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Convexity and Tadpoles

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Abstract. We show that it is possible to produce a convex effective potential using the tadpole method of calculation, both at zero temperature and in real time finite-temperature quantum field theory. We point out, however, that this does not evade the failure of the loop expansion in a theory with a non-convex classical potential at a temperature where the minima of the effective potential have moved in to the classical points of inflection.

Introduction

The formal convexity property of the effective potential [1] is in conflict with the result of explicit loop calculations for theories with a spontaneously broken (i.e. non-convex) classical potential. However, Fujimoto et al. [2] suggested that, for certain critical values of the external current J, more than one saddle point could contribute to the calculation of the path integral. This has the effect of linearly interpolating across non-convex portions of the incorrectly calculated effective potential produced in a standard loop expansion. The "flat-bottomed bucket" shape that results is in agreement with lattice calculations [3] and the application of Wilson recursion relations [4].

In this paper we are interested in a convex effective potential at finite temperature and, to this end, we consider the tadpole method of calculation [5], which is especially convenient in the framework of real time finite-temperature quantum field theory (Thermo-Field Dynamics or T.F.D). This formalism has recently aroused much interest because of its calculational convenience compared with the older imaginary time formalism [6], [7]. A two-loop calculation of the finite temperature effective potential for a pure scalar theory has been carried out, for an unbroken potential by Matsumoto et al. [8], who remarked that the interpolation formula of Fujimoto et al. would suffice to cure the ills of the non-convex case. We shall see that this is only partially true, as was indeed pointed out by Rivers [9], in the context of an imaginary time formalism calculation.

Tadpoles at Zero Temperature

We follow the suggestion of Fujimoto et al. that the effects of subsidiary saddle points in the path integral should be taken into account, but instead of considering a vacuum graph expansion to calculate the effective potential V, we use tadpoles. To find the critical values of J for which more than one saddle point may be important we note that, in graphical terms and with constant fields, J is just the slope of the tangent line to the curve V. Its intersection with the vertical axis, $V - J\phi$, gives the vacuum energy density of the system [10]. If we consider a spontaneously broken $\lambda\phi^4$ theory we can see from Fig. 1 that for $J \neq 0$ only one point on the curve gives a minimum energy density. However, for $J = 0$ both $\langle\phi\rangle$ and $-\langle\phi\rangle$ give a minimum, and we should find some way of incorporating this into our calculational scheme.

When just one minimum in the energy (i.e. saddle point in the path integral) contributes we can consider the usual expansion for V

$$V(\phi) = \sum \frac{1}{n!} \Gamma^n(p_i = 0)(\phi - \langle\phi\rangle)^n \tag{1}$$

where ϕ is the (constant) classical field, $\langle\phi\rangle$ its vacuum expectation value and $\Gamma^n(p_i = 0)$ is the n-point 1PI vertex at zero momentum. The differential of V evaluated at its minimum is given by the tadpole Γ^1. Thus to obtain V we can calculate the tadpole and integrate it with respect to $\langle\phi\rangle$ (see 2(b)).

$$\text{(a)}\ \left.\frac{\partial V}{\partial\phi}\right|_{\langle\phi\rangle} = \Gamma^1_{\langle\phi\rangle} \quad \text{therefore} \quad \text{(b)}\ V = \int d\langle\phi\rangle \Gamma^1_{\langle\phi\rangle}. \tag{2}$$

At $J = 0$ we ought to take into account the equal contributions of both minima by including both the tadpole at $\langle\phi\rangle$ and that at $-\langle\phi\rangle$ (the quantum corrections preserve the Z_2 symmetry of the classical potential U in that of V). We therefore write

$$\Gamma^1_{\text{total}} = \Gamma^1_{\langle\phi\rangle} + \Gamma^1_{-\langle\phi\rangle}. \tag{3}$$

In general, if the Lagrangian has some symmetry group G the various minima, say $\langle \phi_a \rangle$, can all the written as

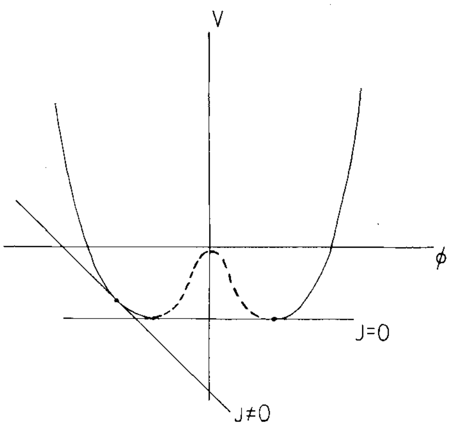

Fig. 1. The Spontaneously broken effective potential for a $\lambda\phi^4$ theory showing both $J=0$ and $J=0$ tangent lines. The non-convex portion that appears, incorrectly, in a standard loop expansion is shown dotted

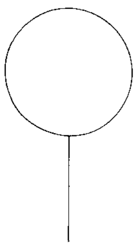

Fig. 2. The one-loop tadpole for $\lambda\phi^4$

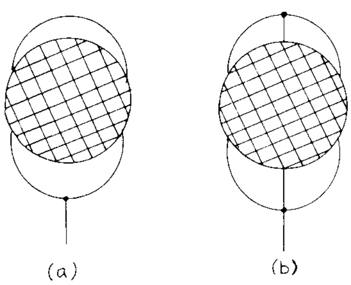

Fig. 3a, b. Multiloop tadpoles must be one of these two general forms

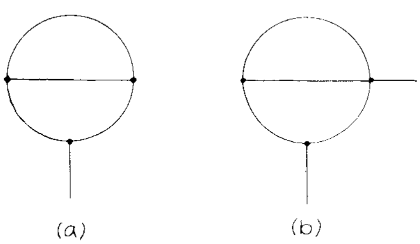

Fig. 4. a Shows a two-loop tadpole which contains a factor $f.f^2$ whereas **b**, which is proportional to $f.fg$, has 2 external legs

the group transform of one particular minimum, say $\langle a \rangle$, so Γ^1_{total} can be thought of as a function of $\langle a \rangle$ for the purpose of integrating the tadpole to obtain V.

To see the implications of (3) for the form of the effective potential consider again a spontaneously broken $\lambda\phi^4$ theory. For convenience in calculation we use dimensional regularization and follow the methods of Lee and Sciaccaluga [11]. To calculate the tadpole we must work order by order in h, to any desired degree of accuracy. The "zero-loop tadpole" is just the differential of the classical potential U. If we choose

$$U = -\tfrac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \tag{4}$$

then the tadpole is

$$U' = -m^2\phi + \tfrac{1}{6}\lambda\phi^4. \tag{5}$$

For $J \neq 0$ only one of the minima contributes and, upon integration, we recover U. However, at $J=0$, (3) applies and we have

$$\Gamma^1_{\text{total}} = U'_{\langle\phi\rangle} + U'_{-\langle\phi\rangle} = 0 \tag{6}$$

Thus our zero-loop effective potential at $J=0$, from the integration of (6), is just a constant, giving the desired linear interpolation.

If we now consider the one-loop tadpole in Fig. 2 we find it is given by

$$\Gamma^1 = -\frac{if}{32\pi^4}\int d^4k \frac{1}{k^2 - M^2} \tag{7}$$

where $f = U'''$ and $M^2 = U''$. After dimensional regularization and minimal subtraction this becomes

$$\Gamma^1 = \frac{f}{32\pi^2}\ln\frac{M^2}{\mu^2} \tag{8}$$

where μ^2 is the arbitrary mass-squared introduced in the renormalization. Because $f(-\phi) = -f(\phi)$ and $M^2(-\phi) = M^2(\phi)$ the contributions from $\langle\phi\rangle$ and $-\langle\phi\rangle$ when $J=0$ to the total tadpole will cancel out, giving a constant contribution to V and preserving the flat-bottomed bucket shape. When $J \neq 0$ we recover the standard one-loop effective potential upon integration of (8)

$$V = \frac{1}{64\pi^2}M^4\left(\ln\frac{M^2}{\mu^2} - \frac{1}{2}\right). \tag{9}$$

We can see on general grounds that this behaviour will be maintained to all orders. The tadpoles must be of the form shown in Fig. 3. The tadpole in Fig. 3a is proportional to f and that in Fig. 3b to $f\lambda$. Inside the blobs the f's must always occur in pairs; otherwise we would have more than one external leg, as we can see in Fig. 4b. All the other elements occurring in the tadpole, such as M^2, are symmetric under $\langle\phi\rangle \leftrightarrow -\langle\phi\rangle$ interchange so, with a tadpole proportional to f, a sum over the two minima will give zero whatever the order in the loop expansion.

Tadpoles at Finite Temperature

If we consider the calculation of a finite temperature effective potential using T.F.D. we obtain a similar convexification. The only mixing between the physical "1" fields and the thermal doublet "2" fields in T.F.D. occurs in the propagators and the vertices differ only in sign (not in structure). To obtain the required graphs for a finite temperature loop calculation of the tadpole one takes the zero temperature graph, fixes the external leg to be a "1" field and then distributes "1" and "2" labels over the ends of the propagators in as many ways as possible. For example, some of the two-loop tadpoles in the $\lambda\phi^4$ theory are shown in Fig. 5. Although the form of the propagators is different at finite temperature

$$\Delta_{11} = \frac{1}{k^2 - M^2 + i\varepsilon} - 2\pi i \frac{\delta(k^2 - M^2)}{e^{\beta\|k_0\|} - 1} \tag{10}$$

$$\Delta_{12} = \Delta_{21} = -2\pi i \delta(k^2 - M^2) \frac{e^{\beta\|k_0\|/2}}{e^{\beta\|k_0\|} - 1} \tag{11}$$

$$\Delta_{22} = -\frac{1}{k^2 - M^2 - i\varepsilon} - 2\pi i \frac{\delta(k^2 - M^2)}{e^{\beta\|k_0\|} - 1} \tag{12}$$

they are still $\langle\phi\rangle \leftrightarrow -\langle\phi\rangle$ symmetric. We can therefore apply the diagrammatic arguments of the previous section to each tadpole of Fig. 5 in turn and observe that the $J = 0$ contributions from both minima cancel.

Tadpoles for the Abelian Higgs Model

As a simple example of a spontaneously broken gauge theory we consider an Abelian Higgs model. In order to use the same cancellation mechanism as in the $\lambda\phi^4$ case we work in an R_ξ gauge. Despite the initial claim by Weinberg [5] that effective potential calculations are not possible in such gauges and the reservations by Jackiw about the possible gauge dependence of such calculations [12], this is perfectly feasible, as was shown by Fukuda and Kugo [13], providing the appropriate Nielsen identities [14] are satisfied. The full gauge-fixed Lagrangian is

$$\begin{aligned} L = & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial A + \xi\varepsilon_{ij}\langle\phi\rangle_i\phi_j)^2 \\ & + \tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - e\varepsilon_{ij}(\partial_\mu\phi_i)\phi_j A^\mu \\ & + \tfrac{1}{2}e^2A^2\phi^2 + \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \\ & + \partial_\mu c\partial^\mu\bar{c} - e^2\xi\bar{c}c\varepsilon_{ij}\langle\phi\rangle_i\phi_j \end{aligned} \tag{13}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1, \varepsilon_{11} = \varepsilon_{22} = 0$ and $\phi^2 = \phi_1^2 + \phi_2^2$. With the given gauge fixing the allowed directions of symmetry breaking are constrained by the Nielsen identities to be parallel or antiparallel to $\langle\phi\rangle$ [15], picking out two points on the ring of minima of the potential (see Fig. 6). As in the $\lambda\phi^4$ theory only one of the minima will contribute to an evaluation of the effective potential when $J \neq 0$, whereas for $J = 0$ both

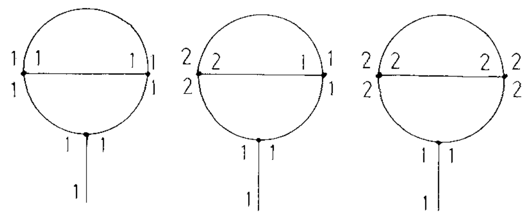

Fig. 5. Some two loop finite-temperature tadpoles for $\lambda\phi^4$

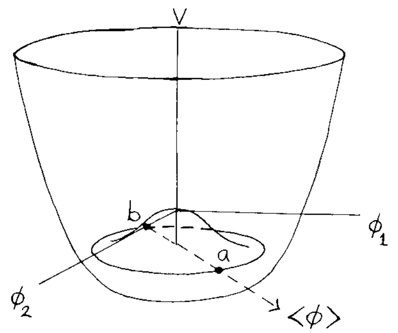

Fig. 6. The potential for an Abelian Higgs model. The direction of the symmetry breaking $\langle\phi\rangle$ introduced in the gauge-fixing is marked, along with its intersection with the ring of minima

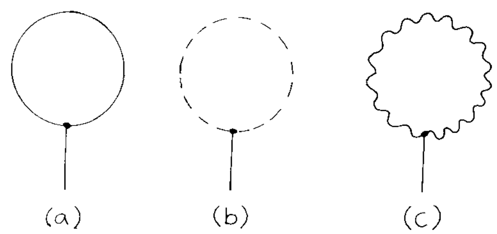

Fig. 7a–c. The one-loop tadpoles for the Abelian Higgs model

points a and b contribute, giving a constant effective potential.

We can see this at the one-loop level (the zero-loop is identical to the pure scalar) by considering the diagrams in Fig. 7. The propagators and vertices necessary to evaluate these are listed in Appendix A and the results are

$$\text{Fig. 7a} = \frac{\lambda}{3}\langle\phi\rangle_i \int d^4k \left(\frac{1}{k^2 - m_1^2} + \frac{k^2 - \xi^2 e^2 \langle\phi\rangle^2}{D_n} \right) \tag{14}$$

$$\text{Fig. 7b} = e^2\xi\langle\phi\rangle_i\varepsilon_{ij}\int d^4k \frac{1}{k^2} \tag{15}$$

$$\text{Fig. 7c} = 2e^2\langle\phi\rangle_i \int d^4k \left(\frac{3}{k^2 - e^2\langle\phi\rangle^2} + \frac{\xi(k^2 - m_2^2 - e^2\xi\langle\phi\rangle^2)}{D_n} \right) \tag{16}$$

where $D_n = k^4 - k^2(m_2^2 - 2e^2\xi\langle\phi\rangle^2) + e^2\langle\phi\rangle^2(e^2\xi^2 \cdot$

$\langle\phi\rangle^2+\xi m_2^2)$, $m_1^2=(\lambda/2)\langle\phi\rangle^2-m^2$ and $m_2^2=(\lambda/6)\langle\phi\rangle^2-m^2$. The proportionality to $\langle\phi\rangle$ which ensures the cancellation of tadpoles from the two minima is still present and the generalization of the one-loop result to all orders and finite temperature follows the same path as the scalar case.

Discussion

We have seen that it is possible to produce a finite-temperature, convex effective potential using T.F.D. tadpoles. However, as the temperature increases the minima of the effective potential creep inwards. At some temperature $\langle\phi\rangle$ will reach the points of inflection of the classical potential where $M^2=U''=0$, and the loop expansion will break down. This is almost obvious from the presence of M^2 as the mass term in the propagators of T.F.D., and explicit calculations provide confirmation. For instance the two-loop effective potential of the pure scalar theory in the $T\to\infty$, $M^2\to 0$ limit is given by (see appendix B)

$$V=-\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4+\frac{\lambda\phi^2}{48}T^2+\frac{\lambda^2\phi^2}{256\pi^4}\ln\frac{M^2}{\mu^2}T^2 \tag{17}$$

which not only diverges but also emphasizes terms in the expansion of higher order in h. As Rivers has pointed out this problem will not arise for a theory with Coleman–Wienberg type symmetry breaking [16] because there $M^2>0$ for all values of ϕ.

We conclude, therefore, that for a theory with a non-convex classical potential even an interpolated loop expansion is only valid up to the temperature at which $\langle\phi\rangle$ reaches the classical points of inflection. Calculations of critical temperatures can not be performed using the loop expansion in such cases.

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Appendix A

The propagators necessary to evaluate the tadpoles in Fig. 7 are listed below.

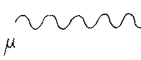
Scalar

$$=\frac{i(k^2-e^2\xi^2\langle\phi\rangle^2)}{D_n}(\delta_{ij}-\eta_i\eta_j)+\frac{i\eta_i\eta_j}{k^2-m_1^2} \tag{A1}$$

Ghost

$$=\frac{i}{k^2} \tag{A2}$$

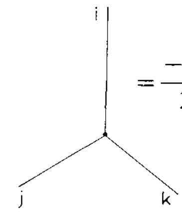
Vector



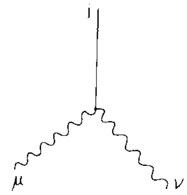
$$=\frac{-i}{k^2-e^2\langle\phi\rangle^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)-\frac{i\xi(k^2-m_2^2-e^2\xi\langle\phi\rangle^2)k_\mu k_\nu}{D_n\quad k^2} \tag{A3}$$

where $\eta=(1,0)$, $\langle\phi\rangle_1=0$, $\langle\phi\rangle_2=\langle\phi\rangle$ and we have assumed that we are evaluating the propagators at the minimum of V.

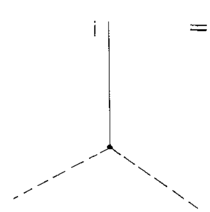
The required vertices are given by



$$=\frac{-i\lambda}{3}(\delta_{ij}\langle\phi\rangle_k+\delta_{jk}\langle\phi\rangle_i+\delta_{ki}\langle\phi\rangle_j) \tag{A4}$$



$$=2ie^2\langle\phi\rangle_i g_{\mu\nu} \tag{A5}$$



$$=-ie^2\xi\langle\phi\rangle_j\varepsilon_{ji} \tag{A6}$$

Appendix B

The two-loop zero temperature effective potential for a $\lambda\phi^4$ theory is given by [11]

$$V_0=\frac{1}{256\pi^4}\left[f^2M^2\left(\frac{1}{4}\ln^2\frac{M^2}{\mu^2}-\frac{9}{7}\left(\ln\frac{M^2}{\mu^2}-1\right)\right)\right]+\frac{\lambda M^4}{256\pi^4}\left(\frac{1}{4}\ln^2\frac{M^2}{\mu^2}+\frac{1}{4}\ln\frac{M^2}{\mu^2}-\frac{79}{28}\right) \tag{B1}$$

and the two-loop finite temperature effective potential by [8]

$$V_\beta=\frac{\lambda M^4}{32\pi^4}F_1^2(\beta M)+\frac{f^2M^2}{128\pi^4}\int dx\,dy\,F(\beta M)G(x,y)+\frac{\lambda M^4}{128\pi^4}\left(\frac{1}{2}+\ln\frac{M^2}{\mu^2}\right)F_1(\beta M)+\frac{fM^2}{128\pi^4}\left(\frac{1}{2}+\frac{\pi}{\sqrt{3}}+\ln\frac{M^2}{\mu^2}\right)F_1(\beta M) \tag{B2}$$

where

$$F_1(\beta M)=\int dx\sqrt{(x^2-1)}\frac{1}{e^{\beta Mx}-1},$$

$$F=\frac{1}{(e^{\beta Mx}-1)(e^{\beta My}-1)} \quad \text{and}$$

$$G(x,y)=\ln\frac{[(1+2\sqrt{(x^2-1)(y^2-1)})^2-4x^2y^2]}{[(1-2\sqrt{(x^2-1)(y^2-1)})^2-4x^2y^2]}.$$

Now as β, $M^2\to 0$ $F_1(\beta M)\to(\pi^2/6)(1/\beta^2M^2)$, so in this limit the leading contribution comes from (B2) and is given by

$$V\approx\frac{f^2}{128\pi^4}\cdot\ln\frac{M^2}{\mu^2}\cdot\frac{\pi^2}{6\beta^2} \tag{B3}$$

which on substituting $f=\lambda\phi$, $\beta=1/T$ and adding the one-loop result gives (17).

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