

A complete realisability semantics for intersection types and arbitrary expansion variables

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Abstract. *Expansion* was introduced at the end of the 1970s for calculating *principal typings* for λ -terms in intersection type systems. *Expansion variables* (E-variables) were introduced at the end of the 1990s to simplify and help mechanise expansion. Recently, E-variables have been further simplified and generalised to also allow calculating other type operators than just intersection. There has been much work on semantics for intersection type systems, but only one such work on intersection type systems with E-variables. That work established that building a semantics for E-variables is very challenging. Because it is unclear how to devise a space of meanings for E-variables, that work developed instead a space of meanings for types that is hierarchical in the sense of having many degrees (denoted by indexes). However, although the indexed calculus helped identify the serious problems of giving a semantics for expansion variables, the sound realisability semantics was only complete when one single E-variable is used and furthermore, the universal type ω was not allowed. In this paper, we are able to overcome these challenges. We develop a realisability semantics where we allow an arbitrary (possibly infinite) number of expansion variables and where ω is present. We show the soundness and completeness of our proposed semantics.

1 Introduction

Expansion is a crucial part of a procedure for calculating *principal typings* and thus helps support compositional type inference. For example, the λ -term $M = (\lambda x.x(\lambda y.yz))$ can be assigned the typing $\Phi_1 = \langle (z : a) \vdash (((a \rightarrow b) \rightarrow b) \rightarrow c) \rightarrow c \rangle$, which happens to be its principal typing. The term M can also be assigned the typing $\Phi_2 = \langle (z : a_1 \sqcap a_2) \vdash (((a_1 \rightarrow b_1) \rightarrow b_1) \sqcap ((a_2 \rightarrow b_2) \rightarrow b_2) \rightarrow c) \rightarrow c \rangle$, and an expansion operation can obtain Φ_2 from Φ_1 . Because the early definitions of expansion were complicated [4], E-variables were introduced in order to make the calculations easier to mechanise and reason about. For example, in System E [2], the above typing Φ_1 is replaced by $\Phi_3 = \langle (z : ea) \vdash e(((a \rightarrow b) \rightarrow b) \rightarrow c) \rightarrow c \rangle$, which differs from Φ_1 by the insertion of the E-variable e at two places, and Φ_2

can be obtained from Φ_3 by substituting for e the *expansion term*:

$$E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2).$$

Carrier and Wells [3] have surveyed the history of expansion and also E-variables. Kamareddine, Nour, Rahli and Wells [13] showed that E-variables pose serious challenges for semantics. In the list of open problems published in 1975 in [6], it is suggested that an arrow type expresses functionality. Following this idea, a type's semantics is given as a set of closed λ -terms with behaviour related to the specification given by the type. In many kinds of semantics, the meaning of a type T is calculated by an expression $[T]_\nu$ that takes two parameters, the type T and a valuation ν that assigns to type variables the same kind of meanings that are assigned to types. In that way, models based on term-models have been built for intersection type systems [7, 14, 11] where intersection types (introduced to type more terms than in the Simply Typed Lambda Calculus) are interpreted by set-theoretical intersection of meanings. To extend this idea to types with E-variables, we need to devise some space of possible meanings for E-variables. Given that a type eT can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (or even arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), the situation is complicated.

This was the main motivation for [13] to develop a space of meanings for types that is hierarchical in the sense of having many degrees. When assigning meanings to types, [13] captured accurately the intuition behind E-variables by ensuring that each use of E-variables simply changes degrees and that each E-variable acts as a kind of capsule that isolates parts of the λ -term being analysed by the typing.

The semantic approach used in [13] is realisability semantics along the lines in Coquand [5] and Kamareddine and Nour [11]. Realisability allows showing *soundness* in the sense that the meaning of a type T contains all closed λ -terms that can be assigned T as their result type. This has been shown useful in previous work for characterising the behaviour of typed λ -terms [14]. One also wants to show the converse of soundness which is called *completeness* (see Hindley [8–10]), i.e., that every closed λ -term in the meaning of T can be assigned T as its result type. Moreover, [13] showed that if more than one E-variable is used, the semantics is not complete. Furthermore, the degrees used in [13] made it difficult to allow the universal type ω and this limited the study to the λI -calculus. In this paper, we are able to overcome these challenges. We develop a realisability semantics where we allow the full λ -calculus, an arbitrary (possibly infinite) number of expansion variables and where ω is present, and we show its soundness and completeness. We do so by introducing an indexed calculus as in [13]. However here, our indices are finite sequences of natural numbers rather than single natural numbers.

In Section 2 we give the full λ -calculus indexed with finite sequences of natural numbers and show the confluence of β , $\beta\eta$ and weak head reduction on the indexed λ -calculus. In Section 3 we introduce the type system for the indexed λ -calculus (with the universal type ω). In this system, intersections and expansions

cannot occur directly to the right of an arrow. In Section 4 we establish that subject reduction holds for \vdash . In Section 5 we show that subject β -expansion holds for \vdash but that subject η -expansion fails. In Section 6 we introduce the realisability semantics and show its soundness for \vdash . In Section 7 we establish the completeness of \vdash by introducing a special interpretation. We conclude in Section 8. Due to space limitations, we omit the details of the proofs. Full proofs however can be found in the expanded version of this article (currently at [12]) which will always be available at the authors' web pages.

2 The pure $\lambda^{\mathcal{L}_{\mathbb{N}}}$ -calculus

In this section we give the λ -calculus indexed with finite sequences of natural numbers and show the confluence of β , $\beta\eta$ and weak head reduction.

Let n, m, i, j, k, l be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We assume that if a metavariable v ranges over a set s then v_i and v', v'' , etc. also range over s . A binary relation is a set of pairs. Let rel range over binary relations. We sometimes write $x \text{ rel } y$ instead of $\langle x, y \rangle \in rel$. Let $\text{dom}(rel) = \{x / \langle x, y \rangle \in rel\}$ and $\text{ran}(rel) = \{y / \langle x, y \rangle \in rel\}$. A function is a binary relation fun such that if $\{\langle x, y \rangle, \langle x, z \rangle\} \subseteq fun$ then $y = z$. Let fun range over functions. Let $s \rightarrow s' = \{fun / \text{dom}(fun) \subseteq s \wedge \text{ran}(fun) \subseteq s'\}$. We sometimes write $x : s$ instead of $x \in s$.

First, we introduce the set $\mathcal{L}_{\mathbb{N}}$ of indexes with an order relation on indexes.

Definition 1. 1. An index is a finite sequence of natural numbers $L = (n_i)_{1 \leq i \leq l}$.

We denote $\mathcal{L}_{\mathbb{N}}$ the set of indexes and \emptyset the empty sequence of natural numbers. We let L, K, R range over $\mathcal{L}_{\mathbb{N}}$.

2. If $L = (n_i)_{1 \leq i \leq l}$, we use $m :: L$ to denote the sequence $(r_i)_{1 \leq i \leq l+1}$ where $r_1 = m$ and for all $i \in \{2, \dots, l+1\}$, $r_i = n_{i-1}$. In particular, $k :: \emptyset = (k)$.
3. If $L = (n_i)_{1 \leq i \leq n}$ and $K = (m_i)_{1 \leq i \leq m}$, we use $L :: K$ to denote the sequence $(r_i)_{1 \leq i \leq n+m}$ where for all $i \in \{1, \dots, n\}$, $r_i = n_i$ and for all $i \in \{n+1, \dots, n+m\}$, $r_i = m_{i-n}$. In particular, $L :: \emptyset = \emptyset :: L = L$.
4. We define on $\mathcal{L}_{\mathbb{N}}$ a binary relation \preceq by:
 $L_1 \preceq L_2$ (or $L_2 \succeq L_1$) if there exists $L_3 \in \mathcal{L}_{\mathbb{N}}$ such that $L_2 = L_1 :: L_3$.

Lemma 1. \preceq is an order relation on $\mathcal{L}_{\mathbb{N}}$.

The next definition gives the syntax of the indexed calculus and the notions of reduction.

Definition 2. 1. Let \mathcal{V} be a countably infinite set of variables. The set of terms \mathcal{M} , the set of free variables $\text{fv}(M)$ of a term $M \in \mathcal{M}$, the degree function $d : \mathcal{M} \rightarrow \mathcal{L}_{\mathbb{N}}$ and the joinability $M \diamond N$ of terms M and N are defined by simultaneous induction as follows:

- If $x \in \mathcal{V}$ and $L \in \mathcal{L}_{\mathbb{N}}$, then $x^L \in \mathcal{M}$, $\text{fv}(x^L) = \{x^L\}$ and $d(x^L) = L$.
- If $M, N \in \mathcal{M}$, $d(M) \preceq d(N)$ and $M \diamond N$ (see below), then $M N \in \mathcal{M}$, $\text{fv}(MN) = \text{fv}(M) \cup \text{fv}(N)$ and $d(M N) = d(M)$.

- If $x \in \mathcal{V}$, $M \in \mathcal{M}$ and $L \succeq d(M)$, then $\lambda x^L.M \in \mathcal{M}$, $\text{fv}(\lambda x^L.M) = \text{fv}(M) \setminus \{x^L\}$ and $d(\lambda x^L.M) = d(M)$.
- 2. – Let $M, N \in \mathcal{M}$. We say that M and N are joinable and write $M \diamond N$ iff for all $x \in \mathcal{V}$, if $x^L \in \text{fv}(M)$ and $x^K \in \text{fv}(N)$, then $L = K$.
 - If $\mathcal{X} \subseteq \mathcal{M}$ such that for all $M, N \in \mathcal{X}$, $M \diamond N$, we write, $\diamond \mathcal{X}$.
 - If $\mathcal{X} \subseteq \mathcal{M}$ and $M \in \mathcal{M}$ such that for all $N \in \mathcal{X}$, $M \diamond N$, we write, $M \diamond \mathcal{X}$.
 The \diamond property ensures that in any term M , variables have unique degrees. We assume the usual definition of subterms and the usual convention for parentheses and their omission (see Barendregt [1] and Krivine [14]). Note that every subterm of $M \in \mathcal{M}$ is also in \mathcal{M} . We let x, y, z , etc. range over \mathcal{V} and M, N, P range over \mathcal{M} and use $=$ for syntactic equality.
- 3. The usual substitution $M[x^L := N]$ of $N \in \mathcal{M}$ for all free occurrences of x^L in $M \in \mathcal{M}$ only matters when $d(N) = L$. Similarly, $M[x_1^{L_1} := N_1, \dots, x_n^{L_n} := N_n]$, the simultaneous substitution of N_i for all free occurrences of $x_i^{L_i}$ in M only matters when for all $i \in \{1, \dots, n\}$, $d(N_i) = L_i$. In a substitution, we sometimes write $(x_i^{L_i} := N_i)_n$ instead of $x_1^{L_1} := N_1, \dots, x_n^{L_n} := N_n$.
- 4. We take terms modulo α -conversion given by: $\lambda x^L.M = \lambda y^L.(M[x^L := y^L])$ where $y^L \notin \text{fv}(M)$. Moreover, we use the Barendregt convention (BC) where the names of bound variables differ from the free ones and where we rewrite terms so that not both λx^L and λx^K co-occur when $L \neq K$.
- 5. A relation rel on \mathcal{M} is compatible iff for all $M, N, P \in \mathcal{M}$:
 - If $M \text{ rel } N$ and $\lambda x^L.M, \lambda x^L.N \in \mathcal{M}$ then $(\lambda x^L.M) \text{ rel } (\lambda x^L.N)$.
 - If $M \text{ rel } N$ and $MP, NP \in \mathcal{M}$ (resp. $PM, PN \in \mathcal{M}$), then $(MP) \text{ rel } (NP)$ (resp. $(PM) \text{ rel } (PN)$).
- 6. The reduction relation \triangleright_β on \mathcal{M} is defined as the least compatible relation closed under the rule: $(\lambda x^L.M)N \triangleright_\beta M[x^L := N]$ if $d(N) = L$
- 7. The reduction relation \triangleright_η on \mathcal{M} is defined as the least compatible relation closed under the rule: $\lambda x^L.(M x^L) \triangleright_\eta M$ if $x^L \notin \text{fv}(M)$
- 8. The weak head reduction \triangleright_h on \mathcal{M} is defined by: $(\lambda x^L.M)NN_1 \dots N_n \triangleright_h M[x^L := N]N_1 \dots N_n$ where $n \geq 0$
- 9. We let $\triangleright_{\beta\eta} = \triangleright_\beta \cup \triangleright_\eta$. For $r \in \{\beta, \eta, h, \beta\eta\}$, we denote by \triangleright_r^* the reflexive transitive closure of \triangleright_r and by \simeq_r the equivalence relation induced by \triangleright_r^* .

The next theorem states that reductions preserve the free variables and the degree of a term.

Theorem 1. *Let $M \in \mathcal{M}$ and $r \in \{\beta, \beta\eta, h\}$.*

1. *If $M \triangleright_\eta^* N$, then $N \in \mathcal{M}$, $\text{fv}(N) = \text{fv}(M)$ and $d(M) = d(N)$.*
2. *If $M \triangleright_r^* N$, then $N \in \mathcal{M}$, $\text{fv}(N) \subseteq \text{fv}(M)$ and $d(M) = d(N)$.*

Proof. 1. By induction on $M \triangleright_\eta^* N$.

2. Case $r = \beta$. By induction on $M \triangleright_\beta^* N$.

Case $r = \beta\eta$, by the β and η cases.

Case $r = h$, by the β case. □

As expansions change the degree of a term, indexes in a term need to increase/decrease.

Definition 3. Let $i \in \mathbb{N}$ and $M \in \mathcal{M}$.

1. We define M^{+i} by:
 - $(x^L)^{+i} = x^{i::L}$ • $(M_1 M_2)^{+i} = M_1^{+i} M_2^{+i}$ • $(\lambda x^L.M)^{+i} = \lambda x^{i::L}.M^{+i}$
2. If $d(M) = i :: L$, we define M^{-i} by:
 - $(x^{i::K})^{-i} = x^K$
 - $(M_1 M_2)^{-i} = M_1^{-i} M_2^{-i}$ • $(\lambda x^{i::K}.M)^{-i} = \lambda x^K.M^{-i}$

Normal forms are defined as usual.

Definition 4. 1. $M \in \mathcal{M}$ is in β - (resp. $\beta\eta$ -, h -) normal form if there is no $N \in \mathcal{M}$ such that $M \triangleright_\beta N$ (resp. $M \triangleright_{\beta\eta} N$, $M \triangleright_h N$).

2. $M \in \mathcal{M}$ is β -normalising (resp. $\beta\eta$ -normalising, h -normalising) if there is an $N \in \mathcal{M}$ such that $M \triangleright_\beta^* N$ (resp. $M \triangleright_{\beta\eta} N$, $M \triangleright_h N$) and N is in β -normal form (resp. $\beta\eta$ -normal form, h -normal form).

Finally, β , $\beta\eta$ and h reductions are confluent on the indexed lambda calculus:

Theorem 2 (Confluence). Let $M, M_1, M_2 \in \mathcal{M}$ and $r \in \{\beta, \beta\eta, h\}$.

1. If $M \triangleright_r^* M_1$ and $M \triangleright_r^* M_2$, then there is M' such that $M_1 \triangleright_r^* M'$ and $M_2 \triangleright_r^* M'$.
2. $M_1 \simeq_r M_2$ iff there is a term M such that $M_1 \triangleright_r^* M$ and $M_2 \triangleright_r^* M$.

Proof. We establish the confluence using the standard parallel reduction method. Full details can be found in [12]. \square

3 Typing system

This paper studies a type system for the indexed λ -calculus with the universal type ω . In this type system, in order to get subject reduction and hence completeness, intersections and expansions cannot occur directly to the right of an arrow (see \mathbb{U} below).

The next two definitions introduce the type system.

Definition 5. 1. Let a countably infinite set \mathcal{A} of atomic types and $\mathcal{E} = \{e_0, e_1, \dots\}$ a countably infinite set of expansion variables. We define sets of types \mathbb{T} and \mathbb{U} , such that $\mathbb{T} \subseteq \mathbb{U}$, and a function $d : \mathbb{U} \rightarrow \mathcal{L}_{\mathbb{N}}$ by:

- If $a \in \mathcal{A}$, then $a \in \mathbb{T}$ and $d(a) = \emptyset$.
- If $U \in \mathbb{U}$ and $T \in \mathbb{T}$, then $U \rightarrow T \in \mathbb{T}$ and $d(U \rightarrow T) = \emptyset$.
- If $L \in \mathcal{L}_{\mathbb{N}}$, then $\omega^L \in \mathbb{U}$ and $d(\omega^L) = L$.
- If $U_1, U_2 \in \mathbb{U}$ and $d(U_1) = d(U_2)$, then $U_1 \sqcap U_2 \in \mathbb{U}$ and $d(U_1 \sqcap U_2) = d(U_1) = d(U_2)$.
- $U \in \mathbb{U}$ and $e_i \in \mathcal{E}$, then $e_i U \in \mathbb{U}$ and $d(e_i U) = i :: d(U)$.

Note that d remembers the number of the expansion variables e_i in order to keep a trace of these variables.

We let T range over \mathbb{T} , and U, V, W range over \mathbb{U} . We quotient types by taking \sqcap to be commutative (i.e. $U_1 \sqcap U_2 = U_2 \sqcap U_1$), associative (i.e. $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$) and idempotent (i.e. $U \sqcap U = U$), by assuming the distributivity of expansion variables over \sqcap (i.e. $e_i(U_1 \sqcap U_2) = e_i U_1 \sqcap e_i U_2$) and by having ω^L as a neutral (i.e. $\omega^L \sqcap U = U$). We denote $U_n \sqcap U_{n+1} \dots \sqcap U_m$ by $\sqcap_{i=n}^m U_i$ (when $n \leq m$). We also assume that for all $i \geq 0$ and $K \in \mathcal{L}_{\mathbb{N}}$, $e_i \omega^K = \omega^{i::K}$.

2. We denote $e_{i_1} \dots e_{i_n}$ by \mathbf{e}_K , where $K = (i_1, \dots, i_n)$ and $U_n \sqcap U_{n+1} \dots \sqcap U_m$ by $\sqcap_{i=n}^m U_i$ (when $n \leq m$).

Definition 6. 1. A type environment is a set $\{x_1^{L_1} : U_1, \dots, x_n^{L_n} : U_n\}$ such that for all $i \in \{1, \dots, n\}$, $d(U_i) = L_i$ and for all $i, j \in \{1, \dots, n\}$, if $x_i^{L_i} = x_j^{L_j}$ then $U_i = U_j$. We let Γ, Δ range over environments and write $()$ for the empty environment. We define $\text{dom}(\Gamma) = \{x^L / x^L : U \in \Gamma\}$. If $\text{dom}(\Gamma_1) \cap \text{dom}(\Gamma_2) = \emptyset$, we write Γ_1, Γ_2 for $\Gamma_1 \cup \Gamma_2$. We write $\Gamma, x^L : U$ for $\Gamma, \{x^L : U\}$ and $x^L : U$ for $\{x^L : U\}$. We denote $x_1^{L_1} : U_1, \dots, x_n^{L_n} : U_n$ by $(x_i^{L_i} : U_i)_n$.

2. If $M \in \mathcal{M}$ and $\text{fv}(M) = \{x_1^{L_1}, \dots, x_n^{L_n}\}$, we denote env_M^ω the type environment $(x_i^{L_i} : \omega^{L_i})_n$.
3. Let $\Gamma_1 = (x_i^{L_i} : U_i)_n, \Gamma'_1, \Gamma_2 = (x_i^{L_i} : U'_i)_n, \Gamma'_2$ and $\text{dom}(\Gamma'_1) \cap \text{dom}(\Gamma'_2) = \emptyset$. We denote $\Gamma_1 \sqcap \Gamma_2$ the type environment $(x_i^{L_i} : U_i \sqcap U'_i)_n, \Gamma'_1, \Gamma'_2$. Note that $\text{dom}(\Gamma_1 \sqcap \Gamma_2) = \text{dom}(\Gamma_1) \cup \text{dom}(\Gamma_2)$ and that, on environments, \sqcap is commutative, associative and idempotent.
4. Let $\Gamma = (x_i^{L_i} : U_i)_{1 \leq i \leq n}$ and $e_j \in \mathcal{E}$. We denote $e_j \Gamma = (x_i^{j::L_i} : e_j U_i)_{1 \leq i \leq n}$. Note that $e_j(\Gamma_1 \sqcap \Gamma_2) = e_j \Gamma_1 \sqcap e_j \Gamma_2$.
5. We write $\Gamma_1 \diamond \Gamma_2$ iff $x^L \in \text{dom}(\Gamma_1)$ and $x^K \in \text{dom}(\Gamma_2)$ implies $K = L$.
6. We follow [3] and write type judgements as $M : \langle \Gamma \vdash U \rangle$ instead of the traditional format of $\Gamma \vdash M : U$, where \vdash is our typing relation. The typing rules of \vdash are given on the left hand side of Figure 6. In the last clause, the binary relation \sqsubseteq is defined on \mathbb{U} by the rules on the right hand side of Figure 6. We let Φ denote types in \mathbb{U} , or environments Γ or typings $\langle \Gamma \vdash U \rangle$. When $\Phi \sqsubseteq \Phi'$, then Φ and Φ' belong to the same set (\mathbb{U} /environments/typings).
7. If $L \in \mathcal{L}_{\mathbb{N}}$, $U \in \mathbb{U}$ and $\Gamma = (x_i^{L_i} : U_i)_n$ is a type environment, we say that:
 - $d(\Gamma) \succeq L$ if and only if for all $i \in \{1, \dots, n\}$, $d(U_i) = L_i \succeq L$.
 - $d(\langle \Gamma \vdash U \rangle) \succeq L$ if and only if $d(\Gamma) \succeq L$ and $d(U) \succeq L$.

To illustrate how our indexes type system works, we give an example:

Example 1. Let $U = e_3(e_2(e_1((e_0 b \rightarrow c) \rightarrow (e_0(a \sqcap (a \rightarrow b)) \rightarrow c)) \rightarrow d) \rightarrow ((e_2 d \rightarrow a) \sqcap b) \rightarrow a))$ where $a, b, c, d \in \mathcal{A}$,

$L_1 = 3 :: \emptyset \preceq L_2 = 3 :: 2 :: \emptyset \preceq L_3 = 3 :: 2 :: 1 :: 0 :: \emptyset$
and $M = \lambda x^{L_2}. \lambda y^{L_1}. (y^{L_1} (x^{L_2} \lambda u^{L_3}. \lambda v^{L_3}. (u^{L_3} (v^{L_3} v^{L_3}))))$.

We invite the reader to check that $M : \langle () \vdash U \rangle$.

Just as we did for terms, we decrease the indexes of types, environments and typings.

$\frac{}{x^\circ : \langle (x^\circ : T) \vdash T \rangle} \text{ (ax)}$	$\frac{}{\Phi \sqsubseteq \Phi} \text{ (ref)}$
$\frac{}{M : \langle \text{env}_M^\omega \vdash \omega^{\text{d}(M)} \rangle} \text{ (\omega)}$	$\frac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3} \text{ (tr)}$
$\frac{M : \langle \Gamma, (x^L : U) \vdash T \rangle}{\lambda x^L. M : \langle \Gamma \vdash U \rightarrow T \rangle} \text{ (\rightarrow_I)}$	$\frac{}{U_1 \sqcap U_2 \sqsubseteq U_1} \text{ (\sqcap_E)}$
$\frac{M : \langle \Gamma \vdash T \rangle \quad x^L \notin \text{dom}(\Gamma)}{\lambda x^L. M : \langle \Gamma \vdash \omega^L \rightarrow T \rangle} \text{ (\rightarrow'_I)}$	$\frac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2} \text{ (\sqcap)}$
$\frac{M_1 : \langle \Gamma_1 \vdash U \rightarrow T \rangle \quad M_2 : \langle \Gamma_2 \vdash U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle} \text{ (\rightarrow_E)}$	$\frac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \rightarrow T_1 \sqsubseteq U_2 \rightarrow T_2} \text{ (\rightarrow)}$
$\frac{M : \langle \Gamma \vdash U_1 \rangle \quad M : \langle \Gamma \vdash U_2 \rangle}{M : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle} \text{ (\sqcap_I)}$	$\frac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2} \text{ (\sqsubseteq_e)}$
$\frac{M : \langle \Gamma \vdash U \rangle}{M^{+j} : \langle e_j \Gamma \vdash e_j U \rangle} \text{ (e)}$	$\frac{U_1 \sqsubseteq U_2}{\Gamma, y^L : U_1 \sqsubseteq \Gamma, y^L : U_2} \text{ (\sqsubseteq_c)}$
$\frac{M : \langle \Gamma \vdash U \rangle \quad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{M : \langle \Gamma' \vdash U' \rangle} \text{ (\sqsubseteq)}$	$\frac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash U_1 \rangle \sqsubseteq \langle \Gamma_2 \vdash U_2 \rangle} \text{ (\sqsubseteq_\diamond)}$

Fig. 1. Typing rules / Subtyping rules

- Definition 7.** 1. If $d(U) \succeq L$, then if $L = \circ$ then $U^{-L} = U$ else $L = i :: K$ and we inductively define the type U^{-L} as follows:
- $$(U_1 \sqcap U_2)^{-i::K} = U_1^{-i::K} \sqcap U_2^{-i::K} \quad (e_i U)^{-i::K} = U^{-K}$$
- We write U^{-i} instead of $U^{-(i)}$.
2. If $\Gamma = (x_i^{L_i} : U_i)_k$ and $d(\Gamma) \succeq L$, then for all $i \in \{1, \dots, k\}$, $L_i = L :: L'_i$ and Γ^{-L} denote $(x_i^{L'_i} : U_i^{-L})_k$. We write Γ^{-i} instead of $\Gamma^{-(i)}$.
3. If U is a type and Γ is a type environment such that $d(\Gamma) \succeq K$ and $d(U) \succeq K$, then we denote $(\langle \Gamma \vdash U \rangle)^{-K} = \langle \Gamma^{-K} \vdash U^{-K} \rangle$.

The next lemma is informative about types and their degrees.

- Lemma 2.** 1. If $T \in \mathbb{T}$, then $d(T) = \circ$.
2. Let $U \in \mathbb{U}$. If $d(U) = L = (n_i)_m$, then $U = \omega^L$ or $U = e_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and for all $i \in \{1, \dots, p\}$, $T_i \in \mathbb{T}$.
3. Let $U_1 \sqsubseteq U_2$.
- $d(U_1) = d(U_2)$.
 - If $U_1 = \omega^K$ then $U_2 = \omega^K$.
 - If $U_1 = e_K U$ then $U_2 = e_K U'$ and $U \sqsubseteq U'$.
 - If $U_2 = e_K U$ then $U_1 = e_K U'$ and $U \sqsubseteq U'$.
 - If $U_1 = \sqcap_{i=1}^p e_K (U_i \rightarrow T_i)$ where $p \geq 1$ then $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^q e_K (U'_j \rightarrow T'_j)$ where $q \geq 1$ and for all $j \in \{1, \dots, q\}$, there exists $i \in \{1, \dots, p\}$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.

4. If $U \in \mathbb{U}$ such that $d(U) = L$ then $U \sqsubseteq \omega^L$.
5. If $U \sqsubseteq U'_1 \sqcap U'_2$ then $U = U_1 \sqcap U_2$ where $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
6. If $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.

The next lemma says how ordering and the decreasing of indexes propagate to environments. Its proof as well as the proof of the previous lemma can be found in [12].

- Lemma 3.**
1. If $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $x^L \notin \text{dom}(\Gamma)$ then $\Gamma, (x^L : U) \sqsubseteq \Gamma', (x^L : U')$.
 2. $\Gamma \sqsubseteq \Gamma'$ iff $\Gamma = (x_i^{L_i} : U_i)_n$, $\Gamma' = (x_i^{L_i} : U'_i)_n$ and for all $i \in \{1, \dots, n\}$, $U_i \sqsubseteq U'_i$.
 3. $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$ iff $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$.
 4. If $\text{dom}(\Gamma) = \text{fv}(M)$, then $\Gamma \sqsubseteq \text{env}_M^\omega$.
 5. If $\Gamma \diamond \Delta$ and $d(\Gamma), d(\Delta) \succeq K$, then $\Gamma^{-K} \diamond \Delta^{-K}$.
 6. If $U \sqsubseteq U'$ and $d(U) \succeq K$ then $U^{-K} \sqsubseteq U'^{-K}$.
 7. If $\Gamma \sqsubseteq \Gamma'$ and $d(\Gamma) \succeq K$ then $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.

The next lemma shows that we do not allow weakening in \vdash .

- Lemma 4.**
1. For every Γ and M such that $\text{dom}(\Gamma) = \text{fv}(M)$ and $d(M) = K$, we have $M : \langle \Gamma \vdash \omega^K \rangle$.
 2. If $M : \langle \Gamma \vdash U \rangle$, then $\text{dom}(\Gamma) = \text{fv}(M)$.
 3. If $M_1 : \langle \Gamma_1 \vdash U \rangle$ and $M_2 : \langle \Gamma_2 \vdash U \rangle$ then $\Gamma_1 \diamond \Gamma_2$ iff $M_1 \diamond M_2$.

Proof. 1. By ω , $M : \langle \text{env}_M^\omega \vdash \omega^K \rangle$. By Lemma 3.4, $\Gamma \sqsubseteq \text{env}_M^\omega$. Hence, by \sqsubseteq and $\sqsubseteq_{\langle \rangle}$, $M : \langle \Gamma \vdash \omega^K \rangle$.

2. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$.

3. If) Let $x^L \in \text{dom}(\Gamma_1)$ and $x^K \in \text{dom}(\Gamma_2)$ then by Lemma 4.2, $x^L \in \text{fv}(M_1)$ and $x^K \in \text{fv}(M_2)$ so $\Gamma_1 \diamond \Gamma_2$. Only if) Let $x^L \in \text{fv}(M_1)$ and $x^K \in \text{fv}(M_2)$ then by Lemma 4.2, $x^L \in \text{dom}(\Gamma_1)$ and $x^K \in \text{dom}(\Gamma_2)$ so $M_1 \diamond M_2$. \square

The next theorem states that within a typing, degrees are well behaved.

Theorem 3. Let $M : \langle \Gamma \vdash U \rangle$.

1. $d(\Gamma) \succeq d(U) = d(M)$.
2. If $d(U) \succeq K$ then $M^{-K} : \langle \Gamma^{-K} \vdash U^{-K} \rangle$.

Proof. Both cases are by induction on the derivation $M : \langle \Gamma \vdash U \rangle$. Full details can be found in [12]. \square

Finally, here are two derivable typing rules (for proofs see [12]).

- Remark 1.*
1. The rule
$$\frac{M : \langle \Gamma_1 \vdash U_1 \rangle \quad M : \langle \Gamma_2 \vdash U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2 \rangle} \sqcap'_T$$
 is derivable.
 2. The rule
$$\frac{}{x^{d(U)} : \langle (x^{d(U)} : U) \vdash U \rangle} ax'$$
 is derivable.

4 Subject reduction properties

In this section we show that subject reduction holds for \vdash . The proof of subject reduction uses generation and substitution. Hence the next two lemmas. For full proofs see [12].

Lemma 5 (Generation for \vdash).

1. If $x^L : \langle \Gamma \vdash U \rangle$, then $\Gamma = (x^L : V)$ and $V \sqsubseteq U$.
2. If $\lambda x^L.M : \langle \Gamma \vdash U \rangle$, $x^L \in \text{fv}(M)$ and $d(U) = K$, then $U = \omega^K$ or $U = \prod_{i=1}^p e_K(V_i \rightarrow T_i)$ where $p \geq 1$ and for all $i \in \{1, \dots, p\}$, $M : \langle \Gamma, x^L : e_K V_i \vdash e_K T_i \rangle$.
3. If $\lambda x^L.M : \langle \Gamma \vdash U \rangle$, $x^L \notin \text{fv}(M)$ and $d(U) = K$, then $U = \omega^K$ or $U = \prod_{i=1}^p e_K(V_i \rightarrow T_i)$ where $p \geq 1$ and for all $i \in \{1, \dots, p\}$, $M : \langle \Gamma \vdash e_K T_i \rangle$.
4. If $M x^L : \langle \Gamma, (x^L : U) \vdash T \rangle$ and $x^L \notin \text{fv}(M)$, then $M : \langle \Gamma \vdash U \rightarrow T \rangle$.

Proof. 1. By induction on the derivation $x^L : \langle \Gamma \vdash U \rangle$. 2. By induction on the derivation $\lambda x^L.M : \langle \Gamma \vdash U \rangle$. 3. Same proof as that of 2. 4. By induction on the derivation $M x^L : \langle \Gamma, x^L : U \vdash T \rangle$. \square

Lemma 6 (Substitution for \vdash). If $M : \langle \Gamma, x^L : U \vdash V \rangle$, $N : \langle \Delta \vdash U \rangle$ and $\Gamma \diamond \Delta$ then $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash V \rangle$.

Proof. By induction on the derivation $M : \langle \Gamma, x^L : U \vdash V \rangle$. \square

Since \vdash does not allow weakening, we need the next definition since when a term is reduced, it may lose some of its free variables and hence will need to be typed in a smaller environment.

Definition 8. If Γ is a type environment and $\mathcal{U} \subseteq \text{dom}(\Gamma)$, then we write $\Gamma \upharpoonright_{\mathcal{U}}$ for the restriction of Γ on the variables of \mathcal{U} . If $\mathcal{U} = \text{fv}(M)$ for a term M , we write $\Gamma \upharpoonright_M$ instead of $\Gamma \upharpoonright_{\text{fv}(M)}$.

Now we are ready to prove the main result of this section:

Theorem 4 (Subject reduction for \vdash). If $M : \langle \Gamma \vdash U \rangle$ and $M \triangleright_{\beta\eta}^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.

Proof. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$. \square

Corollary 1. 1. If $M : \langle \Gamma \vdash U \rangle$ and $M \triangleright_{\beta}^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.
2. If $M : \langle \Gamma \vdash U \rangle$ and $M \triangleright_h^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.

5 Subject expansion properties

In this section we show that subject β -expansion holds for \vdash but that subject η -expansion fails. Again, full proofs can be found in [12].

The next lemma is needed for expansion.

Lemma 7. *If $M[x^L := N] : \langle \Gamma \vdash U \rangle$, $d(N) = L$ and $x^L \in \text{fv}(M)$ then there exist a type V and two type environments Γ_1, Γ_2 such that $d(V) = L$ and:*
 $M : \langle \Gamma_1, x^L : V \vdash U \rangle \quad N : \langle \Gamma_2 \vdash V \rangle \quad \Gamma = \Gamma_1 \sqcap \Gamma_2$

Proof. By induction on the derivation $M[x^L := N] : \langle \Gamma \vdash U \rangle$. \square

Since more free variables might appear in the β -expansion of a term, the next definition gives a possible enlargement of an environment.

Definition 9. *Let $m \geq n$, $\Gamma = (x_i^{L_i} : U_i)_n$ and $\mathcal{U} = \{x_1^{L_1}, \dots, x_m^{L_m}\}$. We write $\Gamma \uparrow^{\mathcal{U}}$ for $x_1^{L_1} : U_1, \dots, x_n^{L_n} : U_n, x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, \dots, x_m^{L_m} : \omega^{L_m}$. If $\text{dom}(\Gamma) \subseteq \text{fv}(M)$, we write $\Gamma \uparrow^M$ instead of $\Gamma \uparrow^{\text{fv}(M)}$.*

We are now ready to establish that subject expansion holds for β (next theorem) and that it fails for η (Lemma 8).

Theorem 5 (Subject expansion for β). *If $N : \langle \Gamma \vdash U \rangle$ and $M \triangleright_{\beta}^* N$, then $M : \langle \Gamma \uparrow^M \vdash U \rangle$.*

Proof. By induction on the length of the derivation $M \triangleright_{\beta}^* N$ using the fact that if $\text{fv}(P) \subseteq \text{fv}(Q)$, then $(\Gamma \uparrow^P) \uparrow^Q = \Gamma \uparrow^Q$. \square

Corollary 2. *If $N : \langle \Gamma \vdash U \rangle$ and $M \triangleright_h^* N$, then $M : \langle \Gamma \uparrow^M \vdash U \rangle$.*

Lemma 8 (Subject expansion fails for η). *Let a be an element of \mathcal{A} . We have:*

1. $\lambda y^{\circ} . \lambda x^{\circ} . y^{\circ} x^{\circ} \triangleright_{\eta} \lambda y^{\circ} . y^{\circ}$
2. $\lambda y^{\circ} . y^{\circ} : \langle () \vdash a \rightarrow a \rangle$.
3. *It is not possible that: $\lambda y^{\circ} . \lambda x^{\circ} . y^{\circ} x^{\circ} : \langle () \vdash a \rightarrow a \rangle$.*

Hence, the subject η -expansion lemmas fail for \vdash .

Proof. 1. and 2. are easy. For 3., assume $\lambda y^{\circ} . \lambda x^{\circ} . y^{\circ} x^{\circ} : \langle () \vdash a \rightarrow a \rangle$. By Lemma 5.2, $\lambda x^{\circ} . y^{\circ} x^{\circ} : \langle (y : a) \vdash a \rangle$. Again, by Lemma 5.2, $a = \omega^{\circ}$ or there exists $n \geq 1$ such that $a = \prod_{i=1}^n (U_i \rightarrow T_i)$, absurd. \square

6 The realisability semantics

In this section we introduce the realisability semantics and show its soundness for \vdash . Again, full proofs can be found in [12].

Crucial to a realisability semantics is the notion of a saturated set:

Definition 10. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$.*

1. *We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of \mathcal{X} , i.e. $\{\mathcal{Y} / \mathcal{Y} \subseteq \mathcal{X}\}$.*
2. *We define $\mathcal{X}^{+i} = \{M^{+i} / M \in \mathcal{X}\}$.*
3. *We define $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M} / \forall N \in \mathcal{X}. M \diamond N \Rightarrow MN \in \mathcal{Y}\}$.*
4. *We say that $\mathcal{X} \wr \mathcal{Y}$ iff for all $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, there exist $N \in \mathcal{X}$ such that $M \diamond N$.*

5. For $r \in \{\beta, \beta\eta, h\}$, we say that \mathcal{X} is r -saturated if whenever $M \triangleright_r^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.

Saturation is closed under intersection, lifting and arrows:

- Lemma 9.**
1. $(\mathcal{X} \cap \mathcal{Y})^{+i} = \mathcal{X}^{+i} \cap \mathcal{Y}^{+i}$.
 2. If \mathcal{X}, \mathcal{Y} are r -saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is r -saturated.
 3. If \mathcal{X} is r -saturated, then \mathcal{X}^{+i} is r -saturated.
 4. If \mathcal{Y} is r -saturated, then, for every set \mathcal{X} , $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is r -saturated.
 5. $(\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i} \subseteq \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.
 6. If $\mathcal{X}^+ \wr \mathcal{Y}^+$, then $\mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+ \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.

We now give the basic step in our realisability semantics: the interpretations and meanings of types.

Definition 11. Let $\mathcal{V}_1, \mathcal{V}_2$ be countably infinite, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.

1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We define $\mathcal{M}^L = \{M \in \mathcal{M} / d(M) = L\}$.
2. Let $x \in \mathcal{V}_1$. We define $\mathcal{N}_x^L = \{x^L N_1 \dots N_k \in \mathcal{M} / k \geq 0\}$.
3. Let $r \in \{\beta, \beta\eta, h\}$. An r -interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^{\otimes})$ is a function such that for all $a \in \mathcal{A}$:
 - $\mathcal{I}(a)$ is r -saturated
 - and
 - $\forall x \in \mathcal{V}_1. \mathcal{N}_x^{\otimes} \subseteq \mathcal{I}(a)$.
 We extend an r -interpretation \mathcal{I} to \mathbb{U} as follows:
 - $\mathcal{I}(\omega^L) = \mathcal{M}^L$
 - $\mathcal{I}(e_i U) = \mathcal{I}(U)^{+i}$
 - $\mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2)$
 - $\mathcal{I}(U \rightarrow T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$
 Let $r\text{-int} = \{\mathcal{I} / \mathcal{I} \text{ is an } r\text{-interpretation}\}$.
4. Let $U \in \mathbb{U}$ and $r \in \{\beta, \beta\eta, h\}$. Define $[U]_r$, the r -interpretation of U by:

$$[U]_r = \{M \in \mathcal{M} / M \text{ is closed and } M \in \bigcap_{\mathcal{I} \in r\text{-int}} \mathcal{I}(U)\}$$

Lemma 10. Let $r \in \{\beta, \beta\eta, h\}$.

1. (a) For any $U \in \mathbb{U}$ and $\mathcal{I} \in r\text{-int}$, we have $\mathcal{I}(U)$ is r -saturated.
 (b) If $d(U) = L$ and $\mathcal{I} \in r\text{-int}$, then for all $x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$.
2. Let $r \in \{\beta, \beta\eta, h\}$. If $\mathcal{I} \in r\text{-int}$ and $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.

Proof. 1a . By induction on T using lemma 9.

1b. We prove $\forall x \in \mathcal{V}_1. \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$ by induction on U .

2. By induction of the derivation $U \sqsubseteq V$. □

Here is the soundness lemma.

Lemma 11 (Soundness). Let $r \in \{\beta, \beta\eta, h\}$, $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$, $\mathcal{I} \in r\text{-int}$ and for all $j \in \{1, \dots, n\}$, $N_j \in \mathcal{I}(U_j)$. We have $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$.

Proof. By induction on the derivation $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$. □

Corollary 3. Let $r \in \{\beta, \beta\eta, h\}$. If $M : \langle () \vdash U \rangle$, then $M \in [U]_r$.

Proof. By Lemma 11, $M \in \mathcal{I}(U)$ for any r -interpretation \mathcal{I} . By Lemma 4, $\text{fv}(M) = \text{dom}(\langle () \vdash U \rangle) = \emptyset$ and hence M is closed. Therefore, $M \in [U]_r$. □

Lemma 12 (The meaning of types is closed under type operations).

Let $r \in \{\beta, \beta\eta, h\}$. On \mathbb{U} , the following hold:

1. $[e_i U]_r = [U]_r^{+i}$
2. $[U \sqcap V]_r = [U]_r \cap [V]_r$
3. If $U \rightarrow T \in \mathbb{U}$ then for any interpretation \mathcal{I} , $\mathcal{I}(U) \wr \mathcal{I}(T)$.

Proof. 1. and 2. are easy. 3. Let $d(U) = K$, $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and $x \in \mathcal{V}_1$ such that for all L , $x^L \notin \text{fv}(M)$, hence $M \diamond x^K$ and $x^K \in \mathcal{I}(U)$. \square

The next definition and lemma put the realisability semantics in use.

Definition 12 (Examples). Let $a, b \in \mathcal{A}$ where $a \neq b$. We define:

- $Id_0 = a \rightarrow a$, $Id_1 = e_1(a \rightarrow a)$ and $Id'_1 = e_1 a \rightarrow e_1 a$.
- $D = (a \sqcap (a \rightarrow b)) \rightarrow b$.
- $Nat_0 = (a \rightarrow a) \rightarrow (a \rightarrow a)$, $Nat_1 = e_1((a \rightarrow a) \rightarrow (a \rightarrow a))$,
and $Nat'_0 = (e_1 a \rightarrow a) \rightarrow (e_1 a \rightarrow a)$.

Moreover, if M, N are terms and $n \in \mathbb{N}$, we define $(M)^n N$ by induction on n :
 $(M)^0 N = N$ and $(M)^{m+1} N = M ((M)^m N)$.

- Lemma 13.**
1. $[Id_0]_\beta = \{M \in \mathcal{M}^\circ / M \triangleright_\beta^* \lambda y^\circ y^\circ\}$.
 2. $[Id_1]_\beta = [Id'_1]_\beta = \{M \in \mathcal{M}^{(1)} / M \triangleright_\beta^* \lambda y^{(1)}.y^{(1)}\}$. (Note that $Id'_1 \notin \mathbb{U}$.)
 3. $[D]_\beta = \{M \in \mathcal{M}^\circ / M \triangleright_\beta^* \lambda y^\circ.y^\circ y^\circ\}$.
 4. $[Nat_0]_\beta = \{M \in \mathcal{M}^\circ / M \triangleright_\beta^* \lambda f^\circ.f^\circ$ or $M \triangleright_\beta^* \lambda f^\circ.\lambda y^\circ.(f^\circ)^n y^\circ$ where $n \geq 1\}$.
 5. $[Nat_1]_\beta = \{M \in \mathcal{M}^{(1)} / M \triangleright_\beta^* \lambda f^{(1)}.f^{(1)}$ or $M \triangleright_\beta^* \lambda f^{(1)}.\lambda x^{(1)}.(f^{(1)})^n y^{(1)}$ where $n \geq 1\}$. (Note that $Nat'_1 \notin \mathbb{U}$.)
 6. $[Nat'_0]_\beta = \{M \in \mathcal{M}^\circ / M \triangleright_\beta^* \lambda f^\circ.f^\circ$ or $M \triangleright_\beta^* \lambda f^\circ.\lambda y^{(1)}.f^\circ y^{(1)}\}$.

7 The completeness theorem

In this section we set out the machinery and prove that completeness holds for \vdash . Again, full proofs can be found in [12].

We need the following partition of the set of variables $\{y^L/y \in \mathcal{V}_2\}$.

Definition 13. 1. Let $L \in \mathcal{L}_\mathbb{N}$. We define $\mathbb{U}^L = \{U \in \mathbb{U} / d(U) = L\}$ and $\mathcal{V}^L = \{x^L / x \in \mathcal{V}_2\}$.

2. Let $U \in \mathbb{U}$. We inductively define a set of variables \mathbb{V}_U as follows:

- If $d(U) = \circ$ then:
 - \mathbb{V}_U is an infinite set of variables of degree \circ .
 - If $U \neq V$ and $d(U) = d(V) = \circ$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.
 - $\bigcup_{U \in \mathbb{U}^\circ} \mathbb{V}_U = \mathcal{V}^\circ$.
- If $d(U) = L$, then we put $\mathbb{V}_U = \{y^L / y^\circ \in \mathbb{V}_{U-L}\}$.

Lemma 14. 1. If $d(U), d(V) \succeq L$ and $U^{-L} = V^{-L}$, then $U = V$.

2. If $d(U) = L$, then \mathbb{V}_U is an infinite subset of \mathcal{V}^L .

3. If $U \neq V$ and $d(U) = d(V) = L$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.
4. $\bigcup_{U \in \mathbb{U}^L} \mathbb{V}_U = \mathcal{V}^L$.
5. If $y^L \in \mathbb{V}_U$, then $y^{i::L} \in \mathbb{V}_{e_i U}$.
6. If $y^{i::L} \in \mathbb{V}_U$, then $y^L \in \mathbb{V}_{U^{-i}}$.

Proof. 1. If $L = (n_i)_m$, we have $U = e_{n_1} \dots e_{n_m} U'$ and $V = e_{n_1} \dots e_{n_m} V'$. Then $U^{-L} = U'$, $V^{-L} = V'$ and $U' = V'$. Thus $U = V$. 2. 3. and 4. By induction on L and using 1. 5. Because $(e_i U)^{-i} = U$. 6. By definition. \square

Our partition of the set \mathcal{V}_2 as above will enable us to give in the next definition useful infinite sets which will contain type environments that will play a crucial role in one particular type interpretation.

Definition 14. 1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We denote $\mathbb{G}^L = \{(y^L : U) / U \in \mathbb{U}^L \text{ and } y^L \in \mathbb{V}_U\}$ and $\mathbb{H}^L = \bigcup_{K \succeq L} \mathbb{G}^K$. Note that \mathbb{G}^L and \mathbb{H}^L are not type environments because they are infinite sets.

2. Let $L \in \mathcal{L}_{\mathbb{N}}$, $M \in \mathcal{M}$ and $U \in \mathbb{U}$, we write:

- $M : \langle \mathbb{H}^L \vdash U \rangle$ if there is a type environment $\Gamma \subset \mathbb{H}^L$ where $M : \langle \Gamma \vdash U \rangle$
- $M : \langle \mathbb{H}^L \vdash^* U \rangle$ if $M \triangleright_{\beta\eta}^* N$ and $N : \langle \mathbb{H}^L \vdash U \rangle$

Lemma 15. 1. If $\Gamma \subset \mathbb{H}^L$ then $e_i \Gamma \subset \mathbb{H}^{i::L}$.

2. If $\Gamma \subset \mathbb{H}^{i::L}$ then $\Gamma^{-i} \subset \mathbb{H}^L$.

3. If $\Gamma_1 \subset \mathbb{H}^L$, $\Gamma_2 \subset \mathbb{H}^K$ and $L \preceq K$ then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$.

Proof. 1. and 2. By Lemma 14. 3. First note that $\mathbb{H}^K \subseteq \mathbb{H}^L$. Let $(x^R : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^R : U_1) \in \Gamma_1 \subset \mathbb{H}^L$ and $(x^R : U_2) \in \Gamma_2 \subset \mathbb{H}^K \subseteq \mathbb{H}^L$, then $d(U_1) = d(U_2) = R$ and $x^R \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by Lemma 14, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^L$. \square

For every $L \in \mathcal{L}_{\mathbb{N}}$, we define the set of terms of degree L which contain some free variable x^K where $x \in \mathcal{V}_1$ and $K \succeq L$.

Definition 15. For every $L \in \mathcal{L}_{\mathbb{N}}$, let $\mathcal{O}^L = \{M \in \mathcal{M}^L / x^K \in \text{fv}(M), x \in \mathcal{V}_1 \text{ and } K \succeq L\}$. It is easy to see that, for every $L \in \mathcal{L}_{\mathbb{N}}$ and $x \in \mathcal{V}_1$, $\mathcal{N}_x^L \subseteq \mathcal{O}^L$.

Lemma 16. 1. $(\mathcal{O}^L)^{+i} = \mathcal{O}^{i::L}$.

2. If $y \in \mathcal{V}_2$ and $(My^K) \in \mathcal{O}^L$, then $M \in \mathcal{O}^L$.

3. If $M \in \mathcal{O}^L$, $M \diamond N$ and $L \preceq K = d(N)$, then $MN \in \mathcal{O}^L$.

4. If $d(M) = L$, $L \preceq K$, $M \diamond N$ and $N \in \mathcal{O}^K$, then $MN \in \mathcal{O}^L$.

The crucial interpretation \mathbb{I} for the proof of completeness is given as follows:

Definition 16. 1. Let $\mathbb{I}_{\beta\eta}$ be the $\beta\eta$ -interpretation defined by: for all type variables a , $\mathbb{I}_{\beta\eta}(a) = \mathcal{O}^\circ \cup \{M \in \mathcal{M}^\circ / M : \langle \mathbb{H}^\circ \vdash^* a \rangle\}$.

2. Let \mathbb{I}_β be the β -interpretation defined by: for all type variables a , $\mathbb{I}_\beta(a) = \mathcal{O}^\circ \cup \{M \in \mathcal{M}^\circ / M : \langle \mathbb{H}^\circ \vdash a \rangle\}$.

3. Let \mathbb{I}_{eh} be the h -interpretation defined by: for all type variables a , $\mathbb{I}_h(a) = \mathcal{O}^\circ \cup \{M \in \mathcal{M}^\circ / M : \langle \mathbb{H}^\circ \vdash a \rangle\}$.

The next crucial lemma shows that \mathbb{I} is an interpretation and that the interpretation of a type of order L contains terms of order L which are typable in these special environments which are parts of the infinite sets of Definition 14.

Lemma 17. *Let $r \in \{\beta\eta, \beta, h\}$ and $r' \in \{\beta, h\}$*

1. *If $\mathbb{I}_r \in r\text{-int}$ and $a \in \mathcal{A}$ then $\mathbb{I}_r(a)$ is r -saturated and for all $x \in \mathcal{V}_1, \mathcal{N}_x^\circ \subseteq \mathbb{I}_r(a)$.*
2. *If $U \in \mathbb{U}$ and $d(U) = L$, then $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L / M : \langle \mathbb{H}^L \vdash^* U \rangle\}$.*
3. *If $U \in \mathbb{U}$ and $d(U) = L$, then $\mathbb{I}_{r'}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L / M : \langle \mathbb{H}^L \vdash U \rangle\}$.*

Now, we use this crucial \mathbb{I} to establish completeness of our semantics.

Theorem 6 (Completeness of \vdash). *Let $U \in \mathbb{U}$ such that $d(U) = L$.*

1. $[U]_{\beta\eta} = \{M \in \mathcal{M}^L / M \text{ closed, } M \triangleright_{\beta\eta}^* N \text{ and } N : \langle () \vdash U \rangle\}$.
2. $[U]_\beta = [U]_h = \{M \in \mathcal{M}^L / M : \langle () \vdash U \rangle\}$.
3. $[U]_{\beta\eta}$ is stable by reduction. I.e., If $M \in [U]_{\beta\eta}$ and $M \triangleright_{\beta\eta}^* N$ then $N \in [U]_{\beta\eta}$.

Proof. Let $r \in \{\beta, h, \beta\eta\}$.

1. Let $M \in [U]_{\beta\eta}$. Then M is a closed term and $M \in \mathbb{I}_{\beta\eta}(U)$. Hence, by Lemma 17, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}^L / M : \langle \mathbb{H}^L \vdash^* U \rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}^L / M : \langle \mathbb{H}^L \vdash^* U \rangle\}$ and so, $M \triangleright_{\beta\eta}^* N$ and $N : \langle \Gamma \vdash U \rangle$ where $\Gamma \subset \mathbb{H}^L$. By Theorem 1, N is closed and, by Lemma 4.2, $N : \langle () \vdash U \rangle$. Conversely, take M closed such that $M \triangleright_{\beta\eta}^* N$ and $N : \langle () \vdash U \rangle$. Let $\mathcal{I} \in \beta\text{-int}$. By Lemma 11, $N \in \mathcal{I}(U)$. By Lemma 10.1, $\mathcal{I}(U)$ is $\beta\eta$ -saturated. Hence, $M \in \mathcal{I}(U)$. Thus $M \in [U]$.
2. Let $M \in [U]_\beta$. Then M is a closed term and $M \in \mathbb{I}_\beta(U)$. Hence, by Lemma 17, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}^L / M : \langle \mathbb{H}^L \vdash U \rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}^L / M : \langle \mathbb{H}^L \vdash U \rangle\}$ and so, $M : \langle \Gamma \vdash U \rangle$ where $\Gamma \subset \mathbb{H}^L$. By Lemma 4.2, $N : \langle () \vdash U \rangle$. Conversely, take M such that $M : \langle () \vdash U \rangle$. By Lemma 4.2, M is closed. Let $\mathcal{I} \in \beta\text{-int}$. By Lemma 11, $M \in \mathcal{I}(U)$. Thus $M \in [U]_\beta$.
It is easy to see that $[U]_\beta = [U]_h$.
3. Let $M \in [U]$ such that $M \triangleright_{\beta\eta}^* N$. By 1, M is closed, $M \triangleright_{\beta\eta}^* P$ and $P : \langle () \vdash U \rangle$. By confluence Theorem 2, there is Q such that $P \triangleright_{\beta\eta}^* Q$ and $N \triangleright_{\beta\eta}^* Q$. By subject reduction Theorem 4, $Q : \langle () \vdash U \rangle$. By Theorem 1, N is closed and, by 1, $N \in [U]$. \square

8 Conclusion

Expansion may be viewed to work like a multi-layered simultaneous substitution. Moreover, expansion is a crucial part of a procedure for calculating principal typings and helps support compositional type inference. Because the early definitions of expansion were complicated, expansion variables (E-variables) were introduced to simplify and mechanise expansion. The aim of this paper is to give a complete semantics for intersection type systems with expansion variables.

The only earlier attempt (see Kamareddine, Nour, Rahli and Wells [13]) at giving a semantics for expansion variables could only handle the λI -calculus, did not allow a universal type, and was incomplete in the presence of more than one expansion variable. This paper overcomes these difficulties and gives a complete semantics for an intersection type system with an arbitrary (possibly infinite) number of expansion variables using a calculus indexes with finite sequences of natural numbers.

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