanishing cosmological constant. These groups are structurally quite diverse: some are direct products of real, simple Lie groups, some are semi-direct products and one is a complex Lie group. As a result, the techniques used in the literature on 3d gravity differ widely, with different approaches taken for different signatures and values of the cosmological constant, which makes it difficult to relate these different cases and to establish a unified and coherent picture of the theory.

This paper is motivated by the desire for one framework encompassing all the different signatures and values of the cosmological constant. The unified description of the isometry groups and their Lie algebras in [4] in terms of a ring which depends on the cosmological constant suggests that this may be possible. Here we begin by describing the local isometry groups in terms of quaternions and the ring introduced in [4]. The quaternionic description of the local isometry groups generalises the well-known fact that $SU(2)$ is isomorphic to the unit quaternions and that $SL(2, \mathbb{C})$ is the complexification of $SU(2)$, i.e. isomorphic to the unit quaternions with complex coefficients. For the Lorentzian setting one uses a Lorentzian version of the quaternions, called split quaternions, and the fact that the group of unit split quaternions is isomorphic to $SL(2, \mathbb{R})$; it is then easy to see that $SL(2, \mathbb{C})$ can also be viewed as the set of unit split quaternions with complex coefficients. To obtain the remaining groups one needs to generalise $\mathbb{C}$ to a ring which, depending on the sign of the cosmological constant and the signature, is isomorphic to $\mathbb{C}$, the so-called dual numbers or the split complex numbers. The upshot of this construction is a unified description of the local isometry groups with the cosmological constant appearing as a deformation parameter.

The formulation of 3d gravity not only requires a choice of the local isometry group but also of an invariant inner product on the Lie algebra of this group. This is most readily apparent in the Chern-Simons formulation of the theory, where triad and spin-connection are combined into a Chern-Simons gauge field and the inner product enters the action explicitly. The inner product ultimately determines the symplectic structure of the phase space. For universes of topology $\mathbb{R} \times S$, where S is a surface of arbitrary genus and possibly with a number of punctures, the phase space can be studied very effectively in the Hamiltonian or combinatorial approach [5, 6, 7]. In this approach, the phase space is realised as a quotient of an auxiliary, finite dimensional space. The auxiliary space has a Poisson structure which is determined by a classical r-matrix whose symmetric part equals the inner product used in the definition of the Chern-Simons action. We will not review the details of this construction and its application to 3d gravity [8, 9, 10, 11] here, but to motivate the second half of this paper we note that the classical r-matrix can be used to define a variety of Poisson structures, three of which play a fundamental role in the combinatorial approach.

First of all there is the so-called Sklyanin bracket [12] on the isometry group, endowing it with the structure of a Poisson-Lie group; in this paper we will describe this structure in the unified, quaternionic language introduced above for all the isometry groups arising in 3d gravity with the exception of $SU(2) \times SU(2)$, where the required r-matrix does not exist. Every Poisson-Lie group has a dual Poisson-Lie group, where Lie brackets and Poisson brackets are, in a suitable sense, interchanged. The Poisson-Lie groups arising in 3d gravity have the special property that in each case the dual Poisson-Lie group is diffeomorphic to the original one. As a result, one can define a second Poisson bracket, sometimes called the dual or Semenov-Tian-Shansky bracket [13], on the original group. The symplectic leaves of the dual Poisson structure are the conjugacy classes of the original group. In the combinatorial description of the phase space of Chern-Simons theory this second Poisson structure is associated with the punctures on the surface S, which represent gravitationally interacting massive point particles with spin. Finally, the third Poisson structure which arises in the combinatorial description is the called the Heisenberg double structure [13]; it is a symplectic Poisson structure defined on two copies of the original group and is associated with handles of the surface S [6].

While our work is motivated by its potential use in 3d gravity, it also has interesting ramifications in pure mathematics. One is related to the fact that all Poisson-Lie groups discussed in this paper are classical doubles. Elements in a neighbourhood of the identity (and in some cases in the entire group) can be written uniquely as a product of two elements belonging to a pair of Poisson-Lie subgroups. One of the subgroups is isomorphic to $SU(2)$ in the Euclidean cases and isomorphic to $SL(2,\mathbb{R})$ in the Lorentzian cases; the other subgroup is either $\mathbb{R}^3$ or isomorphic to the group of 2×2 matrices of the form

$$\begin{pmatrix} e^{\alpha} & x+iy \\ 0 & e^{-\alpha} \end{pmatrix}, \qquad \alpha, x, y \in \mathbb{R}. \tag{1.1}$$

The group of such matrices will be denoted by $AN(2)$ in this paper. The cases where one factor is $\mathbb{R}^3$ are degenerate and rather trivial limits of the generic situation where one of the factor groups is isomorphic to $AN(2)$. We therefore consider the generic situation in this paper, and study the mutual dressing actions of the Poisson-Lie groups $AN(2)$ on the one hand and $SU(2)$ or, respectively, $SL(2,\mathbb{R})$ on the other. The dressing actions are defined by comparing the factorisation in one order with the factorisation in the other order, and the orbits under the dressing actions give the symplectic leaves of the Poisson structures. Our results relate the mutual dressing actions of $SU(2)$ and $AN(2)$, discussed in many textbooks [12, 14] to the mutual dressing actions of $SL(2,\mathbb{R})$ and $AN(2)$. In particular, our treatment leads to a unified description of the symplectic leaves of $SU(2)$ and $SL(2,\mathbb{R})$ for the various choices of Poisson-Lie structures.

The plan of the paper is as follows. In Sect. 2 we introduce basic notation and review the result of [4] that the Lie algebras of the local isometry groups arising in 3d gravity can be obtained by tensoring $\mathfrak{su}(2)$ or $\mathfrak{sl}(2)$ with a ring R_Λ which depends on the cosmological constant. In Sect. 3 we give a unified description of the local isometry groups in terms of (split) quaternions and the ring R_Λ; we introduce an involution on these groups and show that the fixed point sets of this involution are isomorphic to the model spacetimes arising in 3d gravity. Sect. 4 contains a description of double structure of the isometry groups and a fundamental theorem about the factorisation of a general element into elements of the two subgroups described above. In Sect. 5 we use the factorisation theorem to define dressing actions of the subgroups on each other, and we study the geometry of dressing orbits. The final Sect. 6 contains a unified description of the Sklyanin, dual and Heisenberg double Poisson structures associated to the Poisson-Lie groups arising in 3d gravity. It also contains the characterisation of the dressing orbits studied in Sect. 5 as symplectic leaves of Poisson-Lie group structures on $AN(2)$ and $SU(2)$ respectively $SL(2,\mathbb{R})$.

2 The Lie algebras of 3d gravity

2.1 Notation and conventions

Throughout the paper we use Einstein's summation convention. Indices are raised and lowered with either the three-dimensional Euclidean metric $\eta^E = \mathrm{diag}(1,1,1)$ or the three-dimensional Minkowski metric $\eta^L = \mathrm{diag}(1,-1,-1)$. Where necessary, we specify the signature by a superscript E for Euclidean and L for Lorentzian signature, which we omit in formulas valid for both signatures. In particular, we write

$$\boldsymbol{p}\cdot\boldsymbol{q} = \eta_{ab}p^a q^b, \qquad \text{with} \quad \boldsymbol{p} = (p^0, p^1, p^2), \boldsymbol{q} = (q^0, q^1, q^2) \in \mathbb{R}^3, \tag{2.1}$$

where η is either the three-dimensional Euclidean or the three-dimensional Minkowski metric. We also sometimes write simply $\boldsymbol{pq}$ for $\boldsymbol{p}\cdot\boldsymbol{q}$ and $\boldsymbol{p}^2$ for $\boldsymbol{p}\cdot\boldsymbol{p}$.

We denote by J_a^E, $a = 0,1,2$, and J_a^L, $a = 0,1,2$, respectively, the generators of the three-dimensional rotation algebra $\mathfrak{su}(2)$ and the three-dimensional Lorentz algebra $\mathfrak{sl}(2,\mathbb{R})$. In terms of these generators the Lie bracket and Killing form are

$$[J_a, J_b] = \epsilon_{abc}J^c, \qquad \kappa(J_a, J_b) = \eta_{ab}, \tag{2.2}$$

where indices are raised and lowered with the metrics $\eta = \eta^E$ or $\eta = \eta^L$, and ϵ denotes the fully antisymmetric tensor in three dimensions with the convention $\epsilon_{012} = \epsilon^{012} = 1$ (for both signatures).

2.2 Lie algebras over a ring

In this subsection, we assemble some well-known facts and definitions for the Lie algebras occurring in 3d gravity as well as some more recent results from [4]. As shown by Witten [3] the Lie algebras arising in 3d gravity can be expressed in a common form in which the cosmological constant Λ_c appears as a parameter in the Lie bracket. Defining

$$\Lambda = \begin{cases} \Lambda_c & \text{for Euclidean signature} \\ -\Lambda_c & \text{for Lorentzian signature} \end{cases} \tag{2.3}$$

these Lie algebras, in the following denoted by $\mathfrak{h}_\Lambda$, are the six-dimensional Lie algebras with generators J_a, P_a, $a = 0, 1, 2$, and Lie brackets[3]

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P^c, \qquad [P_a, P_b] = \Lambda \epsilon_{abc} J^c. \tag{2.4}$$

Again, indices are raised and lowered with the three-dimensional Euclidean metric $\eta = \eta^E$ or with the Minkowski metric $\eta = \eta^L$. For $\Lambda = 0$, the bracket of the generators P_a vanishes, and the Lie algebra $\mathfrak{h}_\Lambda$ is the three-dimensional Euclidean and Poincaré algebra. For $\Lambda < 0$, one can obtain the bracket (2.4) via the identification $P_a = i\sqrt{|\Lambda|} J_a$, which yields the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, realised as the complexification of its compact real form $\mathfrak{su}(2)$ and its normal real form $\mathfrak{sl}(2, \mathbb{R})$ for Euclidean and Lorentzian signature, respectively . For $\Lambda > 0$, one can introduce an alternative set of generators

$$J^\pm_a = \tfrac{1}{2}(J_a \pm \tfrac{1}{\sqrt{\Lambda}} P_a), \tag{2.5}$$

in terms of which the Lie bracket takes the form of a direct sum

$$[J^\pm_a, J^\pm_b] = \epsilon_{abc} J^c_\pm, \qquad [J^\pm_a, J^\mp_b] = 0. \tag{2.6}$$

Hence, depending on the signature and the sign of Λ, the Lie algebra $\mathfrak{h}_\Lambda$ is given by

$$\mathfrak{h}^E_\Lambda = \begin{cases} \mathfrak{iso}(3) & \Lambda = 0 \\ \mathfrak{su}(2) \oplus \mathfrak{su}(2) & \Lambda > 0 \\ \mathfrak{sl}(2, \mathbb{C}) & \Lambda < 0, \end{cases} \qquad \mathfrak{h}^L_\Lambda = \begin{cases} \mathfrak{iso}(2, 1) & \Lambda = 0 \\ \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) & \Lambda > 0 \\ \mathfrak{sl}(2, \mathbb{C}) & \Lambda < 0. \end{cases} \tag{2.7}$$

For all values of Λ and both signatures, the space of Ad-invariant symmetric bilinear forms on $\mathfrak{h}_\Lambda$ is two dimensional and a basis is given by the forms $t, s : \mathfrak{h}_\Lambda \times \mathfrak{h}_\Lambda \to \mathbb{R}$ defined via

$$t(J_a, J_b) = 0, \qquad t(P_a, P_b) = 0, \qquad t(J_a, P_b) = \eta_{ab}, \tag{2.8}$$
$$s(J_a, J_b) = \eta_{ab}, \qquad s(J_a, P_b) = 0, \qquad s(P_a, P_b) = \Lambda \eta_{ab}. \tag{2.9}$$

It was shown in [4] that the Lie algebras $\mathfrak{h}_\Lambda$ can be described in a common framework by identifying them with the three-dimensional rotation and Lorentz algebra over a commutative ring defined as follows.

Definition 2.1 *(Ring R_Λ [4])*

$R_\Lambda = (\mathbb{R}^2, +, \cdot)$ is the commutative ring obtained from $\mathbb{R}^2$ with the usual addition by defining the Λ-dependent multiplication law

$$(a, b) \cdot (c, d) = (ac + \Lambda bd\,,\, ad + bc) \qquad \forall a, b, c, d \in \mathbb{R}. \tag{2.10}$$

In the following we parametrise elements of R_Λ in terms of a formal parameter θ as $a + \theta b$, $a, b \in \mathbb{R}$ and denote by Im_θ, Re_θ their components

$$Re_\theta(a + \theta b) = a \qquad Im_\theta(a + \theta b) = b \qquad \forall a, b \in \mathbb{R}. \tag{2.11}$$

[3] Our parameter Λ is called λ in [3].

We define a $\mathbb{R}$-linear involution $^*: R_\Lambda \to R_\Lambda$, in the following referred to as conjugation, via

$$(a + \theta b)^* = a - \theta b. \tag{2.12}$$

R_Λ is actually more than a ring: it is an algebra over $\mathbb{R}$ since multiplication by real numbers is also defined. However, since multiplication by real numbers can be seen as a special case of multiplication in R_Λ we do not emphasise this aspect, and continue to refer to R_Λ as a ring. Note that the multiplication law (2.10) follows from the formal relation $\theta^2 = \Lambda$. The ring R_Λ can therefore be viewed as a generalisation of the complex numbers. For $\Lambda < 0$, the relation $\theta = i\sqrt{|\Lambda|}$ identifies the ring R_Λ with the field $\mathbb{C}$. For $\Lambda = 0$ and $\Lambda > 0$ the ring R_Λ has zero divisors and can be identified with the dual numbers [15] and split complex or hyperbolic numbers [16], respectively. In the case of $\Lambda = 0$ the zero divisors are the elements of the form θa, $a \in \mathbb{R}$ satisfying

$$\theta a \cdot \theta b = 0 \qquad \forall a, b \in \mathbb{R}, \tag{2.13}$$

for $\Lambda > 0$, the zero divisors are the elements of the form $\frac{a}{2}(1 \pm \frac{\theta}{\sqrt{\Lambda}})$, $a \in \mathbb{R}$, which satisfy

$$\tfrac{a}{2}(1 \pm \tfrac{\theta}{\sqrt{\Lambda}}) \cdot \tfrac{b}{2}(1 \pm \tfrac{\theta}{\sqrt{\Lambda}}) = \tfrac{ab}{2}(1 \pm \tfrac{\theta}{\sqrt{\Lambda}}) \qquad \tfrac{a}{2}(1 + \tfrac{\theta}{\sqrt{\Lambda}}) \cdot \tfrac{b}{2}(1 - \tfrac{\theta}{\sqrt{\Lambda}}) = 0 \qquad \forall a, b \in \mathbb{R}. \tag{2.14}$$

Lemma 2.2 *[4] Consider the three-dimensional rotation and Lorentz algebra with generators J_a, $a = 0, 1, 2$, and with Lie bracket and Killing form given by* (2.2)*. Extend Lie bracket and Killing form bilinearly to R_Λ. With the identification*

$$P_a = \theta J_a \tag{2.15}$$

one recovers the Lie bracket (2.4) *and the* Ad*-invariant symmetric bilinear forms* (2.8),(2.9) *as the real and θ component of κ.*

3 The Lie groups of 3d gravity

3.1 Quaternionic structure

The local isometry groups arising in 3d gravity are obtained by exponentiating the Lie algebras $\mathfrak{h}_\Lambda^{E,L}$ (2.7). The fact that these can be viewed as Lie algebras over the ring R_Λ does not, by itself, guarantee that the corresponding Lie groups inherit some kind of algebraic structure over the ring R_Λ. In this section we shall explain that this does, however, happen for the isometry groups of 3d gravity. The basic reason for this is best explained in the context of the Clifford algebras. Recall [17] that the Clifford algebra $C\ell(V, \eta)$ associated to a real, n-dimensional vector space V with inner product η of signature (r, s) is the associative algebra over $\mathbb{R}$ generated by the elements of an orthonormal basis $\{e_0, \ldots, e_{n-1}\}$ of V subject to the relations

$$e_a e_b + e_b e_a = -2\eta(e_a, e_b)1. \tag{3.1}$$

The Clifford algebra contains, as subsets, the original vector space V, the double cover $\mathrm{Spin}(r,s)$ of the identity component of its isometry group $SO(r,s)$, and the Lie algebra $so(r,s)$, the latter being realised as the span of elements $e_a e_b$, with $a \neq b$. As explained, for example, in [18], the group $\mathrm{Spin}(r,s)$ is realised as a certain subset of elements in $C\ell(V,\eta)$ obeying an algebraic condition. When tensoring $C\ell(V,\eta)$ with the ring R_Λ one obtains, in particular, the Lie algebra $so(r,s) \otimes R_\Lambda$ as the R_Λ-span of the degree two elements. We shall now show that, at least in three dimensions, one also obtains corresponding Lie groups by simply interpreting the algebraic constraint defining $\mathrm{Spin}(r,s)$ as an equation in R_Λ. We have found it convenient to express our argument in the language of quaternions, which exploits the identification of the degree one Clifford elements e_a with the degree two Clifford elements $\frac{1}{2}\epsilon_{abc}e_b e_c$ in three dimensions. While this identification, and hence the quaternionic language, are only possible in three dimensions, the corresponding construction in the Clifford algebra seems to be possible in any dimension.

Definition 3.1 *((Split) Quaternions) The set of quaternions $\mathbb{H}^E$ is the associative algebra over $\mathbb{R}$ generated by elements e_a, $a = 0,1,2$, and the identity element 1 subject to the relations*

$$e_a e_b = -\eta^E_{ab}\, 1 + \epsilon_{abc} e_c, \tag{3.2}$$

where η^E denotes the three-dimensional Euclidean metric.

The set of split quaternions $\mathbb{H}^L$ is the associative algebra over $\mathbb{R}$ generated by the elements e_a, $a = 0,1,2$, and the identity element subject to the corresponding relations for the three-dimensional Minkowski metric

$$e_a e_b = -\eta^L_{ab}\, 1 + \epsilon_{abc} e^c. \tag{3.3}$$

Quaternions are discussed in many textbooks on linear algebra, for example [19]. The Lorentzian version, called split (or co- or para-) quaternions was introduced in [20], and is less commonly discussed, but [21] contains a detailed and elementary treatment. Elements of $\mathbb{H}^E$ and $\mathbb{H}^L$ can be parametrised as

$$q = q_3\, 1 + \boldsymbol{q}\cdot\boldsymbol{e} = q_3\, 1 + q^a e_a, \qquad q^0, q^1, q^2, q_3 \in \mathbb{R}, \tag{3.4}$$

and multiplication is defined by bi-linear extension of (3.2) and, respectively, (3.3). We will mostly omit the identity quaternion 1 in the following and write $q = q_3 + \boldsymbol{q}\cdot\boldsymbol{e}$. The algebra of (split) quaternions is equipped with an $\mathbb{R}$-linear conjugation defined by

$$\bar{q} = q_3 - q^a e_a. \tag{3.5}$$

From the identity

$$q\bar{q} = q_3^2 + \boldsymbol{q}^2 \quad \text{with} \quad \boldsymbol{q} = (q^0, q^1, q^2) \tag{3.6}$$

it follows that the quaternions $\mathbb{H}^E$ form a division algebra, i. e. every quaternion $q \neq 0$ has a multiplicative inverse. This does not hold for the split quaternions, but in both cases the set of unit (split) quaternions satisfying $q\bar{q} = 1$ forms a group:

$$\mathbb{H}_1^E = \{q \in \mathbb{H}^E \mid q\bar{q} = 1\} = \{q = q_3 + q^a e_a^E \mid q_3^2 + q_0^2 + q_1^2 + q_2^2 = 1\} \cong Spin(3) \cong SU(2), \quad (3.7)$$
$$\mathbb{H}_1^L = \{q \in \mathbb{H}^L \mid q\bar{q} = 1\} = \{q = q_3 + q^a e_a^L \mid q_3^2 + q_0^2 - q_1^2 - q_2^2 = 1\} \cong Spin(2,1) \cong SL(2,\mathbb{R}).$$

For the corresponding Lie algebras we have

$$\mathfrak{h}^E := \{q = q^a e_a^E \in \mathbb{H}^E \mid q^a \in \mathbb{R}\} \cong \mathfrak{so}(3) \cong \mathfrak{su}(2), \tag{3.8}$$
$$\mathfrak{h}^L := \{q = q^a e_a^L \in \mathbb{H}^L \mid q^a \in \mathbb{R}\} \cong \mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R}). \tag{3.9}$$

These isomorphisms can be made explicit via representations. In the Euclidean case, a representation of the algebra $\mathbb{H}^E$ is given by

$$\rho_E(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_E(e_a^E) = -i\sigma_a, \tag{3.10}$$

where σ_a are the Pauli matrices. For Lorentzian signature, two representations are relevant. The first induces a group isomorphism $\mathbb{H}_1^L \to SL(2,\mathbb{R})$ and is given by

$$\rho_L(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_L(e_0^L) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_L(e_1^L) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_L(e_2^L) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.11}$$

while the second identifies $\mathbb{H}_1^L$ with $SU(1,1)$ and takes the form

$$\tilde{\rho}_L(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\rho}_L(e_0^L) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{\rho}_L(e_1^L) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{\rho}_L(e_2^L) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.12}$$

To show how quaternions give a unified description of the local isometry groups in 3d gravity we need to consider quaternions over the commutative ring R_Λ. Formally, this means we consider the tensor product

$$\mathbb{H}^{E,L}(R_\Lambda) := \mathbb{H}^{E,L} \otimes_{\mathbb{R}} R_\Lambda. \tag{3.13}$$

Elements can be parametrised according to

$$g = q_3 + \theta k_3 + (\boldsymbol{q} + \theta \boldsymbol{k}) \cdot \boldsymbol{e}, \qquad q_3, k_3 \in \mathbb{R}, \boldsymbol{q}, \boldsymbol{k} \in \mathbb{R}^3, \tag{3.14}$$

and it is easy to check that $\mathbb{H}^{E,L}(R_\Lambda)$ is an algebra over $\mathbb{R}$.

Both the Lie groups and the Lie algebras arising in 3d gravity can be realised as subsets of $\mathbb{H}^{E,L}(R_\Lambda)$. In order to state this claim precisely, we introduce the projection operator

$$\Pi : \mathbb{H}^{E,L}(R_\Lambda) \to \mathbb{H}^{E,L}, \quad \Pi : p_3 + \boldsymbol{p} \cdot \boldsymbol{e} \mapsto p_3. \tag{3.15}$$

Theorem 3.2 *The local isometry groups in 3d gravity are isomorphic to the multiplicative group*

$$\mathbb{H}_1^{E,L}(R_\Lambda) := \{g \in \mathbb{H}^{E,L}(R_\Lambda) | g\bar{g} = 1\} \tag{3.16}$$

of unit (split) quaternions over the commutative ring R_Λ. We have the following identifications:

$$\begin{array}{ll} \mathbb{H}_1^E(R_{\Lambda>0}) \cong SU(2) \times SU(2), & \mathbb{H}_1^L(R_{\Lambda>0}) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), \\ \mathbb{H}_1^E(R_{\Lambda=0}) \cong SU(2) \ltimes \mathbb{R}^3, & \mathbb{H}_1^L(R_{\Lambda=0}) \cong SL(2,\mathbb{R}) \ltimes \mathbb{R}^3, \\ \mathbb{H}_1^E(R_{\Lambda<0}) \cong SL(2,\mathbb{C}), & \mathbb{H}_1^L(R_{\Lambda<0}) \cong SL(2,\mathbb{C}). \end{array} \tag{3.17}$$

The Lie algebras listed in (2.7) *are realised as the set of (split) quaternions over R_Λ with vanishing unit component. In terms of the projection* (5.9),

$$\mathfrak{h}_\Lambda^{E,L} = \{g \in \mathbb{H}^{E,L}(R_\Lambda) | \Pi(g) = 0\}. \tag{3.18}$$

Notational convention: In the following we will omit the superscript E or L on $\mathbb{H}$ if the statement being made is valid for either choice of signature.

Proof: It is easy to check that $\mathbb{H}_1(R_\Lambda)$ is a group under multiplication. For Euclidean and Lorentzian signature and $\Lambda < 0$ the identities (3.17) can be verified directly by setting $\theta = i\sqrt{|\Lambda|}$ and extending the representations (3.10), (3.11), (3.12) of the quaternions linearly to $\mathbb{C}$, which defines three algebra isomorphisms $\rho_E, \rho_L, \tilde{\rho}_L : \mathbb{H}(R_{\Lambda<0}) \to M(2,\mathbb{C})$. A general quaternion $g \in \mathbb{H}(R_\Lambda)$ parametrised as in (3.14) is a unit quaternion over R_Λ if and only if the parameters q_3, k_3, q^a, k^a satisfy the conditions

$$q_3 k_3 + \boldsymbol{q}\boldsymbol{k} = 0, \qquad\qquad q_3^2 + \Lambda k_3^2 + \boldsymbol{q}^2 + \Lambda \boldsymbol{k}^2 = 1. \tag{3.19}$$

Using formulas (3.10), (3.11), (3.12) it can be shown by direct calculation that this is equivalent to the conditions $\det(\rho_E(g)) = 1$, $\det(\rho_L(g)) = 1$, $\det(\tilde{\rho}_L(g)) = 1$. This implies that the algebra isomorphisms $\rho_E, \rho_L, \tilde{\rho}_L : \mathbb{H}(R_{\Lambda<0}) \to M(2,\mathbb{C})$ restrict to group isomorphisms from $\mathbb{H}_1(R_{\Lambda<0})$ to $SL(2,\mathbb{C})$.

To prove the corresponding statements for the case of $\Lambda > 0$, we note that we can express a general element $g \in \mathbb{H}(R_{\Lambda>0})$ parametrised as in (3.14) as

$$\begin{aligned} g &= \tfrac{1}{2}(1 + \tfrac{\theta}{\sqrt{\Lambda}})u_+(g) + \tfrac{1}{2}(1 - \tfrac{\theta}{\sqrt{\Lambda}})u_-(g), \quad \text{with} \\ u_\pm(g) &= (q_3 + q^a e_a) \pm \sqrt{\Lambda}\,(k_3 + k^a e_a)\,. \end{aligned} \tag{3.20}$$

A direct calculation then shows that g satisfies the condition (3.19) if and only if both elements $u_+(g), u_-(g) \in \mathbb{H}$ are unit quaternions $u_+(g), u_-(g) \in \mathbb{H}_1$. Moreover, identity (2.14) implies

$$u_\pm(gh) = u_\pm(g) \cdot u_\pm(h) \qquad \forall g, h \in \mathbb{H}_1(R_{\Lambda>0}). \tag{3.21}$$

By setting

$$\Phi(g) = (\rho(u_+(g))\,,\, \rho(u_-(g)))\,, \tag{3.22}$$

where ρ is one of the representations (3.10), (3.11), (3.12), we then obtain group isomorphisms $\Phi_E : \mathbb{H}_1^E(R_{\Lambda>0}) \to SU(2)\times SU(2)$, $\Phi_L : \mathbb{H}_1^L(R_{\Lambda>0}) \to SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, $\tilde{\Phi}_L : \mathbb{H}_1^L(R_{\Lambda>0}) \to SU(1,1)\times SU(1,1)$.

When $\Lambda = 0$, the second condition in (3.19) reduces to the requirement that the element $q = q_3 + q^a e_a \in \mathbb{H}$ is a unit quaternion. Moreover, we note that elements of the form

$$(1+\theta v^a e_a)\cdot q, \qquad \boldsymbol{v}\in\mathbb{R}^3, q\in\mathbb{H}_1 \tag{3.23}$$

are unit quaternions over R_0 and that any element $g\in\mathbb{H}_1(R_0)$ can be expressed uniquely as

$$g = (1+\theta v^a(g)e_a)\cdot u(g) \qquad \text{with} \qquad \begin{aligned} u(g) &= q_3 + q^a e_a, \\ v^a(g) &= q_3 k^a - k_3 q^a + \epsilon^{abc} q_b k_c. \end{aligned} \tag{3.24}$$

As the multiplication relations (3.2),(3.3) imply

$$u(gh) = u(g)\cdot u(h), \qquad v^a(gh)e_a = v^a(g)e_a + u(g)v^b(h)e_b u(g)^{-1} \qquad \forall g,h\in\mathbb{H}_1(R_0), \tag{3.25}$$

the definition

$$\Phi(g) = (\rho(u(g)), \boldsymbol{v}(g)), \tag{3.26}$$

with ρ given by (3.10), (3.11) or (3.12), gives rise to group isomorphisms $\Phi_E : \mathbb{H}_1^E(R_0) \to SU(2)\ltimes\mathbb{R}^3$, $\Phi_L : \mathbb{H}_1^L(R_0) \to SL(2,\mathbb{R})\ltimes\mathbb{R}^3$, $\tilde{\Phi}_L : \mathbb{H}_1^L(R_0) \to SU(1,1)\ltimes\mathbb{R}^3$. □

3.2 Involutions

The quaternionic conjugation $^-$ defined in (3.5) can be extended R_Λ-linearly to an involution

$$^-: \mathbb{H}(R_\Lambda)\to\mathbb{H}(R_\Lambda), \quad (q_3+\theta k_3) + (\boldsymbol{q}+\theta\boldsymbol{k})\cdot\boldsymbol{e} \mapsto (q_3+\theta k_3) - (\boldsymbol{q}+\theta\boldsymbol{k})\cdot\boldsymbol{e}. \tag{3.27}$$

Similarly, the conjugation * defined in (2.12) can be extended to an involution

$$^* : \mathbb{H}(R_\Lambda)\to\mathbb{H}(R_\Lambda), \quad (q_3+\theta k_3) + (\boldsymbol{q}+\theta\boldsymbol{k})\cdot\boldsymbol{e} \mapsto (q_3-\theta k_3) + (\boldsymbol{q}-\theta\boldsymbol{k})\cdot\boldsymbol{e}. \tag{3.28}$$

We also often need to consider the combination of the two involutions $^-$ and *, and define

$$g^\circ = \bar{g}^* \tag{3.29}$$

for $g\in\mathbb{H}(R_\Lambda)$. The involution $^\circ$ generalises the notion of taking the adjoint of a matrix. Setting $\theta = i\sqrt{|\Lambda|}$ for $\Lambda<0$, using the representation (3.10) and extending it linearly to

$\mathbb{H}(R_\Lambda)$, one recovers the usual Hermitian conjugation $^\dagger: M(2,\mathbb{C}) \to M(2,\mathbb{C})$. Doing the same with the representations (3.11), (3.12) yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\circ = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix} \qquad \text{for representation (3.11),} \tag{3.30}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\circ = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix} \qquad \text{for representation (3.12),} \tag{3.31}$$

which are the usual anti-algebra automorphisms that characterise $SL(2,\mathbb{R})$ and $SU(1,1)$, respectively, via the condition $g^\circ = g^{-1}$.

Generally, the subsets of $\mathbb{H}_1(R_\Lambda)$ defined by the conditions $g^\circ = g$ and $g^\circ = g^{-1}$ play an important role in the following. The set of elements satisfying $g^\circ = g^{-1}$ is simply the group of unit quaternions:

$$\mathbb{H}_1 = \{q \in \mathbb{H}_1(R_\Lambda) \mid q^* = q\} = \{q \in \mathbb{H}_1(R_\Lambda) \mid q^\circ = q^{-1}\}. \tag{3.32}$$

The set

$$W = \{w \in \mathbb{H}_1(R_\Lambda) | w = w^\circ\} \tag{3.33}$$

provides a unified description of the model spacetimes arising in 3d gravity. To see this, parametrise elements as

$$w = w_3 + \theta \boldsymbol{w} \cdot \boldsymbol{e}, \tag{3.34}$$

for $w_0, w_1, w_2, w_3 \in \mathbb{R}$ satisfying the constraint $w_3^2 + \Lambda \boldsymbol{w}^2 = 1$; in this parametrisation we therefore necessarily have

$$\Lambda \boldsymbol{w}^2 \leq 1. \tag{3.35}$$

The manifolds

$$W_\Lambda = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 + \Lambda \boldsymbol{w}^2 = 1\} \tag{3.36}$$

parametrising the elements (3.34) are isomorphic to various classical geometries for different choices of Λ and the signature E or L. We again suppress the signature label if we refer to either case, and attach it if we refer to a specific signature. In the Euclidean case and $\Lambda < 0$ we have, with $\Lambda = -1$ for definiteness,

$$W_{-1}^E = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 - w_0^2 - w_1^2 - w_2^2 = 1\}, \tag{3.37}$$

which is the two-sheeted hyperboloid embedded in 3+1-dimensional Minkowski space, i.e. isomorphic to two copies of 3-dimensional hyperbolic space. For Euclidean signature and $\Lambda > 0$, we obtain the three-sphere embedded in four-dimensional Euclidean space

$$W_1^E = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 + w_0^2 + w_1^2 + w_2^2 = 1\}, \tag{3.38}$$

and for Euclidean signature with $\Lambda = 0$ two copies of three-dimensional Euclidean space as hyperplanes in four-dimensional Euclidean space

$$W_0^E = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 = 1\}. \tag{3.39}$$

In the Lorentzian case with $\Lambda < 0$ we have

$$W_{-1}^L = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 - w_0^2 + w_1^2 + w_2^2 = 1\}, \tag{3.40}$$

which is the single-sheeted hyperboloid, again embedded in (3+1)-dimensional Minkowski space; this space is isomorphic to the double cover of (2+1)-dimensional de Sitter space. In the Lorentzian case with $\Lambda > 0$ we have

$$W_1^L = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 + w_0^2 - w_1^2 - w_2^2 = 1\}, \tag{3.41}$$

which is isomorphic to the double cover of (2+1)-dimensional anti-de Sitter space[4]. Finally, for $\Lambda = 0$ and Lorentzian signature, we obtain

$$W_0^L = \{(w_3, \boldsymbol{w}) \in \mathbb{R}^4 | w_3^2 = 1\}, \tag{3.42}$$

which is simply two copies of (2+1)-dimensional Minkowski space realised as hyperplanes inside (3+1)-dimensional Minkowski space. Thus, for each signature and value of the cosmological constant, we obtain (in some cases two copies of) the corresponding model space of 3d gravity.

For each signature and value of Λ, elements g of the group $\mathbb{H}_1(R_\Lambda)$ act on the set W via

$$\mathrm{I}(g) : W \to W, \qquad w \mapsto g w g^\circ. \tag{3.43}$$

Geometrically, this is the natural action of each of the local isometry groups (3.17) arising in 3d gravity on the (double covers of) the corresponding model spacetimes. In particular we therefore obtain actions of the unit (split) quaternions $\mathbb{H}_1$ on W, which we will need later in this paper. For $v \in \mathbb{H}_1$, the action

$$\mathrm{I}(v) : W \to W, \quad w \mapsto v w \bar{v} \tag{3.44}$$

is the natural action of the rotation group $SO(3)$ or the orthochronous Lorentz group $SO^+(2,1)$ on the model spacetimes. In the following we shall use the notation $\mathrm{I}(v)$ for the action of $v \in \mathbb{H}_1$ on both the set $W \subset \mathbb{H}(R_\Lambda)$ and the spaces $W_\Lambda \subset \mathbb{R}^4$ used to parametrise W.

4 The classical double

In this section we show how to equip $\mathbb{H}_1(R_\Lambda)$ with the Poisson-Lie structure of a classical double, and study its group structure in detail. Our construction works for arbitrary values of Λ in the Lorentzian case, but only for $\Lambda \leq 0$ in the Euclidean case. We begin by exhibiting Lie-bialgebra structures associated to $\mathfrak{h}_\Lambda$.

[4]Some authors refer to the spaces W_{-1}^L and W_1^L as de Sitter and anti-de Sitter space; we use the conventions of [22].

4.1 Bialgebra structures and classical r-matrices

Our results about the bialgebra structures on $\mathfrak{h}_\Lambda$ follow from a purely Lie-algebraic observation:

Lemma 4.1 *Let $\boldsymbol{n} = (n^0, n^1, n^2)$ be a vector in $\mathbb{R}^3$ satisfying*

$$\boldsymbol{n}^2 = \eta_{ab} n^a n^b = -\Lambda. \tag{4.1}$$

In terms of the generators

$$J_a = \tfrac{1}{2} e_a, \qquad S_a = \tfrac{\theta}{2} e_a + \tfrac{1}{2} \epsilon_{abc} n^b e^c = P_a + \epsilon_{abc} n^b J^c \tag{4.2}$$

of $\mathfrak{h}_\Lambda$, the Lie brackets then take the form

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, S_b] = \epsilon_{abc} S^c + n_b J_a - \eta_{ab}(n^c J_c), \qquad [S_a, S_b] = n_a S_b - n_b S_a. \tag{4.3}$$

Proof: This is a direct calculation using the relations for the quaternions, the condition $\boldsymbol{n}^2 = -\Lambda$ and the following identity for the epsilon-tensor

$$\epsilon_{abc}\epsilon^{cde} = \delta_a^d \delta_b^e - \delta_a^e \delta_b^d. \qquad \square \tag{4.4}$$

The lemma shows in particular that both $\mathfrak{h}$ and the span of $\{S_0, S_1, S_2\}$ form Lie subalgebras of $\mathfrak{h}_\Lambda$. The latter depends on the choice of the vector $\boldsymbol{n}$. For $\boldsymbol{n} = 0$, which requires $\Lambda = 0$, it is simply $\mathbb{R}^3$ with the trivial Lie bracket. For $\boldsymbol{n} \neq 0$, we will denote this Lie algebra by $\mathfrak{an}(2)_{\boldsymbol{n}}$ or, when the dependence of $\boldsymbol{n}$ need not be emphasised, simply by $\mathfrak{an}(2)$ in the following. The reason for the notation $\mathfrak{an}(2)$ is that the decomposition

$$\mathfrak{h}_\Lambda = \mathfrak{h} \oplus \mathfrak{an}(2)_{\boldsymbol{n}} \tag{4.5}$$

implied by Lemma 4.1 generalises the Iwasawa decomposition of $\mathfrak{sl}(2,\mathbb{C})$ into a compact part $\mathfrak{su}(2)$ and a "real abelian + nilpotent part" $\mathfrak{an}(2)$.

To exhibit the structure of $\mathfrak{an}(2)_{\boldsymbol{n}}$ more clearly and to prepare for calculations later in this paper we introduce new generators. Consider first the case $\Lambda \neq 0$, and pick a vector $\boldsymbol{m}$ orthogonal to $\boldsymbol{n}$ but otherwise arbitrary. Let

$$N = -\frac{2}{\Lambda} \boldsymbol{n} \cdot \boldsymbol{S} = -\frac{1}{\theta} \boldsymbol{n} \cdot \boldsymbol{e} \qquad \text{and} \qquad Q = \boldsymbol{m} \cdot \boldsymbol{S} = \frac{\theta}{2} \boldsymbol{m} \cdot \boldsymbol{e} + \frac{1}{2} \boldsymbol{m} \wedge \boldsymbol{n} \cdot \boldsymbol{e}. \tag{4.6}$$

Using $n_a n^a = -\Lambda$ one checks furthermore that

$$\theta Q = \boldsymbol{m} \wedge \boldsymbol{n} \cdot \boldsymbol{S}, \tag{4.7}$$

so that $\{N, Q, \theta Q\}$ is an alternative basis of the $\mathfrak{an}(2)$ Lie subalgebra. One finds

$$N^2 = 1, \qquad Q^2 = (\theta Q)^2 = 0, \qquad NQ = -QN = Q. \tag{4.8}$$

so that, in particular, the commutators take the form

$$[N, Q] = 2Q, \quad [N, \theta Q] = 2\theta Q, \quad [Q, \theta Q] = 0, \tag{4.9}$$

showing that $\mathfrak{an}(2)$ is isomorphic to the Lie algebra of the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$.

When $\Lambda = 0$ and $\boldsymbol{n}{\cdot}\boldsymbol{n} = 0$ with $\boldsymbol{n} \neq 0$ (i.e. in the Lorentzian case), division by θ is ill-defined, and we need to modify the definition (4.6). We introduce a second light-like vector $\tilde{\boldsymbol{n}}$ which satisfies

$$\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{n}} = 0, \qquad \boldsymbol{n}\cdot\tilde{\boldsymbol{n}} = 1. \tag{4.10}$$

Then the vector

$$\tilde{\boldsymbol{m}} = \tilde{\boldsymbol{n}} \wedge \boldsymbol{n} \tag{4.11}$$

is space-like, with $\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}} = -1$. Now define the following generators of the $\mathfrak{an}(2)$ Lie subalgebra

$$\begin{aligned} N &= 2\tilde{\boldsymbol{n}}\cdot\boldsymbol{S} = \theta\tilde{\boldsymbol{n}}\cdot\boldsymbol{e} + \tilde{\boldsymbol{m}}\cdot\boldsymbol{e}, \\ Q &= \tilde{\boldsymbol{m}}\cdot\boldsymbol{S} = \tfrac{\theta}{2}\tilde{\boldsymbol{m}}\cdot\boldsymbol{e} + \tfrac{1}{2}\boldsymbol{n}\cdot\boldsymbol{e}, \\ \theta Q &= \boldsymbol{n}\cdot\boldsymbol{S} = \tfrac{\theta}{2}\boldsymbol{n}\cdot\boldsymbol{e}. \end{aligned} \tag{4.12}$$

Like their counterparts in the $\Lambda \neq 0$ case, they satisfy (4.8), and therefore in particular the commutation relations (4.9).

The Lie subalgebras $\mathfrak{h}$ and $\mathfrak{an}(2)_{\boldsymbol{n}}$ of $\mathfrak{h}_\Lambda$ have the additional feature that they are both isotropic for the non-degenerate, invariant bilinear inner form $t(\cdot,\cdot)$ i.e. $t(X, Y) = 0$ if $X, Y \in \mathfrak{h}$ or $X, Y \in \mathfrak{an}(2)_{\boldsymbol{n}}$. This means, by definition [12], that $(\mathfrak{h}_\Lambda, \mathfrak{h}, \mathfrak{an}(2)_{\boldsymbol{n}})$ together with the invariant bilinear form $t(\cdot,\cdot)$ is a Manin triple. More generally we have

Corollary 4.2 *For every vector $\boldsymbol{n} \neq 0$ satisfying (4.1), the triple $(\mathfrak{h}_\Lambda, \mathfrak{h}, \mathfrak{an}(2)_{\boldsymbol{n}})$ together with the invariant bilinear form $t(\cdot,\cdot)$ defined in (2.8) is a Manin triple. When $\boldsymbol{n} = 0$ $(\mathfrak{h}_0, \mathfrak{h}, \mathbb{R}^3)$ is a Manin triple, with the same invariant bilinear form $t(\cdot,\cdot)$.*

Proof: The proof for $\boldsymbol{n} \neq 0$ follows from the remarks made before the Corollary. The proof for $\boldsymbol{n} = 0$ is analogous. $\Box$.

As explained in [12], this corollary is equivalent, via standard arguments, to the statement that both $\mathfrak{h}$ and $\mathfrak{an}(2)_{\boldsymbol{n}}$ have the structure of a Lie-bialgebra, and that they are dual as Lie-bialgebras. More generally

$$\mathfrak{h}^* = \begin{cases} \mathfrak{an}(2)_{\boldsymbol{n}} & \text{for} \quad \boldsymbol{n} \neq 0 \\ \mathbb{R}^3 & \text{for} \quad \boldsymbol{n} = 0 \end{cases} \tag{4.13}$$

(where * should not be confused with the conjugation in R_Λ).

Furthermore, the Corollary 4.2 is equivalent to the existence of a special bi-algebra structure on the Lie algebra $\mathfrak{h}_\Lambda$:

Corollary 4.3 *The Lie algebra $\mathfrak{h}_\Lambda$ has a canonical Lie-bialgebra structure, called the classical double, with classical r-matrix*

$$r = S_a \otimes J_a = P_a \otimes J^a + n^a \epsilon_{abc} J^b \otimes J^c \ \in \mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda, \tag{4.14}$$

where $\boldsymbol{n}$ is again assumed to satisfy (4.1)*. In particular r satisfies the classical Yang-Baxter equation.*

Proof: This is a standard construction in the theory of Lie-bialgebras, see e.g. [12], Sect. 1.4. □

The r-matrix (4.14) defines the co-commutator of the Lie-bialgebra $\mathfrak{h}_\Lambda$. Equivalently, if defines a commutator on the dual Lie bi-algebra $\mathfrak{h}_\Lambda^*$. Using the pairing $t(\cdot,\cdot)$ (2.8) to identify $\mathfrak{h}_\Lambda^*$ with $\mathfrak{h}_\Lambda$ as vector spaces, we thus obtain a second Lie bracket on $\mathfrak{h}_\Lambda$, called the dual Lie bracket in the following:

$$[J_a, J_b]^* = \epsilon_{abc} J^c, \qquad [J_a, S_b]^* = 0, \qquad [S_a, S_b]^* = n_b S_a - n_a S_b. \tag{4.15}$$

Note that this Lie algebra is simply the direct sum of the Lie algebras $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{R})$ with the Lie algebra $\mathfrak{an}(2)_{\boldsymbol{n}}$, $\boldsymbol{n} \neq 0$, or $\mathbb{R}^3$ for $\boldsymbol{n} = 0$.

4.2 Group structure of the classical double and factorisation

Most of the results about Lie-bialgebras have analogues at the group level, which we briefly review. A Manin triple or, equivalently, a classical double with its canonical r-matrix, exponentiates to a Poisson-Lie group which is (locally) factorisable [12]. Specifically, for the Manin triples $(\mathfrak{h}_\Lambda, \mathfrak{h}, \mathfrak{an}(2)_{\boldsymbol{n}})$ arising in 3d gravity, we obtain a Poison-Lie structure on each of the groups $\mathbb{H}_1(R_\Lambda)$ and the factorisation of elements in a neighbourhood of the identity into elements belonging to the subgroup $\mathbb{H}_1$ obtained by exponentiating $\mathfrak{h}$ and the subgroup $AN(2)_{\boldsymbol{n}}$ obtained by exponentiating $\mathfrak{an}(2)_{\boldsymbol{n}}$. As for the Lie algebra we will omit the label $\boldsymbol{n}$ on the group when the dependence need not be stressed. Since $\mathfrak{h}$ and $\mathfrak{an}(2)_{\boldsymbol{n}}$ are Lie-bialgebras in duality (4.13), the corresponding Lie groups are Poisson-Lie groups in duality

$$\mathbb{H}_1^* = \begin{cases} AN(2)_{\boldsymbol{n}} & \text{for } \ \boldsymbol{n} \neq 0 \\ \mathbb{R}^3 & \text{for } \ \boldsymbol{n} = 0, \end{cases} \tag{4.16}$$

where we again stress that * is not the conjugation in R_Λ. Finally, the dual Lie bi-algebra $\mathfrak{h}_\Lambda$ with Lie brackets (4.15) exponentiates to a Poisson-Lie group which is dual to $\mathbb{H}_1(R_\Lambda)$. From the brackets (4.15) it is obvious that, as a Lie group, the dual group is a direct product:

$$\mathbb{H}_1(R_\Lambda)^* = \begin{cases} \mathbb{H}_1 \times AN(2)_{\boldsymbol{n}} & \text{for } \ \boldsymbol{n} \neq 0 \\ \mathbb{H}_1 \times \mathbb{R}^3 & \text{for } \ \boldsymbol{n} = 0. \end{cases} \tag{4.17}$$

In this subsection, we are mainly concerned with group structure of $\mathbb{H}_1(R_\Lambda)$, leaving the discussion of its Poisson structure for Sect. 6. We will derive explicit expressions for the factorisation of elements in $\mathbb{H}_1(R_\Lambda)$, and show that it is possible to treat all signs of Λ and signatures in a common framework. The case $\boldsymbol{n} = 0$ can be seen as a degenerate limit of our construction. It is less interesting than the generic situation $\boldsymbol{n}$, but we will on occasion highlight some of its features. We will assume $\boldsymbol{n} \neq 0$ unless stated otherwise.

Elements of $AN(2)$ can be parametrised in a number of ways, two of which are important for us in the following. The first is a parametrisation of elements $t \in AN(2)$ in terms of an unconstrained vector $\boldsymbol{q} \in \mathbb{R}^3$:

$$t = \sqrt{1 + (\boldsymbol{qn})^2/4} + \boldsymbol{q}\cdot\boldsymbol{S} = \sqrt{1 + (\boldsymbol{qn})^2/4} + \frac{\theta}{2}\boldsymbol{q}\cdot\boldsymbol{e} + \frac{1}{2}\boldsymbol{q} \wedge \boldsymbol{n}\cdot\boldsymbol{e}. \tag{4.18}$$

In the limit $\boldsymbol{n} \to 0$ this expression reduces to $1 + \frac{\theta}{2}\boldsymbol{q}\cdot\boldsymbol{e}$, which agrees (apart from a factor $\frac{1}{2}$) with the earlier parametrisation (3.23) of elements in $\mathbb{R}^3$.

The second parametrisation for elements in $AN(2)$ makes use of the generators (4.6) and (4.7) for $\Lambda \neq 0$ and (4.12) for $\Lambda = 0$. It takes the form

$$r(\alpha, z) = (1 + zQ)e^{\alpha N}, \tag{4.19}$$

where $z \in R_\Lambda$ and $\alpha \in \mathbb{R}$. To see that this is a valid parametrisation, we relate it to the earlier parametrisation (4.18), focussing on the case $\Lambda \neq 0$ (the calculation for $\Lambda = 0$ is similar). Using the relation

$$e^{\alpha N} = \cosh\alpha + \sinh\alpha\ N \tag{4.20}$$

we deduce

$$r(\alpha, z) = \cosh\alpha + \sinh\alpha\, N + e^{-\alpha} z\, Q. \tag{4.21}$$

In particular, using (4.6) and (4.7), and expanding $z = \xi + \theta\eta$ we can therefore write

$$r(\alpha, z) = \cosh\alpha - \frac{2}{\Lambda}\sinh\alpha\ \boldsymbol{n}\cdot\boldsymbol{S} + e^{-\alpha}\xi\ \boldsymbol{m}\cdot\boldsymbol{S} + e^{-\alpha}\eta\ (\boldsymbol{m} \wedge \boldsymbol{n})\cdot\boldsymbol{S} \quad \text{for} \quad \Lambda \neq 0. \tag{4.22}$$

Comparing with (4.18) we find the following relation between the parameters $\boldsymbol{q}$ and (α, z):

$$\boldsymbol{q} = -\frac{2}{\Lambda}\sinh\alpha\ \boldsymbol{n} + e^{-\alpha}\xi\ \boldsymbol{m} + e^{-\alpha}\eta\ (\boldsymbol{m} \wedge \boldsymbol{n}). \tag{4.23}$$

The parametrisation (4.19) is useful for a number of purposes. It makes manifest the semidirect product structure of $AN(2)$. The relation

$$r(\alpha_1, z_1)r(\alpha_2, z_2) = r(\alpha_1 + \alpha_2, z_1 + e^{2\alpha_1}z_2). \tag{4.24}$$

follows directly from $NQ = -QN = Q$, and shows that $AN(2) \simeq \mathbb{R} \ltimes \mathbb{R}^2$. The main use of the parametrisation (4.19) for us is the dressing action of $AN(2)$ on $\mathbb{H}_1$, to be discussed in Sect. 5

The subgroup $AN(2)$ is intimately related to the subset W of $\mathbb{H}(R_\Lambda)$ introduced and discussed in Sect. 3.2, whose elements satisfy $w^\circ = w$ and parametrise the associated model spacetimes. It is clear that for any $g \in \mathbb{H}_1(R_\Lambda)$, $g^\circ g \in W$ and $gg^\circ \in W$. This holds in particular for elements $t \in AN(2)$. We would like to know if, and under which conditions, the maps

$$S : AN(2) \to W, \quad t \mapsto t^\circ t, \tag{4.25}$$
$$\tilde{S} : AN(2) \to W, \quad t \mapsto tt^\circ \tag{4.26}$$

can be inverted. In discussing these maps we shall often not distinguish between the subsets $AN(2)$ and W of $\mathbb{H}_1(R_\Lambda)$ and the sets $\mathbb{R}^3$ and W_Λ (3.36) used to parametrise them. Thus, using the parametrisation (4.18) for $t \in AN(2)$ in terms of $\boldsymbol{q} \in \mathbb{R}^3$, the map S can be written explicitly as

$$S : \mathbb{R}^3 \to W_\Lambda, \quad \boldsymbol{q} \mapsto (1 - \frac{\Lambda}{2}\boldsymbol{q}^2, \sqrt{1 + (\boldsymbol{qn})^2/4}\, \boldsymbol{q} + \frac{1}{2}\boldsymbol{q} \wedge (\boldsymbol{q} \wedge \boldsymbol{n})). \tag{4.27}$$

Similarly, the map $\tilde{S}$ takes the form

$$\tilde{S} : \mathbb{R}^3 \to W_\Lambda, \quad \boldsymbol{q} \mapsto (1 - \frac{\Lambda}{2}\boldsymbol{q}^2, \sqrt{1 + (\boldsymbol{qn})^2/4}\, \boldsymbol{q} - \frac{1}{2}\boldsymbol{q} \wedge (\boldsymbol{q} \wedge \boldsymbol{n})). \tag{4.28}$$

Lemma 4.4 *The map S of* (4.27) *is injective but not, in general, surjective. Its image is*

$$W_\Lambda^+ = \{(w_3, \boldsymbol{w}) \in W_\Lambda | w_3 + \boldsymbol{wn} > 0\}. \tag{4.29}$$

Restricted to this set, the inverse exists and is given by

$$S^{-1} : W_\Lambda^+ \to \mathbb{R}^3, \quad (w_3, \boldsymbol{w}) \mapsto \begin{cases} \frac{1}{\sqrt{w_3 + \boldsymbol{wn}}} \left(\boldsymbol{w} + \frac{1 - w_3}{\Lambda}\boldsymbol{n}\right) & \text{if} \quad \Lambda \neq 0 \\ \frac{1}{\sqrt{1 + \boldsymbol{wn}}} \left(\boldsymbol{w} + \frac{\boldsymbol{w}^2}{2}\boldsymbol{n}\right) & \text{if} \quad \Lambda = 0. \end{cases} \tag{4.30}$$

An analogous statement holds for the map $\tilde{S}$, with $\boldsymbol{n}$ replaced by $-\boldsymbol{n}$.

Proof: Injectivity of S and the formula (4.30) for the inverse of S can be shown by evaluating $S^{-1} \circ S(\boldsymbol{q})$. With $(w_3, \boldsymbol{w}) = S(\boldsymbol{q})$ one checks that

$$w_3 + \boldsymbol{wn} = (\sqrt{1 + (\boldsymbol{qn})^2/4} + \frac{1}{2}\boldsymbol{qn})^2, \tag{4.31}$$

and using this formula it is straightforward to confirm that $S^{-1} \circ S(\boldsymbol{q}) = \boldsymbol{q}$ as required. It is clear from the expression for S^{-1} that it is only defined on $(w_3, \boldsymbol{w})$ if $w_3 + \boldsymbol{wn} > 0$. The proof for $\tilde{S}$ follows by replacing $\boldsymbol{n} \mapsto -\boldsymbol{n}$. □

We make two comments on the formula (4.30). Firstly, note that the expression for $\Lambda = 0$ in (4.30) also follows from the formula for $\Lambda \neq 0$ by Taylor expanding $w_3 = \sqrt{1 - \Lambda \boldsymbol{w}^2}$ in powers of Λ and taking the limit $\Lambda \to 0$. Secondly, the condition $w_3 + \boldsymbol{wn} > 0$ can be interpreted geometrically by saying that it removes a part of the space W_Λ, i. e. the model spacetime, but the details depend on the value of Λ, the signature and also on the choice of $\boldsymbol{n}$. In the Euclidean case, with $\Lambda < 0$, we find, using the Cauchy-Schwarz inequality and $|\boldsymbol{n}| = \sqrt{-\Lambda}$, that

$$|\boldsymbol{w}\boldsymbol{n}|^2 \leq -\Lambda \boldsymbol{w}^2 < w_3^2. \tag{4.32}$$

Hence, both the conditions $w_3 \pm \boldsymbol{w}\boldsymbol{n} > 0$ are automatically fulfilled for all $w_3 > 0$ (upper sheet of the two-sheeted hyperboloid) but never for $w_3 < 0$. Thus the map S establishes a bijection between $\mathbb{R}^3$ and the upper sheet of the two-sheeted hyperboloid in the Euclidean case. In the Lorentzian cases, the condition for invertibility is harder to interpret geometrically.

We are now ready to state and prove one of the main results of this paper:

Theorem 4.5 *A given element g in the gravity Lie groups $\mathbb{H}_1(R_\Lambda)$ can be factorised into*

$$g = u \cdot s, \qquad s \in AN(2),\ u \in \mathbb{H}_1, \tag{4.33}$$

provided $\Pi\left(g^\circ g(1 - \frac{\boldsymbol{n}\boldsymbol{e}}{\theta})\right) > 0$, where Π is defined as in (3.15). The element g can be factorised into

$$g = r \cdot v, \qquad r \in AN(2),\ v \in \mathbb{H}_1, \tag{4.34}$$

provided $\Pi\left(g g^\circ(1 + \frac{\boldsymbol{n}\boldsymbol{e}}{\theta})\right) > 0$. When they exist, the factors are unique and given by

$$s(g) = \frac{1}{2N_-}\left((1 + g^\circ g) + \frac{\boldsymbol{n}\cdot\boldsymbol{e}}{\theta}(1 - g^\circ g) - \tilde{\delta}_\Lambda \tfrac{\theta}{2}\,\boldsymbol{n}\cdot\boldsymbol{e}\left(\frac{1 - g^\circ g}{\theta}\right)^2\right), \tag{4.35}$$

$$r(g) = \frac{1}{2N_+}\left((1 + g g^\circ) - (1 - g g^\circ)\frac{\boldsymbol{n}\cdot\boldsymbol{e}}{\theta} + \tilde{\delta}_\Lambda \tfrac{\theta}{2}\,\boldsymbol{n}\cdot\boldsymbol{e}\left(\frac{1 - g g^\circ}{\theta}\right)^2\right), \tag{4.36}$$

$$u(g) = \frac{1}{2N_-}\left((g + g^*) - (g - g^*)\frac{\boldsymbol{n}\cdot\boldsymbol{e}}{\theta}\right), \tag{4.37}$$

$$v(g) = \frac{1}{2N_+}\left((g + g^*) + \frac{\boldsymbol{n}\cdot\boldsymbol{e}}{\theta}(g - g^*)\right), \tag{4.38}$$

with g-dependent normalisation factors

$$N_+ = \sqrt{\Pi\left(g g^\circ(1 + \tfrac{\boldsymbol{n}\boldsymbol{e}}{\theta})\right)}, \qquad N_- = \sqrt{\Pi\left(g^\circ g(1 - \tfrac{\boldsymbol{n}\boldsymbol{e}}{\theta})\right)}, \tag{4.39}$$

and $\tilde{\delta}_\Lambda = 1$ for $\Lambda = 0$ and $\tilde{\delta}_\Lambda = 0$ for $\Lambda \neq 0$.

The factor $\frac{1}{\theta}$ in (4.35) to (4.39) is defined via $\frac{1}{\theta} = \frac{\theta}{\Lambda}$ for $\Lambda \neq 0$. For $\Lambda = 0$, θ is a zero divisor of the ring R_Λ and division by θ is ill-defined. Expressions (4.35) to (4.39) are nevertheless well-defined for $\Lambda = 0$, since all factors multiplied by $\frac{1}{\theta}$ are of the form θt, where $t \in \mathbb{H}$. Hence, in a slight abuse of notation we set for $\Lambda = 0$ and $w = \theta t$, $t \in \mathbb{H}$

$$\tfrac{1}{\theta} w = \tfrac{1}{\theta}(\theta t) = \mathrm{Im}_\theta(w) = t. \tag{4.40}$$

All terms involving division by θ for $\Lambda = 0$ are to be interpreted in this sense in the following.

Proof: Consider a general element $g \in \mathbb{H}_1(R_\Lambda)$ factorised as in (4.33) and (4.34). The fact that the conjugation operation $\circ$ is an anti-group automorphism which maps elements of $\mathbb{H}_1$ to their inverses implies

$$g^\circ g = s^\circ s = v^{-1} r^\circ r v, \qquad g g^\circ = r r^\circ = u s s^\circ u^{-1}. \tag{4.41}$$

Since both $g^\circ g$ and $g g^\circ$ are elements of the space W (3.33) we can apply Lemma 4.4 to find

$$s = S^{-1}(g^\circ g), \qquad r = \tilde{S}^{-1}(g g^\circ) \tag{4.42}$$

provided the conditions for the existence of the inverse are satisfied. The formulas (4.35) and (4.36) are simply a re-writing of (4.42), using the definition (4.30). To see this, focus on (4.35) and parametrise

$$g^\circ g = w_3 + \theta \boldsymbol{w} \cdot \boldsymbol{e}. \tag{4.43}$$

Then

$$\Pi\left(g^\circ g (1 - \tfrac{\boldsymbol{n}\boldsymbol{e}}{\theta})\right) = w_3 + \boldsymbol{w} \cdot \boldsymbol{n}, \tag{4.44}$$

showing that the condition for existence of the inverse are precisely those of Lemma 4.4. By straightforward calculation one finds

$$\begin{aligned} &\frac{1}{2N_-}\left((1 + g^\circ g) + \frac{\boldsymbol{n}\cdot\boldsymbol{e}}{\theta}(1 - g^\circ g) - \tilde{\delta}_\Lambda \tfrac{\theta}{2}\, \boldsymbol{n}\cdot\boldsymbol{e}\left(\frac{1 - g^\circ g}{\theta}\right)^2\right) \\ &= \begin{cases} \frac{1}{2\sqrt{w_3 + \boldsymbol{w}\boldsymbol{n}}}\left((1 + w_3 + \boldsymbol{w}\boldsymbol{n}) + \theta(\boldsymbol{w} + \frac{1 - w_3}{\Lambda}\boldsymbol{n})\cdot\boldsymbol{e} + \boldsymbol{w} \wedge \boldsymbol{n}\cdot\boldsymbol{e}\right) & \text{if} \quad \Lambda \neq 0 \\ \frac{1}{2\sqrt{2 + \boldsymbol{w}\boldsymbol{n}}}\left((2 + \boldsymbol{w}\boldsymbol{n}) + \theta(\boldsymbol{w} + \frac{\boldsymbol{w}^2}{2}\boldsymbol{n})\cdot\boldsymbol{e} + \boldsymbol{w} \wedge \boldsymbol{n}\cdot\boldsymbol{e}\right) & \text{if} \quad \Lambda = 0. \end{cases} \end{aligned} \tag{4.45}$$

Using again the relation (4.31) one now checks that the right-hand side is

$$s = \sqrt{1 + (\boldsymbol{q}\boldsymbol{n})^2/4} + \frac{\theta}{2}\boldsymbol{q}\cdot\boldsymbol{e} + \frac{1}{2}\boldsymbol{q} \wedge \boldsymbol{n}\cdot\boldsymbol{e}, \tag{4.46}$$

with $\boldsymbol{q} = S^{-1}((w_3, \boldsymbol{w}))$, thus showing that (4.35) is equivalent to (4.42). The calculation showing (4.36) is analogous.

To show the formula (4.37) we note that, with $g = us$, and $u^* = u$ by definition,

$$\frac{1}{2}(g + g^*) - (g - g^*)\frac{\boldsymbol{n}\cdot\boldsymbol{e}}{2\theta} = u\left(\frac{1}{2}(s + s^*) - (s - s^*)\frac{\boldsymbol{n}\cdot\boldsymbol{e}}{2\theta}\right). \tag{4.47}$$

Now parametrise s as in (4.46), and compute

$$\frac{1}{2}(s + s^*) - (s - s^*)\frac{\boldsymbol{n}\cdot\boldsymbol{e}}{2\theta} = \sqrt{1 + (\boldsymbol{q}\cdot\boldsymbol{n})^2/4} + \tfrac{1}{2}\boldsymbol{q}\boldsymbol{n} \tag{4.48}$$

as well as

$$\Pi(g^\circ g(1-\tfrac{\boldsymbol{ne}}{\theta}))=\Pi(s^\circ s(1-\tfrac{\boldsymbol{ne}}{\theta}))=(\sqrt{1+(\boldsymbol{q}\cdot\boldsymbol{n})^2/4}+\tfrac{1}{2}\boldsymbol{qn})^2. \tag{4.49}$$

Now (4.37) follows from (4.47) with the substitutions (4.48) and (4.49). The proof of (4.38) is again entirely analogous. □

As a consequence of the geometrical considerations after the proof of Theorem 4.5 we have the following

Corollary 4.6 *The factorisation of Theorem 4.5 is globally defined in Euclidean case with* $\Lambda < 0$.

The formulas (4.35)-(4.38) also hold for the case $\Lambda = 0$ and $\boldsymbol{n} = 0$. Then $N_+(g) = N_-(g) = 1$ so that the factorisation is globally defined. The factorisation is now simply

$$u(g)=v(g)=\tfrac{1}{2}(g+g^*), \qquad s(g)=\tfrac{1}{2}(1+g^\circ g), \qquad r(g)=\tfrac{1}{2}(1+gg^\circ), \tag{4.50}$$

showing how to factor an element in the semi-direct product $\mathbb{H}_1(R_0)\cong\mathbb{H}_1\ltimes\mathbb{R}^3$ into an $\mathbb{H}_1$ and a $\mathbb{R}^3$ component.

5 Dressing transformations

One important use of Theorem 4.5 is the definition and computation of dressing transformations. The interest of dressing transformations in physics stems from the fact that their orbits give the symplectic leaves of a Poisson space [13, 23]. In order to define dressing transformations in the case at hand, note that the formulas (4.35)-(4.38) allow one to compute the factors $u\in\mathbb{H}_1$ and $s\in AN(2)$, given the factors $v\in\mathbb{H}_1$ and $r\in AN(2)$ and conversely. It follows directly from the definition (4.33) that u, when defined, is obtained by a left action of r on v; similarly the factor s, when defined, is obtained by a right action of v on r. Thus we define the $AN(2)$ left action

$$L_r:\mathbb{H}_1\to\mathbb{H}_1, \qquad v\mapsto L_r(v):=u \tag{5.1}$$

and the $\mathbb{H}_1$ right action

$$R_v: AN(2)\to AN(2), \qquad r\mapsto R_v(r):=s. \tag{5.2}$$

In this way we obtain dressing transformations on the group manifolds $AN(2)$ and $SU(2)$ or $SL(2,\mathbb{R})$. As mentioned at the beginning of Sect. 4.2, the r-matrices (4.14) give rise to Poisson-Lie structures for both of these groups. We will describe the associated Poisson structures explicitly in Sect. 6.

In this section we introduce a quaternionic formalism which gives a simple and unified description of the geometry of dressing transformations In the case $\Lambda = 0$ and $\boldsymbol{n} = 0$ the dressing transformations are very simple: the left action L_r is trivial, and the right action R_v is the adjoint (right) action of $\mathbb{H}_1$. We therefore assume that $\boldsymbol{n}\neq 0$ in the following. The geometrical interpretation of the dressing right action R_v follows directly from Theorem 4.5:

Theorem 5.1 *In terms of the parametrisation* $r = \sqrt{1+(\boldsymbol{qn})^2/4} + \boldsymbol{q}\cdot\boldsymbol{S}$, $\boldsymbol{q} \in \mathbb{R}^3$ *of* $r \in AN(2)$ *and the map* S *defined in* (4.27)*, the dressing right action of* $v \in \mathbb{H}_1$ *is the map*

$$R_v : \boldsymbol{q} \mapsto S^{-1}\left(I(v^{-1})S(\boldsymbol{q})\right), \tag{5.3}$$

where the action I *of* $v \in \mathbb{H}_1$ *is defined as in* (3.44)*.*

Proof: The formula (5.3) is merely a re-writing of the first formula in (4.41). According to that formula, one obtains $s^\circ s \in W$ from $r^\circ r \in W$ by acting with $\mathrm{I}(v^{-1})$. However, in the notation of (4.25), $S(r) = r^\circ r$ and $S^{-1}(s^\circ s) = s$. □

It is clear from our discussion about the invertibility of S following the proof of Lemma 4.4 that the dressing action R_v is globally defined in the Euclidean case, but not in the Lorentzian case: the formula (5.3) only makes sense if $\mathrm{I}(v^{-1})(S(q)$ is in the domain of the map S^{-1}. In order to understand the geometry of dressing orbits in detail, we use the notation $S(\boldsymbol{q}) = (w_3, \boldsymbol{w})$, with the formulas (4.27) for w_3 and $\boldsymbol{w}$. One then finds that

$$\boldsymbol{w}^2 = \boldsymbol{q}^2\left(1 - \frac{\Lambda}{4}\boldsymbol{q}^2\right). \tag{5.4}$$

Both w_3 and $\boldsymbol{w}^2$ are invariant under the adjoint action (3.44) of $\mathbb{H}_1$ on w; When $\Lambda \neq 0$ it follows from $w_3 = 1 - \frac{\Lambda}{2}\boldsymbol{q}^2$ that $\boldsymbol{q}^2$ is also invariant; when $\Lambda = 0$ the invariance of $\boldsymbol{q}^2$ can be inferred from (5.4). It follows that the orbits under the dressing actions are subsets of the level sets $\boldsymbol{q}^2$ =const.

In the Euclidean case it is easy to see that the orbits are actually equal to the level sets. To prove this we define

$$\mathcal{O}^E_\rho = \{\boldsymbol{q} \in \mathbb{R}^3 | \boldsymbol{q}^2 = \rho\}, \qquad \rho \geq 0, \tag{5.5}$$

and show that the dressing action on $\mathcal{O}_\rho$ is transitive. Suppose that $\boldsymbol{q}, \tilde{\boldsymbol{q}} \in \mathcal{O}_\rho$. With $S(\tilde{\boldsymbol{q}}) = (\tilde{w}_3, \tilde{\boldsymbol{w}})$ we then have $\boldsymbol{w}^2 = \tilde{\boldsymbol{w}}^2$ and $w_3 = \tilde{w}_3$. Thus there exists a quaternion $v \in \mathbb{H}_1$ so that $\mathrm{I}(v^{-1})w = \tilde{w}$; this follows from the transitivity of the $SO(3)$ action on the spheres $\boldsymbol{w}^2$ =const. But then, by definition, $\tilde{\boldsymbol{q}} = R_v(\boldsymbol{q})$, which was to be shown. Geometrically, the orbits of the dressing action of $\mathbb{H}_1$ on the vector $\boldsymbol{q} \in \mathbb{R}^3$ are therefore the familiar orbits of $SO(3)$ acting on $\mathbb{R}^3$: a point (the origin) or spheres.

The Lorentzian situation is more complicated because the level sets $\boldsymbol{q}^2$ =const. are not, in general, connected. We have the trivial orbit consisting of the origin, the single-sheeted hyperboloids

$$\mathcal{O}^L_\rho = \{\boldsymbol{q} \in \mathbb{R}^3 | \boldsymbol{q}^2 = \rho\}, \qquad \rho < 0, \tag{5.6}$$

as well as the upper and lower sheet of the two-sheeted hyperboloid

$$\mathcal{O}^{L+}_\rho = \{\boldsymbol{q} \in \mathbb{R}^3 | \boldsymbol{q}^2 = \rho, q_0 > 0\}, \qquad \mathcal{O}^{L-}_\rho = \{\boldsymbol{q} \in \mathbb{R}^3 | \boldsymbol{q}^2 = \rho, q_0 < 0\}, \qquad \rho > 0, \tag{5.7}$$

and the upper and lower lightcone

$$\mathcal{O}_0^{L+} = \{\boldsymbol{q} \in \mathbb{R}^3 | \boldsymbol{q}^2 = 0, q_0 > 0\}, \qquad \mathcal{O}_0^{L-} = \{\boldsymbol{q} \in \mathbb{R}^3 | \boldsymbol{q}^2 = 0, q_0 < 0\}. \tag{5.8}$$

Since $SO^+(2,1)$ is a connected group, each orbit of the dressing action is connected and therefore must be a subset of one of the components listed above. To check if the dressing action is transitive on the orbits we again start with $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}$ in one of the sets (5.6)-(5.8), and compute the corresponding images $S(\boldsymbol{q}) = (w_3, \boldsymbol{w})$ and $S(\tilde{\boldsymbol{q}}) = (\tilde{w}_3, \tilde{\boldsymbol{w}})$. Again we have $\boldsymbol{w}^2 = \tilde{\boldsymbol{w}}^2$, but this only guarantees the existence of a split quaternion v so that $\mathrm{I}(v^{-1})w = \tilde{w}$ if $\boldsymbol{w}^2 < 0$ because both lie on the same single-sheeted hyperboloid in that case. Note that, because of (5.4), this may happen for both $\boldsymbol{q}^2 < 0$ and $\boldsymbol{q}^2 > 0$. Thus we conclude that the sets (5.6)-(5.8) are indeed dressing orbits provided the label ρ satisfies $\rho(1 - \frac{\Lambda}{4}\rho) < 0$. The origin is trivially an orbit; in all other cases further analysis is need to determine the orbit geometry.

Next we turn to the dressing left action L_r of $r \in AN(2)$ on $\mathbb{H}_1$. Here the cases $\Lambda \neq 0$ and $\Lambda = 0$ with $\boldsymbol{n} \neq 0$ (which is necessarily Lorentzian) require slightly different conventions and treatments. We begin with the case $\Lambda \neq 0$ and recall the definitions (4.6) of the quaternion N and Q in terms of the orthogonal vectors $\boldsymbol{n}$ and $\boldsymbol{m}$, with $\boldsymbol{n}^2 = -\Lambda$. It follows that $N^2 = 1$ and that the operators

$$P = \frac{1}{2}(1 + N) \qquad \text{and} \qquad \bar{P} = \frac{1}{2}(1 - N) \tag{5.9}$$

are projection operators. Furthermore, we define $M = \theta \boldsymbol{m}\cdot\boldsymbol{e}$ and $Q = PM$. Then

$$Q^\circ = MP = \frac{1}{2}\theta\boldsymbol{m}\cdot\boldsymbol{e} - \frac{1}{2}\boldsymbol{m} \wedge \boldsymbol{n}\cdot\boldsymbol{e}, \tag{5.10}$$

and it is easy to check the following "projector algebra" of the elements $P, \bar{P}, Q, Q^\circ$:

$$\begin{aligned}
P^2 &= P, \quad \bar{P}^2 = \bar{P}, \quad P\bar{P} = \bar{P}P = 0, \\
PQ &= Q, \quad PQ^\circ = 0, \quad \bar{P}Q = 0, \quad \bar{P}Q^\circ = Q^\circ, \\
QP &= 0, \quad Q^\circ P = Q^\circ, \quad Q\bar{P} = Q, \quad Q^\circ\bar{P} = 0, \\
Q^2 &= 0, \quad (Q^\circ)^2 = 0, \quad QQ^\circ = -\Lambda m^a m_a P, \qquad Q^\circ Q = -\Lambda m^a m_a \bar{P}.
\end{aligned} \tag{5.11}$$

In words: P and $\bar{P}$ are projection operators when acting from left or right, and the nilpotent elements Q and Q° are eigenstates of the left- and right-projections. The last line in (5.11) suggests the normalisation $m^a m_a = -\frac{1}{\Lambda}$. This can be achieved in the Euclidean case when $\Lambda < 0$, and in the Lorentzian case when $\Lambda > 0$; these cases will be referred to collectively as case (I) in the following. In the Lorentzian case when $\Lambda < 0$, however, the vector $\boldsymbol{n}$ is timelike and it is impossible to find a vector orthogonal $\boldsymbol{n}$ which is also timelike. In that case, called (II) in the following, we choose $m^a m_a = \frac{1}{\Lambda}$. To sum up, we have the relations

$$QQ^\circ = \pm P, \qquad Q^\circ Q = \pm\bar{P}, \tag{5.12}$$

where the upper sign refers to (I) and the lower sign to (II). Finally we note the following properties with respect to conjugations:

$$P^* = \bar{P}, \qquad P^\circ = P, \qquad \bar{Q} = -Q, \qquad Q^* = -Q^\circ. \tag{5.13}$$

An element v of the subgroup $\mathbb{H}_1$ of unit (split) quaternions can be parametrised in terms of the projector algebra elements P and Q as

$$v = xP + yQ + x^*\bar{P} - y^*Q^\circ, \tag{5.14}$$

with $x, y \in R_\Lambda$ satisfying the constraint

$$xx^* \pm yy^* = 1, \tag{5.15}$$

where the upper sign refers to case (I), and the lower to case (II). It is clear that the element v in (5.14) satisfies $v^* = v$, and easy to check that $v\bar{v} = 1$ is implied by (5.14). To see that any element in $\mathbb{H}_1$ can be written in the form (5.14), write

$$x = a + \theta b, \qquad y = c + \theta d \tag{5.16}$$

and find that (5.14) is equivalent to

$$v = a - b\,\boldsymbol{n}\cdot\boldsymbol{e} + c\,(\boldsymbol{m}\wedge\boldsymbol{n})\cdot\boldsymbol{e} + \Lambda d\,\boldsymbol{m}\cdot\boldsymbol{e}. \tag{5.17}$$

This is the expansion of a quaternion in the orthogonal (but not orthonormal) basis $\{1, \boldsymbol{n}\cdot\boldsymbol{e}, (\boldsymbol{m}\wedge\boldsymbol{n})\cdot\boldsymbol{e}, \boldsymbol{m}\cdot\boldsymbol{e},\}$; any element in $\mathbb{H}_1$ can be written in this way, provided

$$a^2 - \Lambda b^2 \pm (c^2 - \Lambda d^2) = 1,$$

which is precisely the condition (5.15).

Putting the formulas of Theorem 4.5 together with notation of this subsection, we arrive at the following geometric characterisation of dressing transformations.

Theorem 5.2 *With the notation of Theorem 4.5, and assuming that $\Lambda \neq 0$, the dressing left action of an element $r \in AN(2)$ on an element $v \in \mathbb{H}_1$ is*

$$L_r(v) = \frac{1}{N_-}\left(rvP + r^*v\bar{P}\right). \tag{5.18}$$

In terms of the parametrisations (4.19) *of r and* (5.14) *of v the action L_r is the map*

$$(x, y) \mapsto \frac{1}{N_-}(e^\alpha x \mp ze^{-\alpha}y^*, e^{-\alpha}y), \tag{5.19}$$

with

$$N_- = \sqrt{(e^\alpha x \mp ze^{-\alpha}y^*)(e^\alpha x^* \mp z^*e^{-\alpha}y) \pm e^{-2\alpha}yy^*}. \tag{5.20}$$

Note that the dressing action L_r, like the dressing action R_v, is globally defined in the Euclidean case where the factor N_- is always non-zero. In the Lorentzian case N_- may vanish, so L_r is not globally defined.

Proof: The result (5.18) is the formula (4.37) written in terms of the projection operators P and $\bar{P}$. Re-writing the parametrisation (4.19) for r in the form

$$r(\alpha, z) = e^{\alpha} P + e^{-\alpha}(\bar{P} + zQ), \tag{5.21}$$

using (5.14) for v and the relations (5.11), one computes

$$rvP = (e^{\alpha} x \mp z e^{-\alpha} y^*)P - e^{-\alpha} y^* Q^0. \tag{5.22}$$

Now observe that, since $v^* = v$, the second term in the final expression (5.18) is the $*$-conjugate of the first. Thus

$$L_r(v) = \frac{1}{N_-}\left((e^{\alpha} x \mp e^{-\alpha} z y^*)P + yQ \quad + \quad *\text{-conjugate}\right). \tag{5.23}$$

Comparing with the parametrisation (5.14), and noting that the factor $1/N_-$ merely ensures that $L_r(v)$ is a unit (split) quaternion, we conclude that, in terms of the coordinates $x, y \in R_\Lambda$ in (5.14), L_r is the map (5.19). $\square$

The formula (5.19) gives a simple geometrical description of orbits under the dressing transformation L_r. The key observation is that the coordinate y is only multiplied by a real factor. Thus the direction of the (split) complex number in the (split) complex plane remains unchanged, and the product yy^* is multiplied by a positive number. The orbit is the set of all $x \in R_\Lambda$ satisfying the constraint (5.15) as yy^* is rescaled by an arbitrary positive number. The geometry of this set depends on the signature and the value of Λ.

Euclidean signature, $\Lambda < 0$: This is a much studied case, see e.g. [14]. The constraint $xx^* + yy^* = 1$ defines a three-sphere S^3_Λ embedded in $\mathbb{R}^4$ (and squashed if $\Lambda \neq -1$). There are two kinds of orbits, depending on whether $y = 0$ or $y \neq 0$. In the former case the orbits consist of points $(x, 0)$ inside S^3_Λ. In the second case we automatically have $xx^* < 1$, so the orbits consist of discs in the x-plane, and are labelled by the argument of y (which is unchanged by the scaling).

Lorentzian signature, $\Lambda > 0$: The constraint $xx^* + yy^* = 1$ defines the double cover $\widetilde{AdS}^3_\Lambda$ of three-dimensional anti-de Sitter space embedded in $\mathbb{R}^4$ (and squashed if $\Lambda \neq 1$). There are three kinds of orbits, depending on whether yy^* is positive, zero or negative. Since $xx^* = 1 - yy^*$, the product xx^* is correspondingly less than 1, equal to 1 or bigger than one. Since the equation $xx^* = 1$ defines a hyperbola with two branches in the x-plane, the dressing orbit is the region between the two branches of the hyperbola when $yy^* > 0$ and the region outside the branches when $yy^* < 0$. When $yy^* = 0$ the orbit is the Cartesian product of the hyperbola (both branches) in the x-plane with the double lightcone in the y-plane.

Lorentzian signature $\Lambda < 0$: The constraint $xx^* - yy^* = 1$ again defines the double cover of three-dimensional anti-de Sitter space embedded in $\mathbb{R}^4$ (and squashed if $\Lambda \neq -1$). This time $yy^* \geq 0$, so there are only two kinds of orbits. If $y = 0$ the orbits consist of points $(x, 0)$ inside $\widetilde{AdS}^3_\Lambda$. If $y \neq 0$, we automatically have $xx^* > 1$, so in each case the orbit is the complement of a disc, and is labelled by the argument of y.

Turning finally to the case $\Lambda = 0$, we use the vectors $\tilde{\boldsymbol{n}}$ and $\tilde{\boldsymbol{m}}$ defined in (4.10) and (4.11) to parametrise elements $v \in \mathbb{H}_1$ via

$$v = a + b\tilde{\boldsymbol{m}}\cdot\boldsymbol{e} + \gamma\boldsymbol{n}\cdot\boldsymbol{e} + \tilde{\gamma}\tilde{\boldsymbol{n}}\cdot\boldsymbol{e} \tag{5.24}$$

in terms of $a, b, \gamma, \tilde{\gamma} \in \mathbb{R}$. The condition $v\bar{v} = 1$ is equivalent to

$$a^2 - b^2 + 2\gamma\tilde{\gamma} = 1. \tag{5.25}$$

Using the parametrisations

$$r(\alpha, z) = (1 + zQ)e^{\alpha N}, \tag{5.26}$$

with N and Q as in (4.12), we compute $L_r(v) = u$ according to (4.37), i.e.

$$L_r(v) = \frac{1}{2N_-}\left((rv + r^*v) - (rv - r^*v)\frac{\boldsymbol{n}\cdot\boldsymbol{e}}{\theta}\right). \tag{5.27}$$

With $\Lambda = 0$ we have not been able to introduce the analogue of the projector algebra (5.11), which simplified the calculation in the $\Lambda \neq 0$ case. However, one can still evaluate (5.27):

Theorem 5.3 *In the case $\Lambda = 0$, $\boldsymbol{n} \neq 0$, the dressing left action L_r of $r \in AN(2)$ on $v \in \mathbb{H}_1$ is conveniently expressed in terms of the parametrisation* (5.26) *for v and the R_0 coordinates*

$$x = (a + b) - \theta\gamma, \qquad y = \tilde{\gamma} + \theta b \tag{5.28}$$

for v, written as in (5.24)*. In terms of these coordinates, the left action L_r is the map*

$$(x, y) \mapsto \frac{1}{N_-}(e^{\alpha}x - ze^{-\alpha}y^*, e^{-\alpha}y), \tag{5.29}$$

where

$$N_- = \sqrt{Re_\theta((e^{\alpha}x - ze^{-\alpha}y^*)^2) - 2Im_\theta((e^{\alpha}x - zy^*e^{-\alpha})e^{-\alpha}y)}. \tag{5.30}$$

Proof: This is a direct, somewhat tedious calculation. The formula for the normalisation factor follows from $v\bar{v} = a^2 - b^2 + 2\tilde{\gamma}\gamma = \mathrm{Re}_\theta(x^2) - 2\mathrm{Im}_\theta(xy)$. □

The geometry of the dressing orbits is more difficult to understand in this case. The constraint (5.25) again defines the double cover of three dimensional anti-de Sitter space embedded in $\mathbb{R}^4$, with the coordinates γ and $\tilde{\gamma}$ playing the role of light-cone coordinates. Under the dressing action, the R_0 coordinate $y = \tilde{\gamma} + \theta b$ is rescaled by a positive, real number. In the general case, where $b \neq 0$ and $\tilde{\gamma} \neq 0$, the equation (5.25) defines a one-parameter family of parabolas in the (a, γ)-plane as y is rescaled. In special cases, the geometry is simpler. In the trivial case $y = 0$ the orbits again consist of a point (a, γ). When $\tilde{\gamma} = 0$ and $b \neq 0$ we have $a^2 = 1 + b^2$, so that a ranges over $\mathbb{R} \setminus [-1, 1]$. Since γ is unconstrained the orbit is the Cartesian product $(\mathbb{R} \setminus [-1, 1]) \times \mathbb{R}$ in the (a, γ)-plane. If $b = 0$ and $\tilde{\gamma} \neq 0$ we obtain the family of parabolas defined by $a^2 + 2\tilde{\gamma}\gamma - 1 = 0$ in the (a, γ)-plane.

6 Poisson structures associated to the 3d gravity groups

In this section we consider Poisson structures and Poisson-Lie structures associated to the Lie groups in 3d gravity, more precisely the Sklyanin Poisson-Lie structure, the dual Poisson-Lie structure and the Heisenberg double Poisson structure. We derive explicit expressions for the Poisson brackets in terms of a set of natural coordinates derived from their factorisation into subgroups $\mathbb{H}_1$ and $AN(2)$ (or $\mathbb{H}_1$ and $\mathbb{R}^3$) of and their identification with the set of unit quaternions $\mathbb{H}_1(R_\Lambda)$.

A strong motivation for considering these Poisson structures is their role in the description of the phase space of 3d gravity. It was shown by Fock and Rosly [5] that the phase space and Poisson structure of Chern-Simons theory with gauge group G on manifolds of topology $\mathbb{R}\times S_{g,n}$, where $S_{g,n}$ is an orientable two-surface of genus g with n punctures, can be described in terms of an auxiliary Poisson structure on the manifold G^{n+2g}. This Poisson structure is defined uniquely in terms of a classical r-matrix for the group G. Moreover, it was demonstrated by Alekseev and Malkin [6] that the contribution of different handles and punctures to this Poisson structure can be decoupled and related to two well-known Poisson structures from the theory of the Poisson-Lie groups: each puncture corresponds to a copy of the dual Poisson-Lie structure on G, while each handle is characterised by a copy of the associated Heisenberg double Poisson structure. For semidirect product groups of the form $G \ltimes \mathfrak{g}^*$ an explicit expression for the decoupling map and the resulting Poisson structures is given in [10, 11].

In the application to 3d gravity in its formulation as a Chern-Simons gauge theory with gauge group $\mathbb{H}_1(R_\Lambda)$, this implies that Fock and Rosly's auxiliary Poisson structure of the theory on a general manifold on manifold $\mathbb{R} \times S_{g,n}$ is given as the direct product

$$\underbrace{\mathbb{H}_1(R_\Lambda)_D \times \ldots \times \mathbb{H}_1(R_\Lambda)_D}_{n\times} \times \underbrace{D_+(\mathbb{H}_1(R_\Lambda)) \times \ldots \times D_+(\mathbb{H}_1(R_\Lambda))}_{g\times}, \tag{6.1}$$

where $\mathbb{H}_1(R_\Lambda)_D$ is the group $\mathbb{H}_1(R_\Lambda)$ with the dual Poisson-Lie structure and $D_+(\mathbb{H}_1(R_\Lambda))$ the manifold $\mathbb{H}_1(R_\Lambda) \times \mathbb{H}_1(R_\Lambda)$ with the Heisenberg double Poisson structure. Hence, determining the dual Poisson-Lie structure and the associated Heisenberg double Poisson structure for the Lie groups arising in 3d gravity amounts to giving a complete description of the phase space and its Poisson structure for spacetimes of general genus g and with n punctures representing massive, spinning particles.

In the following we restrict attention to the Lorentzian 3d gravity with general cosmological constant and the Euclidean case with $\Lambda \leq 0$, where the Lie groups $\mathbb{H}_1(R_\Lambda)$ have the structure of a classical double. The key idea is to combine the factorisation of the groups derived in Theorem 4.5 and the identification of the Lie groups in 3d gravity with the set of unit quaternions $\mathbb{H}_1(R_\Lambda)$ to obtain a natural set of coordinates, in which these brackets are of a particularly simple form. More specifically, for $\boldsymbol{n} \neq 0$ we consider the functions $\tilde{p}^a \in \mathcal{C}^\infty(\mathbb{H}_1)$, $\tilde{q}^a \in \mathcal{C}^\infty(AN(2))$ defined by

$$\tilde{p}^a(u) = -2\Pi\left(u \cdot e^a\right), \qquad \tilde{q}^a(s) = -2\mathrm{Im}_\theta\, \Pi\left(s \cdot e^a\right) \qquad \forall u \in \mathbb{H}_1, s \in AN(2) \tag{6.2}$$

and apply them to the group elements obtained by factorising $g \in \mathbb{H}_1(R_\Lambda)$ as $g = u(g) \cdot s(g)$, $u \in \mathbb{H}_1$, $s \in AN(2)$ as in (4.33). In the case $\boldsymbol{n} = 0$ we obtain functions $q^a \in \mathcal{C}^\infty(\mathbb{R}^3)$ via the same formula as in (6.2) but now with $s \in \mathbb{R}^3$ (compare our comment after (4.18)). As the factorisation is not global except for the case $\boldsymbol{n} = 0$ and the Euclidean case with $\Lambda < 0$, the resulting functions are not defined globally but only on the subset of factorisable elements of $\mathbb{H}_1(R_\Lambda)$. We obtain the following lemma.

Lemma 6.1 *Consider the set of factorisable group elements*

$$F(\boldsymbol{n}) = \{g \in \mathbb{H}_1(R_\Lambda) \mid \Pi\left(g^\circ g(1 - \tfrac{\boldsymbol{ne}}{\theta})\right) > 0\} \subset \mathbb{H}_1(R_\Lambda), \tag{6.3}$$

and set, for $g \in F(\boldsymbol{n})$, $a = 0, 1, 2$,

$$p^a(g) = -2\Pi\left(u(g) \cdot e^a\right), \quad q^a(g) = -2Im_\theta\, \Pi\left(s(g) \cdot e^a\right), \tag{6.4}$$

with $u(g)$, $s(g)$ given by (4.35). The coordinate functions $p^a, q^a \in \mathcal{C}^\infty(F(\boldsymbol{n}))$, $a = 0, 1, 2$, determine group elements $g \in F(\boldsymbol{n})$ completely up to a minus sign.

Proof: This follows directly from the uniqueness of the factorisation and the definition of the coordinate functions (6.2). Using the parametrisation (4.18) and the standard parametrisation of the quaternions, we find that the coordinate functions $\tilde{q}^a, \tilde{p}^a$ are

$$\tilde{q}^a :\ \sqrt{1 + (\boldsymbol{qn})^2/4} + \boldsymbol{q} \cdot \boldsymbol{S} \mapsto q^a, \qquad \tilde{p}^a :\ p_3 + \boldsymbol{p} \cdot \boldsymbol{J} \mapsto p^a. \tag{6.5}$$

This determines elements of $AN(2)$ (when $\boldsymbol{n} \neq 0$) or $\mathbb{R}^3$ (when $\boldsymbol{n} = 0$), uniquely and elements of $\mathbb{H}$ up to a choice of minus sign arising from the relation $p_3^2 = 1 - \boldsymbol{p}^2$. □

We will now demonstrate that these coordinates give rise to a unified description of the Sklyanin and the dual Poisson-Lie structure on $\mathbb{H}_1(R_\Lambda)$ in which the Poisson bracket takes a rather simple form and the structural similarities for the different signature and signs of Λ are readily apparent. In particular, we have the following theorem.

Theorem 6.2 *In terms of the coordinate functions $p^a, q^a \in \mathcal{C}^\infty(F(\boldsymbol{n}))$, the Sklyanin bracket $\{\,,\,\}_S$ and the dual bracket $\{\,,\,\}_D$ on the gravity Lie groups $\mathbb{H}_1(R_\Lambda)$ associated to the classical r-matrix (4.14) are*

$$\begin{aligned}
&\{p^a, p^b\}_S = p_3(n^a p^b - p^a n^b), \quad \{q^a, q^b\}_S = q_3 \epsilon^{abc} q_c, \quad \{p^a, q^b\}_S = 0, \\
&\{p^a, p^b\}_D = -p_3(n^a p^b - p^a n^b), \quad \{q^a, q^b\}_D = q_3 \epsilon^{abc} q_c, \quad \{p^a, q^b\}_D = q_3 \epsilon^{abc} p_c + p_3(\boldsymbol{qn}\, \eta^{ab} - n^a q^b),
\end{aligned} \tag{6.6}$$

where $q_3, p_3 \in \mathcal{C}^\infty(F(\boldsymbol{n}))$ are given by

$$q_3 = \sqrt{1 + (\boldsymbol{qn})^2/4}, \qquad p_3 = \pm\sqrt{1 - \boldsymbol{p}^2/4}. \tag{6.7}$$

The functions

$$Im_\theta(\Pi(u \cdot s)) = -\tfrac{1}{4} p_a q^a, \qquad Re_\theta(\Pi(u \cdot s)) = p_3 q_3 - \tfrac{1}{4}\epsilon_{abc} n^a p^b q^c \tag{6.8}$$

are Casimir functions for the dual bracket $\{\,,\,\}_D$.

Proof: The general formula for the Sklyanin-Poisson structure on a Lie group G with associated quasitriangular Lie algebra $\mathfrak{g}$ can be found for instance in [12]. The dual Poisson structure is introduced and discussed in [13], see also [6] and [7]. In terms of a basis X_a, $a = 1, \ldots, \dim(\mathfrak{g})$, of $\mathfrak{g}$ and its classical r-matrix $r = r^{ab} X_a \otimes X_b$, the Sklyanin Poisson-Lie structure and its dual on G can be characterised by the Poisson bivectors

$$B_S = \tfrac{1}{2} r^{ab} \left(X_a^R \wedge X_b^R - X_a^L \wedge X_b^L \right), \tag{6.9}$$
$$B_D = \tfrac{1}{2} r^{ab} \left(X_a^R \wedge X_b^R + X_a^L \wedge X_b^L \right) + r^{ab} X_a^R \wedge X_b^L, \tag{6.10}$$

where X_a^L, X_a^R are the right- and left-invariant vector fields associated to the generators via

$$X_a^L f(g) = \frac{d}{dt}|_{t=0} f(e^{-tX_a} g), \qquad X_a^R f(g) = \frac{d}{dt}|_{t=0} f(g e^{tX_a}) \qquad \forall g \in G, f \in \mathcal{C}^\infty(G). \tag{6.11}$$

To obtain these Poisson structures for the gravity Lie groups $\mathbb{H}_1(R_\Lambda)$, we insert the classical r-matrix (4.14) into (6.9), (6.10). Since the coordinate functions $p^a, q^a \in \mathcal{C}^\infty(F(\boldsymbol{n}))$ are invariant, respectively, under right multiplication with elements of $AN(2) \subset \mathbb{H}_1(R_\Lambda)$ and left multiplication with elements of $\mathbb{H}_1 \subset \mathbb{H}_1(R_\Lambda)$, we find that their Poisson brackets are

$$\{p^a, p^b\}_D = -\{p^a, p^b\}_S = \tfrac{1}{2} \left(S_c^L p^a J_L^c p^b - S_c^L p^b J_L^c p^a \right), \tag{6.12}$$
$$\{q^a, q^b\}_D = \{q^a, q^b\}_S = \tfrac{1}{2} \left(S_c^R q^a J_R^c q^b - S_c^R q^b J_R^c q^a \right), \tag{6.13}$$
$$\{p^a, q^b\}_D = -\tfrac{1}{2} \left(J_c^L p^a S_L^c q^b + J_c^R p^a S_R^c q^b \right) - J_c^L p^a S_R^c q^b, \tag{6.14}$$
$$\{p^a, q^b\}_S = \tfrac{1}{2} \left(J_c^L p^a S_L^c q^b - J_c^R p^a S_R^c q^b \right), \tag{6.15}$$

where J_a^L, S_a^L and J_a^R, S_a^R are the right-and left-invariant vector fields on $\mathbb{H}_1(R_\Lambda)$ defined as in (6.11).

To evaluate (6.12) to (6.15), we need to determine the action of the vector fields J_a^L, S_a^L, J_a^R, S_a^R on the coordinate functions $p^a, q^a \in \mathcal{C}^\infty(F(\boldsymbol{n}))$. The first step is to determine the action of the left- and right-invariant vector fields on the groups $\mathbb{H}_1$ and $AN(2)$ on the coordinate functions $\tilde{p}^a \in \mathcal{C}^\infty(\mathbb{H}_1)$, $\tilde{q}^a \in \mathcal{C}^\infty(AN(2))$ and $\tilde{q}^a \in \mathcal{C}^\infty(\mathbb{R}^3)$ for $\boldsymbol{n} = 0$. Using the standard parametrisation of the quaternions for $\mathbb{H}_1$ and the definition (6.2), we find

$$L_{J_a} \tilde{p}^b(u) = \frac{d}{dt}|_{t=0} \tilde{p}^b(e^{-tJ_a} \cdot u) = -\eta^{ab} \tilde{p}_3 + \tfrac{1}{2} \epsilon^{abc} \tilde{p}_c, \tag{6.16}$$
$$R_{J_a} \tilde{p}^b(u) = \frac{d}{dt}|_{t=0} \tilde{p}^b(u \cdot e^{tJ_a}) = \eta^{ab} \tilde{p}_3 + \tfrac{1}{2} \epsilon^{abc} \tilde{p}_c, \tag{6.17}$$

with

$$\tilde{p}_3(u) = \Pi(u) = \pm\sqrt{1 - \tilde{\boldsymbol{p}}^2/4(u)}. \tag{6.18}$$

The corresponding calculation for $AN(2)$ and $\mathbb{R}^3$ using the parametrisation (4.18) and definition (6.2) yields

$$L_{S_a} \tilde{q}^b(s) = \frac{d}{dt}|_{t=0} \tilde{q}^b(e^{-tS_a} \cdot s) = -(\tilde{q}_3 - \tfrac{1}{2} \tilde{\boldsymbol{q}}\boldsymbol{n})\, \eta^{ab} - \tfrac{1}{2} n^a \tilde{q}^b, \tag{6.19}$$
$$R_{S_a} \tilde{q}^b(s) = \frac{d}{dt}|_{t=0} \tilde{q}^b(s \cdot e^{tS_a}) = (\tilde{q}_3 + \tfrac{1}{2} \tilde{\boldsymbol{q}}\boldsymbol{n})\, \eta^{ab} - \tfrac{1}{2} n^a \tilde{q}^b, \tag{6.20}$$

with

$$\tilde{q}_3(s) = \sqrt{1 + (\tilde{\boldsymbol{q}}\boldsymbol{n})^2/4(s)}. \tag{6.21}$$

This determines the action of the vector fields J^L_a, S^R_a on the coordinate functions $p^a, q^a \in \mathcal{C}^\infty(F(\boldsymbol{n}))$:

$$J^L_a p^b = -\eta^{ab} p_3 + \tfrac{1}{2}\epsilon^{abc} p_c, \qquad J^L_a q^b = 0, \tag{6.22}$$

$$S^R_a q^b = \left(q_3 + \tfrac{1}{2}\boldsymbol{q}\boldsymbol{n}\right)\eta^{ab} - \tfrac{1}{2} n^a q^b, \qquad S^R_a p^b = 0. \tag{6.23}$$

To obtain the corresponding expressions for the vector fields J^R_a, S^L_a we combine (6.16), (6.17) and (6.19), (6.20) with infinitesimal dressing transformations. Inserting

$$v = e^{tJ_a}, \qquad r = q_3 + q^a S_a \tag{6.24}$$

into (4.35) and (4.36), using the expressions (4.2) of the generators J_a, S_a in terms of the quaternions and the multiplication relations (3.2), (3.3) and expanding the resulting expressions in t we obtain

$$u = 1 + t\left(\delta^c_a + \frac{q_a n^c - (\boldsymbol{q}\boldsymbol{n})\delta^c_a}{q_3 + \frac{1}{2}\boldsymbol{q}\boldsymbol{n}}\right) J_c + O(t^2), \tag{6.25}$$

$$s = \left(1 + t\frac{\epsilon_{abc} q^b n^d}{2(q_3 + \frac{1}{2}\boldsymbol{q}\boldsymbol{n})}\right) r + t\epsilon_a{}^{cd}\left(q_d - \frac{\boldsymbol{q}^2 n^d}{2(q_3 + \frac{1}{2}\boldsymbol{q}\boldsymbol{n})}\right) S_c + O(t^2). \tag{6.26}$$

Combining (6.25) with (6.17) then yields expressions for the action of the vector field J^R_a on the coordinate functions p^a, q^a

$$J^R_a p^b = \frac{q_3 - \frac{1}{2}\boldsymbol{q}\boldsymbol{n}}{q_3 + \frac{1}{2}\boldsymbol{q}\boldsymbol{n}}\left(p_3\delta^b_a + \tfrac{1}{2}\epsilon_a{}^{bc} p_c\right) + \frac{q_a}{q_3 + \frac{1}{2}\boldsymbol{q}\boldsymbol{n}}\left(p_3 n^b + \tfrac{1}{2}\epsilon^{bcd} p_c n_d\right), \tag{6.27}$$

$$J^R_a q^b = \frac{q_3 \epsilon_a{}^{bc} q_c - \frac{1}{2} q^a \epsilon^{bcd} n_c q_d}{q_3 + \frac{1}{2}\boldsymbol{q}\boldsymbol{n}}. \tag{6.28}$$

To determine the action of the vector field S^L_a, we insert

$$r = e^{-tS_a}, \qquad v = p_3 + p^a J_a \tag{6.29}$$

into (4.35), (4.37) and expand the resulting expressions in t

$$u = \left(1 - t\left(p_3(p_a n^c - p^c n_a) + \tfrac{1}{2}\boldsymbol{p}\boldsymbol{n}\,\epsilon_a{}^{cd} p_d - \tfrac{1}{2}\boldsymbol{p}^2 \epsilon_a{}^{cd} n_d\right) J_c\right)\cdot v + O(t^2), \tag{6.30}$$

$$s = 1 - t\left(-p_3\epsilon_a{}^{cd} p_d + \tfrac{1}{2} p_a p^c + \left(1 - \tfrac{1}{2}\boldsymbol{p}^2\right)\delta^c_a\right) S_c + O(t^2). \tag{6.31}$$

Combining (6.30) with (6.16) and applying the identity

$$x^b \epsilon^{acd} y_c x_d - x^a \epsilon^{bcd} y_c x_d = (\boldsymbol{x}\boldsymbol{y})\epsilon^{abc} x_c - (\boldsymbol{x}^2)\epsilon^{abc} y_c \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \tag{6.32}$$

we then obtain the action of the vector field S^L_a on the coordinate functions p^a and q^a:

$$S^L_a p^b = \tfrac{p_3}{2} p^b \epsilon_{acd} p^c n^d + p_3^2 n_a p^b + \tfrac{1}{4}\boldsymbol{p}\boldsymbol{n} p_a p^b - p_a n^b, \tag{6.33}$$

$$S^L_a q^b = -\left(\left(1 - \tfrac{1}{2}\boldsymbol{p}^2\right)\eta_{ac} + \tfrac{1}{2} p_a p_c - p_3 \epsilon_{acd} p^d\right)\left(\left(q_3 - \tfrac{1}{2}\boldsymbol{q}\boldsymbol{n}\right)\eta^{bc} + \tfrac{1}{2} n^c q^b\right). \tag{6.34}$$

Inserting expressions (6.22), (6.23), (6.27), (6.28), (6.33), (6.34) for the action of the vector fields J_a^L, J_a^R, S_a^L, S_a^R on the coordinate functions into formulas (6.12) to (6.15) yields (6.6). It can then be verified by direct calculation that the functions (6.8) Poisson commute with the coordinate functions p^a and q^a for the dual bracket. $\square$

Comparing formulas (6.6) for the Poisson bracket with formulas (4.3) and (4.15), we find that, up to factors p_3, q_3 and a minus sign, the dual Poisson bracket $\{\,,\,\}_D$ and the Sklyanin bracket $\{\,,\,\}_S$ agree, respectively, with the Lie bracket (4.3) and dual Lie bracket (4.15). Moreover, the two Casimir functions of the dual bracket $\{\,,\,\}_D$ are of a particularly simple form and have a interpretation as the real and imaginary part of the group element's unit coefficient in its description as a quaternion over R_Λ.

The factorisation of the gravity Lie groups in Theorem 4.5 does not only give rise to a natural set of coordinates in which the dual Poisson structure and the Sklyanin bracket on the subset $F(\boldsymbol{n}) \subset \mathbb{H}_1(R_\Lambda)$ take a particularly simple form, but also provides useful information about the Poisson structures of the subgroups $\mathbb{H}_1$ and $AN(2)_{\boldsymbol{n}}$ (or $\mathbb{R}^3$ when $\boldsymbol{n} = 0$). As discussed after (4.16) these are mutually dual Poisson-Lie groups. The Sklyanin bracket $\{\,,\,\}_S$ on the double $\mathbb{H}_1(R_\Lambda)$ encodes the Poisson structures of both these subgroups. Furthermore, the dressing transformations studied in Sect. 5 determine the symplectic leaves of the Poisson-Lie groups $\mathbb{H}_1$ and $AN(2)_{\boldsymbol{n}}$ (respectively $\mathbb{R}^3$ when $\boldsymbol{n} = 0$). The results can be summarised as follows.

Theorem 6.3 *In terms of the coordinate functions $\tilde{p}^a$ on $\mathbb{H}_1$ and $\tilde{q}^a$ on $(AN(2)_{\boldsymbol{n}})$ ($\mathbb{R}^3$ when $\boldsymbol{n} = 0$) defined in* (6.5)*, the Poisson brackets on the Poisson-Lie groups $\mathbb{H}_1$ and $AN(2)_{\boldsymbol{n}}$ ($\mathbb{R}^3$ when $\boldsymbol{n} = 0$) take the form*

$$\{\tilde{p}^a, \tilde{p}^b\} = \tilde{p}_3(n^a\tilde{p}^b - \tilde{p}^a n^b) \quad \text{and} \quad \{\tilde{q}^a, \tilde{q}^b\} = \tilde{q}_3 \epsilon^{abc}\tilde{q}_c, \tag{6.35}$$

with $\tilde{p}_3$ and $\tilde{q}_3$ defined in terms of $\tilde{p}^a$ and $\tilde{q}^a$ as in (6.7)*. The symplectic leaves of the Poisson manifold $AN(2)_{\boldsymbol{n}}$ ($\mathbb{R}^3$ when $\boldsymbol{n} = 0$) are the orbits under the dressing action* (5.3) *and the symplectic leaves of the Poisson manifold $\mathbb{H}_1$ are the orbits of the dressing action* (5.19) *for $\Lambda \neq 0$ and* (5.29) *for $\Lambda = 0$.*

Proof: This is an application of standard results in the theory of Poisson-Lie groups. The formula for the Poisson brackets follows from the fact that, as a Poisson manifold, a classical double equipped with the Sklyanin bracket is a direct product of the factor groups equipped with their respective Poisson structures; see e.g. Proposition 8.4.5 in [14]. The characterisation of symplectic leaves as orbits of dressing transformations goes back to [13], see [23] for a pedagogical exposition and further references. $\square$

The other Poisson structure relevant to the description of the phase space of 3d gravity is the Heisenberg double Poisson structure introduced by Semenov-Tian-Shansky [13] which is related to the contributions of the handles. The Heisenberg double $D_+(G)$ of a Poisson-Lie group G is the group $G \times G$ equipped with the unique Poisson structure such that the canonical embeddings $G \to G \times G$ and $G^* \to G \times G$ of the Poisson-Lie group and its dual are Poisson maps. The former is simply the diagonal map $G \ni g \mapsto (g,g) \in G \times G$, the latter is defined in terms of the factorisation of G into the two subgroups associated to its classical r-matrix. Denoting these subgroups by H and K, one has $G^* = H \times K$ as a group, and the embedding of G^* into $G \times G$ is given by $H \times K \ni (h,k) \mapsto (h, k^{-1})$. It is shown in [13] that the Heisenberg double Poisson structure $D_+(G)$ is given by the Poisson bivector

$$\begin{aligned} B_{D_+} =& \tfrac{1}{2} r^{ba} \left(X_a^{1R} \wedge X_b^{1R} + X_a^{1L} \wedge X_b^{1L} \right) + \tfrac{1}{2} r^{ba} \left(X_a^{2R} \wedge X_b^{2R} + X_a^{2L} \wedge X_b^{2L} \right) \\ &+ r^{ba} \left(X_a^{1R} \wedge X_b^{2R} + X_a^{1L} \wedge X_b^{2L} \right), \end{aligned} \tag{6.36}$$

where X_a, $a = 1, \ldots, \dim(\mathfrak{g})$, is a basis of $\mathfrak{g} = \mathrm{Lie}(G)$, X_a^{iR}, X_a^{iL} denote the associated left- and right-invariant vector fields on the two components of the group and $r = r^{ab} X_a \otimes X_b$ its classical r-matrix.

For the gravity Lie groups $G = \mathbb{H}_1(R_\Lambda)$, we have $H = \mathbb{H}_1$ and $K = AN(2)$ (or $K = \mathbb{R}^3$ when $\boldsymbol{n} = 0$), and the canonical embeddings of $\mathbb{H}_1(R_\Lambda)$ and $\mathbb{H}_1(R_\Lambda)^* = \mathbb{H}_1 \times AN(2)$ (or $\mathbb{H}_1(R_\Lambda)^* = \mathbb{H}_1 \times \mathbb{R}^3$) are given by

$$\mathbb{H}_1(R_\Lambda) \to \mathbb{H}_1(R_\Lambda) \times \mathbb{H}_1(R_\Lambda), \qquad g \mapsto (g, g), \tag{6.37}$$

$$\mathbb{H}_1(R_\Lambda)^* \to \mathbb{H}_1(R_\Lambda) \times \mathbb{H}_1(R_\Lambda), \qquad (u, s) \mapsto (u, s^{-1}). \tag{6.38}$$

Applying Semenov-Tian-Shansky's Poisson bivector (6.36) to the coordinate functions defined above, we obtain the following theorem characterising the Heisenberg double of the Poisson-Lie groups $\mathbb{H}_1(R_\Lambda)$.

Theorem 6.4 *Writing elements of $\mathbb{H}_1(R_\Lambda) \times \mathbb{H}_1(R_\Lambda)$ as (g_1, g_2), with $g_1, g_2 \in \mathbb{H}_1(R_\Lambda)$, and using the coordinate functions*

$$p^a((g_1,g_2)) = -2\Pi(u(g_1) \cdot e^a), \qquad k_a((g_1,g_2)) = -2\Pi(u(g_2) \cdot e^a), \tag{6.39}$$

$$q^a((g_1,g_2)) = -2Im_\theta \Pi(s(g_1) \cdot e^a), \qquad l_a((g_1,g_2)) = -2Im_\theta \Pi(s(g_2) \cdot e^a), \tag{6.40}$$

with u, s defined as in (4.35), (4.37), *and the associated functions p_3, q_3, k_3, l_3 defined as in* (6.7), *the Heisenberg double Poisson structure associated to the classical r-matrix* (4.14) *takes the form*

$$\begin{aligned} &\{p^a, p^b\}_{D_+} = p_3 \left(n^a p^b - p^a n^b \right), \qquad \{q^a, q^b\}_{D_+} = -q_3\, \epsilon^{abc} q_c, \\ &\{p^a, q^b\}_{D_+} = p_3 q_3\, \eta^{ab} + \tfrac{1}{2} p_3 \left(n^a q^b - \boldsymbol{qn}\, \eta^{ab} \right) - \tfrac{1}{2} q_3 \epsilon^{abc} p_c - \tfrac{1}{4} \epsilon^{adf} p_f \left(n^d q^b - \boldsymbol{qn}\, \eta^{db} \right), \end{aligned} \tag{6.41}$$

$$\begin{aligned} &\{k^a, k^b\}_{D_+} = k_3 \left(n^a k^b - k^a n^b \right), \qquad \{l^a, l^b\}_{D_+} = -l_3\, \epsilon^{abc} l_c, \\ &\{k^a, l^b\}_{D_+} = k_3 l_3\, \eta^{ab} + \tfrac{1}{2} k_3 \left(n^a l^b - \boldsymbol{ln}\, \eta^{ab} \right) - \tfrac{1}{2} l_3 \epsilon^{abc} k_c - \tfrac{1}{4} \epsilon^{adf} k_f \left(n^d l^b - \boldsymbol{ln}\, \eta^{db} \right), \end{aligned}$$

$$
\begin{aligned}
&\{q^a, k^b\}_{D_+} = 0,\\
&\{p^a, k^b\}_{D_+} = \tfrac{1}{2}\epsilon^{acd}p_c k_d\left(\tfrac{1}{4}\boldsymbol{kn}k^b - n^b\right) + \tfrac{1}{2}p_3^2 k^b \epsilon^{acd}p_c n_d - \tfrac{1}{2}p_3 k_3 k^b \epsilon^{acd}k_c n_d + \tfrac{1}{4}k^a k^b(k_3\boldsymbol{pn} - p_3\boldsymbol{kn})\\
&\qquad\quad - n^a k^b(p_3^3 + \tfrac{1}{4}k_3\boldsymbol{pn}) + p_3 k^a n^b,\\
&\{q^a, l^b\}_{D_+} = \frac{l_3 + \frac{1}{2}\boldsymbol{ln}}{q_3 + \frac{1}{2}\boldsymbol{qn}}(\tfrac{1}{2}q^b\epsilon^{acd}q_c n_d - q_3\epsilon^{abc}q_c) - l^b\epsilon^{acd}q_c n_d,\\
&\{p^a, l^b\}_{D_+} = \frac{l_3 + \frac{1}{2}\boldsymbol{ln}}{q_3 + \frac{1}{2}\boldsymbol{qn}}(p_3 n^a q^b + (q_3 - \tfrac{1}{2}\boldsymbol{qn})(p_3\eta^{ab} - \tfrac{1}{2}\epsilon^{abc}p_c) + \tfrac{1}{2}q^b\epsilon^{acd}p_c n_d) - \frac{q_3 p_3 n^a l^b}{q_3 + \frac{1}{2}\boldsymbol{qn}}\\
&+ (l_3 - \tfrac{1}{2}\boldsymbol{ln})((p_3(1 - \tfrac{1}{2}\boldsymbol{k}^2) + \tfrac{k_3}{2}\boldsymbol{p}^2)\eta^{ab} - (k_3 p_3 - k_3^2 + \tfrac{1}{2})\epsilon^{abc}p_c + \tfrac{1}{2}p_3 k^a k^b - \tfrac{1}{2}k_3 p^a p^b - \tfrac{1}{4}k^b\epsilon^{acd}p_c k_d)\\
&+ (p_3 k_3^2 - \tfrac{p_3}{2})n^a l^b + \tfrac{p_3}{2}\boldsymbol{kn}k^a l^b + k_3(p_3 - k_3)l^b\epsilon^{acd}p_c n_d - \tfrac{1}{8}\boldsymbol{knl}^b\epsilon^{acd}p_c k_d.
\end{aligned}
$$

Proof: The formulas in the theorem can be checked by straightforward but lengthy computation. One inserts the classical r-matrix (4.14) into (6.36) and applies the resulting expression to the coordinate functions p^a, q^a, k^a, l^a on the two copies of $\mathbb{H}_1(R_\Lambda)$. This yields

$$\{p^a, p^b\}_{D_+} = \tfrac{1}{2}\left(J_c^L p^a S_L^c p^b - J_c^L p^b S_L^c p^a\right) = \{p^a, p^b\}_S = -\{p^a, p^b\}_D, \tag{6.42}$$
$$\{q^a, q^b\}_{D_+} = \tfrac{1}{2}\left(J_c^R q^a S_R^c q^b - J_c^R q^b S_R^c q^a\right) = -\{q^a, q^b\}_S = -\{q^a, q^b\}_D, \tag{6.43}$$
$$\{p^a, q^b\}_{D_+} = \tfrac{1}{2}\left(J_c^R p^a S_R^c q^b + J_c^L p^a S_L^c q^b\right), \tag{6.44}$$
$$\{p^a, k^b\}_{D_+} = J_c^L p^a S_L^c k^b, \qquad \{p^a, l^b\}_{D_+} = J_c^L p^a S_L^c l^b + J_c^R p^a S_R^c l^b, \tag{6.45}$$
$$\{q^a, l^b\}_{D_+} = J_c^R q^a S_R^c l^b, \qquad \{q^a, k^b\}_{D_+} = 0. \tag{6.46}$$

To evaluate these expressions further one makes use of the formulas (6.22), (6.23), (6.27), (6.28), (6.33), (6.34) for the action of the left- and right-invariant vector fields on $\mathbb{H}_1(R_\Lambda)$ on the coordinate functions . □

Theorem 6.2 and Theorem 6.4 provide explicit expressions for the Sklyanin Poisson-Lie structure, the dual Poisson-Lie structure and the Heisenberg double Poisson structure associated to the local isometry groups arising in 3d gravity for Lorentzian signature and the Euclidean case with $\Lambda \leq 0$.

From the viewpoint of 3d gravity this amounts to a complete parametrisation of phase space and Poisson structure for spacetimes of arbitrary genus and with an arbitrary number of massive spinning particles. Not only are the resulting expressions for the Poisson structure of a rather simple form, but the structural similarities of the theory for different signatures and signs of the cosmological constant are readily apparent. The resulting expressions for the Poisson structure take the same form and the dependence on the cosmological constant is explicit and encoded in the vector $\boldsymbol{n}$ satisfying $\boldsymbol{n}^2 = -\Lambda$. The description thus unifies the description of phase space and Poisson structure for different signatures and signs of the cosmological constant and allows one to investigate all cases in a common framework in which the cosmological constant appears as a parameter.

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