Pair approximations for spatial structures?

Denis Mollison

This work explores the success of pair approximations in capturing local correlations and the spatial structure of population contact networks, especially in respect of the rate of spread of epidemics.

Networks of interest range from the local extreme where interactions are only between nearest neighbours in some low dimensional space, and the infinite-dimensional 'mean-field' extreme where all interact equally with all [?, ?, ?, ?]. Intermediate cases of practical interest include 'small-world' and meta-population models [?, ?, ?].

One of the obvious distinctions between homogeneous mixing and spatial population structures lies in their local correlations: if 'AB' means 'A is a neighbour of B', then $P(AC|AB, BC) \gg P(AC)$ for the spatial case.

Pair approximation differential equations (PAs), that add second order variables such as [SI], the mean number of (S,I) pairs of neighbours to a standard SIR differential equation model [?, ?], have recently been widely used to approximate spatial ecological and epidemic processes [?, ?]. How well do they do this?

There are theoretical reasons why PAs should be better at approximating mean-field than spatial networks. Figure 1 shows how PAs provide excellent approximations to mean-field SIRs for a wide range of the correlation parameter ϕ. In

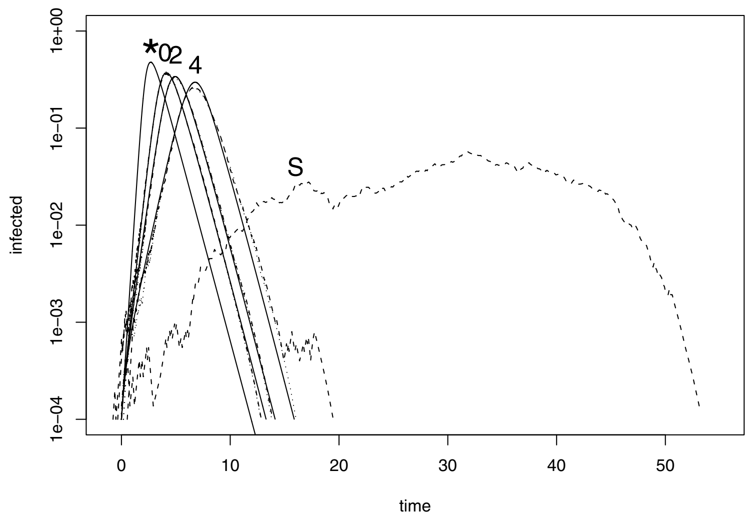

Figure 1. Comparison between stochastic SIRs on simple random graphs, constrained to have varying correlation parameter ϕ ($= 0, 0.2, 0.4$; 2 simulations each, dashed curves), and PAs of the same ϕ (solid curves). Also shown are the standard SIR DE ('$\star$') and a simulation of a spatial stochastic SIR ('S') – see Figure 2.

particular, the duration of the epidemic is of order $\log(N)$, where N is the population size.

Now for a spatial SIR with local contacts – Figure 2 shows a nearest-neighbour SIR on a sphere – the duration is of order $\sqrt{N}$, so the PA cannot be expected to

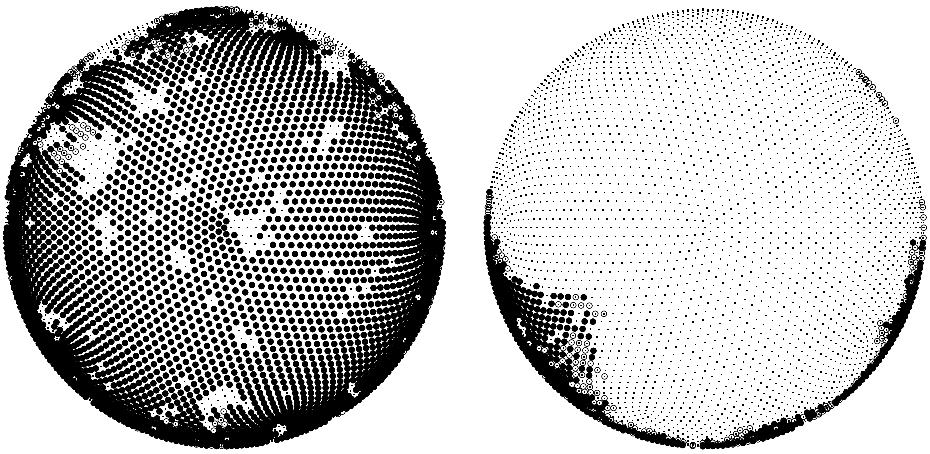

FIGURE 2. Simulation of a nearest-neighbour SIR on a hexagonal$^{(\star)}$ lattice on a sphere: $\cdot$ susceptible, $\odot$ infectious, $\bullet$ removed. This outbreak started at the north pole (left), and has just reached the southern hemisphere (right).
[($\star$) Note: An exact hexagonal lattice on a sphere is not possible; here there are 12 sites that each have only 5 neighbours.]

approximate this well, as is confirmed by the time plot for the spatial SIR (curve 'S' in Figure 1), which is very different from the PA with the same value of the correlation parameter ($\phi = 0.4$).

There may seem to be a paradox here, in that the spatial network is an element of the set of random graphs $G(N, \phi)$ that have the same number of sites and the same value of the correlation parameter, although members of that set can generally be assumed to be mean-field in character. The resolution of the paradox is that, within $G(N, \phi)$, such spatial or near-spatial networks are of almost infinite improbability, what we might call 'Adams-improbable' [**?**].

More broadly, this work in progress tends to support the generalisation that spatial processes need explicit spatial modelling [**?**, **?**].

REFERENCES

[1] Adams, D (1979) *The Hitchhiker's Guide to the Galaxy*, Pan Books.
[2] Ball, FG, Mollison, D and Scalia-Tomba, G-P (1997) 'Epidemics in populations with two levels of mixing', *Ann. Appl. Prob.* **7**, 46-89.

[3] Bolker, B (ed.) (1995) *Spatial dynamics group report*, in *Ecology of Infectious Diseases in Natural Populations* (eds. BT Grenfell and AP Dobson), Cambridge UP, 399-420.

[4] Barbour, A and Mollison, D (1990) 'Epidemics and random graphs', in *Stochastic Processes in Epidemic Theory* (eds. J-P Gabriel, C Lefèvre and P Picard), Springer-Verlag, 86-89.

[5] Durrett, R and Levin, SA (1994) 'The importance of being discrete (and spatial)', *Theor Pop Biol* **46**, 363-394.

[6] Ferguson, N, Donnelly, C, and Anderson, R (2001) 'The Foot-and-Mouth epidemic in Great Britain: pattern of spread and impact of interventions', *Science* **292**, 1155-1160.

[7] Keeling, MJ (1999) 'The effects of local spatial structure on epidemiological invasions', *Proc R Soc Lond* **B 266**, 859-867.

[8] Mollison, D (1977) 'Spatial contact models for ecological and epidemic spread' (with Discussion), *J Roy Statist Soc* **B 39**, 283-326.

[9] Mollison, D and Levin, SA (1995) 'Spatial dynamics of parasitism', in *Ecology of Infectious Diseases in Natural Populations* (eds. BT Grenfell and AP Dobson), Cambridge UP, 384-398.

[10] Morris, AJ (1997) *Representing spatial interactions in simple epidemiological models*, PhD Thesis, Warwick University.

[11] Newman, MEJ (2000) 'Models of the Small World: A Review', http://www.arxiv.org/abs/cond-mat/0001118.

[12] Rand, DA (1999) 'Correlation equations and pair approximations for spatial ecologies', in *Advanced Ecological Theory* (ed. Jacqueline McGlade), 100-142.

[13] Watts, DJ, and Strogatz, SH (1998) 'Collective dynamics of 'small-world' networks', *Nature* **393**, 440-442.