The $3D$ Plaquette Ising Model

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Radboud Uni 2020

Plan of talk

Phase transitions (on a computer)

String Theory (on a computer)

3D Plaquette Ising (Gonihedric) Model - a cautionary tale

Gauging/Subsystem symmetry/Fractons

Phase Transitions (on a computer)

First and Second Order Transitions

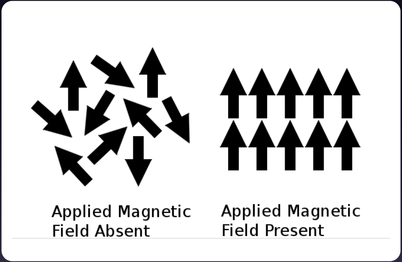
First-order phase transitions are those that involve a latent heat.

Second-order transitions are also called continuous phase transitions. They are characterized by a divergent susceptibility, an infinite correlation length, and a power-law decay of correlations near criticality.

Transitions - Examples

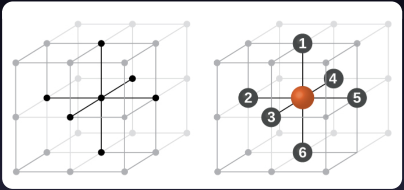
First order - melting

Second order - Curie



Transitions - on/in a computer

Spins interact with nearest neighbours



Low-T, like to align - ordered phase

High-T, disordered phase

Transitions - on a computer

Hamiltonian q-state Potts, $\sigma = 1 \dots q$

$$\mathcal{H}_q = -\sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}$$

Evaluate a partition function, $\beta = 1/k_b T$

$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta \mathcal{H}_q)$$

Derivatives of free energy give observables (energy, magnetization..)

$$F(\beta) = \ln Z(\beta)$$

Measure 1001 Different Observables

Order parameter

$$M = (q \max\{n_i\} - N)/(q-1)$$

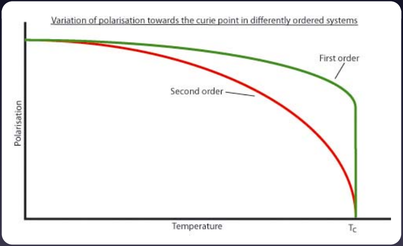
Per-site quantities denoted by $e = E/N$ and $m = M/N$

$$\begin{aligned} u(\beta) &= \langle E \rangle / N, \\ C(\beta) &= \beta^2 N[\langle e^2 \rangle - \langle e \rangle^2]. \end{aligned}$$

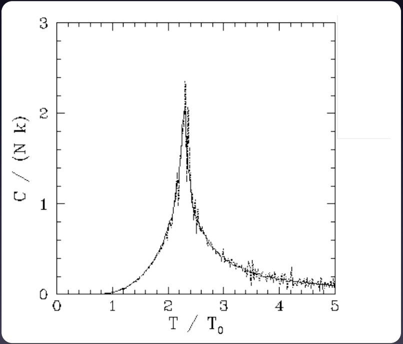
$$\begin{aligned} m(\beta) &= \langle |m| \rangle, \\ \chi(\beta) &= \beta N[\langle m^2 \rangle - \langle |m| \rangle^2] \end{aligned}$$

First and Second Order Transitions - Characteristics

First order - discontinuities in magnetization, energy (latent heat)



Second order - divergences in specific heat, susceptibility



Continuous Transitions - Critical exponents

(Continuous) Phase transitions characterized by critical exponents

Define $t = |T - T_c|/T_c$

Then in general, $\xi \sim t^{-\nu}$, $M \sim t^{\beta}$, $C \sim t^{-\alpha}$, $\chi \sim t^{-\gamma}$

Can be rephrased in terms of the linear size of a system L

$\xi \sim L$, $M \sim L^{-\beta/\nu}$, $C \sim L^{\alpha/\nu}$, $\chi \sim L^{\gamma/\nu}$

2*nd* Order Transitions - Continuum Limits

At a second order transition, correlation length diverges, lattice “washes out”

Define a continuum limit at this point

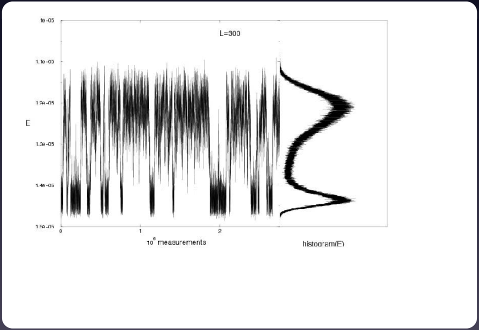
e.g. $2D$ Ising described by CFT at transition point (ditto $3, 4$ state Potts)

Use a suitable discretized model with a continuous transition to define the theory we are interested in by taking such a continuum limit

1st Order FSS: Heuristic two-phase model

A fraction $W_{\rm o}$ in q ordered phase(s), energy $e_{\rm o}$

A fraction $W_{\rm d} = 1 - W_{\rm o}$ in disordered phase, energy $e_{\rm d}$



1st Order FSS: Energy moments

Energy moments become

$$\langle e^n \rangle = W_\mathrm{o} e_\mathrm{o}^n + (1 - W_\mathrm{o}) e_\mathrm{d}^n$$

And the specific heat then reads:

$$C_V(\beta, L) = L^d \beta^2 \left(\left\langle e^2 \right\rangle - \langle e \rangle^2 \right) = L^d \beta^2 W_\mathrm{o} (1 - W_\mathrm{o}) \Delta e^2$$

Max of $C_V^{\mathrm{max}} = L^d (\beta^\infty \Delta e / 2)^2$ at $W_\mathrm{o} = W_\mathrm{d} = 0.5$

Volume scaling

1st Order FSS: Specific Heat peak shift

Probability of being in any of the states

$$W_o \sim q \, \exp(-\beta L^d f_o)\,, \; W_d \sim \exp(-\beta L^d f_d)$$

Take logs, expand around β^∞

$$\begin{aligned} \ln(W_o/W_d) &= \ln q + \beta L^d (f_d - f_o) \\ &= \ln q + L^d \Delta e(\beta - \beta^\infty) \end{aligned}$$

Solve for specific heat peak $W_o = W_d$, $\ln(W_o/W_d) = 0$

$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta e} + \ldots$$

1st Order FSS: summary

Peaks grow as L^d (volume)

Transition point estimates shift as $1/L^d$ (1/volume)

Should be true for all first order PTs

String Theory (on a computer)

String worldsheets - Random Surfaces

Particle action - proper length

$$S \sim Length \sim \int d\tau$$

String action - proper area

$$S \sim Area \sim \int dA = \int d\sigma d\tau \sqrt{g}$$

Polyakov action

$$S \sim \int d^2\sigma \sqrt{\det g}\, g^{ab}\, \partial_a X_\mu \partial_b X^\mu$$

String worldsheets - Random Surfaces

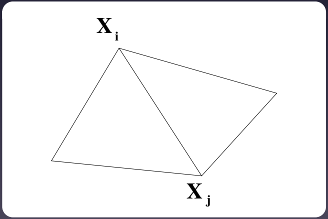
Game is to calculate a partition function

$$Z = \int DgDX \exp(-S_E)$$

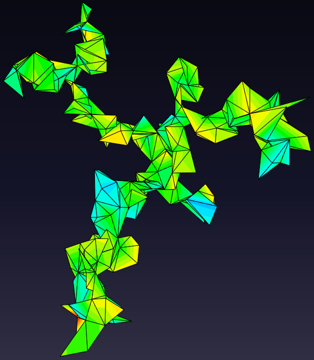
Triangulated Surfaces - String Theory on a Computer

Discretize worldsheet with triangles - sum over metrics become sum over triangulations

$$S \sim \sum_{ij} (X^{\mu}(i) - X^{\mu}(j))^2$$



Typical Surfaces

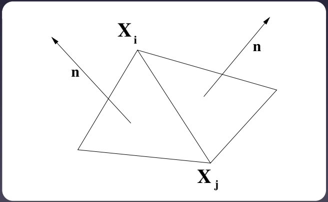


Collapsed, branch-polymer like

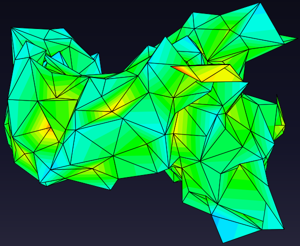
Modifying the Gaussian Action

- Add extrinsic curvature term

$$
\begin{aligned}
S &= \sum_{ij} (X^{\mu}(i) - X^{\mu}(j))^2 \\
&+ \lambda \sum_{\Delta_i, \Delta_j} (1 - \vec{n}_i \cdot \vec{n}_j)
\end{aligned}
$$



Smoothed Surfaces



Go hunting for continuum limit at a (continuous) transition between phases

Bad news - doesn't seem to work

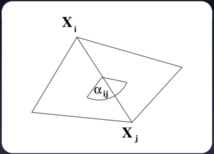
3D Plaquette Ising

The Gonihedric action

$$S = \sum_{ij} |X^{\mu}(i) - X^{\mu}(j)|\, \theta_{ij}, \qquad \theta_{ij} = ||\pi - \alpha_{ij}||$$

Gonia: angle

Hedra: face



Go hunting for continuum limit at a (continuous) transition between phases

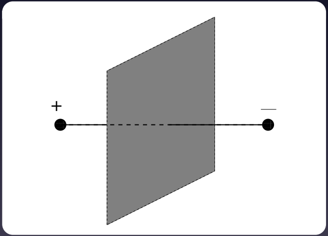
Bad news - doesn't seem to work

Spins Cluster Boundaries as Surface Models

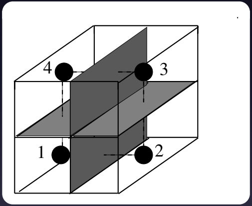
Spin cluster boundaries $\leftrightarrow$ surfaces

Edge spins: $U_{ij} = -1$

Vertex spins: $\sigma_i \sigma_j = -1$



Counting configurations with spins (areas and intersections)



Ising/Surface correspondence

Allow energy from areas, edges and intersections (*A. Cappi, P Colangelo, G. Gonella and A. Maritan*)

$$\mathcal{H} = \sum(\beta_A n_A + \beta_C n_C + \beta_I n_I)$$

$$\beta_A = 2J_1 + 8J_2, \quad \beta_C = 2J_3 - 2J_2, \quad \beta_I = -4J_2 - 4J_3$$

$$\mathcal{H} \;=\; J_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \sigma_i \sigma_j + J_3 \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

Gonihedric $\rightarrow$ Tune Out Area Term

One parameter family of "Gonihedric" Ising models (Savvidy, Wegner)

$$\mathcal{H}^{\kappa} = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

$\kappa = 0$ pure plaquette

$$\mathcal{H} \ = \ - \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

Plaquette Ising/Gonihedric model

Hamiltonian

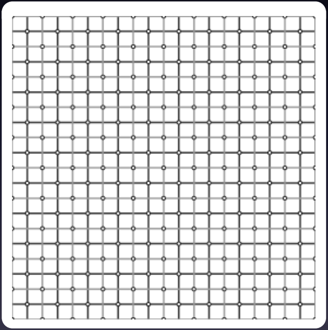
$$\mathcal{H} = -\sum_{\Box} \sigma_i \sigma_j \sigma_k \sigma_l$$

Spins at vertices of 3D cubic lattice

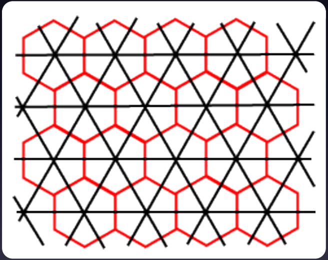
Strong first order phase transition (so no use for continuum limits)

Duality - geometric

Square lattice - self dual



Triangle - hexagon dual



Duality - spin models

Spins on orginal lattice $\leftrightarrow$ spins on dual lattice

High-T $\leftrightarrow$ Low-T

$$\tanh \beta = e^{-2\beta_{dual}}$$

Ising, square lattice

$$\begin{aligned} Z(\beta) &\simeq (1 + N \tanh(\beta)^4 + \ldots) \\ Z(\beta_{dual}) &\simeq (1 + N \exp(-2\beta_{dual})^4 + \ldots) \end{aligned}$$

Duality - plaquette spin model

Plaquette Ising model

$$\mathcal{H} \;=\; -\sum_{\square} \sigma_i \sigma_j \sigma_k \sigma_l$$

Dual to this

$$\mathcal{H}_{dual} = -\sum_{\langle ij \rangle_x} \sigma_i \sigma_j - \sum_{\langle ij \rangle_y} \tau_i \tau_j - \sum_{\langle ij \rangle_z} \sigma_i \sigma_j \tau_i \tau_j \,,$$

Exercise for a starting PhD student (Marco Mueller)

Simulate 3d plaquette model and dual

Do a better job than, ahem, before at determining transition point of both

Obtain consistent estimates of transition point

A Problem

Determine critical point(s) $L = 8 \ldots 27$, periodic bc, $1/L^3$ fits

Original model:

$$\beta^\infty = 0.549994(30)$$

Dual model:

$$\beta^\infty_{dual} = 1.31029(19)$$

Translate back with $\tanh \beta = e^{-2\beta_{dual}}$ giving

$$\beta^\infty = 0.55317(11)$$

Estimates are about 30 error bars apart

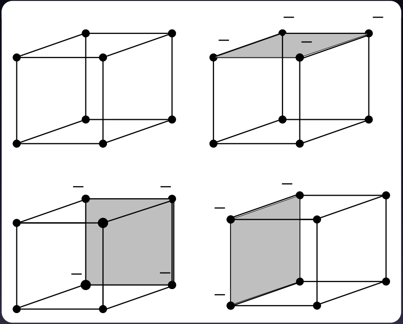
(Non) Solutions

Blame the student (yours truly....)

Blame the student (yours truly....)

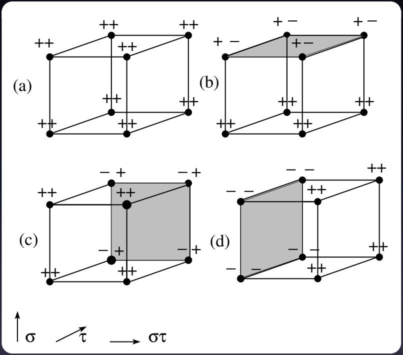
Think - What is special about plaquette model?

Groundstates: Plaquette



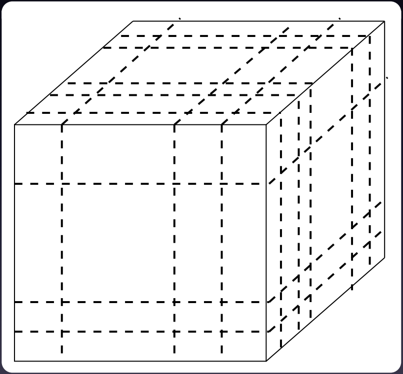
Persists into low temperature phase: degeneracy 2^{3L}

Groundstates: Dual



Dual degeneracy

Typical Ground state - subsystem symmetry



(subextensive) exponential degeneracy $\sim 2^{3L}$

1st Order FSS with Exponential Degeneracy

Normally q is constant

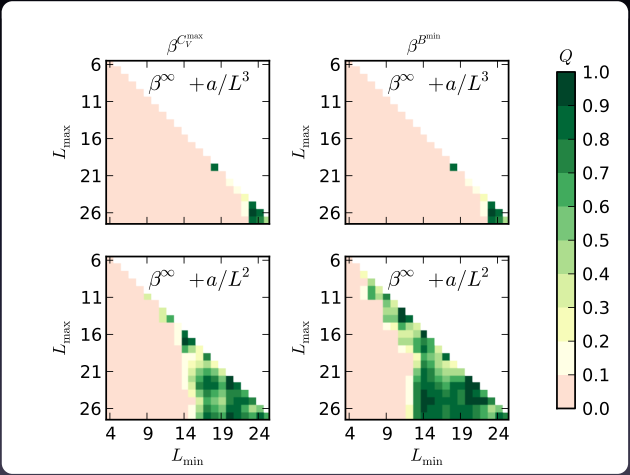
If $q \propto e^{L}$ ($q = e^{(3 \ln 2) L}$), as in Gonihedric model

$$\beta^{C_V^{\max}}(L) = \beta^{\infty} - \frac{\ln q}{L^d \Delta e} + \dots$$

becomes

$$\beta^{C_V^{\max}}(L) = \beta^{\infty} - \frac{3 \ln 2}{L^{d-1} \Delta e} + \dots$$

Scaling L^2 not L^3



Standard $1/L^3$ gives much poorer quality

Gauging/Subsystem Symmetry/Fractons

Gauging: From here to there:

Here - Quantum Transverse Ising:

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

Gauge the global $\mathbb{Z}_2$ symmetry

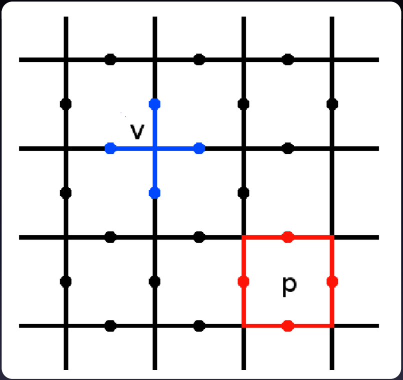
$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \sigma_i^z \tau_{ij}^z \sigma_j^z - h \sum_i \sigma_i^x - J_p \sum_{\square} \tau_i^z \tau_j^z \tau_k^z \tau_l^z$$

$\beta \to 0$, gauge invariance: $\sigma_i^x \prod_{i \in v} \tau_i^x = 1$

There - Toric Code:

$$\mathcal{H} = -h \sum_v A_v - J_p \sum_p B_p$$

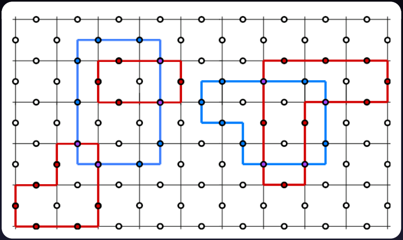
Toric Code



$$A_v = \prod_{i \in v} \tau_i^x, \qquad B_p = \prod_{i \in p} \tau_i^z$$

$$\mathcal{H} = -J_v \sum_v A_v - J_p \sum_p B_p$$

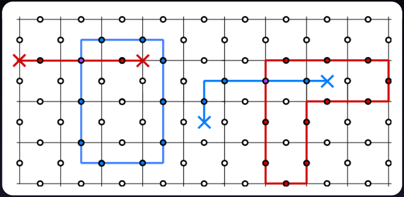
Toric Code: Ground State



$$|\xi_0\rangle = \prod_v \frac{1}{\sqrt{2}} (\mathbb{1}_v + A_v) \underbrace{|0\rangle \otimes \ldots \otimes |0\rangle}_{N_e \text{times}}$$

A “Loop Soup”

Toric Code: Excitations



Defects (i.e. quasiparticles) appear on the end of strings

$$W_e = \prod \tau_z, \quad W_m = \prod \tau_x$$

Braiding excitations reveals anyonic behavior

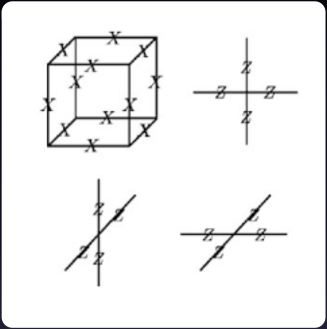
Toric Code: Anyons

e, m bosonic w.r.t. themselves

Take e for a walk around m, gives -1 phase $\implies$ anyons

Other interesting properties, topological degeneracy of ground state etc

The X-cube Model

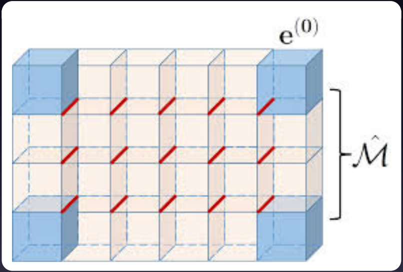


$$A_c = \prod_{i \in \text{cube}} \tau_i^x, \quad B_i^{xy,yz,xz} = \prod_{j \in +,i} \tau_j^z$$

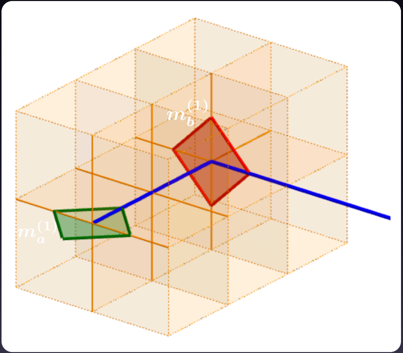
$$\mathcal{H} = -J_{\text{cube}} \sum A_{\text{cube}} - J_{xy} \sum_i B_i^{xy} - J_{yz} \sum_i B_i^{yz} - J_{xz} \sum_i B_i^{xz}$$

Toric Code: Fractons

Electric excitations τ_z



Magnetic excitations τ_x



Pics c/o Vijay et.al.

From here to there: Subsystem Symmetry Gauging

Here - Quantum Transverse Plaquette Ising:

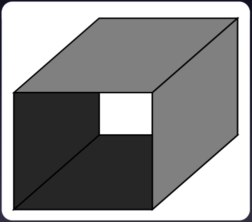
$$\mathcal{H} = -\beta \sum_{\square} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z - h \sum_i \sigma_i^x$$

Gauge the $\mathbb{Z}_2$ subsystem symmetry

$$\mathcal{H} = -\beta \sum_{\square} \tau_{\square}^z \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z - h \sum_i \sigma_i^x + \ldots$$

From here to there: Subsystem Symmetry Gauging II

Equivalent of plaquette flux term in 2D is matchbox (not cube)



Gives $B_i^{xy,yz,xz} = \prod_{j\in +,i} \sigma_j^z$ flux terms

From here to there: Subsystem Symmetry Gauging

There (almost)

$$\mathcal{H} = -\beta \sum_{\square} \tau_{\square}^{z} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z - h \sum_i \sigma_i^x$$
$$-J_{xy} \sum_i B_i^{xy} - J_{yz} \sum_i B_i^{yz} - J_{xz} \sum_i B_i^{xz}$$

$\beta \rightarrow 0$, gauge invariance: $\sigma_i^x \prod_i \tau_i^x = 1$

There

$$\mathcal{H} = -h \sum A_{\text{cube}} - J_{xy} \sum_i B_i^{xy} - J_{yz} \sum_i B_i^{yz} - J_{xz} \sum_i B_i^{xz}$$

Gauging

Gauge global $\mathbb{Z}_2$, get Toric Code, anyons etc

Gauge subsystem symmetry, get fractons (reduced mobility anyons)

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