The paradoxes and the infinite dazzled ancient mathematics and continue to do so today

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***Abstract*—This paper looks at how ancient mathematicians (and especially the Pythagorean school) were faced by problems/paradoxes associated with the infinite which led them to juggle two systems of numbers: the discrete whole/rationals which were handled arithmetically and the continuous magnitude quantities which were handled geometrically. We look at how approximations and mixed numbers (whole numbers with fractions) helped develop the arithmetization of geoemtry and the development of mathematical analysis and real numbers.**

***Index Terms*—Euclid, Mathematical Analysis, Infinitesimals.**

I. Why did it take so long to develop Real Numbers and Analysis?

> God made the integers; all else is the work of man.
> Kronecker

The concepts and language of mathematics have been under development slowly but surely since ancient times. Despite the obstacles, this development uncovered fascinating results, which include as late as the 20th century, a sound foundation of the theory of the infinitesimal (which is in essence the foundation of mathematics) and the theory of the computable. Well before then, Leibniz (1646–1717) conceived of *automated deduction* where he wanted to find a language L and a method that could carry out proof checking/finding to determine the correctness of statements in L.[1] Leibniz was frustrated by the limitations in expressing thoughts:

> If we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in Arithmetic/Geometry.
> Leibniz

But at the time of Leibniz, expressibility in Arithmetic was far from complete and the real numbers were still not developed. The later development of real analysis[2] would be based on the real numbers and the arithmetisation of geometry.

[1]Now we know, due to later results by Gödel, Church and Turing, that such a method can not work for every statement.

[2]Thanks to Euler who converted the calculus of Newton and Leibniz from a geometrical field to a field where mathematical formulae are analysed.

A. From naturals to intergers and rationals

Natural numbers were long understood, but it may come as a surprise that as late as the 14th century, negative numbers were not known in Europe. In Italy, a double entry bookkeeping system compensated for their absence. Accounts in which debits may be greater than credits were compared without using negative integers. If c and d are in $\mathbb{N}^+$, then *account* $c \ominus d$ has credit c and debit d. Define accounts $= \{m \ominus n | m, n, p, q \in \mathbb{N}^+\}$. Just like the arithmetic $(\mathbb{N}^+, =, +, \cdot, 1)$ on natural numbers $\mathbb{N}^+ = \{1, 2, \cdots\}$ is defined with equality =, addition +, multiplication $\cdot$, and identity element 1 for $\cdot$, we define (accounts, $\cong, +_c, \cdot_c$) by:

- $m \ominus n \cong p \ominus q$ iff $m + q = n + p$.
- $(m \ominus n) +_c (p \ominus q) = (m + p) \ominus (n + q)$.
- $(m \ominus n) \cdot_c (p \ominus q) = (mp + nq) \ominus (mq + np)$.

The integers $(\mathbb{Z}, +_i, \cdot_i, 0_i, 1_i, -\alpha)$ are then defined from the equivalence classes: $[m \ominus n] = \{p \ominus q : p \ominus q \cong m \ominus n\}$ by:

- $\mathbb{Z} = \{[m \ominus n] \mid m, n \in \mathbb{N}^+\}$.
- $[(m \ominus n)] +_i [(p \ominus q)] = [(m \ominus n) +_c (p \ominus q)]$.
- $[(m \ominus n)] \cdot_i [(p \ominus q)] = [(m \ominus n) \cdot_c (p \ominus q)]$.
- Identity 0_i for $+_i$: for any m, n in $\mathbb{N}^+$, $[m \ominus m] = [n \ominus n]$.
- Identity 1_i for $\cdot_i$: take $1_i = [(p+1) \ominus p]$ for any $p \in \mathbb{N}^+$.
- Inverse for $+_i$: if $\alpha = [m \ominus n]$, then $-\alpha = [n \ominus m]$.

Like we defined $(\mathbb{Z}, +_i, \cdot_i, 0_i, 1_i, -\alpha)$ from (accounts, $\cong, +_c, \cdot_c$) which were defined from $(\mathbb{N}^+, =, +, \cdot, 1)$, we can define positive rational numbers $(\mathbb{Q}^+, +_r, \cdot_r, 1_r, \mathbf{a}^{-1})$ from fractions $= \{\frac{m}{n} \mid m, n \in \mathbb{N}^+\}$ where the arithmetic of (fractions, $\asymp, +_f, \cdot_f$) is defined from $(\mathbb{N}^+, =, +, \cdot, 1)$ by:

- $\frac{m}{n} \asymp \frac{p}{q}$ if and only if $mq = np$,
- $\frac{m}{n} +_f \frac{p}{q} = \frac{mq + np}{nq}$ and $\frac{m}{n} \cdot_f \frac{p}{q} = \frac{mp}{nq}$.

Then, we define $(\mathbb{Q}^+, +_r, \cdot_r, 1_r, \mathbf{a}^{-1})$ from equivalence classes $\left[\frac{m}{n}\right] = \left\{\frac{p}{q} \mid \frac{p}{q} \asymp \frac{m}{n}\right\}$ as follows:

- $\mathbb{Q}^+ = \{\left[\frac{m}{n}\right] \mid m, n \in \mathbb{N}^+\}$.
- $\left[\frac{m}{n}\right] +_r \left[\frac{p}{q}\right] = \left[\frac{m}{n} +_f \frac{p}{q}\right]$ and $\left[\frac{m}{n}\right] \cdot_r \left[\frac{p}{q}\right] = \left[\frac{m}{n} \cdot_f \frac{p}{q}\right]$.
- Identity 1_r for $\cdot_r$: take $1_r = \left[\frac{1}{1}\right]$.
- Inverse for $+_r$: $\left[\frac{m}{n}\right]^{-1} = \left[\frac{n}{m}\right]$.

The steps to build $(\mathbb{Z}, +_i, \cdot_i, 0_i, 1_i, -\alpha)$ and $(\mathbb{Q}^+, +_r, \cdot_r, 1_r, \mathbf{a}^{-1})$ lead to a generalisation as follows:

Definition 1: If $(S, \circ)$ is a Commutative Cancellation Semigroup[3] (CCS), then build $(S \times S, \approx, *)$ as follows:

- Define congruence $\approx$ on $S \times S$ based on $(S, \circ)$ by: $(x, y) \approx (u, v)$ iff $x \circ v = y \circ u$.
- The operation $*$ on $S \times S$ inherited from $\circ$ is defined by $(x, y) * (u, v) = (x \circ u, y \circ v)$.

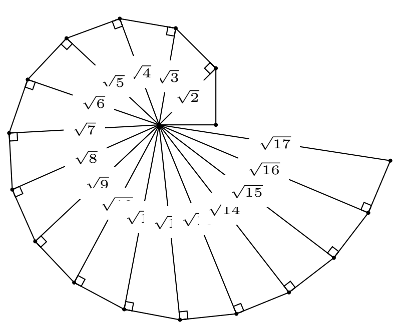
Then, define $[(x, y)] = \{(u, v) : (u, v) \approx (x, y)\}$ and $S_d = \{[(x, y)] : x, y \in S\}$, and build $(s_d, \circ_d, e_d, \mathfrak{a}^{-1})$ as follows:

- Define $[(x, y)]e_d[(u, v)] = [(x, y) * (u, v)] = [(x \circ u, y \circ v)]$. Note that $(S_d, \circ_d)$ is a CCS.
- Note that if $x \in S$, then $x_d = [(y \circ x, y)] \in S_d$.
- Identity: Define e_d to be $[(x, x)]$ for some x in S. For all $\mathfrak{a}$, we have $e_d \circ_d \mathfrak{a} = \mathfrak{a} \circ_d e_d = \mathfrak{a}$.
- Inverses: If $\mathfrak{a} = [(x, y)]$, define $\mathfrak{a}^{-1}$ to be $[(y, x)]$. We have $\mathfrak{a} \circ_d \mathfrak{a}^{-1} = e_d = \mathfrak{a}^{-1} \circ_d \mathfrak{a}$.

Comparing the theory of fractions and the theory of accounts suggests that we can define a unified theory for adding inverses and, if none is present, identity elements.

CCS	$(\mathbb{N}^+, +)$	$(\mathbb{N}^+, \cdot)$
inverses	$\times$	$\times$
Identity element	$\times$	$\surd$
CCS with identity and inverses	$(\mathbb{Z}, +_i)$ $\surd$	$(\mathbb{Q}^+, \cdot_r)$ $\surd$

Just like we built $(\mathbb{Z}, +)$ with identity 0_i and inverses $-\alpha$ from $(\mathbb{N}^+, +)$, we can build $(\mathbb{Q}, +_{r_i})$ with identity and inverses from $(\mathbb{Q}^+, +_r)$. But we cannot build $\mathbb{R}$ this way. The real numbers need to be constructed (using approximations and limits like Dedekind cuts, Cauchy sequences, etc.). This brings us to what is the foundations of mathematics? The foundation of mathematics is reasoning about whether the infinitesimal is sound. Euclid's Elements developed mathematics in geometric terms and anything not expressible in such terms was excluded. Geometry could accommodate the whole numbers and their ratios as well as irrational magnitudes. Think for example of the spiral of Theodorus of Cyrene which established that the square roots of non square integers from 3 to 17 are irrationals.



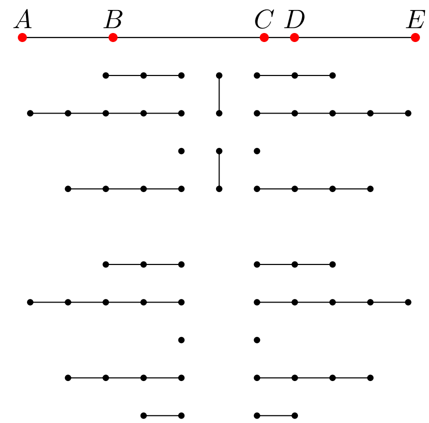
B. Proofs by Pebbles/Diagrams

Knorr [5] suggests that the original proofs were proofs as diagrams using *pebble diagrams*. It is known that the ancient Greeks did arithmetic by counting with pebbles, and pebble diagrams give these calculations by representing the pebbles by using small circles.

Example 1: Here are some statements and their proofs:

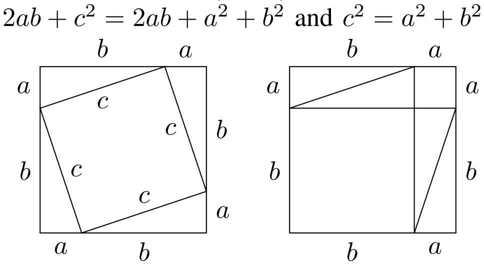
- The square of an odd number is 1 + a multiple of 4. The square of an even number is a multiple of 4.

- *If as many odd numbers as we please be added together, and their multitude be even, then the sum is even.*



The Greeks also mastered the use of geometric proofs:

Example 2: The geometric proof of the Pythagorean Theorem: $c^2 = a^2 + b^2$. The left square shows $(a+b)^2 = 2ab + c^2$ while the right one shows $(a + b)^2 = 2ab + a^2 + b^2$. Hence, $2ab + c^2 = 2ab + a^2 + b^2$ and $c^2 = a^2 + b^2$.



C. Proofs by Contradiction

According to Knorr [5], the change from proofs using diagrams/pebbles to proofs as sequences of statements occurred with the discovery of incommensurability:

Theorem 1: There is no unit which measures exactly the side and diagonal of a square.

Key results needed for the incommensurability proof relate to Pythagorean triples and the theory of Odd/Even Numbers:

Definition 2: Pythagorean triples are triples of positive whole numbers representing the lengths of two legs and the hypotenuse of a right triangle. I.e., a Pythagorean triple is a triple of positive integers (a, b, c) if and only if $a^2 + b^2 = c^2$. E.g. (3, 4, 5), (6, 8, 10), (5, 12, 13), (9, 12, 15), (8, 15, 17).

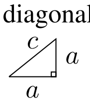
[3] I.e., $\circ$ satisfies closure, commutativity, associativity and cancellation law on S where cancellation means that $a \circ b = a \circ c$ implies $b = c$.

The following are the results needed to prove incommensurability theorem 1. Assume (a, b, c) is a Pythagorean triple.

1. If c is even, then both a and b are even.
2. If c is even, then $(\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$ is also a Pythagorean triple.
3. If c is a multiple of four, then so are a and b.
4. If c is odd, then one of a, b is odd and the other is even.
5. If any two of a, b, c is even, then the third is also even.
6. If one of a, b, c is odd, then two are odd and one is even.

$1 \cdots 6$ above can be shown using diagrams/pebbles. However, theorem 1 itself needs a proof by contradiction:

Proof. Suppose there is such a unit in terms of which, the side of the square is a and the diagonal is c.

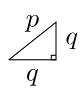
Then, we have a right triangle 

and so (a, a, c) is a Pythagorean triple. Now c must either be even or odd.

- Suppose c even. Then, by 1., a is even. So by 2., we can double the unit and halve all the dimensions. Clearly, we cannot do this indefinitely, since otherwise the unit will grow larger than a.
- So we must have a Pythagorean triple of the form (a, a, c) in which c is odd. But then, by 4., a is both even and odd, a contradiction. □

The proof of incommensurability is believed to be the first proof by contradiction in the history of mathematical proofs. The proof cannot be "seen" by looking at a diagram: it is necessary to follow a sequence of sentences with reasons.

Theorem 1 implies that $\sqrt{2}$ is not a rational number.
Proof: Assume $\sqrt{2} = \frac{p}{q}$, then $2q^2 = p^2$. Hence (q, q, p)



forms a Pythagorean triple. Hence there is a unit which measures exactly the side and diagonal of a square. This contradicts the incommensurability theorem. □

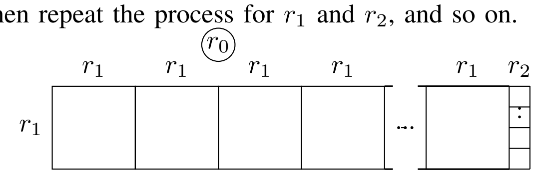
D. Numbers and Magnitudes

With the incommensurability results, the notion of "number" as a discrete collection of units (e.g., naturals or rationals) was no longer enough. There arose a need for numbers that are continuous. The Greeks did not know how to handle these continuous quantities. The main problem was that they treated mathematical objects as given and did not conceive of constructing them. And so, they juggled with two notions:

- Their notion of "numbers" (as a multitude of units, Definition 2 of Book VII).
- The so-called "magnitudes" (which include things like lines and areas and volumes, etc.).

The Greeks developed arithmetic for their numbers, but treated their magnitudes geometrically. However, although they had not thought of constructing new mathematical objects, they did introduce a procedure for approximating ratios. Such approximations were helpful for the much later constructions of magnitudes (e.g., the real numbers).

Before explaining how the Greeks developed approximations, we explain the *anthyphairesis* concept. Anthyphairesis is composed of two Greek terms: $\upsilon\phi\alpha\iota\rho\epsilon\omega$ (meaning *subtract*) and $\alpha\nu\tau\iota$ (meaning *alternating/reciprocal*) and hence $\alpha\nu\theta\upsilon\phi\alpha\iota\rho\epsilon\sigma\iota\varsigma$ stands for *alternated/reciprocal subtraction*. So, given whole numbers r_0 and r_1, repeatedly subtract r_1 from r_0, $r_0 - r_1$, $r_0 - r_1 - r_1$, $\cdots$ until $r_2 < r1$ remains, then repeat the process for r_1 and r_2, and so on.



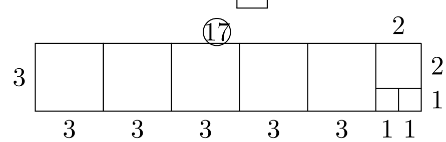
Euclid used anthyphairesis to check whether two numbers are prime to one another. He proved that anthyphairesis applied to two relatively prime numbers leads to the unit.

> PROPOSITION 1. OF BOOK VII OF THE *Elements*
> Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number left never measures the one before it until a unit is left, the original numbers will be prime to one another.

Example 3: Here is why 17 and 3 are prime to one another.

17	=	5	×	3	+	2
3	=	1	×	2	+	①
2	=	2	×	1	+	0

The ratio and continued fraction are respectively:
$[5, 1, 2]$ and $\frac{17}{3} = \boxed{5} + \frac{1}{\boxed{1} + \frac{1}{\boxed{2}}}$.



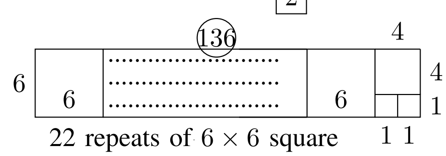
Euclid proved that anthyphairesis applied to non relatively prime numbers gives their greatest common divisor (GCD).

> PROPOSITION 2. OF BOOK VII OF THE *Elements*
> Given two numbers not prime to one another, to find their greatest common measure.

Example 4: As we see below, 136 and 6 are not prime to one another and their greatest common divider is 2.

136	=	22	×	6	+	4.
6	=	1	×	4	+	②.
4	=	2	×	2	+	0.

The ratio and continued fraction are respectively:
$[22, 1, 2]$ and $\frac{136}{6} = \boxed{22} + \frac{1}{\boxed{1} + \frac{1}{\boxed{2}}}$.

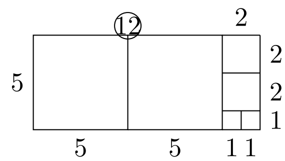


Example 5:

- 12 and 5 are prime to one another.

12	=	2	×	5	+	2
5	=	2	×	2	+	①
2	=	2	×	1	+	0

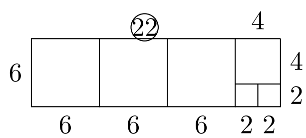
The ratio and continued fraction are respectively: $[2,2,2]$ and $\frac{12}{5} = \boxed{2} + \cfrac{1}{\boxed{2} + \cfrac{1}{\boxed{2}}}$.



- 22 and 6 are not prime to one another and their greatest common divider is 2.

22	=	3	×	6	+	4.
6	=	1	×	4	+	②.
4	=	2	×	2	+	0.

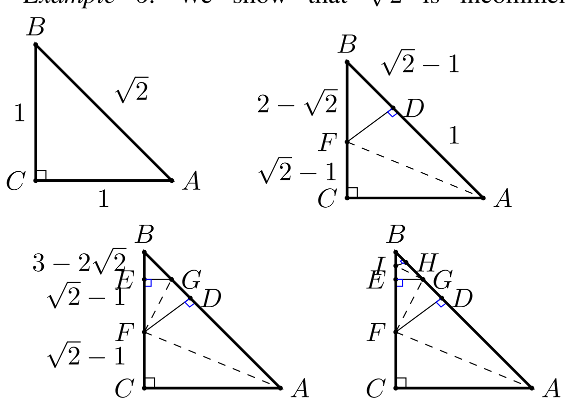
The ratio and continued fraction are respectively: $[3,1,2]$ and $\frac{22}{6} = \boxed{3} + \cfrac{1}{\boxed{1} + \cfrac{1}{\boxed{2}}}$.



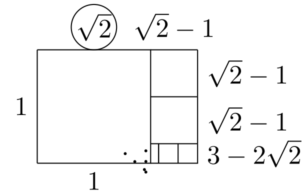
The Greeks also applied anthyphairesis to magnitudes. They showed that two magnitudes are commensurable if and only if anthyphairesis terminates and that if the anthyphairesis procedure of finding the ratio or GCD of two numbers is applied to incommensurable magnitudes, it will not terminate.

PROPOSITION 2 OF BOOK X OF THE *Elements*.
If, when the less of two unequal magnitudes is continuously subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

Example 6: We show that $\sqrt{2}$ is incommensurable.



Geometrically, we assume the isosceles rectangular triangle BCA below and take BD of length $\sqrt{2}-1$. From D we draw the perpendicular to AB meeting BC on F. We get an isosceles recltangular triangle BDF. We repeat the process obtaining isosceles recltangular triangles BEG, BHI, and so on. In this repetition, the less of two unequal magnitudes is continuously subtracted in turn from the greater, yet what is left never measures the one before it. This can be repeated infinitely and $\sqrt{2}$ is incommensurable. Using anthyphairesis:



The ratio of $\sqrt{2}$ to 1 is $[1,2,2,\cdots]$ and the continued fraction is $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}$.

$\sqrt{2}$ is called a quadratic irrational because it is the solution to the quadratic equation $x^2 - 2 = 0$. Note that these continued fractions provide an approximation to $\sqrt{2}$ as follows:

- $\sqrt{2} \approx 1$,
- $\sqrt{2} \approx 1 + \frac{1}{2} = 1.5$,
- $\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2}} = 1.4$,
- $\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 1.417$,
- $\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = 1.4139$ etc.

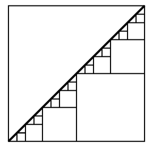
Infinite repetitions/approximations were a useful part of Greek's Mathematics but, anthyphairesis had its limitations. E.g., the obvious theorem below cannot be proved with it:

If the ratio of A to C is the same as the ratio of B to C, then $A = B$.

To overcome the problems, Eudoxus, defined *proportion* (having the same ratio) for magnitudes instead of ratios. He invented the method of exhaustion which was used by Archimedes and Euclid (see Sections I-G and I-H). Theodorus of Cyrene used Eudoxus approximation in his spiral of irrational numbers pictured earlier.

E. The Greeks' problems with infinitesimals/limits

The Greeks were puzzled by limits and infinitesimals. They needed approximations but faced obstacles they could not explain. For example, in the diagram below, the length of the stepped line is clearly $2s$ no matter how many steps there are. But as the number of steps increases, the stepped line seems to approach the diagonal whose length is $\sqrt{2}s \neq 2s$.



They demonstrated many paradoxes like the following:

> **Zeno's Dichotomy Paradox** There is no motion, because what moves must arrive at the middle of its course before it reaches the end.

For example, to leave the room, you first have to get halfway to the door, then halfway from that point to the door, etc. No matter how close you are to the door, you have to go half the remaining distance. Hence, there is no finite motion because always going half way while in motion is infinite.

Suppose the distance is 1 meter and the object moves at 1 meter per second. It must reach halfway ($\frac{1}{2}$ meter from the starting point) in $a_1 = \frac{1}{2}$ second. Let $t_1 = a_1$. From this halfway point, the object moves halfway to the end, which is $a_2 = \frac{1}{4}$ meters. The total time so far is $t_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4}$. We clearly have the following infinite sequences:

$$a_1, a_2, a_3, \ldots = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

$t_1, t_2, \ldots = \dfrac{1}{2}, \dfrac{3}{4}, \dfrac{7}{8}, \ldots$ where each $t_n = a_1 + a_2 + \cdots + a_n$.

Zeno concluded that the total time which is the sum of an infinite sequence must be infinite and we can never reach our destination. This is incorrect since we can reach our destination in a finite time. So, where did Zeno get it wrong?

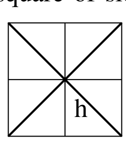
In modern notation, we see that:

- $t_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} < 1$ and $\lim_{n\to\infty} t_n = 1$.
- $2\Sigma_{n=1}^{\infty} a_n = 2a_1 + 2\Sigma_{n=2}^{\infty}\frac{1}{2^n} = 1 + \Sigma_{n=1}^{\infty}\frac{1}{2^n} = 1 + \Sigma_{n=1}^{\infty} a_n$.
- Hence, $\Sigma_{n=1}^{\infty} a_n = 1$ and $lim_{n\to\infty} t_n = \Sigma_{n=1}^{\infty} a_n = 1$.

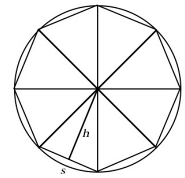
Despite the complications of limits, the Greeks continued to use them to measure magnitudes. Both Archimedes and Euclid (see Sections I-G and I-H) used Eudoxus theory of proportions which is a geometric method based on exhaustive approximations designed to overcome the difficulties obtained from the discovery of the irrationals.

F. The area a regular polygon

For both Archimedes' theorem and Euclid's theorem, we need a general formula for the area of a regular polygon (i.e., a polygon where all angles (resp. all sides) are equal). Let us start with the area of a square of side s.



Instead of simply taking s^2, take the bottom of the 4 triangles obtained by the diagonals. Note that the altitude $h = \frac{1}{2}s$. The area A of the square = 4× area of triangle = $4 \times \frac{1}{2}hs = \frac{1}{2}h(4s) = \frac{1}{2}hp$. where p is the perimeter of the square. Note that $A = \frac{1}{2}hp = \frac{1}{2}\frac{s}{2}(4s) = s^2$.



Now let us consider a regular octagon. If we divide it into triangles the same way, we get eight triangles, each of whose areas is $\frac{1}{2}hs$. If we take all eight triangles and note that here $p = 8s$, we get for the area $A = \frac{1}{2}h(8s) = \frac{1}{2}hp$.

We saw this for the square and the regular octagon, but it holds for every regular polygon:

> The area of any regular polygon is one-half the altitude to a side times the perimeter, or $\frac{1}{2}hp$.

Now we come to the area of a circle. Note that the above polygon was inscribed in the circle with circumference C. If we keep increasing the number of sides, the perimeter will approach the circumference C and the altitude will approach the radius r. By the above, this suggests that the formula for the area of a circle should be

$$A = \frac{1}{2}rC.$$

And since π is defined to be the ratio of the circumference of a circle to twice its radius, we have

$$\pi = \frac{C}{2r},$$

Hence

$$A = \frac{1}{2}r(2\pi r) = \pi r^2$$

This must have seemed obvious to the ancient Greeks from an early period in the history of their geometry. But how could they prove it? At one time some of them argued that a circle is a regular polygon with infinitely many sides, but they eventually decided that this kind of reasoning is not immune to attacks by sophists. For just because regular polygons with an increasing number of sides seems to be approaching a circle, does not automatically justify in deducing this formula for the area of a circle. They found evidence like this to be misleading. Recall the stepped line which wrongly gave the impression that $\sqrt{2}s = 2s$.

G. Euclid on Areas of Circles and Squares

It took a long time for the proof that $A = \frac{1}{2}rC$ to be given. Although this was obvious to the Greeks, a proof was hard to find. Before that proof was given (by Archimedes), Euclid proved that the areas of circles have the same proportion as the squares on their diameters (Proposition 2 of Book XII of *Elements*). The proof uses Proposition 1 of Book XII.

Proposition 1 of Book XII of the *Elements*.
Similar polygons inscribed in circles are to one another as the squares on the diameters of the circles.

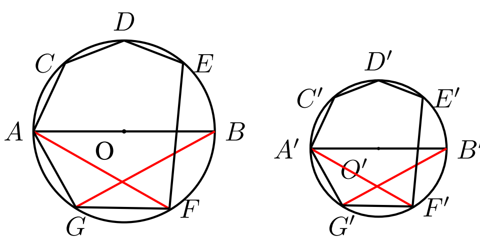
Similar figures are those which have the same shape. In similar polygons the corresponding angles are equal and the corresponding sides all have the same proportion.

The areas A of similar polygons are proportional to:

- The squares of their altitudes h.
- The squares of their perimeters p.
- The squares of any of their linear parts.

$\frac{p_1}{p_2} = \frac{h_1}{h_2}$ and $\frac{A_1}{A_2} = \frac{h_1}{h_2}\frac{h_1}{h_2} = \frac{h_1^2}{h_2^2} = \frac{p_1^2}{p_2^2}$.

The proof of Proposition 1 of Book XII uses the above and the fact that AGB is similar to $A'G'B'$ below and hence $(\frac{AB}{A'B'})^2 = (\frac{AG}{A'G'})^2 = \frac{A_1}{A_2}$.

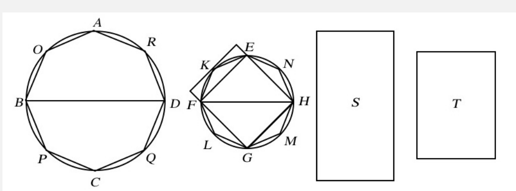


Now we look at Euclid's proposition 2 and its proof:

Proposition 2 of Book XII of the *Elements*.
Circles are to one another as the squares on the diameters.

Euclid starts his proof as follows:

Let $ABCD$, $EFGH$ be circles, and BD, FH their diameters; I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the square on BD to the square on FH.



For, if the square on BD is not to the square on FH as the circle $ABCD$ is to the circle $EFGH$, then, as the square on BD is to the square on FH, so will the circle $ABCD$ be either to some less area than the circle $EFGH$ or to a greater.

Euclid's strategy is to prove his result by contradiction. In fact, it will be a double proof by contradiction. He will first assume that it will be in the ratio to a smaller area S, derive a contradiction from that, then assume that it will be in the ratio to a larger area S, and then derive a contradiction from that area as well. As a result, the only possibility left will be the result stated in the proposition.

We will not repeat the proof here (see [2, 4]). We must mention however that Euclid's method is based on Eudoxus exhaustion which infinitely inscribes and circumscribes polygons inside the circles. First, Euclid assumes it to be in that ratio to a less area S and shows that the square $EFGH$ *inscribed* in the circle $EFGH$ is greater than half of the circle $EFGH$. He shows this by noting that the *circumscribed* square, which includes area outside the circle, has twice the area of the inscribed square.

Then, he bisects the circumference EF, FG, GH, HE at the points K, L, M, N and joins EK, KF, FL, LG, GM, MH, HN, NE and proves that the new circumference (in effect inscribing a new regular polygon with twice the number of sides as the previous one), is more than half the area inside the circle but outside the previous polygon. By bisecting the remaining circumferences and joining straight lines, and by doing this continually, one is left with some segments of the circle which will be less than the excess by which the circle EFGH exceeds the area S.

In modern notation, let the circles have areas a and b respectively, and let the ratio of the squares of their diameters be k. Let the areas of the polygons inscribed in the circle with area a (resp. b) have areas $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$). We have $0 < a_1 < a_2 < \ldots < a_n < \ldots < a$ and $0 < b_1 < b_2 < \ldots < b_n < \ldots < b$.

- For each n, we have
  - $k = \frac{a_n}{b_n}$, so that $\frac{a_n}{k} = b_n$.
  - $(a - a_{n+1}) < \frac{1}{2}(a - a_n)$ and $(b - b_{n+1}) < \frac{1}{2}(b - b_n)$.
- We want to prove $k = \frac{a}{b}$.
- If $k \neq \frac{a}{b}$, then $k = \frac{a}{S}$, where $S < b$ or $S > b$.
  - Suppose $S < b$. Choose N so that $b - b_N < b - S$. The number N represents the number of times the number of sides of the inscribed polygon was doubled. Then $S < b_N$. But $S = \frac{a}{k} > \frac{a_N}{k} = b_N$, a contradiction.
  - Suppose $S > b$. This is similar to the above case with a and b reversed.

It follows that $k = \frac{a}{b}$. □

H. Archimedes' Measurement of a Circle

Archimedes used Eudoxus' exhaustion to prove the following proposition (and hence its corollary that the area of a circle of circumference C and radius r is $A = \frac{1}{2}rC$).

Proposition 1 of Archimedes's Book "Measurement of a Circle".
The area of any circle is equal to a right-angled triangle in which one of the sides about the right triangle is equal to the radius, and the other to the circumference of the circle.

As we see from the begin of its proof, an infinite number of polygons will be inscribed/circumscribed in the circle.

> Let $ABCD$ be the given circle, K the triangle described.
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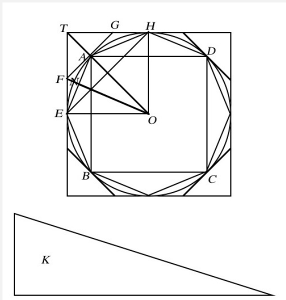
> 
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>
> Then, if the circle is not equal to K, it must be either greater or less.
> I. If possible, let the circle be greater than K.
> Inscribe a square $ABCD$, bisect the arcs AB, BC, CD, DA, and then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the circle over K.

Let us write the proof in modern notation.
Let $K = \frac{1}{2}rC$ (the area of the triangle). If $A \neq K$, then:

I. Suppose $A > K$.
   - Inscribe a square with side s_1, altitude to the side h_1, and perimeter p_1. The area of the square is $a_1 = \frac{1}{2}h_1p_1$.
   - Now, double the number of sides of the inscribed polygon, and keep doubling it. For polygon n with side s_n, altitude to the side h_n, and perimeter p_n, the area is $a_n = \frac{1}{2}h_np_n$.
   - From the geometry of the situation, we have that $h_1 < h_2 < \ldots < h_n < \ldots r$, $p_1 < p_2 < \ldots < p_n < \ldots < C$, and $a_1 < a_2 < \ldots < a_n < \ldots < A$.
   - Now choose N so that $A - a_N < A - \frac{1}{2}rC$. It follows that $\frac{1}{2}rC < a_N$.
   - But since $h_N < r$, $p_N < C$, and $a_N = \frac{1}{2}h_Np_N$, we have $a_N < \frac{1}{2}rC$, a contradiction.

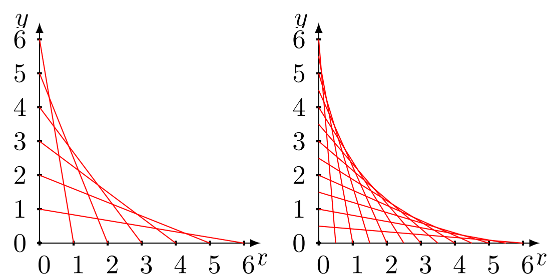
II. Suppose, on the contrary, that $A < K$.
   - Circumscribe a square with perimeter P_1; then the area is $A_1 = \frac{1}{2}rP_1$.
   - Double the number of sides of the circumscribed figure, and keep doing it. If, for the nth polygon, the perimeter is P_n, then the area is $A_n = \frac{1}{2}rP_n$.
   - From the geometry, we have $C < \ldots < P_n < \ldots < P_2 < P_1$ and $A < \ldots A_n < \ldots < A_2 < A_1$.
   - Choose N where $A_N - A < \frac{1}{2}rC - A$. Then $A_N < \frac{1}{2}rC$.
   - But $C < P_N$ and $A_N = \frac{1}{2}rP_N$, so $\frac{1}{2}rC < A_N$, another contradiction.

It follows that $A = K = \frac{1}{2}rC$. □

I. Eudoxus, the infinitesimal and the limit

We saw the use of Eudoxus' exhaustion method in the proofs of Euclid and Archimedes. This method infinitely constructs new objects that would eventually only differ in infinitesimal amounts. It can be used to develop a definition of the limit of a sequence and a function. Historically, the development of calculus and analysis in European mathematics occured before a definition of the real numbers. At the time of Descartes, Leibniz and Newton, it had not even been settled whether or not there were infinitely small quantities. For centuries before and after, infinitesimals oscillated between being accepted and being rejected. They were *introduced* in 450 BC, *banned* by Eucledian mathematicians because of the problems they faced with them, used by Kepler to calculate the area of an ellipse as the infinite sum of vertical lines contained in the ellipse, *banned again* in the 1630s by religious clerics in Rome. They still *flourished* in the 17th century[4] and were *crucial* for the development of calculus by Newton and Leibniz. They were **thought** to exist by Cauchy who used them in his approach to calculus, then they were *abandoned again* in the 19th century due to their unclear logical status to be *revived again* in the 20th century especially in Robinson's non-standard analysis. Nowadays, they take center stage in the foundations of mathematics which many people define as a sound theory of infinitesimals.

The next graph demonstrates how a curved line is made of infinitely small straight line segments.



The next example explains a cleric position on infinitesimals.

Example 7: To find the derivative $f'(2)$ at $x = 2$ of $y = f(x) = x^2$, we assume $x \neq 2$. Then we calculate:

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 2^2}{x - 2} = \frac{(x+2)(x-2)}{x-2} = x + 2.$$

Since we are only able to conclude that the quotient is equal to x + 2 on the assumption that $x \neq 2$, we appear to have taken an illegal step. We justify this by saying that we are taking its limit as $x \to 2$ and write: $\frac{dy}{dx} = \lim_{x \to 2} \frac{\Delta y}{\Delta x}$. Newton calls $\frac{dy}{dx} = \lim_{x \to 2} \frac{\Delta y}{\Delta x}$, *ultimate value* or *value at instant of disappearance*. Sarcastically, this is called *the*

[4] In the ideas that a curved line is made of infinitely small straight line segments, and quantities that differ by an infinitely small quantity are equal.

ghosts of a departed quantity in a critique [1] by Bishop Berkeley addressed to a certain "Infidel Mathematician". [1] examined whether the object and principles of the modern Analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith.

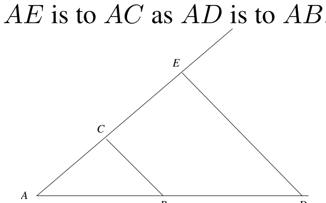
J. Infinitesimals and the birth of analysis

At school, after studying arithmetic and elementary algebra, you are introduced to geometry (*the study of shapes*) and trigonometry (*the study of side lengths and angles of triangles*) and then you move to a *pre-calculus course* which combines advanced algebra and geometry with trigonometry. After all this, you are introduced to *calculus*. Calculus (originally called *infinitesimal calculus*) is the mathematical study of continuous change. The infinitesimal part is important. It is believed that if Descartes had expressed rather than supressed the infinitesimals and infinites in his method, he would have invented the calculus before Newton and Leibniz.

Calculus formalizes the study of continuous change, while analysis provides it with a rigorous foundation in logic. As we saw, the Greeks dealt with discrete numbers arithmetically and with continuous magnitudes geometrically. But continuous systems can be subdivided indefinitely, and their description requires the real numbers. This infinite subdivision was influenced by Eudoxus' and Archimedes' approximations. The real numbers were not present in the historic approach to define limits and develop the calculus. The ancient Greeks separated whole and rational numbers, which are discrete, from continuous magnitudes. They had different kinds of magnitudes for lengths, areas, volumes, angles, etc., and never multiplied two lengths to get another length. The beginning of algebra and the reduction of geometrical problems into algebraic and arithmetical ones in the 9th century [6, 7] paved the way for Descartes innovative ruler-and-compass construction for multiplying two lengths to get a length. This allowed Algebra to be a science concerned with numbers rather than geometric magnitudes.

Here is how the ruler-and-compass construction works:

Example 8: The length of AB is a. On a line AC through A and at an angle to AB, let the length of AC be a unit, and construct E on the same line so that the length of AE is b. Join C and B with line segment BC, and construct a line through E parallel to BC; let this line intersect the extension of AB at D. Then triangles ABC and ADE are similar. Hence, AE is to AC as AD is to AB. I.e., $AD = ab$.



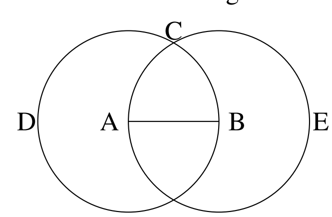
The move to generalise the geometric concepts and methods of the calculus to more algebraic forms continued into the 18th century. But the field was still rife with disagreements on the need and use of infinitesimals and mathematicians began to worry about the lack of rigorous foundations of the calculus (recall that the foundations of mathematics is a sound reasoning about the infinitesimal). This would change due Cauchy's ideas of *function and limit* which led to a more *rigorous* formulation of the calculus, limit/continuity/*real numbers*. And, due to the emerging *exact definition of real numbers* the rules for reasoning with real numbers became even more precise. However, all this historical background of the development of analysis is rarely reflected in the modern teaching of the subject. Instead, students are introduced to methods that they find challenging, like the $\epsilon-\delta$/ $\epsilon-N$ proofs of limits without background material on why limits, infinites, approximations and infinitesimals were developed. From our experience, an evolutionary and somewhat historic approach is helpful. This is why we embarked on a book [4] that introduces mathematical analysis by employing the evolution of this area of mathematics to first develop fundamental concepts of mathematical analysis and to only introduce formal definitions after the concepts are understood.

The landscape of mathematics would change forever during the 19th century and the commitment to rigorous foundations would lead to the discovery of computability and its limits. Rigorous foundations also shed light on the holes that started to appear in Euclid's historic work which led some to question the deductive structure of the Elements. Such logical inaccuracies have been addressed in the work of Hilbert [3] who wrote 20 postulates adequate to prove all the theorems in the Elements. Here we go through some of these holes.

- Look at Proposition 1 of Book I of Euclid's Elements:

  > *To construct an equilateral triangle on a given finite straight line.*

  Let AB be the finite straight line. The proof draws two circles with radius AB, and center A (resp. B). The circles intersect at C and the triangle ABC is equilateral.

  There is a problem in this proof. At first glance, there does not appear to be any doubt that the construction given there constructs the desired equilateral triangle and that the proof proves that it is an equilateral triangle. However, there is a gap in the proof. There is, in fact, no proof that the point C exists. We can construct a *model* of geometry in which all of the postulates and axioms are satisfied but Proposition 1 is not.
- Euclid's Postulate 5 (the parallel postulates) is less obvious than the other postulates.
- Euclid used a number of statements as facts in his Elements even though they had neither been proved nor been introduced as postulates. For example:

A straight line that intersects one side of a triangle but does not pass through any vertex of the triangle must intersect one and only one of the other sides.

Based on this statement, Pasch proved that Euclid's formulation was not complete in the sense that there are statements that should hold but which cannot be proven from Euclid's formulation.

1) A straight line passing through the center of a circle must intersect the circle.
2) Given 3 different points on the same line, one of them is between the other two.

Having introduced the discrete (natural, rational and integer) numbers, and having emphasised the historical treatment of continuous magnitudes and the need for real numbers in the development of analysis, we now discuss the real numbers.

K. What are the real numbers?

Recall Proposition 2. of Book VII of the *Elements* and the approximations for $\sqrt{2}$ in Section I-D. You can think of $\sqrt{2}$ as all the rational numbers strictly less than it. I.e., as: $\{1, 1+\frac{1}{2}, 1+\cfrac{1}{2+\cfrac{1}{2}}, 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}, \cdots\}$. All irrational numbers have infinitely distinct approximations like $\sqrt{2}$. Hence, the real numbers can be defined as non empty subsets of the rationals which satisfy some properties (see below). Real numbers will be defined as elements of a complete ordered field. Hence the following definitions.

Definition 3: A *field* is a set $(S, +, \cdot)$ such that S is closed under $+$ and $\cdot$ and satisfies distributivity $a(b+c) = ab+ac$, commutativity and associativity of $+$ and $\cdot$, and existence of identity elements 0 and 1 ($a+0 = a$ and $a \cdot 1 = a$) and inverses $-a$ and a^{-1} (for each a except for 0 under $\cdot$).

Example 9: None of $\mathbb{N}^+$ or $\mathbb{Z}$ is a field but $\mathbb{Q}$ is a field.

Definition 4: A field is *ordered* (by $<$) if for all a, b, c:

- exactly one of $a < b$, $a = b$, and $b < a$ holds.
- if $a < b$ and $b < c$, then $a < c$.
- if $0 < a$ and $0 < b$, then $0 < a+b$ and $0 < ab$.
- $a < b$ if and only if $0 < b + (-a)$.

The next axiom is important for the real numbers.

AXIOM OF COMPLETENESS [AC]
Every nonempty set of quantities that has an upper bound has a least upper bound.

Now we give the definition of the **Real Numbers** $\mathbb{R}$.

Definition 5: Our quantities form an ordered field that satisfies the Axiom of Completeness AC. We will refer to them as *real numbers* and denote their collection by $\mathbb{R}$.
Recall that the real numbers are continuous whereas the natural/integer/rational numbers are discrete. The following help us to see some differences between these numbers.

ARCHIMEDES LAW [AL]
For any two quantities a and b where $b > a > 0$, there is a positive integer n such that $b < an$.

Definition 6: An ordered field which also satisfies AL is called an *Archimedean ordered field*.

Example 10: $\mathbb{Q}$ is an Archimedean ordered field.

- **Completeness implies the Archimedean Property** Assume a and b are real numbers such that $a > 0$. There is a positive integer n such that $an > b$.
- We can approximate real numbers by rational numbers. **Density of rationals** If a and b are any two real numbers with $a < b$, then there is a rational number r such that $a < r < b$.

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