SUSY, Spin Chains (and ASEPS?)

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Congratulations (and thanks)!

Symbols of Armenia

Plan of talk

SUSY

(XXZ) Spin Chains/SUSY - Fendley, Hagendorf

The missing punchline - ASEPs (ASAPs) - Ayyer, Mallick

NOT Johnston

SUSY

Originally.....

$$Q|Boson\rangle = |Fermion\rangle$$

$$Q^\dagger|Fermion\rangle = |Boson\rangle$$

$$Q^2 = 0, \ \left(Q^\dagger\right)^2 = 0$$

In particular, Super-Poincaré algebra in 4d

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$$

Getting rid of divergences in QFT

Lattice discretisation difficult (but see Catterall et.al.)

SUSY QM

Hamiltonian

$$H\psi_n(x) = \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2\right)\psi_n(x) = E_n\psi_n(x)$$

Define

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x); \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$

Two SUSY Hamiltonians

$$H^{(1)} = A^\dagger A = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x)$$

$$H^{(2)} = AA^\dagger = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x)$$

SUSY QM II

SUSY algebra

$$\begin{aligned} [H, Q] &= [H, Q^\dagger] = 0 \\ \{Q, Q^\dagger\} &= H \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \end{aligned}$$

$$\begin{aligned} H &= \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(2)} \end{pmatrix} \\ Q &= \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \\ Q^\dagger &= \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \end{aligned}$$

XXZ Spin Chain

XXZ Hamiltonian acts on $V^{\otimes N}$ where $V \simeq \mathbb{C}^2$.

$$\begin{aligned} H^{(N)} &= -\frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}\sigma_i^z\sigma_{i+1}^z\right) \\ &\quad -\frac{1}{4}(\sigma_1^z + \sigma_N^z) + \frac{3N-1}{4}\mathbb{I} \end{aligned}$$

Write as sum of local terms, $h_{ij} : V \otimes V \to V \otimes V$ in bulk

$$H^{(N)} = \sum_{\langle ij \rangle} h_{ij} + h_1 + h_N$$

XXZ Spin Chain II

Local Hamiltonian (in the bulk)

$$h_{ij} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{1}{2}\sigma^z \otimes \sigma^z + \frac{3}{4}\mathbb{I}$$

In matrix form, standard basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

XXZ Spin Chain/SUSY

Possible to define (Fendley, Yang)

$$Q^{(N)} : V^{\otimes N} \rightarrow V^{\otimes N-1}; \;\; Q^{(N)\dagger} : V^{(N-1)} \rightarrow V^{(N)}$$

Properties of Q: nilpotent

$$Q^{(N-1)} Q^{(N)} = 0, \quad Q^{\dagger(N+1)} Q^{\dagger(N)} = 0,$$

Hamiltonian length N

$$H^{(N)} = Q^{\dagger(N)} Q^{(N)} + Q^{(N+1)} Q^{(N+1)\dagger}.$$

XXZ Spin Chain/Supercharge I

Length-changing operator

$$H^{(N-1)}Q^{(N)} = Q^{(N)}H^{(N)}, \quad H^{(N)}Q^{\dagger(N)} = Q^{\dagger(N)}H^{(N-1)}.$$

Local expressions (Hagendorf)

$$Q^{(N)} = \sum_{i=1}^{N-1}(-1)^{i+1}q_{i,i+1}, \quad Q^{\dagger(N)} = \sum_{i}^{N-1}(-1)^{i+1}q_i^{\dagger}$$

Local action - "dynamical SUSY"

$$q_i^{\dagger} : V \to V \otimes V; \quad q_{i,i+1} : V \otimes V \to V$$

XXZ Spin Chain/Supercharge II

In terms of the local supercharge, nilpotency $(Q^\dagger)^2 = 0$ becomes

$$(q^\dagger \otimes 1 - 1 \otimes q^\dagger) q^\dagger = 0$$

Or, more accurately - including a sort of "gauge" freedom

$$\left[(q^\dagger \otimes 1) q^\dagger - (1 \otimes q^\dagger) q^\dagger \right] |m\rangle = |\chi\rangle \otimes |m\rangle - |m\rangle \otimes |\chi\rangle$$

For all $|m\rangle \in V$, for a particular $|\chi\rangle \in V \to V \otimes V$

XXZ Spin Chain/Supercharge III

Substitute $Q, Q^\dagger$ expression into Hamiltonian, most terms vanish:

$$q_i^\dagger q_j + q_{j+1} q_i^\dagger = 0, \quad 1 \leq i < j-1 \leq N-1$$

Leaving

$$\begin{aligned} h &= -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q \\ &\quad + \frac{1}{2}\left(q\, q^\dagger \otimes 1 + 1 \otimes q\, q^\dagger\right) \end{aligned}$$

With boundary terms

$$h_B = \frac{1}{2} q\, q^\dagger$$

XXZ Spin Chain/Supercharge IIIbis

The game of "does my Hamiltonian have dynamical supersymmetry"?

1) Pick a $q^\dagger$ - (4×2)

2) Construct h - (4×4) using:

$$
\begin{aligned}
h &= -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q \\
&\quad + \frac{1}{2}\left(q\, q^\dagger \otimes 1 + 1 \otimes q\, q^\dagger\right)
\end{aligned}
$$

XXZ Spin Chain/Supercharge IV

$q^\dagger$ is remarkably simple for XYZ (at combinatorial point)

$$q^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Action

$$q^\dagger|0\rangle = \emptyset \; ; \quad q^\dagger|1\rangle = |00\rangle$$

XXZ Spin Chain/Supercharge V

Could start with $|1\rangle$ instead of $|0\rangle$, i.e.

$$\bar{q}^\dagger|1\rangle = \emptyset \; ; \;\; \bar{q}^\dagger|0\rangle = |11\rangle$$

$\bar{q}^\dagger$ is still simple

$$\bar{q}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stitch together $q, \bar{q}$, using “gauge” freedom

$$q^\dagger(y) = x \begin{pmatrix} -2y & 1 \\ -y^2 & -y \\ -y^2 & -y \\ y^3 & -2y^2 \end{pmatrix} \; ; \;\; x = (1 + |y|^6)^{-1/2}$$

XXZ Spin Chain/Supercharge VI

With $q^\dagger(y)$ still true that

$$h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boundaries become ($y = \rho e^{i\theta}$)

$$h_{\mathrm{B}}(y) = \left(\frac{1 + 5\rho^2 + \rho^4}{4(1 - \rho^2 + \rho^4)} \right) \mathbb{I} + \sum_{j=1}^{3} \lambda_j \sigma^j,$$

where

$$\lambda_1 = -\frac{\rho \cos\theta}{1+\rho^2}, \quad \lambda_2 = -\frac{\rho \sin\theta}{1+\rho^2}, \quad \lambda_3 = -\frac{1}{4}\left(\frac{1-\rho^2}{1+\rho^2}\right).$$

XXZ Spin Chain/Supercharge VII

So far - SUSY for XXZ with off-diagonal BCs same at boths ends

$$h_{ij} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{1}{2}\sigma^z \otimes \sigma^z + \frac{3}{4}\mathbb{I}$$

$$h_{\mathrm{B}}(y) = \left(\frac{1 + 5\rho^2 + \rho^4}{4(1 - \rho^2 + \rho^4)} \right) \mathbb{I} + \sum_{j=1}^{3} \lambda_j \sigma^j ,$$

XXZ Spin Chain/Supercharge VIII

Can also arrange different BCs, take

$$|\phi(y)\rangle = \sum_{m=0}^{1} \phi_m(y)|m\rangle, \quad \phi_m(y) = -\frac{y^{m+1}}{\sqrt{\{m+1\}}}.$$

$$|\xi_k(y)\rangle = x(|\phi(y)\rangle - |\phi(q^{2(k+1)}y)\rangle), \quad q = -\exp(-i\pi/3)$$

Boundary vectors

$$q(y)|\xi_k(y)\rangle = |\xi_k(y)\rangle \otimes |\xi_k(y)\rangle, \quad k = 0, 1, 2$$

Then

$$Q_{j,k}(y)|\psi\rangle = |\xi_j(y)\rangle \otimes |\psi\rangle + (-1)^{N-1}|\psi\rangle \otimes |\xi_k(y)\rangle + Q(y)|\psi\rangle.$$

XXZ Spin Chain/Supercharge IX

Now - SUSY for XXZ with off-diagonal BCs different at boths ends

$$h_{ij} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{1}{2}\sigma^z \otimes \sigma^z + \frac{3}{4}\mathbb{I}$$

$$h_{\mathrm{B}}(y) = \left(\frac{1 + 5\rho^2 + \rho^4}{4(1 - \rho^2 + \rho^4)} \right) \mathbb{I} + \sum_{j=1}^{3} \lambda_j \sigma^j,$$

$$h_{\mathrm{B}}^{(k)}(y) = h_{\mathrm{B}}(q^{2(k+1)} y)$$

$$q = -\exp(-i\pi/3)$$

Random additional Comments

XYZ too

$$h_{ij} = (1-z)\sigma^x \otimes \sigma^x + (1+z)\sigma^y \otimes \sigma^y - \frac{1-z^2}{2}\sigma^z \otimes \sigma^z$$

$q^\dagger$ in this case

$$q^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -z \end{pmatrix}$$

Hard fermion models: M_2, M_l - Hagendorf, Huijse

$Gl(n|m)$ - Meidinger, Mitev

Story so Far

XXZ (and XYZ) spin chain has a “dynamical” supersymmetry

Relates chains of different length

Can be formulated for both open and closed chains

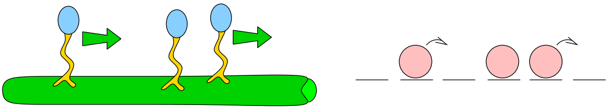
Can be formulated for diagonal, off-diagonal, unequal BCs

ASEPs

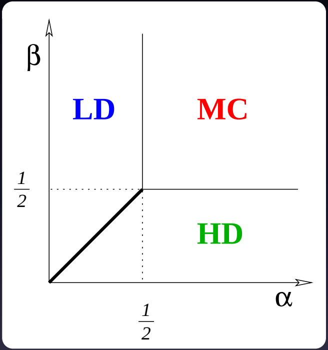
Originally came from biology

Model for transport in cells

Kinesins moving along a microtubule



The Phase Diagram ($q = 0$)



Roll of honour (TASEP $q = 0$): Derrida, Evans, Hakim Pasquier
Roll of honour (PASEP $q \neq 0$): Blythe, Evans, Colaiori, Essler

What's this got to do with XXZ?

Markov matrix M

$$\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = M|P(t)\rangle,$$

Related to H_{XXZ}

$$M = \sqrt{pq}\, U_\mu^{-1} H_{XXZ} U_\mu$$

Where

$$\begin{aligned} U_\mu &= \bigotimes_{i=1}^{L} \begin{pmatrix} 1 & 0 \\ 0 & \mu Q^{j-1} \end{pmatrix}, \\ H_{\mathrm{XXZ}} &= -\frac{1}{2}\sum_{j=1}^{L-1} \Big[\sigma_j^{\mathrm{x}}\sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}}\sigma_{j+1}^{\mathrm{y}} - \Delta\sigma_j^{\mathrm{z}}\sigma_{j+1}^{\mathrm{z}} \\ &\qquad + h(\sigma_{j+1}^{\mathrm{z}} - \sigma_j^{\mathrm{z}}) + \Delta \Big] + B_1 + B_L \end{aligned}$$

What's this got to do with *XXZ*??

Various *XXZ* parameters related to (P)ASEP

$$
\begin{aligned}
B_1 &= \frac{1}{2\sqrt{pq}}\left(\alpha+\gamma+(\alpha-\gamma)\sigma_1^z-2\alpha\mu e^{\lambda}\sigma_1^- -2\gamma\mu^{-1}e^{-\lambda}\sigma_1^+\right)\\
B_L &= \frac{1}{2\sqrt{pq}}\left(\beta+\delta-(\beta-\delta)\sigma_L^z-2\delta\mu Q^{L-1}\sigma_L^-\right.\\
&- \left.2\beta\mu^{-1}Q^{-L+1}\sigma_L^+\right)\\
\Delta &= -\frac{1}{2}(Q+Q^{-1}),\quad h=\frac{1}{2}(Q-Q^{-1}),\quad Q=\sqrt{\frac{q}{p}}.
\end{aligned}
$$

And SUSY??

Well.....“transfer matrix” symmetry (Ayyer, Mallick)

For ASEP and ASAP

$$T_{L,L+1} M_L = M_{L+1} T_{L,L+1}$$

Relates *M*s of different length

$$M_L = \sum_{i=1}^{L-1} M_{i,i+1} + R + L$$

Where, for ASAP

$$10 \rightarrow 01 \text{ with rate } 1$$
$$11 \rightarrow 00 \text{ with rate } \lambda$$

And SUSY???

ASAP local Hamiltonian

$$
\begin{aligned}
h_{i,i+1} &= \sigma_i^+\sigma_{i+1}^- + \lambda\sigma_i^+\sigma_{i+1}^+ + \frac{1-\lambda}{4}\sigma_i^z\sigma_{i+1}^z + \frac{1+\lambda}{4}\sigma_i^z \\
&- \frac{1-\lambda}{4}\sigma_{i+1}^z - \frac{1+\lambda}{4}\mathbb{I} \\
L &= \alpha\left(\sigma_1^- + \lambda\sigma_1^+ - \frac{1-\lambda}{2}\sigma_1^z - \frac{1+\lambda}{2}\right) \\
R &= \beta\left(\sigma_L^+ - \frac{1-\sigma_L^z}{2}\right)
\end{aligned}
$$

So??

Is the transfer matrix symmetry related to SUSY?

Related spin chains have unequal, off-diagonal BCs

NOT at combinatorial point $<<<<$

References

Xiao Yang and Paul Fendley, Non-local spacetime supersymmetry on the lattice, *J. Phys. A*, 37(38):8937–8948, 2004.

Christian Hagendorf and Jean Liénardy, Open spin chains with dynamic lattice supersymmetry, *J. Phys. A*, 50(18):185202, 32, 2017.

Robert Weston and Junye Yang, Lattice Supersymmetry in the Open XXZ Model: An Algebraic Bethe Ansatz Analysis, [arXiv:1709.00442]