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N=4 Supersymmetric Yang-Mills Theory on a Kähler Surface

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We study $N = 4$ supersymmetric Yang-Mills theory on a Kähler manifold with $b_2^+ \geq 3$. Adding suitable perturbations we show that the partition function of the $N = 4$ theory is the sum of contributions from two branches: (i) instantons, (ii) a special class of Seiberg-Witten monopoles. We determine the partition function for the theories with gauge group $SU(2)$ and $SO(3)$, using S-duality. This leads us to a formula for the Euler characteristic of the moduli space of instantons.

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1. Introduction

In the paper [1] Vafa and Witten presented strong evidence for S-duality of $N = 4$ super-Yang-Mills theory [2]. They used a topological, twisted version of the $N = 4$ theory [3][4] and were able to determine the partition function of $N = 4$ super-Yang-Mills theory on certain Kähler manifolds. In particular they identified the partition function with the Euler characteristic of the moduli space of instantons, provided certain vanishing theorems hold.

This paper is an elaboration and generalisation of the work of Vafa and Witten. We want to determine the partition function for a general compact Kähler surface X with $b_2^+ \geq 3$. The twisted version of $N = 4$ super-Yang-Mills theory studied by Vafa and Witten is an important example of a balanced topological field theory as defined in [5] (see also [6][7][8][9]). These topological field theories carry two topological supercharges. On a Kähler manifold the number of topological charges is extended to four, and these charges may be interpreted as the differentials of a Dolbeault version of balanced $\mathcal{G}$-equivariant cohomology [10], where $\mathcal{G}$ is the group of gauge transformations. Our computation of the partition function involves a a series of perturbations which break the supersymmetry down to $N = 2$ and $N = 1$ (topological) supersymmetry.

The perturbation down to $N = 2$ is achieved by adding a bare mass term for the hypermultiplet. Geometrically, this term may be viewed as the equivariant momentum map of a $\mathcal{G} \times S^1$-action on the hypermultiplet. As a result of its inclusion in the action, the path integral is localised on the fixed point set of the $\mathcal{G} \times S^1$-action, which consists of two branches: (i) the moduli space of anti-self-dual connections, (ii) the moduli space of a certain class of Seiberg-Witten monopoles [11]. Perturbing further down to $N = 1$, following [12], leads to the factorization of the Seiberg-Witten classes contributing to branch (ii).

Specializing to gauge groups $SU(2)$ and $SO(3)$ we propose a formula for the branch (ii) contribution on a general Kähler manifold with $b_2^+ \geq 3$. Then S-duality of $N = 4$ super-Yang-Mills theory enables us to determine the entire partition function. As a corollary we obtain a formula for the Euler characteristic of the moduli space of instantons (branch (i)). We consider the pure $N = 2$ limit and obtain the essential part of Witten's formula for Donaldson invariants [11]. In the final section we add brief comments on relation with a $\mathcal{N}_{ws} = (2, 2)$ gauged linear sigma model and its applications.

2. Prelude

Here we briefly review the notion of a balanced topological Yang-Mills theory [5], which is a twisted $N = 4$ theory [1]. The topological field theory computing the Euler characteristic of certain moduli spaces was studied in [6] and [7]. (See [8] for the precise relation of the earlier work [6] with [5].) We also recall salient features of topological Yang-Mills theory [3] on a Kähler manifold [10][13].

2.1. Balanced Topological Yang-Mills Theory

Let X be a compact Riemann four manifold and E be a $SU(n)$-bundle over X. The bundle E is classified by the instanton number

$$k = \frac{1}{8\pi^2}\int_X \mathrm{Tr}\, F\wedge F, \tag{2.1}$$

where Tr is the trace in the fundamental representation of $SU(n)$ and $F \in \Omega^2_X(\mathfrak{g}_E)$ is the adjoint valued curvature 2-form on M. We denote the group of gauge transformation by $\mathcal{G}$, i.e. elements of $\mathcal{G}$ are maps $g : X \to SU(N)$.

We introduce two global supercharges $Q_\pm$ carrying an additive quantum number (ghost number) $U = \pm 1$. They satisfy the following commutation relations:

$$Q_+^2 = \delta_{\phi_{++}}, \qquad \{Q_+, Q_-\} = \delta_{\phi_{+-}}, \qquad Q_-^2 = \delta_{\phi_{--}}. \tag{2.2}$$

where δ_ϕ denotes the gauge transformation generated by an adjoint scalar $\phi \in \Omega^0_X(\mathfrak{g}_E)$. The charges $Q_\pm$ can be interpreted physically as the twisted supercharges of $N = 4$ super-Yang-Mills theory [1] or mathematically as the differentials of balanced $\mathcal{G}$-equivariant cohomology [5].

To have a complete representation of this algebra one needs to impose a consistency condition (Bianchi identity) which is conveniently summarised in the following quintet of fields

$$\begin{array}{ccccc}
U = +2 & & \phi_{++} & & \\
 & & & \searrow & \\
U = +1 & & & & \boldsymbol{\eta}_+ \\
 & & & \nearrow & \\
U = 0 & \text{consistency} & \phi_{+-} & & \\
 & & & \searrow & \\
U = -1 & & & & \boldsymbol{\eta}_- \\
 & & & \nearrow & \\
U = -2 & & \phi_{--} & &
\end{array} \;. \tag{2.3}$$

Here $\nearrow$ and $\searrow$ represent the action of Q_+ and Q_- respectively. The diagram then expresses the requirement that acting with Q_+ and Q_- on the triplet $(\phi_{++}, \phi_{+-}, \phi_{--})$ one generates in the indicated manner a space which is spanned by the doublet $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-)$. Since our global supercharges are scalars we have $\boldsymbol{\eta}_\pm \in \Omega^0_X(\mathfrak{g}_E)$. We will sometimes use the notation $\phi_{++} = \phi$, $\phi_{+-} = C$ and $\phi_{--} = \bar{\phi}$.

Next we have to choose the basic field of our theory. Here we take a connection 1-form A. Acting with the supercharges will then generate three further fields $\boldsymbol{\psi}_\pm, H \in \Omega^1_X(\mathfrak{g}_E)$. This leads to the basic quartet of any balanced topological field theory. To define the theory we want to study here we also need a self-dual adjoint-valued two-form B, which leads to another quartet. These two multiplets are summarised in the following picture:

$$
\begin{array}{cccccccccccc}
U=+1 & & & & \boldsymbol{\psi}_+ & & & & & \boldsymbol{\chi}_+ & & \\
 & & & \nearrow & & \searrow & & & \nearrow & & \searrow & \\
U=0 & \text{fields} & A & & & & H\,, & \text{equations} \quad B & & & & H^+\,, \\
 & & & \searrow & & \nearrow & & & \searrow & & \nearrow & \\
U=-1 & & & & \boldsymbol{\psi}_- & & & & & \boldsymbol{\chi}_- & &
\end{array}
\tag{2.4}
$$

where $\boldsymbol{\chi}_\pm, H^+ \in \Omega^{2+}_X(\mathfrak{g}_E)$. A balanced topological field theory is uniquely determined by its field content and the balanced algebra, and moreover has the pleasant property that there is no U-number anomaly: any fermionic zero-mode has a partner with the opposite U-number.

The action functional can be written as

$$
S = \frac{1}{2}(Q_+Q_- - Q_-Q_+)\,(\mathcal{F}_0 + \mathcal{F}_1) \tag{2.5}
$$

where

$$
\begin{aligned}
\mathcal{F}_0 &= -\frac{4}{e^2}\int_X \mathrm{Tr}\left(B^{\mu\nu}\left(F^+_{\mu\nu} + \frac{1}{12}[B_{\mu\rho}, B_{\nu\sigma}]g^{\rho\sigma}\right)\right), \\
\mathcal{F}_1 &= -\frac{4}{e^2}\int_X \mathrm{Tr}\left(\boldsymbol{\chi}_+ \wedge \boldsymbol{\chi}_- - \boldsymbol{\psi}_+ \wedge *\boldsymbol{\psi}_- + \boldsymbol{\eta}_+ * \boldsymbol{\eta}_-\right),
\end{aligned} \tag{2.6}
$$

The bosonic part of the action functional has the following explicit form

$$
\begin{aligned}
S_{bose} = \frac{4}{e^2}\int_X \mathrm{Tr}\Bigg(& H^+\wedge H^+ - H^{\mu\nu +}\left(F^+_{\mu\nu} + \frac{1}{4}[B_{\mu\rho}, B_{\nu\sigma}]g^{\rho\sigma}\right) + H\wedge *H - B\wedge d_A^+ H \\
& - \frac{1}{4}d_A C\wedge *d_A C - \frac{1}{4}[C,B]^2 + [\phi,\bar\phi]^2 + d_A\phi * d_A\bar\phi + [\phi,B]\wedge[\bar\phi,B] \\
& + [\phi, C] * [\bar\phi, C]\Bigg),
\end{aligned} \tag{2.7}
$$

where d_A^+ denotes the projection of d_A to the self-dual part. We integrate out H^+ and H using the algebraic equation of motions

$$
H^+_{\mu\nu} = \frac{1}{2}\left(F^+_{\mu\nu} + \frac{1}{4}[B_{\mu\rho}, B_{\nu\sigma}]g^{\rho\sigma}\right), \qquad H = \frac{1}{2}d_A^* B, \tag{2.8}
$$

and a Weitzenböck formula. We get

$$
\begin{aligned}
S_{bose} = -\frac{1}{e^2}\int_X \mathrm{Tr}\Big(& F^+\wedge F^+ + \frac{1}{4}d_A B\wedge * d_A B + d_A C\wedge * d_A C + \frac{1}{16}[B_{\mu\rho},B_{\nu\rho}][B_{\mu\sigma},B_{\nu\sigma}] \\
& + [C,B^+]^2 + \frac{1}{4}B_{\mu\nu}\left(\frac{1}{6}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})R + W^+_{\mu\nu\rho\sigma}\right)B_{\rho\sigma} + 4[\phi,\bar\phi]^2 \\
& - 4d_A\phi * d_A\bar\phi - 4[\phi,B]\wedge[\bar\phi,B] - 4[C,\phi]*[C,\bar\phi]\Big),
\end{aligned}
\tag{2.9}
$$

where R is the scalar curvature of X and W^+ is the self-dual part of the Weyl tensor. The resulting action is invariant under $Q_\pm$ if we modify the transformation laws of $\boldsymbol{\chi}_\pm$ according to the replacement (2.8). Due to the fixed point theorem of Witten for global supersymmetry the path integral is localized on the fixed point locus of $Q_\pm$. This locus is given by the following equations:

$$
\begin{aligned}
& F^+_{\mu\nu} + \frac{1}{4}[B_{\mu\rho},B_{\nu\sigma}]g^{\rho\sigma} = 0,, \qquad && [C,B] = [\phi,\bar\phi] = [C,\phi_{\pm\pm}] = [\phi_{\pm\pm},B] = 0, \\
& d_A^* B = 0, && d_A C = d_A\phi_{\pm\pm} = 0.
\end{aligned}
\tag{2.10}
$$

These equations are equivalent to the equations in section 2.4 of [1] and we call them the Vafa-Witten equations. The equations for fermionic zero-modes are just the linearization of the fixed point equation (2.10) and the condition that they are orthogonal to gauge orbits. Due to the balanced structure each fermionic zero-mode has a partner with the opposite U-number. Thus there is no ghost-number anomaly and the partition function is well-defined.

2.2. Topological Yang-Mills Theory

Both Q_+ and Q_- individually satisfy the $N_T = 1$ supersymmetry algebra; mathematically both may thus be thought of as differentials of $\mathcal{G}$-equivariant cohomology. Concentrating on Q_+ we obtain the algebra for the original topological Yang-Mills theory [3].

$$
Q_+^2 = \delta_{\phi_{++}}, \qquad A \longrightarrow \boldsymbol{\psi}_+, \quad \phi_{--}\longrightarrow \boldsymbol{\eta}_-, \quad \boldsymbol{\chi}_- \longrightarrow H^+, \tag{2.11}
$$

Now we no longer have a balanced structure and expect U-number anomalies. This model has well-known non-trivial observables representing Donaldson's map [14]

$$
\begin{aligned}
& \mathcal{O}_0 = \frac{1}{8\pi^2}\int_X \mathrm{Tr}\,(F\wedge F), && \mathcal{O}_1 = \frac{1}{4\pi^2}\int_{\gamma_3}\mathrm{Tr}\,(\boldsymbol{\psi}_+\wedge F), \\
& \mathcal{O}_2 = \frac{1}{4\pi^2}\int_{\gamma_2}\mathrm{Tr}\,(\phi_{++}F + \boldsymbol{\psi}_+\wedge\boldsymbol{\psi}_+), && \mathcal{O}_3 = \frac{1}{4\pi^2}\int_{\gamma_1}\mathrm{Tr}\,(\phi_{++}\boldsymbol{\psi}_+), \\
& \mathcal{O}_4 = \frac{1}{8\pi^2}\mathrm{Tr}\,(\phi_{++}^2), &&
\end{aligned}
\tag{2.12}
$$

where $\gamma_i \in H_i(X)$ and the subscript denotes the U-number. The expectation values of the observables define Donaldson invariants.

We shall now see that one can obtain topological Yang-Mills theory from balanced topological Yang-Mills theory by introducing mass terms for $\boldsymbol{\psi}_-, \phi_{+-}, \boldsymbol{\eta}_+, B^+, \boldsymbol{\chi}_+$, provided the underlying four-manifold is Kähler.

2.3. Topological Yang-Mills theory on a Kähler manifold

Now let X be a Kähler surface with the Kähler form ω. For a $SU(n)$ bundle E over X we have the space $\mathcal{A}$ of all connections, which is an affine space. A tangent vector on $\mathcal{A}$ can be represented by an element of $\Omega^1_X(\mathfrak{g}_E)$. Picking a complex structure I in X we can introduce a complex structure on $\mathcal{A}$ by the decomposition $T\mathcal{A} = T^{1,0}\mathcal{A} \oplus T^{0,1}\mathcal{A}$ and the identification $\delta A^{0,1} \in T^{1,0}\mathcal{A}$ where $\delta A^{0,1} \in \Omega^{0,1}_X(\mathfrak{g}_E)$. We also have a Kähler structure on $\mathcal{A}$ with the Kähler form

$$\tilde{\omega} = \frac{1}{4\pi^2}\int_X \mathrm{Tr}\,(\delta A^{1,0} \wedge \delta A^{0,1}) \wedge \omega. \tag{2.13}$$

Any self-dual two-form α^+ can be decomposed as

$$\alpha^+ = \alpha^{2,0} + \alpha^0\omega + \alpha^{0,2}, \tag{2.14}$$

where $\alpha^{0,2} \in \Omega^{0,2}_X$ and $\alpha^0 \in \Omega^0(X)$. In particular, an anti-self-dual connection A satisfies

$$F^{2,0}_A = 0, \qquad F_A \wedge \omega = 0, \qquad F^{0,2}_A = 0. \tag{2.15}$$

We denote the moduli space of anti-self-dual connections by $\mathcal{M}$.

Since a Kähler surface has $U(2)$ holonomy, the number N_T of topological supersymmetries is doubled. Thus N_T is identical to the number N of physical supersymmetries of the underlying super-Yang-Mills theory. So we will use N to denote both the number of topological and physical supersymmetries. Then the supersymmetry generator Q_+ (2.11) can be decomposed as $Q_+ = \mathbf{s}_+ + \bar{\mathbf{s}}_+$, with respect to the complex structure on $\mathcal{A}$, with the commutation relation

$$\mathbf{s}_+^2 = 0, \qquad \{\mathbf{s}_+, \bar{\mathbf{s}}_+\} = \delta_{\phi_{++}}, \qquad \bar{\mathbf{s}}_+^2 = 0. \tag{2.16}$$

The supercharges $\mathbf{s}_+$ and $\bar{\mathbf{s}}_+$ are the differentials of the Dolbeault version of $\mathcal{G}$-equivariant cohomology [10]. There is a natural bi-grading (p, q) such that the original ghost number U is given by $U = (p + q)$ and there is a new ghost number R given by $R = (p - q)$. The $N = 2$ supercharges $\mathbf{s}_+$ and $\bar{\mathbf{s}}_+$ are assigned the following degrees

$$\mathbf{s}_+ : (1, 0), \qquad \bar{\mathbf{s}}_+ : (0, 1). \tag{2.17}$$

The connection one-form A has the degree $(0,0)$. Indicating the action of $\mathbf{s}_+$ by $\nearrow$ and the action of $\bar{\mathbf{s}}_+$ by $\searrow$, the field contents and R-number assignments are

$$\begin{array}{cccccccccccc}
R=+1 & & & \psi_+^{0,1} & & & \eta_- & & & \bar{\chi}_-^{0,2} & & \\
 & & \nearrow & & & \nearrow & & \searrow & & & \searrow & \\
R=0 & A & & & ,\quad \phi_{++}, \quad \phi_{--} & & & & H^0\,, & & & H^{2,0}, H^{0,2}\,. \\
 & & \searrow & & & \searrow & & \nearrow & & & \nearrow & \\
R=-1 & & & \bar{\psi}_+^{1,0} & & & \bar{\eta}_- & & & \chi_-^{2,0} & &
\end{array} \tag{2.18}$$

Here $\psi_+^{0,1} \in \Omega_X^{0,1}(\mathfrak{g}_E)$, $\bar{\psi}_+^{1,0} \in \Omega_X^{1,0}(\mathfrak{g}_E)$, $\eta_-, \bar{\eta}_-, H^0 \in \Omega_X^0(\mathfrak{g}_E)$, $\chi_-^{2,0}, H^{2,0} \in \Omega_X^{2,0}(\mathfrak{g}_E)$, and $\bar{\chi}_-^{0,2}, H^{0,2} \in \Omega_X^{0,2}(\mathfrak{g}_E)$. This diagram shows that the model is balanced in terms of the R-number. The net U-number, however, is generally non-vanishing and equals the formal dimension of the moduli space of anti-self-dual connections. The action functional of the theory is defined by

$$\begin{aligned}
S_2 = &-\frac{\mathbf{s}_+}{e^2}\int_X \operatorname{Tr}\chi_-^{2,0}\wedge F^{0,2} - \frac{\bar{\mathbf{s}}_+}{e^2}\int_X \operatorname{Tr}\bar{\chi}_-^{0,2}\wedge F^{2,0} \\
&+\frac{\mathbf{s}_+\bar{\mathbf{s}}_+}{e^2}\int_X \operatorname{Tr}\Big(2i\phi_{--}F\wedge\omega + \chi_-^{2,0}\wedge\bar{\chi}_-^{0,2} - 2\eta_- * \bar{\eta}_-\Big)
\end{aligned} \tag{2.19}$$

One can further break half of the $N=2$ symmetry by introducing mass for the $N=1$ matter multiplet ($N=1$ chiral multiplet) [12]. The required perturbation makes use of a holomorphic two-form $\omega^{0,2} \in \Omega_X^{0,2}$ and was essential in the development of Donaldson and Seiberg-Witten theory. One can also deform topological Yang-Mills theory to holomorphic Yang-Mills theory [10].

3. $N=4$ Theory

In this section we study $N=4$ super-Yang-Mills theory on a Kähler surface X.

3.1. $N=4$ Supersymmetry

Naturally we can carry out the decomposition of Q_+ also for Q_-. Thus we write $Q_\pm = \mathbf{s}_\pm + \bar{\mathbf{s}}_\pm$ and have

$$\mathbf{s}_\pm^2 = 0, \qquad \{\mathbf{s}_\pm, \bar{\mathbf{s}}_\pm\} = \delta_{\phi_{\pm\pm}}, \qquad \bar{\mathbf{s}}_\pm^2 = 0. \tag{3.1}$$

We assign the degree (p,q) to the super-charges $\mathbf{s}_\pm, \bar{\mathbf{s}}_\pm$ by

$$\begin{aligned}
&\mathbf{s}_+ : (+1,0), \qquad \bar{\mathbf{s}}_+ : (0,+1), \\
&\mathbf{s}_- : (0,-1), \qquad \bar{\mathbf{s}}_- : (-1,0).
\end{aligned} \tag{3.2}$$

We also introduce another bigraded additive quantum number $(J_L, J_R)_{\mathcal{R}}$, which we refer to $U(1)_{\mathcal{R}}$ charge.[1]

$$\begin{aligned} \mathbf{s}_+ &: (+1,0), \qquad & \bar{\mathbf{s}}_+ &: (-1,0), \\ \mathbf{s}_- &: (0,-1), \qquad & \bar{\mathbf{s}}_- &: (0,+1). \end{aligned} \tag{3.3}$$

Both the degree and $U(1)_{\mathcal{R}}$ charge originate from the unbroken part of $SO(5,1)_I$ symmetry of physical $N=4$ SYM theory after twisting on a Kähler manifold [1].

The degree above represent the form degree in the Dolbeault equivariant cohomology. The operators $\mathfrak{s} = \mathbf{s}_+ \oplus \mathbf{s}_-$ and $\bar{\mathfrak{s}} = \bar{\mathbf{s}}_+ \oplus \bar{\mathbf{s}}_-$ may be interpreted as the holomorphic and the anti-holomorphic differentials on iterated superspace [5]. Then, similarly to (2.16), we have

$$\mathfrak{s}^2 = 0, \qquad \{\mathfrak{s}, \bar{\mathfrak{s}}\} = \delta_{\Phi}, \qquad \bar{\mathfrak{s}}^2 = 0, \tag{3.4}$$

where Φ is the gauge transformation generator. According to our grading, we can decompose Φ into components $\Phi = \phi_{++} \oplus \sigma \oplus \bar{\sigma} \oplus \phi_{--}$ with the degree $(1,1) \oplus (0,0) \oplus (0,0) \oplus (-1,-1)$ and the $U(1)_{\mathcal{R}}$ charge $(0,0) \oplus (1,1) \oplus (-1,-1) \oplus (0,0)$, respectively. Now (3.4) reduces to

$$\begin{aligned} &\{\mathbf{s}_+, \bar{\mathbf{s}}_+\} = \delta_{\phi_{++}}, \qquad \{\mathbf{s}_+, \bar{\mathbf{s}}_-\} = \delta_{\sigma}, \qquad \{\mathbf{s}_+, \mathbf{s}_-\} = 0, \qquad \mathbf{s}_\pm^2 = 0, \\ &\{\mathbf{s}_-, \bar{\mathbf{s}}_-\} = \delta_{\phi_{--}}, \qquad \{\mathbf{s}_-, \bar{\mathbf{s}}_+\} = \delta_{\bar{\sigma}}, \qquad \{\bar{\mathbf{s}}_+, \bar{\mathbf{s}}_-\} = 0. \qquad \bar{\mathbf{s}}_\pm^2 = 0. \end{aligned} \tag{3.5}$$

Note that

$$\{Q_+, Q_-\} = \{\mathbf{s}_+, \bar{\mathbf{s}}_-\} + \{\mathbf{s}_-, \bar{\mathbf{s}}_+\} = \delta_{\phi_{+-}}, \tag{3.6}$$

leading to $\sigma + \bar{\sigma} = \phi_{+-}$. We decompose the self-dual 2-form $B \in \Omega_X^{2+}(\mathfrak{g}_E)$ as $B = B^{2,0} + B_0\omega + B^{0,2}$. We combined the scalar B_0 with ϕ_{+-} into the complex scalar

$$\sigma \equiv \frac{1}{2}\phi_{+-} + \frac{i}{2}B^0. \tag{3.7}$$

3.2. The Fields and Action Functional

It is convenient to represent the action of various supercharges by two-dimensional vectors as follows

$$\begin{array}{ccc} & \uparrow{\scriptstyle \bar{s}_+} & \\ \xleftarrow{s_-} & & \xrightarrow{s_+} \\ & \downarrow{\scriptstyle \bar{s}_-} & \end{array} \tag{3.8}$$

We represent the various fields by points in the plane. Then, the commutation relations (3.5) can be represented by simple diagrammatic rules.

1 This terminology will be clarified in a later section.

In order to obtain all the components of the $N = 4$ superfields it is necessary and sufficient to introduce two basic fields, the connection A and an adjoint $(2,0)$-form $B^{2,0}$. We decompose the connection 1-form A, $A = A^{1,0}+A^{0,1}$ according to the complex structure on X. Now we consider the space $\mathcal{A}$ of all connections and the space of $\Omega^{2,0}_X(\mathfrak{g}_E)\oplus\Omega^{0,2}_X(\mathfrak{g}_E)$ all $B^{2,0}$ and $B^{0,2}$. We introduce a complex structure on these space by declaring $A^{0,1}$ and $B^{2,0}$ as the holomorphic coordinates. Then we have

$$\bar{\mathfrak{s}}A^{0,1} = 0, \qquad \bar{\mathfrak{s}}B^{2,0} = 0, \tag{3.9}$$

where $\bar{\mathfrak{s}} = \bar{\mathbf{s}}_+ \oplus \bar{\mathbf{s}}_-$ as defined earlier. From the above condition and the commutation relations (3.5) we build up two holomorphic quartets as follows;

$$\begin{array}{ccccccccccc}
\psi^{0,1}_- & \xleftarrow{s_-} & A^{0,1} & \xrightarrow{s_+} & \psi^{0,1}_+ & \qquad & \chi^{2,0}_- & \xleftarrow{s_-} & B^{2,0} & \xrightarrow{s_+} & \chi^{2,0}_+ \\
 & & & & & \text{and} & & & & & \\
\psi^{0,1}_- & \xrightarrow{s_+} & H^{0,1} & \xleftarrow{s_-} & \psi^{0,1}_+ & & \chi^{2,0}_- & \xrightarrow{s_+} & H^{2,0} & \xleftarrow{s_-} & \chi^{2,0}_+,
\end{array} \tag{3.10}$$

where $\psi_\pm, H^{1,0} \in \Omega^{1,0}_X(\mathfrak{g}_E)$ and $\chi^{0,2}_\pm, H^{0,2} \in \Omega^{0,2}_X(\mathfrak{g}_E)$. It follows that

$$\mathfrak{s}A^{1,0} = 0, \qquad \mathfrak{s}B^{0,2} = 0, \tag{3.11}$$

which lead to two anti-holomorphic quartets

$$\begin{array}{ccccc}
\bar{\psi}^{1,0}_+ & \bar{\psi}^{1,0}_+ & & \bar{\chi}^{0,2}_+ & \bar{\chi}^{0,2}_+ \\
\Big\uparrow{\scriptstyle \bar{s}_+} & \Big\downarrow{\scriptstyle \bar{s}_-} & & \Big\uparrow{\scriptstyle \bar{s}_+} & \Big\downarrow{\scriptstyle \bar{s}_-} \\
A^{1,0} & H^{1,0} & \text{and} & B^{0,2} & H^{0,2} \\
\Big\downarrow{\scriptstyle \bar{s}_-} & \Big\uparrow{\scriptstyle \bar{s}_+} & & \Big\downarrow{\scriptstyle \bar{s}_-} & \Big\uparrow{\scriptstyle \bar{s}_+} \\
\bar{\psi}^{1,0}_-\;, & \bar{\psi}^{1,0}_-\;, & & \bar{\chi}^{0,2}_-\;, & \bar{\chi}^{0,2}_-\;.
\end{array} \tag{3.12}$$

Imposing the commutation relations (3.5) we obtain the following consistency nine-plet:

$$\begin{array}{ccccc}
\bar{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & \phi_{++} \\
\Big\downarrow{\scriptstyle \bar{s}_-} & & \Big\downarrow{\scriptstyle \bar{s}_-} & & \Big\downarrow{\scriptstyle \bar{s}_-} \\
\bar{\eta}_- & \xrightarrow{s_+} & H^0 & \xleftarrow{s_-} & \bar{\eta}_+ \\
\Big\uparrow{\scriptstyle \bar{s}_+} & & \Big\uparrow{\scriptstyle \bar{s}_+} & & \Big\uparrow{\scriptstyle \bar{s}_+} \\
\phi_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma
\end{array} \tag{3.13}$$

We may call the above multiplet twisted holomorphic since

$$\begin{cases} (\mathbf{s}_+ \oplus \bar{\mathbf{s}}_-)\sigma = 0, \\ (\bar{\mathbf{s}}_+ \oplus \mathbf{s}_-)\bar{\sigma} = 0, \end{cases} \qquad \begin{cases} (\mathbf{s}_+ \oplus \bar{\mathbf{s}}_+)\phi_{++} = 0, \\ (\mathbf{s}_- \oplus \bar{\mathbf{s}}_-)\phi_{--} = 0. \end{cases} \tag{3.14}$$

The explicit form of the superalgebra is worked out in the Appendix A. The $U(1)_{\mathcal{R}}$ charges $(J_L, J_R)_{\mathcal{R}}$ and the degrees (p,q) of the various fields are given by the following tables

Table 1

Bose	ϕ_{++}	ϕ_{--}	σ	$\bar{\sigma}$	$B^{2,0}$	$B^{0,2}$	H_0	$H^{0,2}$	$H^{2,0}$	A	$H^{1,0}$	$H^{0,1}$
J_L	0	0	+1	−1	−1	+1	0	0	0	0	−1	+1
J_R	0	0	+1	−1	+1	−1	0	0	0	0	+1	−1
p	1	−1	0	0	−1	+1	0	0	0	0	−1	+1
q	1	−1	0	0	+1	−1	0	0	0	0	+1	−1

Table 2

Fermi	η_+	η_-	$\bar{\eta}_+$	$\bar{\eta}_-$	$\chi_+^{2,0}$	$\chi_-^{2,0}$	$\bar{\chi}_+^{0,2}$	$\bar{\chi}_-^{0,2}$	$\bar{\psi}_+^{1,0}$	$\bar{\psi}_-^{1,0}$	$\psi_+^{0,1}$	$\psi_-^{0,1}$
J_L	0	+1	0	−1	0	−1	0	+1	−1	0	+1	0
J_R	−1	0	+1	0	+1	0	−1	0	0	+1	0	−1
p	+1	0	0	−1	0	−1	+1	0	0	−1	+1	0
q	0	−1	+1	0	+1	0	0	−1	−1	0	0	−1

The action functional should have degree $(0,0)$ and $U(1)_{\mathcal{R}}$ charge $(0,0)$ and be annihilated by all four supercharges. Those requirements (almost) fix the action functional in the following form

$$S_4 = \mathbf{s}_+\bar{\mathbf{s}}_+\mathbf{s}_-\bar{\mathbf{s}}_-\mathcal{F} + \mathbf{s}_+\mathbf{s}_-\mathcal{W} + \bar{\mathbf{s}}_+\bar{\mathbf{s}}_-\overline{\mathcal{W}} \tag{3.15}$$

with

$$\begin{aligned} \mathcal{F} =& \frac{1}{e^2}\int_X \mathrm{Tr}\left(\kappa F\wedge F + B^{2,0}\wedge B^{0,2} - 2\sigma * \bar{\sigma}\right),\\ \mathcal{W} =& \frac{1}{e^2}\int_X \mathrm{Tr}\left(B^{2,0}\wedge F^{0,2}\right), \end{aligned} \tag{3.16}$$

where κ is a Kähler potential of X and e is the Yang-Mills coupling constant. Note that $\bar{\mathbf{s}}_\pm\mathcal{W} = 0$ and $\mathbf{s}_\pm\overline{\mathcal{W}} = 0$ which ensures the manifest $N=4$ supersymmetry of the action functional S_4.

Before moving on, we make two comments

(1). Let $\mathcal{A}$ be the space of all connections on a Kähler surface X with Kähler form $\omega = -i\partial\bar{\partial}\kappa$. Using the complex and the Kähler structures on X we can introduce the corresponding structures on $\mathcal{A}$. In particular on $\mathcal{A}$ we have a Käher potential $\tilde{\kappa}$ defined via

$$\tilde{\kappa} = \frac{1}{8\pi^2}\int_X \kappa \mathrm{Tr}\, F \wedge F \tag{3.17}$$

We assert that our supersymmetry generators can be viewed as the differentials of Dolbeault equivariant cohomology on $\mathcal{A} \oplus \mathcal{B}$. For instance we have the following relation

$$\mathbf{s}_-^2 = 0, \qquad \{\mathbf{s}_-, \bar{\mathbf{s}}_-\} = \delta_{\phi_{--}}, \qquad \bar{\mathbf{s}}_-^2 = 0. \tag{3.18}$$

This implies that we have the equivariant Kähler identity

$$-i\mathbf{s}_-\bar{\mathbf{s}}_-\tilde{\kappa} = \frac{1}{4\pi^2}\int_X \mathrm{Tr}\,\phi_{--}F \wedge \omega - \frac{1}{4\pi^2}\int_X \mathrm{Tr}\,\psi_-^{0,1} \wedge \bar{\psi}_-^{1,0} \wedge \omega, \tag{3.19}$$

where we used integration by parts and the Bianchi identity $\partial_A F^{0,2} + \bar{\partial}_A F^{1,1} = 0$. The term proportional to ϕ_{--} is the $\mathcal{G}$-momentum-map on $\mathcal{A}$ and the second term is the Kähler form on $\mathcal{A}$.

(2). Now let us compare the field contents of our model with the field contents of $N = 4$ super-Yang-Mills theory. The latter may be viewed as a $N = 2$ theory with an adjoint hypermultiplet or as a $N = 1$ theory with three adjoint $N = 1$ chiral supermultiplets. To determine the twisted $N = 2$ vetor and hyper multiplets we have to choose a $N = 2$ subalgebra (*the Dolbault $\mathcal{G}$-equivariant cohomology*). The commutations relations (3.5) implies that we have four sets of *equivalent* $N = 2$ subalgebra generated eigther by $\mathbf{s}_+ \oplus \bar{\mathbf{s}}_+$, or $\mathbf{s}_- \oplus \bar{\mathbf{s}}_-$, or $\mathbf{s}_+ \oplus \bar{\mathbf{s}}_-$, or $\mathbf{s}_- \oplus \bar{\mathbf{s}}_+$. For any choice we can identifiy $\mathbf{s}$ and $\bar{\mathbf{s}}$, omitting $\pm$ indicies, with holomorphic and anti-holomophic differentials, respectively, of Dolbeault $\mathcal{G}$-equivariant cohomology;

$$\mathbf{s}^2 = 0, \qquad \{\mathbf{s}, \bar{\mathbf{s}}\} = \delta_\varphi, \qquad \bar{\mathbf{s}}^2 = 0. \tag{3.20}$$

In a later section we will perturb the $N = 4$ theory to $N = 2$ theory by introducing a bare mass for the hypermultiplet. In the pure $N = 2$ limit any one of the above four cases lead to the Donalson-Witten theory.

In this paper we choose the $N = 2$ subalgebra generated by $\mathbf{s}_+ \oplus \bar{\mathbf{s}}_+$, satisfying the commuation relation (2.16). Then the twisted $N = 2$ vector multiplet is given by

$$(A^{1,0}, A^{0,1}), (\psi_+^{0,1}, \bar{\psi}_+^{1,0}), (\phi, \bar{\phi}), (\eta_-, \bar{\eta}_-), (\chi_-^{2,0}, \bar{\chi}_-^{0,2}), (H, H^{2,0}, H^{0,2}), \tag{3.21}$$

where $\phi := \phi_{++}$ and $\bar{\phi} := \phi_{--}$. The remainder corresponds to the twisted $N = 2$ hypermultiplet:

$$(B^{2,0}, B^{0,2}), (\sigma, \bar{\sigma}), (\psi_-^{0,1}, \bar{\psi}_-^{1,0}), (\eta_+, \bar{\eta}_+), (\chi_+^{2,0}, \bar{\chi}_+^{0,2}), (H^{1,0}, H^{0,1}) \tag{3.22}$$

3.3. The Localization

Now we return to the action functional. Collecting the terms in the action which depend on the auxiliary fields:

$$\begin{aligned} S_4 =& \frac{1}{e^2}\int_X \mathrm{Tr}\Big(-H^{2,0}\wedge H^{0,2} - H^{2,0}\wedge F^{0,2} - H^{0,2}\wedge F^{2,0} + H\left(2F\wedge\omega - i[B^{2,0}, B^{0,2}]\right) \\ & - 2H * H + B^{2,0}\wedge\bar{\partial}_A H^{0,1} + B^{0,2}\wedge\partial_A H^{1,0} - 2H^{1,0}\wedge * H^{0,1}\Big) + \ldots \end{aligned} \tag{3.23}$$

we can integrate $H^{2,0}, H^{0,2}, H^0$ using the following replacement

$$\begin{aligned} H^{2,0} &= F^{2,0}, \\ H\omega^2 &= F\wedge\omega - \frac{i}{2}[B^{2,0}, B^{0,2}], \\ H^{0,2} &= F^{0,2}, \end{aligned} \tag{3.24}$$

Similarly $H^{1,0}, H^{0,1}$ are integrated out by the replacements

$$\begin{aligned} H^{1,0} &= \frac{1}{2}\partial_A^* B^{2,0}, \\ H^{0,1} &= \frac{1}{2}\bar{\partial}_A^* B^{0,2}. \end{aligned} \tag{3.25}$$

The resulting action functional is invariant under $\mathbf{s}_\pm$ and $\bar{\mathbf{s}}_\pm$ if we modify the transformation laws $(A.3)$, $(A.4)$ and $(A.5)$ in the Appendix according to (3.24) and (3.25).

According to the fixed point theorem of Witten the path integral is localized on the fixed point locus of all four supersymmetries. Collecting all the simultaneous fixed points equations from the transformation laws for fermionic fields we have

$$\begin{aligned} & F^{0,2} = 0, \\ & F\wedge\omega - \frac{i}{2}[B^{2,0}, B^{0,2}] = 0, \\ & [\bar{\sigma}, B^{0,2}] = [\sigma, B^{0,2}] = [\sigma, \bar{\sigma}] = 0, \\ & \bar{\partial}_A^* B^{0,2} = d_A\bar{\sigma} = 0, \end{aligned} \tag{3.26}$$

and

$$[\phi_{++}, \phi_{--}] = 0, \qquad [\phi_{\pm\pm}, \bar{\sigma}] = 0, \qquad [\phi_{\pm\pm}, B^{0,2}] = 0, \qquad d_A\phi_{\pm\pm} = 0. \tag{3.27}$$

These are the Vafa-Witten equations on a Kähler manifold and were derived in a different way in section 2.4 of [1]. Already at this point we could exploit the S^1-action and the resulting localization pointed out in [1]. However, we will find that it is more useful to return to this point in a more general framework.

4. Perturbations to $N=2$ and $N=1$ Theories

The path integral of any cohomological theory is localized on the fixed point locus of the global supersymmetry. We examine how the fixed points are changed when we perturb the original $N=4$ theory to $N=2$ and $N=1$ theories by introducing bare masses for the matter multiplets. The bare mass for the $N=2$ hypermultiplet will be introduced by exploiting a global S^1-action on the adjoint hypermultiplet. The path integral is then localized on the fixed point locus of the unbroken $N=2$ symmetry, which has two branches: branch (i) is the moduli space of anti-self-dual connections, branch (ii) is the moduli space of a special type of abelian Seiberg-Witten monopoles. We will further break the supersymmetry down to $N=1$ to get the perturbed Seiberg-Witten equation presented in branch (ii). The last step is crucial in order to achieve the factorization of the Seiberg-Witten basic class. The procedure in this section should be understood in conjunction with the more physical approach taken by Vafa and Witten in section 5 of [1].

4.1. Perturbation to $N=2$ Theory

We want to perturb the $N=4$ theory to a $N=2$ theory maintaining the two supersymmetry generated by $\mathbf{s}_+$ and $\bar{\mathbf{s}}_+$. To begin with we regard the $N=4$ theory as $N=2$ theory with an adjoint hypermultiplet. In this language the action functional is written as (compare with (2.19))

$$
\begin{aligned}
S_4 = \frac{\mathbf{s}_+}{e^2}\int_X \mathrm{Tr}\left(-\chi_-^{2,0}\wedge F^{0,2} - B^{2,0}\wedge\bar{\partial}_A\psi_-^{0,1}\right) + \frac{\bar{\mathbf{s}}_+}{e^2}\int_X \mathrm{Tr}\left(-\bar{\chi}_-^{0,2}\wedge F^{2,0} - B^{0,2}\wedge\partial_A\bar{\psi}_-^{1,0}\right)\\
+\frac{\mathbf{s}_+\bar{\mathbf{s}}_+}{e^2}\int_X \mathrm{Tr}\Big(\bar{\phi}\left(2iF\wedge\omega + [B^{2,0},B^{0,2}] - 2[\sigma,\bar{\sigma}]\right) + \chi_-^{2,0}\wedge\bar{\chi}_-^{0,2} - 2\eta_-\ast\bar{\eta}_- - 2\psi_-^{0,1}\wedge\ast\bar{\psi}_-^{1,0}\Big)
\end{aligned}
\tag{4.1}
$$

Now we want to perturb the theory to $N=2$ theory by introducing a bare mass term to the hypermultiplet. If we divide the hypermultiplets into

$$
\begin{aligned}
\mathbf{H} &:= (\psi_-^{0,1}, H^{0,1}, \bar{\sigma}, \eta_+, B^{0,2}, \bar{\chi}_+^{0,2}),\\
\overline{\mathbf{H}} &:= (\bar{\psi}_-^{1,0}, H^{1,0}, \sigma, \bar{\eta}_+, B^{2,0}, \chi_+^{2,0}),
\end{aligned}
$$

the algebra as well as the action functional are invariant under the following infinitesimal S^1-action

$$
\delta_m(\mathbf{H},\overline{\mathbf{H}}) = (im\mathbf{H}, -im\overline{\mathbf{H}}), \tag{4.2}
$$

where m is a positive real parameter.

This S^1-action is the unbroken part of the global $SO(4)$ symmetry rotating the components of the two complex bosons of the untwisted $N=2$ hypermultiplet. It is the right

$U(1)_{\mathcal{R}}$ symmetry. Note, from the tables 1 and 2 in Sect. 3.2., that $\mathbf{H}$ and $\overline{\mathbf{H}}$ have $J_R = -1$ and $J_R = +1$ respectively. We want to extend the $\mathcal{G}$-equivariant Dolbeault cohomology to $\mathcal{G} \times S^1$-equivariant one. Then it is natural to assign the degree $(1,1)$ to m. Differentials of the extended equivariant Dobeault cohomology have the following commutation relations

$$\mathbf{s}_+^2 = 0, \qquad \{\mathbf{s}_+, \bar{\mathbf{s}}_+\} = \delta_\phi + \delta_m, \qquad \bar{\mathbf{s}}_+^2 = 0 \tag{4.3}$$

where δ_m is defined in (4.2).

Altogether we thus have the following modified transformation laws for hypermultiplets, denoting $\delta = \bar{\varepsilon}_- \mathbf{s}_+ + \varepsilon_- \bar{\mathbf{s}}_+$

$$\begin{aligned}
\delta\psi_-^{0,1} &= \bar{\epsilon}_- H^{0,1} + \epsilon_- \bar{\partial}_A \bar{\sigma}, & \delta H^{0,1} &= -\epsilon_-([\phi, \psi_-^{0,1}] + im\psi_-^{0,1}) + \epsilon_-[\bar{\sigma}, \psi_+^{0,1}] - \epsilon_- \bar{\partial}_A \eta_+,\\
\delta\bar{\psi}_-^{1,0} &= \epsilon_- H^{1,0} + \bar{\epsilon}_- \partial_A \sigma, & \delta H^{1,0} &= -\bar{\epsilon}_-([\phi, \bar{\psi}_-^{1,0}] - im\bar{\psi}_-^{1,0}) + \bar{\epsilon}_-[\sigma, \bar{\psi}_+^{1,0}] - \bar{\epsilon}_- \partial_A \bar{\eta}_+,
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\delta B^{0,2} &= \epsilon_- \bar{\chi}_+^{0,2}, & \delta\bar{\chi}_+^{0,2} &= -\bar{\epsilon}_-([\phi, B^{0,2}] + imB^{0,2}),\\
\delta B^{2,0} &= \bar{\epsilon}_- \chi_+^{2,0}, & \delta\chi_+^{2,0} &= -\epsilon_-([\phi, B^{2,0}] - imB^{0,2}),
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\delta\bar{\sigma} &= -\bar{\epsilon}_- \eta_+, & \delta\eta_+ &= +\epsilon_-([\phi, \bar{\sigma}] + im\bar{\sigma}),\\
\delta\sigma &= -\epsilon_- \bar{\eta}_+, & \delta\bar{\eta}_+ &= +\bar{\epsilon}_-([\phi, \sigma] - im\sigma).
\end{aligned} \tag{4.6}$$

The action functional is still defined by the formula (4.1), but using the modified transformations laws (4.4), (4.5) and (4.6), one obtains the new expression

$$S_4' = S_4 + \frac{1}{e^2}\int_X \mathrm{Tr}\,\Big(-im\phi_{--}\left[B^{2,0}, B^{0,2}\right] + 2im\phi_{--} * [\sigma, \bar{\sigma}] + m\psi_-^{0,1} \wedge \bar{\psi}_-^{1,0} \wedge \omega\Big), \tag{4.7}$$

which contains new m-dependent terms. Physically, these terms are part of the mass term for the hypermultiplet. To get the remaining terms we digress briefly.

Recall that the bosonic components of the hypermultiplet are $\sigma, \bar{\sigma}$ and $B^{2,0}, B^{0,2}$. Now consider the space $\mathcal{H}$ of all bosonic hypermultiplets. On $\mathcal{H}$ we have a $\mathcal{G} \times S^1$ action. Thus we can introduce a $\mathcal{G} \times S^1$-invariant inner product. We also introduce a compatible complex structure on $\mathcal{H}$ by declaring $\bar{\sigma}$ and $B^{0,2}$ as the holomorphic coordinates. Now we define a Kähler potential on $\mathcal{H}$ as

$$h = \int_X \mathrm{Tr}\,(B^{2,0} \wedge B^{0,2} - 2\sigma * \bar{\sigma}). \tag{4.8}$$

The relation (4.3) implies that $\mathbf{s}_+$ and $\bar{\mathbf{s}}_+$ are the holomorphic and anti-holomorphic differentials of $\mathcal{G} \times S^1$-equivariant Dolbeault cohomology, i.e.,

$$\begin{aligned}
\mathbf{s}_+ \sigma &= 0, & \mathbf{s}_+ B^{0,2} &= 0,\\
\bar{\mathbf{s}}_+ \bar{\sigma} &= 0, & \bar{\mathbf{s}}_+ B^{2,0} &= 0.
\end{aligned}$$

Thus we compute the equivariant Kähler form I_h from the Kähler potential h according to

$$\begin{aligned} I_h &= i\mathbf{s}_+\bar{\mathbf{s}}_+ h \\ &= \int_X \mathrm{Tr}\Big(i\phi_{++}[B^{2,0}, B^{0,2}] - 2i\phi_{++}[\sigma, \bar{\sigma}] + i\chi_+^{2,0} \wedge \bar{\chi}_+^{0,2} - 2i\bar{\eta}_+ * \eta_+ \\ &\qquad + mB^{2,0} \wedge B^{0,2} - 2m\sigma * \bar{\sigma} \Big). \end{aligned} \tag{4.9}$$

Note that the terms proportional to m and to ϕ are the momentum maps of the S^1 and $\mathcal{G}$ actions respectively. The remaining terms are the Kähler form on $\mathcal{H}$.

The equivariant Kähler form I_h can now be used to construct the promised mass terms, which we define to be $\frac{4m}{e^2} I_\kappa$. Now all elements of the hypermultiplet have acquired a bare mass. Putting all the mass terms together we thus get the $N = 2$ symmetric action

$$S_2(m) = S_4' + \frac{m}{e^2} I_h, \tag{4.10}$$

which includes all the bare mass terms for the $N = 2$ hypermultiplet (3.22).

The new $N = 2$ supersymmetric action $S_2(m)$ contains no additional terms in $H^{2,0}, H^{0,2}, H^0$ and $H^{1,0}, H^{0,1}$ compared with S_4. Thus integrating out these fields leads to the same replacements as given in (3.24) and (3.25). With these replacements we can now collect the fixed point equations for the unbroken supersymmetry charges $\mathbf{s}_+$ and $\bar{\mathbf{s}}_+$. From the fixed point equations $\mathbf{s}_+\bar{\eta}_- = \bar{\mathbf{s}}_+\eta_- = 0$ in (A.3) and $\mathbf{s}_+\chi_-^{2,0} = \bar{\mathbf{s}}_+\bar{\chi}_-^{0,2} = 0$ in (A.4) we have

$$\begin{aligned} &F^{0,2} = [\bar{\sigma}, B^{0,2}] = 0, \\ &iF \wedge \omega + \frac{1}{2}[B^{2,0}, B^{0,2}] - \frac{1}{2}[\sigma, \bar{\sigma}]\omega \wedge \omega = 0, \\ &F^{2,0} = [\sigma, B^{2,0}] = 0. \end{aligned} \tag{4.11}$$

From (4.4) we get

$$\begin{aligned} \bar{\partial}_A^* B^{0,2} &= \bar{\partial}_A \bar{\sigma} = 0, \\ \partial_A^* B^{2,0} &= \partial_A \sigma = 0. \end{aligned} \tag{4.12}$$

From $\mathbf{s}_+\bar{\eta}_- = \bar{\mathbf{s}}_+\eta_- = 0$ in $(A.3)$ and from $\mathbf{s}_+\bar{\psi}_+^{1,0} = \bar{\mathbf{s}}_+\psi_+^{0,1} = 0$ in $(A.5)$ we have

$$[\phi, \bar{\phi}] = 0, \qquad d_A\phi = 0. \tag{4.13}$$

From (4.5) and (4.6) we have

$$\begin{aligned} &[\phi, B^{0,2}] + imB^{0,2} = 0, \qquad [\phi, \bar{\sigma}] + im\bar{\sigma} = 0, \\ &[\phi, B^{2,0}] - imB^{2,0} = 0, \qquad [\phi, \sigma] - im\sigma = 0. \end{aligned} \tag{4.14}$$

In studying solutions of these fixed point equations we specialize to the gauge group $SU(2)$. First of all, (4.13) implies that ϕ should be diagonalized in the fixed points. Thus we have two branches

Branch (i):$\phi = 0$, i.e. the gauge symmetry is unbroken. Then (4.14) implies that $B^{2,0}, B^{0,2}$ and $\sigma, \bar{\sigma}$ vanish. So the fixed point equation (4.11) reduces to the anti-self-duality equation for the connection A: $F_A^+ = 0$.

Branch (ii): $\phi \neq 0$, i.e. the gauge symmetry is broken down to $U(1)$. Thus the bundle E splits into line bundles , $E = L \oplus L^{-1}$ with $L \cdot L = -k$, and $\phi = i\phi^a T^a$ takes the form $\phi = iaT^3$. Then the only non-trivial solutions of (4.14) are, with $m + a = 0$:

$$\begin{aligned} B^{0,2} &= \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \qquad & \bar{\sigma} &= \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \\ B^{2,0} &= \begin{pmatrix} 0 & 0 \\ \bar{\beta} & 0 \end{pmatrix}, \qquad & \sigma &= \begin{pmatrix} 0 & 0 \\ \bar{\alpha} & 0 \end{pmatrix}. \end{aligned} \tag{4.15}$$

Then (4.11) reduces to

$$\begin{aligned} F_{L^2}^{0,2} &= \alpha\beta = 0, \\ iF_{L^2} \wedge \omega &= \beta \wedge \bar{\beta} - \alpha\bar{\alpha}\omega^2, \end{aligned} \qquad \bar{\partial}_{L^2}^{*}\beta = \bar{\partial}_{L^2}\alpha = 0. \tag{4.16}$$

Here α is a section of L^2 and β is a section of $K^{-1} \otimes L^2$, with K denoting the canonical line bundle. To make progress it it is useful to regard the above equation as a perturbation of another equation. To achieve this note that

$$F_{K^{-1/2} \otimes L^2} = \frac{1}{2} F_{K^{-1}} + F_{L^2} \quad \rightarrow F_{L^2} = F_{K^{-1/2} \otimes L^2} - \frac{1}{2} F_{K^{-1}}, \tag{4.17}$$

so that we can write

$$\begin{aligned} & F_{L^2}^{0,2} = \alpha\beta = 0, \\ & \frac{i}{2} F_{\zeta} \wedge \omega = \beta \wedge \bar{\beta} - \alpha\bar{\alpha}\omega^2 + \frac{i}{2} F_{K^{-1}} \wedge \omega, \end{aligned} \qquad \bar{\partial}_{L^2}^{*}\beta = \bar{\partial}_{L^2}\alpha = 0. \tag{4.18}$$

This is a perturbation of the Seiberg-Witten equation [11] for a *spin*^{c} structure $\zeta = K^{-1} \otimes L^4$; this fact will be crucial in the next section. For later use we also note that $c_1(\zeta) = w_2(X)$ modulo 2 since $c_1(K) = w_2(X)$ mod 2.

4.2. Perturbation to $N = 1$ Theory

We can further break the remaining $N = 2$ symmetry down to $N = 1$ by introducing a bare mass for the $N = 1$ matter-multiplet. We will do this preserving $\bar{\mathbf{s}}_+$-symmetry.

Note that, among the $N=2$ vector multiplet given by (3.21), the $N=1$ matter multiplet consists of $(\psi_+^{0,1}, \phi, \bar{\phi}, \bar{\eta}_-, \chi_-^{2,0})$.

The required mass term involves an holomorphic two-form $\omega^{2,0}$ and has the form

$$\frac{1}{e^2} I_{\omega^{2,0}} = \frac{1}{2e^2} \int_X \mathrm{Tr}\,(\psi_+^{0,1} \wedge \psi_+^{0,1}) \wedge \omega^{2,0}. \tag{4.19}$$

This term is invariant under $\mathbf{s}_+$-symmetry, but not invariant under the $\bar{\mathbf{s}}_+$ symmetry;

$$\frac{\bar{\mathbf{s}}_+}{e^2} I_{\omega^{2,0}} = -\frac{1}{e^2} \int_X \mathrm{Tr}\,\phi \bar{\partial}_A \psi_+^{0,1} \wedge \omega^{2,0}. \tag{4.20}$$

However, the imposition of the $\chi_-^{2,0}$ equation of motion leads to invariance. The relevant term in the action $S_2(m)$ is $-\int_X \mathrm{Tr}\,\chi_-^{2,0} \wedge \bar{\partial}_A \psi_+^{0,1}$. If we add (4.19) to the action $S_2(m)$ of (4.10) and at the same time change the $\bar{\mathbf{s}}_+$-transformation of $\chi_-^{2,0}$ according to

$$\bar{\mathbf{s}}_+ \chi_-^{2,0} = [\bar{\sigma}, B^{2,0}] \longrightarrow \bar{\mathbf{s}}_+ \chi_-^{2,0} = [\bar{\sigma}, B^{2,0}] - \phi \omega^{2,0}, \tag{4.21}$$

then the new action $S_2(m)' + \frac{1}{e^2} I_{\omega^{2,0}}$ enjoys $\bar{\mathbf{s}}_+$-symmetry. Here $S_2(m)'$ is given by

$$S_2(m)' = S_2(m) - \frac{1}{e^2} \int_X \mathrm{Tr}\,\phi[\sigma, B^{0,2}] \wedge \omega^{2,0}, \tag{4.22}$$

where the additional term is due to the modification (4.21) (see (4.1)). Since $\bar{\mathbf{s}}_+ \phi = 0$, we still have the property $\bar{\mathbf{s}}_+^2 = 0$. In this way the one component $\psi_+^{0,1}$ of the $N=1$ chiral superfield has obtained a mass. To give mass to the remaining components in the $N=1$ matter multiplet we add the following $\bar{\mathbf{s}}_+$-exact terms to the action

$$\begin{aligned} I_{\phi\bar{\phi}} =& -\bar{\mathbf{s}}_+ \int_X \mathrm{Tr}\,\left(\bar{\phi}\chi_-^{2,0}\right) \wedge \omega^{0,2} \\ =& -\int_X \mathrm{Tr}\,\left(\bar{\phi}[\bar{\sigma}, B^{2,0}]\right) \wedge \omega^{0,2} + \int_X \mathrm{Tr}\,\left(\phi\bar{\phi}\right) \omega^{2,0} \wedge \omega^{0,2} \\ & - \int_X \mathrm{Tr}\,\left(\bar{\eta}_- \chi_-^{2,0}\right) \wedge \omega^{0,2} \end{aligned} \tag{4.23}$$

A similar prescription for breaking pure $N=2$ theory down to $N=1$ was given by Witten in [12].

To sum up, the total action

$$S_1(m, \omega^{0,2}) = S_2(m)' + \frac{1}{e^2} I_{\omega^{0,2}} + \frac{1}{e^2} I_{\phi\bar{\phi}}, \tag{4.24}$$

has only $\bar{\mathbf{s}}_+$ supersymmetry and all the matter multiplets have a bare mass.

Now the fixed point equations (4.11) receive an important change due to the modification of the $\bar{\mathbf{s}}_+$ transformation law of $\chi_-^{0,2}$ given by (4.21). The new fixed point equations are

$$
\begin{aligned}
F^{0,2} &= [\bar\sigma, B^{0,2}] - \phi\omega^{0,2} = 0, \\
iF\wedge\omega + \frac{1}{2}[B^{2,0}, B^{0,2}] &- \frac{1}{2}[\sigma,\bar\sigma]\omega\wedge\omega = 0,
\end{aligned}
\qquad \bar\partial_A^* B^{0,2} = \partial_A\bar\sigma = 0, \tag{4.25}
$$

while (4.13) and (4.14) remain unchanged. Thus there are again two branches and it is easy to see that the fixed point equation for branch (i) is unchanged, while the equations for branch (ii) become:

$$
\begin{aligned}
F_\zeta^{0,2} &= \alpha\beta - m\omega^{0,2} = 0, \\
\frac{i}{2}F_\zeta\wedge\omega &= \beta\wedge\bar\beta - \alpha\bar\alpha\omega\wedge\omega - \frac{i}{2}F_{K^{-1}},
\end{aligned} \tag{4.26}
$$

where $\zeta = K^{-1}\otimes L^4$. This is a perturbed version of the Seiberg-Witten equation, containing the perturbation introduced by Witten in [11]. The condition

$$\alpha\beta = m\omega^{0,2}, \tag{4.27}$$

gives the crucial factorization condition of the Seiberg-Witten basic classes.

5. Analysis of Branch (ii)

5.1. A Selection Rule

Our analysis of branch (ii) exploits the relation of the the defining equations with the the Seiberg-Witten equation.

As a first step we need to classify which Seiberg-Witten classes contribute to branch (ii). For an arbitrary $spin^c$ structure x, which can always be written in terms of an arbitrary integral line bundle ξ as

$$x = K^{-1}\otimes\xi^2, \tag{5.1}$$

we have an associated Seiberg-Witten equation. If the square root $\xi^{1/2} = L$ of ξ exists, the Seiberg-Witten equation is identical to the fixed point equation (4.18) of branch (ii). The inclusion of the perturbation in (4.26) further implies that we also have to satisfy the factorization condition $\alpha\beta = \omega^{0,2}$, where we have scaled $m = 1$ in (4.27).

Let the canonical divisor K be given by $K = \sum_i r_i C_i$, where the C_i are irreducible components. The factorization means that

$$K^{1/2} \otimes x^{1/2} = \xi = \sum_i s_i C_i, \tag{5.2}$$

where s_i are integers with $0 \le s_i \le r_i$ and I is the trivial line bundle. Thus, the question of which Seiberg-Witten classes contribute to branch (ii) reduces to finding line bundles L satisfying $L \cdot L = -k$ and

$$2L = \sum_{i=1}^{n} s_i C_i, \qquad 0 \le s_i \le r_i. \tag{5.3}$$

Now let x be a Seiberg-Witten basic class. If $(x^{1/4} \otimes K^{1/4})$ exists as a line bundle, then the associated SW invariant n_x contributes to the path integral in branch (ii). Note that $(x^{1/2} \otimes K^{1/2})$ always exist as a line bundle. The question is thus whether the square root of $(x^{1/2} \otimes K^{1/2})$ exists, which is the case iff

$$\frac{1}{2}[x + K] = 0, \tag{5.4}$$

or, equivalently

$$\frac{1}{2}[x + w_2(X)] = 0, \tag{5.5}$$

where [...] means the mod 2 reduction. Here $w_2(X)$ is the second Stiefel-Whitney class of our Kähler manifold X. In the $SU(2)$ case such a square root may not exist. However, if we repeat the analysis for an $SO(3)$ bundle E, the factorization condition can be met provided the second Stiefel-Whitney class $w_2(E)$ of E satisfies

$$\frac{1}{2}[x + w_2(X)] \equiv w_2(E). \tag{5.6}$$

With the abbreviations $z_0 = w_2(X)$, $z = w_2(E)$ and $2x' = x + K$ the branch (ii) contribution has the form

$$\sum_x n_x \delta_{z,[x']} \times (\ldots), \tag{5.7}$$

where the summation is over all Seiberg-Witten basic classes x. This general form applies to both the $SU(2)$ and the $SO(3)$ case. In principle one could proceed to compute the branch (ii) contribution directly using localization techniques. However, in practice this requires that one starts with a suitable compactification of the moduli space of the Vafa-Witten equations in order to make the integration over the normal bundle of branch (ii) well-defined. Here we are not able to follow that path. Instead we determine the branch (ii) contribution to the partition function by a generalisation of the results of Vafa and Witten in [1].

5.2. Partition Function for Branch (ii)

Consider the case $K = \sum_i^n [C_i]$ where the $[C_i]$ are irreducible and disjoint. Vafa and Witten made a prediction for what we call the contribution to the partition function from branch (ii) (eq. 5.50 in [1]). For the $SU(2)$ case the answer is

$$\left(\frac{G(q^2)}{4}\right)^{\nu/2} \left(\frac{\theta_0}{\eta^2}\right)^{\sum_{i=1}^n (1-g_i)} \sum_{\vec{\varepsilon}} \delta_{0,w_2(\vec{\varepsilon})} \left(\prod_{i=1}^n t_i^{\varepsilon_i}\right) \left(\frac{\theta_1}{\theta_0}\right)^{\sum \varepsilon_i (1-g_i)}, \tag{5.8}$$

where $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\varepsilon_i = 0$ or 1 chosen independently. Here $\nu = (\chi + \sigma)/4$ and χ and σ denote the Euler number and signature of the manifold, respectively.

From our perspective the sum and the delta function $\delta_{0,w_2(\vec{\varepsilon})}$ can be understood as follows. What is called $w_2(\vec{\varepsilon})$ in [1] is a special form of $[x']$, so that the sum in (5.8) is over the same range as the sum (5.7). In our notation, the factorization condition has the form

$$x' = 2L = \sum_{i=1}^n \varepsilon_i [C_i], \qquad 0 \le s_i \le 1. \tag{5.9}$$

From

$$k = -L \cdot L = -\frac{1}{4} x' \cdot x' = -\frac{1}{4} \sum_i s_i^2 (g_i - 1) \tag{5.10}$$

and since $s_i^2 = s_i$ for $s_i = 0$ or 1, we recover the formula

$$k = -\frac{1}{4} \sum_i \varepsilon_i (g_i - 1), \qquad 0 \le \varepsilon_i \le 1. \tag{5.11}$$

given in [1]. Note also that $\sum_{i=1}^n (g_i - 1) = K \cdot K = 2\chi + 3\sigma$.

We now propose a formula for the branch (ii) contributions on general Kähler manifolds with $b_2^+ \ge 3$. We replace

$$\sum_{\vec{\varepsilon}} \delta_{0,w_2(\vec{\varepsilon})} \left(\prod_{i=1}^n t_i^{\varepsilon_i}\right) \left(\frac{\theta_1}{\theta_0}\right)^{\sum \varepsilon_i (1-g_i)} \longrightarrow (-1)^\nu \sum_x \delta_{0,[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x' \cdot x'}, \tag{5.12}$$

where the summation is over all Seiberg-Witten basic classes x with the Seiberg-Witten invariants n_x. Then, (5.8) can be written as

$$(-1)^\nu \left(\frac{G(q^2)}{4}\right)^{\nu/2} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi - 3\sigma} \sum_x \delta_{0,x'} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x' \cdot x'}. \tag{5.13}$$

In the $SO(3)$ case, for a fixed $z = w_2(E)$, we immediately get

$$(-1)^\nu \left(\frac{G(q^2)}{4}\right)^{\nu/2} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x \delta_{z,x'} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} . \tag{5.14}$$

6. The Partition Function

Vafa and Witten make a precise statement about the expected behaviour of the partition function of $N=4$ super-Yang-Mills theory under S-duality [1]. According to [1] the partition function of $N=4$ theory is a modular form invariant under the $\Gamma_0(4)$ subgroup of $SL(2,\mathbb{Z})$. If this is true the total partition function can be determined from the contribution (5.14) of branch (ii) alone, as we shall now show. The basic idea is to examine the terms generated by applying the S-duality transformations corresponding to $\tau \to -1/\tau$ and $\tau \to \tau+1$ to (5.14). Combining the resulting terms in a convenient fashion one gets

$$\begin{aligned} Z_z =& (-1)^\nu \left(\frac{G(q^2)}{4}\right)^{\nu/2} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x \delta_{z,[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1} \left(\frac{G(q^{1/2})}{4}\right)^{\nu/2} \left(\frac{\theta_0+\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1} i^{-z^2} \left(\frac{G(-q^{1/2})}{4}\right)^{\nu/2} \left(\frac{\theta_0-i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-x'\cdot x'} , \end{aligned} \tag{6.1}$$

where the sum $\sum_x$ is over all Seiberg-Witten basic classes, as before. In principle there could be a contribution to the partition function which can not be obtained by performing modular transformations of the contribution of branch (ii). However, such a contribution vanishes for a manifold with $b_2^+ > 1$.

According to [1] the required transformation behaviour under S-duality is:

$$Z_y(-1/\tau) = 2^{-b_2/2}(-1)^\nu \left(\frac{\tau}{i}\right)^{-\chi/2} \sum_z (-1)^{z\cdot y} Z_z(\tau) \tag{6.2}$$

We can check that our proposed expression (6.1) transforms correctly as follows. First we insert (6.1) into the RHS of (6.2) and obtain

$$\begin{aligned} RHS =& \left(\frac{\tau}{i}\right)^{-\chi/2} \Bigg[2^{-b_2/2} \left(\frac{G(q^2)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{z\cdot[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1+b_2/2}(-1)^\nu \left(\frac{G(q^{\frac{1}{2}})}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0+\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_x \delta_{z,[x']} n_x \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1}(-1)^\nu i^{z^2-\frac{\sigma}{2}} \left(\frac{G(-q^{\frac{1}{2}})}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0-i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{[x']\cdot z} i^{x'\cdot x'} n_x \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-x'\cdot x'} \Bigg] \end{aligned} \tag{6.3}$$

where we used

$$\begin{aligned}
&\sum_z (-1)^{z\cdot y}\delta_{z,x'} = (-1)^{y\cdot x'},\\
&\sum_z (-1)^{z\cdot y+z\cdot x'} = 2^{b_2}\delta_{y,x'},\\
&\sum_z (-1)^{z\cdot y} i^{-z^2}(-1)^{z\cdot x'} = 2^{b_2/2} i^{+y^2-\sigma/2+x'\cdot x'}(-1)^{x'\cdot y}.
\end{aligned} \tag{6.4}$$

Carefully taking into account differences in notation these formulae follow from those noted as eq. (5.40) in [1]. In comparing (6.3) with the expression (6.1) evaluated at $-1/\tau$ one finds that the first line and second line in (6.3) equal, respectively, the second and first line in (6.1) evaluated at $-1/\tau$. The third line of (6.3) should thus be compared with the third line in (6.1) at $-1/\tau$. The equality here may require some explanation. Performing $\tau \to -1/\tau$ in the third line on (6.1) one finds, with some rearrangements, that

$$\begin{aligned}
&2^{1-b_1}(-1)^{\nu} i^{z^2-\sigma/2}\left(\frac{\tau}{i}\right)^{-\chi/2}\left(\frac{G(-q^{1/2})}{4}\right)^{\nu/2}\left(\frac{\theta_0-i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma}\\
&\qquad\times\sum_x (-1)^{-z^2+[x']\cdot z} i^{-x'\cdot x'} i^{\sigma}(-1)^{\nu} n_x \left(\frac{\theta_0-i\theta_1}{\theta_0+i\theta_1}\right)^{-x'\cdot x'+2\chi+3\sigma}.
\end{aligned} \tag{6.5}$$

We want to show that the above is identical to the third term in (6.3). A crucial property is that $-x$ is a Seiberg-Witten basic class if x is. Note also that $x' = \frac{1}{2}x + \frac{1}{2}K$. Writing $\bar{x}' = -\frac{1}{2}x + \frac{1}{2}K$ we have $-x'\cdot x' + 2\chi + 3\sigma = \bar{x}'\cdot\bar{x}'$, since a Seiberg-Witten basic class x satisfies $x\cdot x = 2\chi + 3\sigma$. Now the second line of (6.5) can be rewritten as

$$\sum_{-x}(-1)^{-z^2+x^2}(-1)^{[\bar{x}']\cdot z} i^{\bar{x}'\cdot\bar{x}'} n_{-x}\left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-\bar{x}'\cdot\bar{x}'}, \tag{6.6}$$

where we used $n_{-x} = (-1)^{\nu} n_x$ and the fact that $\nu = (\chi+\sigma)/4$ is an integer. The Wu formula implies $(-1)^{-z^2+x^2} = 1$, and we replace the dummy variable $-x, \bar{x}'$ with x, x' to complete the proof.

6.1. A Relation with Strings

Taubes proved that Seiberg-Witten invariants (SW) are equivalent to Gromov-Witten invariants (Gr) for a symplectic 4-manifold of simple type [15]. Here we only consider a Kähler surface. Let ξ be a non-trivial, complex line bundle over X and use ξ to define a $spin^c$ structure $x = K^{-1}\otimes\xi^2$. Then $SW(K^{-1}\otimes\xi^2) = Gr(\xi)$. Consider a line bundle ξ such that $SW(K^{-1}\otimes\xi^2) \neq 0$, then the Poincaré dual of $c_1(\xi)$ is represented by the fundamental class of an embedded, holomorphic submanifold with, say, n irreducible components. Then each component H_i satisfies the adjunction formula $g(H_i) = 1 + H_i \cdot H_i$, where $g(H_i)$ is the genus of H_i. We can define the integer multiplicities a_i by writing $\xi = \sum_{i=1}^n a_i H_i$.

Let the canonical divisor K be given by a union of irreducible components C_i with multiplicities r_i, i.e $K = \sum_i r_i C_i$. The factorization of a Seiberg-Witten basic class x means that

$$K^{1/2} \otimes x^{1/2} = \xi = \sum_i s_i C_i \tag{6.7}$$

where the s_i are integers with $0 \leq s_i \leq r_i$. Consequently, Taubes' result leads to the identifications

$$a_i = s_i, \qquad C_i = H_i. \tag{6.8}$$

Physically speaking this means that the world sheets of the superconducting cosmic strings discussed by Witten in [12] are embedded holomorphic curves.

Recall that for a fixed instanton number k the Seiberg-Witten classes $x = -K + 2x'$, with $x' \cdot x' = -4k$ and $z = [x']$, contribute to the partition function of $N = 4$ theory in branch (ii). From the above discussion we identify x' with holomorphic curve $x' = \sum_{i=1}^n s_i H_i$ and $1 - g(x') = -x' \cdot x'$. So the branch (ii) contribution can be written as the sum of contributions of all holomorphic curves Σ with $[\Sigma] = z$. The summation over the (space-time) instanton numbers is replaced with the summation over the genus of the holomorphic curves (the world-sheet instantons). So our formula (6.1) for the partition function of $N = 4$ theory can also be viewed as a genus expansion:

$$\begin{aligned} Z_z = {} & (-1)^\nu \left(\frac{G(q^2)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_\Sigma \delta_{z,[\Sigma]} Gr(\Sigma) \left(\frac{\theta_1}{\theta_0}\right)^{1-g(\Sigma)} \\ & + 2^{1-b_1} \left(\frac{G\left(q^{1/2}\right)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0+\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_\Sigma (-1)^{[\Sigma]\cdot z} Gr(\Sigma) \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{1-g(\Sigma)} \\ & + 2^{1-b_1} i^{-z^2} \left(\frac{G\left(-q^{1/2}\right)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0-i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_\Sigma (-1)^{[\Sigma]\cdot z} Gr(\Sigma) \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{1-g(\Sigma)} . \end{aligned} \tag{6.9}$$

6.2. The $N = 2$ Limit and Donaldson-Witten Invariants

It is instructive to rewrite the formula (6.1) as follows:

$$
\begin{aligned}
Z_z = & (-1)^\nu \left(\frac{G(q^2)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x \delta_{z,[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} \\
& + 2^{1-b_1+\frac{1}{4}(7\chi+11\sigma)} G\left(q^{1/2}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0+\theta_1}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{-x'\cdot x'} \\
& + 2^{1-b_1+\frac{1}{4}(7\chi+11\sigma)} i^{-z^2} G\left(-q^{1/2}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0-i\theta_1}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-x'\cdot x'}
\end{aligned}
\tag{6.10}
$$

The fact that this formula can naturally be grouped into three terms whereas we classically think of contributions from two branches can be understood physically as follows. The first term is the contribution from branch (ii) and stems from the singularity in the u-plane due to the massless adjoint hypermultiplet. The remaining two terms are the contribution from branch (i), which classically corresponds to the singularity at the origin of the u-plane. Geometrically, the branch (i) contribution is the Euler characteristic of the moduli space of instantons [1]. The fact that this contribution is made up from two terms is due to a quantum effect: the classical singularity at the origin of the u-plane bifurcates quantum mechanically [16].

From the above formula we can recover the Donaldson invariants for gauge groups $SU(2)$ and $SO(3)$ as follows. For a simply connected manifold of simple type, Witten's formula for the generating functional of Donaldson's invariants is [11]

$$
\begin{aligned}
\left\langle e^{\hat{v}+\lambda u}\right\rangle_z = & 2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp(v^2/2+2\lambda) \sum_x (-1)^{[x']\cdot z} n_x e^{v\cdot x} \\
& + 2^{1+\frac{1}{4}(7\chi+11\sigma)} i^{\nu-z^2} \exp(-v^2/2-2\lambda) \sum_x (-1)^{[x']\cdot z} n_x e^{-iv\cdot x}.
\end{aligned}
\tag{6.11}
$$

Here $\hat{v}$ is the observable $\mathcal{O}_2$ (2.12) associated with a two-dimensional cycle v and $u = \mathcal{O}_4$. The expectation value is computed using (twisted) $N = 2$ super-Yang-Mills theory.

To obtain the above formula from the $N = 4$ theory we could turn on the observables (2.12) after breaking the supersymmetry down to $N = 2$ and, following [16][17], take the double scaling limit $m \to \infty$ and $q \to 0$ with $\Lambda^2 = 2q^{1/2}m^2$ being fixed. In this limit the singularity coming from the massless adjoint hypermultiplet (branch (ii)) moves to infinity in the u-plane and no longer contributes to the path integral. On the other hand the two other singularities remain at the points $u = \pm\Lambda^2$ (in Donaldson theory Λ^2 is normalized to 2). Here we are not able to consider general expectation values of observables. However, we can compute the $N=2$ limit of the partition function (6.10) (the $q\to 0$ limit since (6.10) does not depend on m). The leading terms only come from the second and the third lines and are given by

$$2^{1-b_1+\frac{1}{4}(7\chi+11\sigma)}\left(\sum_x(-1)^{[x']\cdot z}n_x+i^{\nu-z^2}\sum_x(-1)^{[x']\cdot z}n_x\right)\left(q^{3\nu/4}+\ldots\right) \tag{6.12}$$

Note that this partition function vanishes unless the dimension of the moduli space of instantons is zero. Since $dim_{\mathbb{C}}\mathcal{M}_k=4k-3\nu$, this occurs when $k=3\nu/4$, thus explaining the leading term $q^{3\nu/4}$ in (6.12). In fact, the expression (6.12) contains all the non-trivial information about Donaldson's invariants.

Recently an important contribution to related issues appeared in [18]. It would be fruitful to apply the physical methods of [18] to the problems addressed here. In that way one could compute the entire generating functional of the $N=2$ theory with a massive adjoint hypermultiplet, and work on more general four-manifolds.

The relation between the S^1 action and the mass term of the hypermultiplet described in this paper were summarized and used by one of us (JSP) in the paper [19]. There the same sequence of perturbations $N=4,2,1$ was used to relate the zero-dimensional reduction of the Vafa-Witten equation ($N=4$) to the ADHM description of instantons ($N=2$) and of torsion-free-sheaves ($N=1$). Subsequently the same S^1-action and its application to the mass perturbation of the $N=4$ theory was also considered in [20].

7. Comments on Sigma-Model Approach

In this paper we formulated a twisted $N=4$ SYM theory on a Kähler manifold as a topological gauge theory based on balanced equivariant Dolbeault cohomology. It turns out that the algebra of balanced Dolbeault cohomology is isomorphic to the algebra of physical $N_{ws}=(2,2)$ supersymmetric gauge theory in two-dimensions. The internal symmetry of our model includes $SO(1,1)$ such that the indicies $\pm$ in $\mathbf{s}_\pm$ and $\bar{\mathbf{s}}_\pm$ can be identified with the $SO(1,1)$ spinor indecies. Then we may have a natural extension of the model to live in a product manifold $X\times\Sigma$, where the $SO(1,1)$ acts on the two-dimensional surface Σ as the Lorentz symmetry. The two equivariant differentials $\mathbf{s}_+$ and $\bar{\mathbf{s}}_+$ can be identified with the two left-moving supercharges on Σ. The other two differentials $\mathbf{s}_-$ and $\bar{\mathbf{s}}_-$ are identified with the two right-moving supercharges on Σ. With the above extension $\phi_{\pm\pm}$ correspond to the components of two-dimensional vector in the light coordinate (or in the complex coordinate).

Now the commutations relations (3.5) are modified as

$$
\begin{array}{llll}
\{\mathbf{s}_+, \bar{\mathbf{s}}_+\} = -\partial_{++} + \delta_{\phi_{++}}, & \{\mathbf{s}_+, \bar{\mathbf{s}}_-\} = \delta_{\sigma}, & \{\mathbf{s}_+, \mathbf{s}_-\} = 0, & \mathbf{s}_\pm^2 = 0, \\
\{\mathbf{s}_-, \bar{\mathbf{s}}_-\} = -\partial_{--} + \delta_{\phi_{--}}, & \{\mathbf{s}_-, \bar{\mathbf{s}}_+\} = \delta_{\bar{\sigma}}, & \{\bar{\mathbf{s}}_+, \bar{\mathbf{s}}_-\} = 0. & \bar{\mathbf{s}}_\pm^2 = 0.
\end{array}
\tag{7.1}
$$

Consequently we may intepret the $N_{ws} = (2,2)$ supercharges in two-dimensions as the differentials of the balanced $P_\Sigma \times \mathcal{G}$-equivariant Dolbeault cohomology, where P_Σ denotes the group of translation along Σ. The supersymmetry transformation laws are obtained from those in Appendix A by replacing $\phi_{\pm\pm}$ with the two-dimensional covariant derivatives $D_{\pm\pm}$. Now we identify $\bar{\epsilon}_-$ and ϵ_- with sections of $K_\Sigma^{-1/2}$ and $\bar{\epsilon}_+$ and ϵ_+ with sections of $\bar{K}_\Sigma^{-1/2}$, where K_Σ denotes the canonical line bundle on Σ. Those generators are scalars on X. The balanced structure is simply the classical chiral symmetry. The $U(1)_\mathcal{R}$ symmetry in Sect. 3.1 and 3.2, with the left and right charges in (3.3) and Tables 1 and 2, becomes the $\mathcal{R}$ symmetry of the $N_{ws} = (2,2)$ theory. The two holomorphic quartets in (3.10) correspond to two $N_{ws} = (2,2)$ chiral matter multiplets. The consistent nine-plet in (3.13) corresponds the $N_{ws} = (2,2)$ vector multiplet.

The action functional is defined as in (3.15) with a slight modification as follows

$$
S = \mathbf{s}_+\bar{\mathbf{s}}_+\mathbf{s}_-\bar{\mathbf{s}}_- \int_\Sigma d\mu \mathcal{F} + \mathbf{s}_+\mathbf{s}_- \int_\Sigma d\mu \mathcal{W} + \bar{\mathbf{s}}_+\bar{\mathbf{s}}_- \int_\Sigma d\mu \overline{\mathcal{W}}, \tag{7.2}
$$

where $d\mu$ is the two-dimensional volume-form. One may identify $\mathcal{W}$ with chiral superpotential. Maintaining the $U(1)$ $\mathcal{R}$ symmetry we might consider more general action functional

$$
S(t,\bar{t}) = S + t\bar{\mathbf{s}}_+\mathbf{s}_- \int_{X\times\Sigma} d\tilde{\mu} \mathrm{Tr}\, \sigma + \bar{t}\mathbf{s}_+\bar{\mathbf{s}}_- \int_{X\times\Sigma} d\tilde{\mu} \mathrm{Tr}\, \bar{\sigma} \tag{7.3}
$$

with $t = \frac{\theta}{2\pi} + ir$. The additional term corresponds to the Fayet-Iluopolos term (if we have a $U(1)$ factor in the gauge group).

The resulting theory may be viewed as an infinite dimensional $\mathcal{N}_{ws} = (2,2)$ supersymmetric gauged linear-sigma model [21][22]. Since our model restricted to X is a topological theory the only scale dependence is from two dimensions. Picking the volume of X being small and taking the infrared limit $vol(\Sigma) \to \infty$ in two-dimensions our model flows to a non-linear sigma-model which target space is the Vafa-Witten moduli space on X. Thus we may practically regard our extended model a two-dimensional $N_{ws} = (2,2)$ theory.

We may regard the sigma-model viewpoints as a unifying framework for the four-dimensional theory. By considering topological sigma models [23], we have string theoretic generalization of Donaldson-Witten and Vafa-Witten theories on a Kähler surface. The Donaldson-Witten theory would be viewed as an low-energy effective space-time theory for the twisted version, i.e., A-model, of our sigma model. The Vafa-Witten theory we studied in this paper would be viewed as an effective space-time theory which computes the Witten index of world-sheet supersymmetry. We may also embedd the Vafa-Witten theory into the half-twisted version of our model. The partition function of the half-twisted model is the elliptic genus which is the index of $\bar{\mathbf{s}}_+$.[2] At present we do not know if further stringy corrections involve new differential-topological information on a four-manifolds beyond the Seiberg-Witten invariants.

The sigma model approach for Käher manifolds with $b_2^+(X) = 1$ with $b_2(X) < 10$ and $b_2^+(X) = 3$ would be very interesting. In the both cases we have an important vanishing theorem that the Vafa-Witten equation reduces to the equations for Yang-Mills instantons [1]. Then our model in the infrared limit flows to supersymmetric non-linear sigma-model where target space is the moduli space of instantons on X. For X is a $K3$ surface ($b_2^+(X) = 3$) our model is closely related to the Higgs branch of the $D_1 - D_5$ system where D_5 branes wrapp around X. We expect that our model flows to $N_{ws} = (4,4)$ superconformal theory in the infrared limit. According to a celebrated conjecture of Maldacena the superconformal theory is dual to type IIB superstrings on $AdS_3 \times S^3 \times K3$ [24].

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[2] Note that the constructive definition of Vafa-Witten theory involves only one supersymmetry, say $\bar{\mathbf{s}}_+$. We may identify the partition function of Vafa-Witten theory as the degeneration of the index of two-dimensional supercharge $\bar{\mathbf{s}}_+$ to the contributions of the constant modes in two-dimensions.

Appendix A. The $N=4$ Algebra

The commutation relations (3.5) for the four supercharges $\mathbf{s}_\pm$ and $\bar{\mathbf{s}}_\pm$ together with the conditions in (3.9) determine the entire transformation laws. In terms of infinitesimal fermionic parameters $\epsilon_-, \bar\epsilon_-$ and $\epsilon_+, \bar\epsilon_+$, we denote $\delta = \bar\epsilon_-\mathbf{s}_+ + \bar\epsilon_+\mathbf{s}_- + \epsilon_+\bar{\mathbf{s}}_- + \epsilon_+\bar{\mathbf{s}}_+$.

We start from the consistency nine-plet (3.13). We have

$$\begin{aligned} \delta\phi_{++} &= \bar\epsilon_+\eta_+ + \epsilon_+\bar\eta_+, &\qquad \delta\sigma &= -\bar\epsilon_+\eta_- - \epsilon_-\bar\eta_+, \\ \delta\phi_{--} &= \bar\epsilon_-\eta_- + \epsilon_-\bar\eta_-, &\qquad \delta\bar\sigma &= -\bar\epsilon_-\eta_+ - \epsilon_+\bar\eta_-, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \delta\eta_+ &= +i\epsilon_+H_0 + \frac{1}{2}\epsilon_+[\sigma,\bar\sigma] + \frac{1}{2}\epsilon_+[\phi_{++},\phi_{--}] + \epsilon_-[\phi_{++},\bar\sigma], \\ \delta\bar\eta_+ &= -i\bar\epsilon_+H_0 - \frac{1}{2}\bar\epsilon_+[\sigma,\bar\sigma] + \frac{1}{2}\bar\epsilon_+[\phi_{++},\phi_{--}] + \bar\epsilon_-[\phi_{++},\sigma], \\ \delta\eta_- &= +i\epsilon_-H_0 - \frac{1}{2}\epsilon_-[\sigma,\bar\sigma] - \frac{1}{2}\epsilon_-[\phi_{++},\phi_{--}] + \epsilon_+[\phi_{--},\sigma], \\ \delta\bar\eta_- &= -i\bar\epsilon_-H_0 + \frac{1}{2}\bar\epsilon_-[\sigma,\bar\sigma] - \frac{1}{2}\bar\epsilon_-[\phi_{++},\phi_{--}] + \bar\epsilon_+[\phi_{--},\bar\sigma], \\ \delta H_0 &= +\frac{i}{2}\bar\epsilon_-[\phi_{++},\eta_-] + \frac{i}{2}\bar\epsilon_-[\sigma,\eta_+] + \frac{i}{2}\bar\epsilon_+[\phi_{--},\eta_+] + \frac{i}{2}\bar\epsilon_+[\bar\sigma,\eta_-] \\ &\quad - \frac{i}{2}\epsilon_-[\phi_{++},\bar\eta_-] - \frac{i}{2}\epsilon_-[\bar\sigma,\bar\eta_+] - \frac{i}{2}\epsilon_+[\phi_{--},\bar\eta_+] - \frac{i}{2}\epsilon_+[\sigma,\bar\eta_-]. \end{aligned} \tag{A.2}$$

For the holomorphic and anti-holomorphic quartets from $B^{2,0}$ and $B^{0,2}$, resprectively, we have

$$\begin{aligned} \delta B^{2,0} &= \bar\epsilon_-\chi_+^{2,0} - \bar\epsilon_+\chi_-^{2,0}, \\ \delta B^{0,2} &= \epsilon_-\bar\chi_+^{0,2} - \epsilon_+\bar\chi_-^{0,2}, \\ \delta\chi_+^{2,0} &= \bar\epsilon_+H^{2,0} - \epsilon_+[\sigma,B^{2,0}] - \epsilon_-[\phi_{++},B^{2,0}], \\ \delta\chi_-^{2,0} &= \bar\epsilon_-H^{2,0} + \epsilon_-[\bar\sigma,B^{2,0}] + \epsilon_+[\phi_{--},B^{2,0}], \\ \delta\bar\chi_+^{0,2} &= \epsilon_+H^{0,2} - \bar\epsilon_+[\bar\sigma,B^{0,2}] - \bar\epsilon_-[\phi_{++},B^{0,2}], \\ \delta\bar\chi_-^{0,2} &= \epsilon_-H^{0,2} + \bar\epsilon_-[\sigma,B^{0,2}] + \bar\epsilon_+[\phi_{--},B^{0,2}], \\ \delta H^{2,0} &= -\epsilon_-[\phi_{++},\chi_-^{2,0}] + \epsilon_-[\eta_+,B^{2,0}] - \epsilon_-[\bar\sigma,\chi_+^{2,0}] \\ &\quad - \epsilon_+[\phi_{--},\chi_+^{2,0}] - \epsilon_+[\eta_-,B^{2,0}] - \epsilon_+[\sigma,\chi_-^{2,0}], \\ \delta H^{0,2} &= -\bar\epsilon_-[\phi_{++},\bar\chi_-^{0,2}] + \bar\epsilon_-[\bar\eta_+,B^{0,2}] - \bar\epsilon_-[\sigma,\bar\chi_+^{0,2}] \\ &\quad - \bar\epsilon_+[\phi_{--},\bar\chi_+^{0,2}] - \bar\epsilon_+[\bar\eta_-,B^{0,2}] - \bar\epsilon_+[\bar\sigma,\bar\chi_-^{0,2}]. \end{aligned} \tag{A.3}$$

For the holomorphic and anti-holomorphic quartets from the connection one-form A

we have

$$
\begin{aligned}
\delta A^{1,0} =&\epsilon_+ \bar{\psi}_-^{1,0} + \epsilon_- \bar{\psi}_+^{1,0},\\
\delta A^{0,1} =&\bar{\epsilon}_+ \psi_-^{0,1} + \bar{\epsilon}_- \psi_+^{0,1},\\
\delta \bar{\psi}_+^{1,0} =& -\epsilon_+ H^{1,0} + \bar{\epsilon}_+ \partial_A \bar{\sigma} + \bar{\epsilon}_- \partial_A \phi_{++},\\
\delta \bar{\psi}_-^{1,0} =& +\epsilon_- H^{1,0} + \bar{\epsilon}_- \partial_A \sigma + \bar{\epsilon}_+ \partial_A \phi_{--},\\
\delta \psi_+^{0,1} =& -\bar{\epsilon}_+ H^{0,1} + \epsilon_+ \bar{\partial}_A \sigma + \epsilon_- \bar{\partial}_A \phi_{++},\\
\delta \psi_-^{0,1} =& +\bar{\epsilon}_- H^{0,1} + \epsilon_- \bar{\partial}_A \bar{\sigma} + \epsilon_+ \bar{\partial}_A \phi_{--},\\
\delta H^{1,0} =& -\bar{\epsilon}_- [\phi_{++}, \bar{\psi}_-^{1,0}] + \bar{\epsilon}_- [\sigma, \bar{\psi}_+^{1,0}] - \bar{\epsilon}_- \partial_A \bar{\eta}_+\\
&+ \bar{\epsilon}_+ [\phi_{--}, \bar{\psi}_+^{1,0}] - \bar{\epsilon}_+ [\bar{\sigma}, \bar{\psi}_-^{1,0}] + \bar{\epsilon}_+ \partial_A \bar{\eta}_-\\
\delta H^{0,1} =& -\epsilon_- [\phi_{++}, \psi_-^{0,1}] + \epsilon_- [\bar{\sigma}, \psi_+^{0,1}] - \epsilon_- \bar{\partial}_A \eta_+\\
&+ \epsilon_+ [\phi_{--}, \psi_+^{0,1}] - \epsilon_+ [\sigma, \psi_-^{0,1}] + \epsilon_+ \bar{\partial}_A \eta_-.
\end{aligned} \tag{A.4}
$$

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