A complete realisability semantics for intersection types and infinite expansion variables

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Abstract. *Expansion* was introduced at the end of the 1970s for calculating *principal typings* for λ-terms in intersection type systems. *Expansion variables* (E-variables) were introduced at the end of the 1990s to simplify and help mechanise expansion. Recently, E-variables have been further simplified and generalised to also allow calculating other type operators than just intersection. There has been much work on semantics for intersection type systems, but only one such work on intersection type systems with E-variables. That work established that building a semantics for E-variables is very challenging. Because it is unclear how to devise a space of meanings for E-variables, that work developed instead a space of meanings for types that is hierarchical in the sense of having many degrees (denoted by indexes). However, although the indexed calculus helped identify the serious problems of giving a semantics for expansion variables, the sound realisability semantics was only complete when one single E-variable is used and furthermore, the universal type ω was not allowed. In this paper, we are able to overcome these challenges. We develop a realisability semantics where we allow an arbitrary (possibly infinite) number of expansion variables and where ω is present. We show the soundness and completeness of our proposed semantics.

1 Introduction

Expansion is a crucial part of a procedure for calculating *principal typings* and thus helps support compositional type inference. For example, the λ-term $M = (\lambda x.x(\lambda y.yz))$ can be assigned the typing $\Phi_1 = \langle (z : a) \vdash (((a \to b) \to b) \to c) \to c \rangle$, which happens to be its principal typing. The term M can also be assigned the typing $\Phi_2 = \langle (z : a_1 \sqcap a_2) \vdash (((a_1 \to b_1) \to b_1) \sqcap ((a_2 \to b_2) \to b_2) \to c) \to c \rangle$, and an expansion operation can obtain Φ_2 from Φ_1. Because the early definitions of expansion were complicated [4], E-variables were introduced in order to make the calculations easier to mechanise and reason about. For example, in System E [2], the above typing Φ_1 is replaced by $\Phi_3 = \langle (z : ea) \vdash e((((a \to b) \to b) \to c) \to c) \rangle$, which differs from Φ_1 by the insertion of the E-variable e at two places, and Φ_2 can be obtained from Φ_3 by substituting for e the *expansion term*:
$E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2)$.

Carlier and Wells [3] have surveyed the history of expansion and also E-variables. Kamareddine, Nour, Rahli and Wells [12] showed that E-variables pose serious challenges for semantics. In the open problems published in the proceedings of the Lecture Notes in Computer Science symposium held in 1975 [6], it is suggested that an arrow type expresses functionality. Following this idea, a type's semantics is given as a set of closed λ-terms with behaviour related to the specification given by the type. In many kinds of semantics, the meaning of a type T is calculated by an expression $[T]_\nu$ that takes two parameters, the type T and a valuation ν that assigns to type variables the same kind of meanings that are assigned to types. In that way, models based on term-models have been built for intersection type systems [7, 13, 11] where intersection types (introduced to type more terms than in the Simply Typed Lambda Calculus) are interpreted by set-theoretical intersection of meanings. To extend this idea to types with E-variables, we need to devise some space of possible meanings for E-variables. Given that a type $e\,T$ can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (or even arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), the situation is complicated.

This was the main motivation for [12] to develop a space of meanings for types that is hierarchical in the sense of having many degrees. When assigning meanings to types, [12] captured accurately the intuition behind E-variables by ensuring that each use of E-variables simply changes degrees and that each E-variable acts as a kind of capsule that isolates parts of the λ-term being analysed by the typing.

The semantic approach used in [12] is realisability semantics along the lines in Coquand [5] and Kamareddine and Nour [11]. Realisability allows showing *soundness* in the sense that the meaning of a type T contains all closed λ-terms that can be assigned T as their result type. This has been shown useful in previous work for characterising the behaviour of typed λ-terms [13]. One also wants to show the converse of soundness which is called *completeness* (see Hindley [8–10]), i.e., that every closed λ-term in the meaning of T can be assigned T as its result type. Moreover, [12] showed that if more than one E-variable is used, the semantics is not complete. Furthermore, the degrees used in [12] made it difficult to allow the universal type ω and this limited the study to the λI-calculus. In this paper, we are able to overcome these challenges. We develop a realisability semantics where we allow the full λ-calculus, an arbitrary (possibly infinite) number of expansion variables and where ω is present, and we show its soundness and completeness. We do so by introducing an indexed calculus as in [12]. However here, our indexes are finite sequences of natural numbers rather than single natural numbers.

In Section 2 we give the full λ-calculus indexed with finite sequences of natural numbers and show the confluence of β, $\beta\eta$ and weak head reduction on the indexed λ-calculus. In Section 3 we introduce the type system for the indexed λ-calculus (with the universal type ω). In this system, intersections and expansions cannot occur directly to the right of an arrow. In Section 4 we establish that subject reduction holds for $\vdash$. In Section 5 we show that subject β-expansion holds for $\vdash$ but that subject η-expansion fails. In Section 6 we introduce the realisability semantics and show its soundness for $\vdash$. In Section 7 we establish the completeness of $\vdash$ by introducing a special interpretation. We conclude in Section 8. Omitted proofs can be found in the appendix.

2 The pure $\lambda^{\mathcal{L}_{\mathbb{N}}}$-calculus

In this section we give the λ-calculus indexed with finite sequences of natural numbers and show the confluence of β, $\beta\eta$ and weak head reduction.

Let n, m, i, j, k, l be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We assume that if a metavariable v ranges over a set s then v_i and v', v'', etc. also range over s. A binary relation is a set of pairs. Let *rel* range over binary relations. We sometimes write $x\ rel\ y$ instead of $\langle x, y\rangle \in rel$. Let $\mathrm{dom}(rel) = \{x\ /\ \langle x, y\rangle \in rel\}$ and $\mathrm{ran}(rel) = \{y\ /\ \langle x, y\rangle \in rel\}$. A function is a binary relation *fun* such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq fun$ then $y = z$. Let *fun* range over functions. Let $s \to s' = \{fun\ /\ \mathrm{dom}(fun) \subseteq s \wedge \mathrm{ran}(fun) \subseteq s'\}$. We sometimes write $x : s$ instead of $x \in s$.

First, we introduce the set $\mathcal{L}_{\mathbb{N}}$ of indexes with an order relation on indexes.

Definition 1. *1. An index is a finite sequence of natural numbers $L = (n_i)_{1 \leq i \leq l}$. We denote $\mathcal{L}_{\mathbb{N}}$ the set of indexes and $\oslash$ the empty sequence of natural numbers. We let L, K, R range over $\mathcal{L}_{\mathbb{N}}$.*

2. If $L = (n_i)_{1 \leq i \leq l}$ and $m \in \mathbb{N}$, we use $m :: L$ to denote the sequence $(r_i)_{1 \leq i \leq l+1}$ where $r_1 = m$ and for all $i \in \{2, \dots, l+1\}$, $r_i = n_{i-1}$. In particular, $k :: \oslash = (k)$.

3. If $L = (n_i)_{1 \leq i \leq n}$ and $K = (m_i)_{1 \leq i \leq m}$, we use $L :: K$ to denote the sequence $(r_i)_{1 \leq i \leq n+m}$ where for all $i \in \{1, \dots, n\}$, $r_i = n_i$ and for all $i \in \{n+1, \dots, n+m\}$, $r_i = m_{i-n}$. In particular, $L :: \oslash = \oslash :: L = L$.

4. We define on $\mathcal{L}_{\mathbb{N}}$ a binary relation $\preceq$ by:
$L_1 \preceq L_2$ (or $L_2 \succeq L_1$) if there exists $L_3 \in \mathcal{L}_{\mathbb{N}}$ such that $L_2 = L_1 :: L_3$.

Lemma 2. *$\preceq$ is an order relation on $\mathcal{L}_{\mathbb{N}}$.*

The next definition gives the syntax of the indexed calculus and the notions of reduction.

Definition 3. *1. Let $\mathcal{V}$ be a countably infinite set of variables. The set of terms $\mathcal{M}$, the set of free variables $\mathrm{fv}(M)$ of a term $M \in \mathcal{M}$, the degree function $d : \mathcal{M} \to \mathcal{L}_{\mathbb{N}}$ and the joinability $M \diamond N$ of terms M and N are defined by simultaneous induction as follows:*

- *If $x \in \mathcal{V}$ and $L \in \mathcal{L}_{\mathbb{N}}$, then $x^L \in \mathcal{M}$, $\mathrm{fv}(x^L) = \{x^L\}$ and $d(x^L) = L$.*
- *If $M, N \in \mathcal{M}$, $d(M) \preceq d(N)$ and $M \diamond N$ (see below), then $M\ N \in \mathcal{M}$, $\mathrm{fv}(MN) = \mathrm{fv}(M) \cup \mathrm{fv}(N)$ and $d(M\ N) = d(M)$.*
- *If $x \in \mathcal{V}$, $M \in \mathcal{M}$ and $L \succeq d(M)$, then $\lambda x^L.M \in \mathcal{M}$, $\mathrm{fv}(\lambda x^L.M) = \mathrm{fv}(M) \setminus \{x^L\}$ and $d(\lambda x^L.M) = d(M)$.*

2.
- *Let $M, N \in \mathcal{M}$. We say that M and N are joinable and write $M \diamond N$ iff for all $x \in \mathcal{V}$, if $x^L \in \mathrm{fv}(M)$ and $x^K \in \mathrm{fv}(N)$, then $L = K$.*
- *If $\mathcal{X} \subseteq \mathcal{M}$ such that for all $M, N \in \mathcal{X}, M \diamond N$, we write, $\diamond\mathcal{X}$.*
- *If $\mathcal{X} \subseteq \mathcal{M}$ and $M \in \mathcal{M}$ such that for all $N \in \mathcal{X}, M \diamond N$, we write, $M \diamond \mathcal{X}$.*

The $\diamond$ property ensures that in any term M, variables have unique degrees.
We assume the usual definition of subterms and the usual convention for parentheses and their omission (see Barendregt [1] and Krivine [13]). Note that every subterm of $M \in \mathcal{M}$ is also in $\mathcal{M}$. We let $x, y, z,$ etc. range over $\mathcal{V}$ and M, N, P range over $\mathcal{M}$ and use $=$ for syntactic equality.

3. The usual substitution $M[x^L := N]$ of $N \in \mathcal{M}$ for all free occurrences of x^L in $M \in \mathcal{M}$ only matters when $d(N) = L$. Similarly, $M[x_1^{L_1} := N_1, \dots, x_n^{L_n} := N_n]$, the simultaneous substitution of N_i for all free occurrences of $x_i^{L_i}$ in M only matters when for all $i \in \{1, \dots, n\}$, $d(N_i) = L_i$. In a substitution, we sometimes write $(x_i^{L_i} := N_i)_n$ instead of $x_1^{L_1} := N_1, \dots, x_n^{L_n} := N_n$.

4. We take terms modulo α-conversion given by:
$\lambda x^L.M = \lambda y^L.(M[x^L := y^L])$ where $y^L \notin \mathrm{fv}(M)$.
Moreover, we use the Barendregt convention (BC) where the names of bound variables differ from the free ones and where we rewrite terms so that not both λx^L and λx^K co-occur when $L \neq K$.

5. A relation rel on $\mathcal{M}$ is compatible iff for all $M, N, P \in \mathcal{M}$:
- *If M rel N and $\lambda x^L.M, \lambda x^L.M \in \mathcal{M}$ then $(\lambda x^L.M)$ rel $(\lambda x^L.N)$.*
- *If M rel N and $MP, NP \in \mathcal{M}$ (resp. $PM, PN \in \mathcal{M}$), then (MP) rel (NP) (resp. (PM) rel (PN)).*

6. The reduction relation $\rhd_\beta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $(\lambda x^L.M)N \rhd_\beta M[x^L := N]$ if $d(N) = L$

7. The reduction relation $\rhd_\eta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $\lambda x^L.(M\ x^L) \rhd_\eta M$ if $x^L \notin \mathrm{fv}(M)$

8. The weak head reduction $\rhd_h$ on $\mathcal{M}$ is defined by:
$(\lambda x^L.M)NN_1 \dots N_n \rhd_h M[x^L := N]N_1 \dots N_n$ where $n \geq 0$

9. *We let* $\rhd_{\beta\eta} = \rhd_\beta \cup \rhd_\eta$. *For* $r \in \{\beta, \eta, h, \beta\eta\}$, *we denote by* $\rhd_r^*$ *the reflexive and transitive closure of* $\rhd_r$ *and by* $\simeq_r$ *the equivalence relation induced by* $\rhd_r^*$.

Theorem 4. *Let* $M \in \mathcal{M}$ *and* $r \in \{\beta, \beta\eta, h\}$.

1. *If* $M \rhd_\eta^* N$, *then* $N \in \mathcal{M}$, $\mathrm{fv}(N) = \mathrm{fv}(M)$ *and* $d(M) = d(N)$.
2. *If* $M \rhd_r^* N$, *then* $N \in \mathcal{M}$, $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$ *and* $d(M) = d(N)$.

As expansions change the degree of a term, indexes in a term need to increase/decrease.

Definition 5. *Let* $i \in \mathbb{N}$ *and* $M \in \mathcal{M}$.

1. *We define* M^{+i} *by:*
   $\bullet (x^L)^{+i} = x^{i::L}$ $\quad\bullet (M_1\ M_2)^{+i} = M_1^{+i}\ M_2^{+i}$ $\quad\bullet (\lambda x^L.M)^{+i} = \lambda x^{i::L}.M^{+i}$
2. *If* $d(M) = i :: L$, *we define* M^{-i} *by:*
   $\bullet (x^{i::K})^{-i} = x^K$ $\quad\bullet (M_1\ M_2)^{-i} = M_1^{-i}\ M_2^{-i}$ $\quad\bullet (\lambda x^{i::K}.M)^{-i} = \lambda x^K.M^{-i}$

Normal forms are defined as usual.

Definition 6. 1. $M \in \mathcal{M}$ *is in* β*-normal form (*$\beta\eta$*-normal form,* h*-normal form resp.) if there is no* $N \in \mathcal{M}$ *such that* $M \rhd_\beta N$ *(*$M \rhd_{\beta\eta} N$, $M \rhd_h N$ *resp.).*
2. $M \in \mathcal{M}$ *is* β*-normalising (*$\beta\eta$*-normalising,* h*-normalising resp.) if there is an* $N \in \mathcal{M}$ *such that* $M \rhd_\beta^* N$ *(*$M \rhd_{\beta\eta} N$, $M \rhd_h N$ *resp.) and* N *is in* β*-normal form (*$\beta\eta$*-normal form,* h*-normal form resp.).*

Theorem 7 (Confluence). *Let* $M, M_1, M_2 \in \mathcal{M}$ *and* $r \in \{\beta, \beta\eta, h\}$.

1. *If* $M \rhd_r^* M_1$ *and* $M \rhd_r^* M_2$, *then there is* M' *such that* $M_1 \rhd_r^* M'$ *and* $M_2 \rhd_r^* M'$.
2. $M_1 \simeq_r M_2$ *iff there is a term* M *such that* $M_1 \rhd_r^* M$ *and* $M_2 \rhd_r^* M$.

3 Typing system

This paper studies a type system for the indexed λ-calculus with the universal type ω. In this type system, in order to get subject reduction and hence completeness, intersections and expansions cannot occur directly to the right of an arrow (see $\mathbb{U}$ below).

The next two definitions introduce the type system.

Definition 8. 1. *Let a countably infinite set* $\mathcal{A}$ *of atomic types and* $\mathcal{E} = \{e_0, e_1, ...\}$ *a countably infinite set of expansion variables. We define sets of types* $\mathbb{T}$ *and* $\mathbb{U}$, *such that* $\mathbb{T} \subseteq \mathbb{U}$, *and a function* $d : \mathbb{U} \to \mathcal{L}_\mathbb{N}$ *by:*
   - *If* $a \in \mathcal{A}$, *then* $a \in \mathbb{T}$ *and* $d(a) = \oslash$.
   - *If* $U \in \mathbb{U}$ *and* $T \in \mathbb{T}$, *then* $U \to T \in \mathbb{T}$ *and* $d(U \to T) = \oslash$.
   - *If* $L \in \mathcal{L}_\mathbb{N}$, *then* $\omega^L \in \mathbb{U}$ *and* $d(\omega^L) = L$.
   - *If* $U_1, U_2 \in \mathbb{U}$ *and* $d(U_1) = d(U_2)$, *then* $U_1 \sqcap U_2 \in \mathbb{U}$ *and* $d(U_1 \sqcap U_2) = d(U_1) = d(U_2)$.
   - $U \in \mathbb{U}$ *and* $e_i \in \mathcal{E}$, *then* $e_iU \in \mathbb{U}$ *and* $d(e_iU) = i :: d(U)$.

   Note that d *remembers the number of the expansion variables* e_i *in order to keep a trace of these variables.*

   We let T *range over* $\mathbb{T}$, *and* U, V, W *range over* $\mathbb{U}$. *We quotient types by taking* $\sqcap$ *to be commutative (i.e.* $U_1 \sqcap U_2 = U_2 \sqcap U_1$*), associative (i.e.* $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$*) and idempotent (i.e.* $U \sqcap U = U$*), by assuming the distributivity of expansion variables over* $\sqcap$ *(i.e.* $e_i(U_1 \sqcap U_2) = e_iU_1 \sqcap e_iU_2$*) and by having* ω^L *as a neutral (i.e.* $\omega^L \sqcap U = U$*). We denote* $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ *by* $\sqcap_{i=n}^m U_i$ *(when* $n \leq m$*). We also assume that for all* $i \geq 0$ *and* $K \in \mathcal{L}_\mathbb{N}$, $e_i\omega^K = \omega^{i::K}$.

$$\overline{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T\rangle}\ (ax)$$

$$\overline{M : \langle env^{\omega}_{M} \vdash \omega^{\mathrm{d}(M)}\rangle}\ (\omega)$$

$$\frac{M : \langle \Gamma, (x^L : U) \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}\ (\to_I)$$

$$\frac{M : \langle \Gamma \vdash T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T\rangle}\ (\to'_I)$$

$$\frac{M_1 : \langle \Gamma_1 \vdash U \to T\rangle \quad M_2 : \langle \Gamma_2\ U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}\ (\to_E)$$

$$\frac{M : \langle \Gamma \vdash U_1\rangle \qquad M : \langle \Gamma \vdash U_2\rangle}{M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}\ (\sqcap_I)$$

$$\frac{M : \langle \Gamma \vdash U\rangle}{M^{+j} : \langle e_j \Gamma \vdash e_j U\rangle}\ (e)$$

$$\frac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}\ (\sqsubseteq)$$

$$\overline{\Phi \sqsubseteq \Phi}\ (ref)$$

$$\frac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ (tr)$$

$$\overline{U_1 \sqcap U_2 \sqsubseteq U_1}\ (\sqcap_E)$$

$$\frac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ (\sqcap)$$

$$\frac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ (\to)$$

$$\frac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2}\ (\sqsubseteq_e)$$

$$\frac{U_1 \sqsubseteq U_2}{\Gamma, y^L : U_1 \sqsubseteq \Gamma, y^L : U_2}\ (\sqsubseteq_c)$$

$$\frac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash U_1\rangle \sqsubseteq \langle \Gamma_2 \vdash U_2\rangle}\ (\sqsubseteq_{\langle\rangle})$$

Fig. 1. Typing rules / Subtyping rules

2. *We denote* $e_{i_1} \ldots e_{i_n}$ *by* $\boldsymbol{e}_K$*, where* $K = (i_1, \ldots, i_n)$ *and* $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ *by* $\sqcap_{i=n}^{m} U_i$ *(when* $n \leq m$*).*

Definition 9. *1. A type environment is a set* $\{x_1^{L_1} : U_1, \ldots, x_n^{L_n} : U_n\}$ *such that for all* $i \in \{1, \ldots, n\}$, $d(U_i) = L_i$ *and for all* $i, j \in \{1, \ldots, n\}$*, if* $x_i^{L_i} = x_j^{L_j}$ *then* $U_i = U_j\}$. *We use* Γ, Δ *to range over environments and write* $()$ *for the empty environment. We define* $\mathrm{dom}(\Gamma) = \{x^L \,/\, x^L : U \in \Gamma\}$*. If* $\mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) = \emptyset$*, we write* Γ_1, Γ_2 *for* $\Gamma_1 \cup \Gamma_2$*. We write* $\Gamma, x^L : U$ *for* $\Gamma, \{x^L : U\}$ *and* $x^L : U$ *for* $\{x^L : U\}$*. We denote* $x_1^{L_1} : U_1, \ldots, x_n^{L_n} : U_n$ *by* $(x_i^{L_i} : U_i)_n$*.*

2. *If* $M \in \mathcal{M}$ *and* $\mathrm{fv}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$*, we denote* env^{ω}_M *the type environment* $(x_i^{L_i} : \omega^{L_i})_n$*.*
3. *Let* $\Gamma_1 = (x_i^{L_i} : U_i)_n, \Gamma'_1$*,* $\Gamma_2 = (x_i^{L_i} : U'_i)_n, \Gamma'_2$ *and* $\mathrm{dom}(\Gamma'_1) \cap \mathrm{dom}(\Gamma'_2) = \emptyset$*. We denote* $\Gamma_1 \sqcap \Gamma_2$ *the type environment* $(x_i^{L_i} : U_i \sqcap U'_i)_n, \Gamma'_1, \Gamma'_2$*. Note that* $\mathrm{dom}(\Gamma_1 \sqcap \Gamma_2) = \mathrm{dom}(\Gamma_1) \cup \mathrm{dom}(\Gamma_2)$ *and that, on environments,* $\sqcap$ *is commutative, associative and idempotent.*
4. *Let* $\Gamma = (x_i^{L_i} : U_i)_{1 \leq i \leq n}$ *and* $e_j \in \mathcal{E}$*. We denote* $e_j \Gamma = (x_i^{j::L_i} : e_j U_i)_{1 \leq i \leq n}$*. Note that* $e_j(\Gamma_1 \sqcap \Gamma_2) = e_j \Gamma_1 \sqcap e_j \Gamma_2$*.*
5. *We write* $\Gamma_1 \diamond \Gamma_2$ *iff* $x^L \in \mathrm{dom}(\Gamma_1)$ *and* $x^K \in \mathrm{dom}(\Gamma_2)$ *implies* $K = L$*.*
6. *We follow [3] and write type judgements as* $M : \langle \Gamma \vdash U\rangle$ *instead of the traditional format of* $\Gamma \vdash M : U$*, where* $\vdash$ *is our typing relation. The typing rules of* $\vdash$ *are given on the left hand side of Figure 6. In the last clause, the binary relation* $\sqsubseteq$ *is defined on* $\mathbb{U}$ *by the rules on the right hand side of Figure 6. We let* Φ *denote types in* $\mathbb{U}$*, or environments* Γ *or typings* $\langle \Gamma \vdash U\rangle$*. When* $\Phi \sqsubseteq \Phi'$*, then* Φ *and* Φ' *belong to the same set (*$\mathbb{U}$*/environments/typings).*
7. *If* $L \in \mathcal{L}_{\mathbb{N}}$*,* $U \in \mathbb{U}$ *and* $\Gamma = (x_i^{L_i} : U_i)_n$ *is a type environment, we say that:*
   - $d(\Gamma) \succeq L$ *if and only if for all* $i \in \{1, \ldots, n\}$*,* $d(U_i) = L_i \succeq L$*.*
   - $d(\langle \Gamma \vdash U\rangle) \succeq L$ *if and only if* $d(\Gamma) \succeq L$ *and* $d(U) \succeq L$*.*

To illustrate how our indexed type system works, we give an example:

Example 10. Let $U = e_3(e_2(e_1((e_0 b \to c) \to (e_0(a \sqcap (a \to b)) \to c)) \to d) \to (((e_2 d \to a) \sqcap b) \to a))$ where $a, b, c, d \in \mathcal{A}$,

$L_1 = 3 :: \oslash \preceq L_2 = 3 :: 2 :: \oslash \preceq L_3 = 3 :: 2 :: 1 :: 0 :: \oslash$

and

$M = \lambda x^{L_2}.\lambda y^{L_1}.(y^{L_1}\ (x^{L_2}\ \lambda u^{L_3}.\lambda v^{L_3}.(u^{L_3}\ (v^{L_3}\ v^{L_3}))))$.

We invite the reader to check that $M : \langle () \vdash U \rangle$.

Just as we did for terms, we decrease the indexes of types, environments and typings.

Definition 11. *1. If $d(U) \succeq L$, then if $L = \oslash$ then $U^{-L} = U$ else $L = i :: K$ and we inductively define the type U^{-L} as follows:*

$(U_1 \sqcap U_2)^{-i::K} = U_1^{-i::K} \sqcap U_2^{-i::K}$ $\qquad$ $(e_i U)^{-i::K} = U^{-K}$

We write U^{-i} instead of $U^{-(i)}$.

2. If $\Gamma = (x_i^{L_i} : U_i)_k$ and $d(\Gamma) \succeq L$, then for all $i \in \{1, \dots, k\}$, $L_i = L :: L'_i$ and we denote $\Gamma^{-L} = (x^{L'_i} : U_i^{-L})_k$.

We write Γ^{-i} instead of $\Gamma^{-(i)}$.

3. If U is a type and Γ is a type environment such that $d(\Gamma) \succeq K$ and $d(U) \succeq K$, then we denote $(\langle \Gamma \vdash U \rangle)^{-K} = \langle \Gamma^{-K} \vdash U^{-K} \rangle$.

The next lemma is informative about types and their degrees.

Lemma 12. *1. If $T \in \mathbb{T}$, then $d(T) = \oslash$.*

2. Let $U \in \mathbb{U}$. If $d(U) = L = (n_i)_m$, then $U = \omega^L$ or $U = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and for all $i \in \{1, \dots, p\}$, $T_i \in \mathbb{T}$.

3. Let $U_1 \sqsubseteq U_2$.

(a) $d(U_1) = d(U_2)$.

(b) If $U_1 = \omega^K$ then $U_2 = \omega^K$.

(c) If $U_1 = \boldsymbol{e}_K U$ then $U_2 = \boldsymbol{e}_K U'$ and $U \sqsubseteq U'$.

(d) If $U_2 = \boldsymbol{e}_K U$ then $U_1 = \boldsymbol{e}_K U'$ and $U \sqsubseteq U'$.

(e) If $U_1 = \sqcap_{i=1}^p \boldsymbol{e}_K(U_i \to T_i)$ where $p \geq 1$ then $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^q \boldsymbol{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and for all $j \in \{1, \dots, q\}$, there exists $i \in \{1, \dots, p\}$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.

4. If $U \in \mathbb{U}$ such that $d(U) = L$ then $U \sqsubseteq \omega^L$.

5. If $U \sqsubseteq U'_1 \sqcap U'_2$ then $U = U_1 \sqcap U_2$ where $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.

6. If $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.

The next lemma says how ordering or the decreasing of indexes propagate to environments.

Lemma 13. *1. If $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $x^L \notin \mathrm{dom}(\Gamma)$ then $\Gamma, (x^L : U) \sqsubseteq \Gamma', (x^L : U')$.*

2. $\Gamma \sqsubseteq \Gamma'$ iff $\Gamma = (x_i^{L_i} : U_i)_n$, $\Gamma' = (x_i^{L_i} : U'_i)_n$ and for every $1 \leq i \leq n$, $U_i \sqsubseteq U'_i$.

3. $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$ iff $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$.

4. If $\mathrm{dom}(\Gamma) = \mathrm{fv}(M)$, then $\Gamma \sqsubseteq env_M^\omega$

5. If $\Gamma \diamond \Delta$ and $d(\Gamma), d(\Delta) \succeq K$, then $\Gamma^{-K} \diamond \Delta^{-K}$.

6. If $U \sqsubseteq U'$ and $d(U) \succeq K$ then $U^{-K} \sqsubseteq U'^{-K}$.

7. If $\Gamma \sqsubseteq \Gamma'$ and $d(\Gamma) \succeq K$ then $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.

The next lemma shows that we do not allow weakening in $\vdash$.

Lemma 14. *1. For every Γ and M such that $\mathrm{dom}(\Gamma) = \mathrm{fv}(M)$ and $d(M) = K$, we have $M : \langle \Gamma \vdash \omega^K \rangle$.*

2. If $M : \langle \Gamma \vdash U \rangle$, then $\mathrm{dom}(\Gamma) = \mathrm{fv}(M)$.

3. If $M_1 : \langle \Gamma_1 \vdash U \rangle$ and $M_2 : \langle \Gamma_2 \vdash U \rangle$ then $\Gamma_1 \diamond \Gamma_2$ iff $M_1 \diamond M_2$.

Proof 1. By ω, $M : \langle env_M^\omega \vdash \omega^K \rangle$. By Lemma 13.4, $\Gamma \sqsubseteq env_M^\omega$. Hence, by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma \vdash \omega^K \rangle$.
2. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$.
3. If) Let $x^L \in \mathrm{dom}(\Gamma_1)$ and $x^K \in \mathrm{dom}(\Gamma_2)$ then by Lemma 14.2, $x^L \in \mathrm{fv}(M_1)$ and $x^K \in \mathrm{fv}(M_2)$ so $\Gamma_1 \diamond \Gamma_2$. Only if) Let $x^L \in \mathrm{fv}(M_1)$ and $x^K \in \mathrm{fv}(M_2)$ then by Lemma 14.2, $x^L \in \mathrm{dom}(\Gamma_1)$ and $x^K \in \mathrm{dom}(\Gamma_2)$ so $M_1 \diamond M_2$. $\square$

The next theorem states that within a typing, degrees are well behaved.

Theorem 15. *Let $M : \langle \Gamma \vdash U \rangle$.*

1. $d(\Gamma) \succeq d(U) = d(M)$.
2. *If* $d(U) \succeq K$ *then* $M^{-K} : \langle \Gamma^{-K} \vdash U^{-K} \rangle$.

Finally, here are two derivable typing rules.

Remark 16. 1. The rule $\dfrac{M : \langle \Gamma_1 \vdash U_1 \rangle \qquad M : \langle \Gamma_2 \vdash U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2 \rangle}\ \sqcap'_I$ is derivable.
2. The rule $\dfrac{}{x^{\mathrm{d}(U)} : \langle (x^{\mathrm{d}(U)} : U) \vdash U \rangle}\ ax'$ is derivable.

4 Subject reduction properties

In this section we show that subject reduction holds for $\vdash$. The proof of subject reduction uses generation and substitution. Hence the next two lemmas.

Lemma 17 (Generation for $\vdash$).

1. *If* $x^L : \langle \Gamma \vdash U \rangle$, *then* $\Gamma = (x^L : V)$ *and* $V \sqsubseteq U$.
2. *If* $\lambda x^L.M : \langle \Gamma \vdash U \rangle$, $x^L \in \mathrm{fv}(M)$ *and* $d(U) = K$, *then* $U = \omega^K$ *or* $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ *where* $p \geq 1$ *and for all* $i \in \{1, \ldots, p\}$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i \rangle$.
3. *If* $\lambda x^L.M : \langle \Gamma \vdash U \rangle$, $x^L \notin \mathrm{fv}(M)$ *and* $d(U) = K$, *then* $U = \omega^K$ *or* $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ *where* $p \geq 1$ *and for all* $i \in \{1, \ldots, p\}$, $M : \langle \Gamma \vdash \boldsymbol{e}_K T_i \rangle$.
4. *If* $M\ x^L : \langle \Gamma, (x^L : U) \vdash T \rangle$ *and* $x^L \notin \mathrm{fv}(M)$, *then* $M : \langle \Gamma \vdash U \to T \rangle$.

Lemma 18 (Substitution for $\vdash$). *If* $M : \langle \Gamma, x^L : U \vdash V \rangle$, $N : \langle \Delta \vdash U \rangle$ *and* $\Gamma \diamond \Delta$ *then* $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash V \rangle$.

Since $\vdash$ does not allow weakening, we need the next definition since when a term is reduced, it may lose some of its free variables and hence will need to be typed in a smaller environment.

Definition 19. *If* Γ *is a type environment and* $\mathcal{U} \subseteq \mathrm{dom}(\Gamma)$, *then we write* $\Gamma \upharpoonright_{\mathcal{U}}$ *for the restriction of* Γ *on the variables of* $\mathcal{U}$. *If* $\mathcal{U} = \mathrm{fv}(M)$ *for a term* M, *we write* $\Gamma \upharpoonright_M$ *instead of* $\Gamma \upharpoonright_{\mathrm{fv}(M)}$.

Now we are ready to prove the main result of this section:

Theorem 20 (Subject reduction for $\vdash$). *If* $M : \langle \Gamma \vdash U \rangle$ *and* $M \rhd^*_{\beta\eta} N$, *then* $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.

Corollary 21. 1. *If* $M : \langle \Gamma \vdash U \rangle$ *and* $M \rhd^*_{\beta} N$, *then* $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.
2. *If* $M : \langle \Gamma \vdash U \rangle$ *and* $M \rhd^*_{h} N$, *then* $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.

5 Subject expansion properties

In this section we show that subject β-expansion holds for $\vdash$ but that subject η-expansion fails.

The next lemma is needed for expansion.

Lemma 22. *If* $M[x^L := N] : \langle \Gamma \vdash U \rangle$, $d(N) = L$ *and* $x^L \in \mathrm{fv}(M)$ *then there exist a type* V *and two type environments* Γ_1, Γ_2 *such that* $d(V) = L$ *and:*
$M : \langle \Gamma_1, x^L : V \vdash U \rangle \qquad N : \langle \Gamma_2 \vdash V \rangle \qquad \Gamma = \Gamma_1 \sqcap \Gamma_2$

Since more free variables might appear in the β-expansion of a term, the next definition gives a possible enlargement of an environment.

Definition 23. *Let $m \geq n$, $\Gamma = (x_i^{L_i} : U_i)_n$ and $\mathcal{U} = \{x_1^{L_1}, ..., x_m^{L_m}\}$. We write $\Gamma\uparrow^{\mathcal{U}}$ for $x_1^{L_1} : U_1, ..., x_n^{L_n} : U_n, x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, ..., x_m^{L_m} : \omega^{L_m}$. If $\mathrm{dom}(\Gamma) \subseteq \mathrm{fv}(M)$, we write $\Gamma\uparrow^M$ instead of $\Gamma\uparrow^{\mathrm{fv}(M)}$.*

We are now ready to establish that subject expansion holds for β (next theorem) and that it fails for η (Lemma 26).

Theorem 24 (Subject expansion for β). *If $N : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta^* N$, then $M : \langle \Gamma\uparrow^M \vdash U \rangle$.*

Corollary 25. *If $N : \langle \Gamma \vdash U \rangle$ and $M \rhd_h^* N$, then $M : \langle \Gamma\uparrow^M \vdash U \rangle$.*

Lemma 26 (Subject expansion fails for η). *Let a be an element of $\mathcal{A}$. We have:*

1. $\lambda y^{\oslash}.\lambda x^{\oslash}.y^{\oslash} x^{\oslash} \rhd_\eta \lambda y^{\oslash}.y^{\oslash}$
2. $\lambda y^{\oslash}.y^{\oslash} : \langle () \vdash a \to a \rangle$.
3. *It is not possible that*
   $\lambda y^{\oslash}.\lambda x^{\oslash}.y^{\oslash} x^{\oslash} : \langle () \vdash a \to a \rangle$.
   Hence, the subject η-expansion lemmas fail for $\vdash$.

Proof 1. and 2. are easy. For 3., assume $\lambda y^{\oslash}.\lambda x^{\oslash}.y^{\oslash} x^{\oslash} : \langle () \vdash a \to a \rangle$. By Lemma 17.2, $\lambda x^{\oslash}.y^{\oslash} x^{\oslash} : \langle (y : a) \vdash \to a \rangle$. Again, by Lemma 17.2, $a = \omega^{\oslash}$ or there exists $n \geq 1$ such that $a = \sqcap_{i=1}^n (U_i \to T_i)$, absurd. □

6 The realisability semantics

In this section we introduce the realisability semantics and show its soundness for $\vdash$.

Crucial to a realisability semantics is the notion of a saturated set:

Definition 27. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$.*

1. *We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of $\mathcal{X}$, i.e. $\{\mathcal{Y} \;/\; \mathcal{Y} \subseteq \mathcal{X}\}$.*
2. *We define $\mathcal{X}^{+i} = \{M^{+i} \;/\; M \in \mathcal{X}\}$.*
3. *We define $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M} \;/\; M\, N \in \mathcal{Y}$ for all $N \in \mathcal{X}$ such that $M \diamond N\}$.*
4. *We say that $\mathcal{X} \wr \mathcal{Y}$ iff for all $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, there exists $N \in \mathcal{X}$ such that $M \diamond N$.*
5. *For $r \in \{\beta, \beta\eta, h\}$, we say that $\mathcal{X}$ is r-saturated if whenever $M \rhd_r^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.*

Saturation is closed under intersection, lifting and arrows:

Lemma 28. *1. $(\mathcal{X} \cap \mathcal{Y})^{+i} = \mathcal{X}^{+i} \cap \mathcal{Y}^{+i}$.*
2. *If $\mathcal{X}, \mathcal{Y}$ are r-saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is r-saturated.*
3. *If $\mathcal{X}$ is r-saturated, then $\mathcal{X}^{+i}$ is r-saturated.*
4. *If $\mathcal{Y}$ is r-saturated, then, for every set $\mathcal{X}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is r-saturated.*
5. *$(\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i} \subseteq \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.*
6. *If $\mathcal{X}^+ \wr \mathcal{Y}^+$, then $\mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+ \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.*

We now give the basic step in our realisability semantics: the interpretations and meanings of types.

Definition 29. *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be countably infinite, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.*

1. *Let $L \in \mathcal{L}_{\mathbb{N}}$. We define $\mathcal{M}^L = \{M \in \mathcal{M} / d(M) = L\}$.*
2. *Let $x \in \mathcal{V}_1$. We define $\mathcal{N}_x^L = \{x^L\, N_1 ... N_k \in \mathcal{M} /\; k \geq 0\}$.*

3. *Let $r \in \{\beta, \beta\eta, h\}$. An r-interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^{\oslash})$ is a function such that for all $a \in \mathcal{A}$:*
   - *$\mathcal{I}(a)$ is r-saturated* and *$\forall x \in \mathcal{V}_1.\ \mathcal{N}_x^{\oslash} \subseteq \mathcal{I}(a)$.*

   We extend an r-interpretation $\mathcal{I}$ to $\mathbb{U}$ as follows:
   - $\mathcal{I}(\omega^L) = \mathcal{M}^L$
   - $\mathcal{I}(e_i U) = \mathcal{I}(U)^{+i}$
   - $\mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2)$
   - $\mathcal{I}(U \to T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$

   Let r-int $= \{\mathcal{I} \ / \ \mathcal{I}$ is an r-interpretation$\}$.
4. *Let $U \in \mathbb{U}$ and $r \in \{\beta, \beta\eta, h\}$. Define $[U]_r$, the r-interpretation of U by:*
   $[U]_r = \{M \in \mathcal{M} \ / \ M$ *is closed and* $M \in \bigcap_{\mathcal{I} \in r\text{-}int} \mathcal{I}(U)\}$

Lemma 30. *Let $r \in \{\beta, \beta\eta, h\}$.*

1. *(a) For any $U \in \mathbb{U}$ and $\mathcal{I} \in r$-int, we have $\mathcal{I}(U)$ is r-saturated.*
   (b) If $d(U) = L$ and $\mathcal{I} \in r$-int, then for all $x \in \mathcal{V}_1$, $\mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$.
2. *Let $r \in \{\beta, \beta\eta, h\}$. If $\mathcal{I} \in r$-int and $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.*

Here is the soundness lemma.

Lemma 31 (Soundness). *Let $r \in \{\beta, \beta\eta, h\}$, $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$, $\mathcal{I} \in r$-int and for all $j \in \{1, \dots, n\}$, $N_j \in \mathcal{I}(U_j)$. We have $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$.*

Corollary 32. *Let $r \in \{\beta, \beta\eta, h\}$. If $M : \langle () \vdash U \rangle$, then $M \in [U]_r$.*

Proof By Lemma 31, $M \in \mathcal{I}(U)$ for any r-interpretation $\mathcal{I}$. By Lemma 14, $\text{fv}(M) = \text{dom}(()) = \emptyset$ and hence M is closed. Therefore, $M \in [U]_r$. □

Lemma 33 (The meaning of types is closed under type operations). *Let $r \in \{\beta, \beta\eta, h\}$. On $\mathbb{U}$, the following hold:*

1. $[e_i U]_r = [U]_r^{+i}$
2. $[U \sqcap V]_r = [U]_r \cap [V]_r$
3. *If $U \to T \in \mathbb{U}$ then for any interpretation $\mathcal{I}$, $\mathcal{I}(U) \wr \mathcal{I}(T)$.*

Proof 1. and 2. are easy. 3. Let $d(U) = K$, $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and $x \in \mathcal{V}_1$ such that for all L, $x^L \notin \text{fv}(M)$, hence $M \diamond x^K$ and $x^K \in \mathcal{I}(U)$. □

The next definition and lemma put the realisability semantics in use.

Definition 34 (Examples). *Let $a, b \in \mathcal{A}$ where $a \neq b$. We define:*

- $Id_0 = a \to a$, $Id_1 = e_1(a \to a)$ *and* $Id'_1 = e_1 a \to e_1 a$.
- $D = (a \sqcap (a \to b)) \to b$.
- $Nat_0 = (a \to a) \to (a \to a)$, $Nat_1 = e_1((a \to a) \to (a \to a))$, *and* $Nat'_0 = (e_1 a \to a) \to (e_1 a \to a)$.

Moreover, if M, N are terms and $n \in \mathbb{N}$, we define $(M)^n\ N$ by induction on n: $(M)^0\ N = N$ and $(M)^{m+1}\ N = M\ ((M)^m\ N)$.

Lemma 35. *1.* $[Id_0]_\beta = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* \lambda y^{\oslash} y^{\oslash}\}$.

2. $[Id_1]_\beta = [Id'_1]_\beta = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd_\beta^* \lambda y^{(1)}.y^{(1)}\}$. *(Note that $Id'_1 \notin \mathbb{U}$.)*
3. $[D]_\beta = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash} y^{\oslash}\}$.
4. $[Nat_0]_\beta = \{M \in \mathcal{M}^{\oslash} \ / \ M \rhd_\beta^* \lambda f^{\oslash}.f^{\oslash}$ *or* $M \rhd_\beta^* \lambda f^{\oslash}.\lambda y^{\oslash}.(f^{\oslash})^n y^{\oslash}$ *where* $n \geq 1\}$.
5. $[Nat_1]_\beta = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd_\beta^* \lambda f^{(1)}.f^{(1)}$ *or* $M \rhd_\beta^* \lambda f^{(1)}.\lambda x^{(1)}.(f^{(1)})^n y^{(1)}$ *where* $n \geq 1\}$. *(Note that $Nat'_1 \notin \mathbb{U}$.)*
6. $[Nat'_0]_\beta = \{M \in \mathcal{M}^{\oslash} \ / M \rhd_\beta^* \lambda f^{\oslash}.f^{\oslash}$ *or* $M \rhd_\beta^* \lambda f^{\oslash}.\lambda y^{(1)}.f^{\oslash} y^{(1)}\}$.

7 The completeness theorem

In this section we set out the machinery and prove that completeness holds for $\vdash$.

We need the following partition of the set of variables $\{y^L/y \in \mathcal{V}_2\}$.

Definition 36. *1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We define $\mathbb{U}^L = \{U \in \mathbb{U}/d(U) = L\}$ and $\mathcal{V}^L = \{x^L/x \in \mathcal{V}_2\}$.*

2. Let $U \in \mathbb{U}$. We inductively define a set of variables $\mathbb{V}_U$ as follows:

- *If $d(U) = \oslash$ then:*
  - *$\mathbb{V}_U$ is an infinite set of variables of degree $\oslash$.*
  - *If $U \neq V$ and $d(U) = d(V) = \oslash$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.*
  - *$\bigcup_{U \in \mathbb{U}^\oslash} \mathbb{V}_U = \mathcal{V}^\oslash$.*
- *If $d(U) = L$, then we put $\mathbb{V}_U = \{y^L \;/\; y^\oslash \in \mathbb{V}_{U^{-L}}\}$.*

Lemma 37. *1. If $d(U), d(V) \succeq L$ and $U^{-L} = V^{-L}$, then $U = V$.*

2. If $d(U) = L$, then $\mathbb{V}_U$ is an infinite subset of $\mathcal{V}^L$.

3. If $U \neq V$ and $d(U) = d(V) = L$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.

4. $\bigcup_{U \in \mathbb{U}^L} \mathbb{V}_U = \mathcal{V}^L$.

5. If $y^L \in \mathbb{V}_U$, then $y^{i::L} \in \mathbb{V}_{e_i U}$.

6. If $y^{i::L} \in \mathbb{V}_U$, then $y^L \in \mathbb{V}_{U^{-i}}$.

Proof 1. If $L = (n_i)_m$, we have $U = e_{n_1}...e_{n_m}U'$ and $V = e_{n_1}...e_{n_m}V'$. Then $U^{-L} = U'$, $V^{-L} = V'$ and $U' = V'$. Thus $U = V$. 2. 3. and 4. By induction on L and using 1. 5. Because $(e_i U)^{-i} = U$. 6. By definition. □

Our partition of the set $\mathcal{V}_2$ as above will enable us to give in the next definition useful infinite sets which will contain type environments that will play a crucial role in one particular type interpretation.

Definition 38. *1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We denote $\mathbb{G}^L = \{(y^L : U) \;/\; U \in \mathbb{U}^L$ and $y^L \in \mathbb{V}_U\}$ and $\mathbb{H}^L = \bigcup_{K \succeq L} \mathbb{G}^K$. Note that $\mathbb{G}^L$ and $\mathbb{H}^L$ are not type environments because they are infinite sets.*

2. Let $L \in \mathcal{L}_{\mathbb{N}}$, $M \in \mathcal{M}$ and $U \in \mathbb{U}$, we write:

- *$M : \langle \mathbb{H}^L \vdash U \rangle$ if there is a type environment $\Gamma \subset \mathbb{H}^L$ where $M : \langle \Gamma \vdash U \rangle$*
- *$M : \langle \mathbb{H}^L \vdash^* U \rangle$ if $M \rhd^*_{\beta\eta} N$ and $N : \langle \mathbb{H}^L \vdash U \rangle$*

Lemma 39. *1. If $\Gamma \subset \mathbb{H}^L$ then $e_i \Gamma \subset \mathbb{H}^{i::L}$.*

2. If $\Gamma \subset \mathbb{H}^{i::L}$ then $\Gamma^{-i} \subset \mathbb{H}^L$.

3. If $\Gamma_1 \subset \mathbb{H}^L$, $\Gamma_2 \subset \mathbb{H}^K$ and $L \preceq K$ then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$.

Proof 1. and 2. By lemma 37. 3. First note that $\mathbb{H}^K \subseteq \mathbb{H}^L$. Let $(x^R : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^R : U_1) \in \Gamma_1 \subset \mathbb{H}^L$ and $(x^R : U_2) \in \Gamma_2 \subset \mathbb{H}^K \subseteq \mathbb{H}^L$, then $\mathrm{d}(U_1) = \mathrm{d}(U_2) = R$ and $x^R \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by lemma 37, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^L$. □

For every $L \in \mathcal{L}_{\mathbb{N}}$, we define the set of terms of degree L which contain some free variable x^K where $x \in \mathcal{V}_1$ and $K \succeq L$.

Definition 40. *For every $L \in \mathcal{L}_{\mathbb{N}}$, let $\mathcal{O}^L = \{M \in \mathcal{M}^L \;/\; x^K \in \mathrm{fv}(M)$, $x \in \mathcal{V}_1$ and $K \succeq L\}$. It is easy to see that, for every $L \in \mathcal{L}_{\mathbb{N}}$ and $x \in \mathcal{V}_1$, $\mathcal{N}^L_x \subseteq \mathcal{O}^L$.*

Lemma 41. *1. $(\mathcal{O}^L)^{+i} = \mathcal{O}^{i::L}$.*

2. If $y \in \mathcal{V}_2$ and $(My^K) \in \mathcal{O}^L$, then $M \in \mathcal{O}^L$

3. If $M \in \mathcal{O}^L$, $M \diamond N$ and $L \preceq K = d(N)$, then $MN \in \mathcal{O}^L$.

4. If $d(M) = L$, $L \preceq K$, $M \diamond N$ and $N \in \mathcal{O}^K$, then $MN \in \mathcal{O}^L$.

The crucial interpretation $\mathbb{I}$ for the proof of completeness is given as follows:

Definition 42. *1. Let $\mathbb{I}_{\beta\eta}$ be the $\beta\eta$-interpretation defined by: for all type variables a, $\mathbb{I}_{\beta\eta}(a) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \;/\; M : \langle \mathbb{H}^\oslash \vdash^* a \rangle\}$.*

2. *Let $\mathbb{I}_\beta$ be the β-interpretation defined by: for all type variables a, $\mathbb{I}_\beta(a) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash a\rangle\}$.*
3. *Let $\mathbb{I}_{eh}$ be the h-interpretation defined by: for all type variables a, $\mathbb{I}_h(a) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash a\rangle\}$.*

The next crucial lemma shows that $\mathbb{I}$ is an interpretation and that the interpretation of a type of order L contains terms of order L which are typable in these special environments which are parts of the infinite sets of Definition 38.

Lemma 43. *Let $r \in \{\beta\eta, \beta, h\}$ and $r' \in \{\beta, h\}$*

1. *If $\mathbb{I}_r \in r$-int and $a \in \mathcal{A}$ then $\mathbb{I}_r(a)$ is r-saturated and for all $x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathbb{I}_r(a)$.*
2. *If $U \in \mathbb{U}$ and $d(U) = L$, then $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U\rangle\}$.*
3. *If $U \in \mathbb{U}$ and $d(U) = L$, then $\mathbb{I}_{r'}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U\rangle\}$.*

Proof 1. We do two cases:
Case $r = \beta\eta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{O}^\oslash \subseteq \mathbb{I}_{\beta\eta}(a)$. Now we show that $\mathbb{I}_{\beta\eta}(a)$ is $\beta\eta$-saturated. Let $M \rhd_{\beta\eta}^* N$ and $N \in \mathbb{I}_{\beta\eta}(a)$.

- If $N \in \mathcal{O}^\oslash$ then $N \in \mathcal{M}^\oslash$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in \text{fv}(N)$. By theorem 4.2, $\text{fv}(N) \subseteq \text{fv}(M)$ and $\text{d}(M) = \text{d}(N)$, hence, $M \in \mathcal{O}^\oslash$
- If $N \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* a\rangle\}$ then $N \rhd_{\beta\eta}^* N'$ and $\exists \Gamma \subset \mathbb{H}^\oslash$, such that $N' : \langle \Gamma \vdash a\rangle$. Hence $M \rhd_{\beta\eta}^* N'$ and since by theorem 4.2, $\text{d}(M) = \text{d}(N')$, $M \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* a\rangle\}$.

Case $r = \beta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{O}^\oslash \subseteq \mathbb{I}_\beta(a)$. Now we show that $\mathbb{I}_\beta(a)$ is β-saturated. Let $M \rhd_\beta^* N$ and $N \in \mathbb{I}_\beta(a)$.

- If $N \in \mathcal{O}^\oslash$ then $N \in \mathcal{M}^\oslash$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in \text{fv}(N)$. By theorem 4.2, $\text{fv}(N) \subseteq \text{fv}(M)$ and $\text{d}(M) = \text{d}(N)$, hence, $M \in \mathcal{O}^\oslash$
- If $N \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash a\rangle\}$ then $\exists \Gamma \subset \mathbb{H}^\oslash$, such that $N : \langle \Gamma \vdash a\rangle$. By theorem 24, $M : \langle \Gamma\uparrow^M \vdash a\rangle$. Since by theorem 4.2, $\text{fv}(N) \subseteq \text{fv}(M)$, let $\text{fv}(N) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ and $\text{fv}(M) = \text{fv}(N) \cup \{x_{n+1}^{L_{n+1}}, \ldots, x_{n+m}^{L_{n+m}}\}$. So $\Gamma\uparrow^M = \Gamma, (x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, \ldots, x_{n+m}^{L_{n+m}} : \omega^{L_{n+m}})$. $\forall n+1 \leq i \leq n+m$, let U_i such that $x_i \in \mathbb{V}_{U_i}$. Then $\Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \ldots, x_{n+m}^{L_{n+m}} : U_{n+m}) \subset \mathbb{H}^\oslash$ and by $\sqsubseteq$, $M : \langle \Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \ldots, x_{n+m}^{L_{n+m}} : U_{n+m}) \vdash a\rangle$. Thus $M : \langle \mathbb{H}^\oslash \vdash a\rangle$ and since by theorem 4.2, $\text{d}(M) = \text{d}(N)$, $M \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash a\rangle\}$.

2. By induction on U.

- $U = a$: By definition of $\mathbb{I}_{\beta\eta}$.
- $U = \omega^L$: By definition, $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{M}^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* \omega^L\rangle\} \subseteq \mathbb{I}_{\beta\eta}(\omega^L)$.
  Let $M \in \mathbb{I}_{\beta\eta}(\omega^L)$ where $\text{fv}(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$. We have $M : \langle (x_i^{L_i} : \omega^{L_i})_n \vdash \omega^L\rangle$ and $M \in \mathcal{M}^L$. $\forall\ 1 \leq i \leq n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 14, $M : \langle \Gamma \vdash \omega^L\rangle$. Hence $M : \langle \mathbb{H}^L \vdash \omega^L\rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* \omega^L\rangle\}$.
  We deduce $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* \omega^L\rangle\}$.
- $U = e_i V$: $L = i :: K$ and $\text{d}(V) = K$. By IH and lemma 41, $\mathbb{I}_{\beta\eta}(e_i V) = (\mathbb{I}_{\beta\eta}(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V\rangle\})^{+i} =$
  $\mathcal{O}^{i::L} \cup (\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V\rangle\})^{+i}$.
  - If $M \in \mathcal{M}^K$ and $M : \langle \mathbb{H}^K \vdash^* V\rangle$, then $M \rhd_{\beta\eta}^* N$ and $N : \langle \Gamma \vdash V\rangle$ where $\Gamma \subset \mathbb{H}^K$. By e, lemmas 46 and 39, $N^{+i} : \langle e_i\Gamma \vdash e_i V\rangle$, $M^{+i} \rhd_{\beta\eta}^* N^{+i}$ and $e_i\Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash^* U\rangle$.

- If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash^* U \rangle$, then $M \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash U \rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas 46, 13, and 39, $M^{-i} \rhd^*_{\beta\eta} N^{-i}$, $N^{-i} : \langle \Gamma^{-i} \vdash V \rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by lemma 46, $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V \rangle\}$.

Hence $(\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V \rangle\})^{+i} = \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U \rangle\}$ and $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U \rangle\}$.

– $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_{\beta\eta}(U_1 \sqcap U_2) = \mathbb{I}_{\beta\eta}(U_1) \cap \mathbb{I}_{\beta\eta}(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_1 \rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_2 \rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_1 \rangle\} \cap \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_2 \rangle\})$.

- If $M \in \mathcal{M}^L$, $M : \langle \mathbb{H}^L \vdash^* U_1 \rangle$ and $M : \langle \mathbb{H}^L \vdash^* U_2 \rangle$, then $M \rhd^*_{\beta\eta} N_1$, $M \rhd^*_{\beta\eta} N_2$, $N_1 : \langle \Gamma_1 \vdash U_1 \rangle$ and $N_2 : \langle \Gamma_2 \vdash U_2 \rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. By confluence theorem 7 and subject reduction theorem 20, $\exists M'$ such that $M \rhd^*_{\beta\eta} M'$, $M' : \langle \Gamma_1 \upharpoonright_{M'} \vdash U_1 \rangle$ and $M' : \langle \Gamma_2 \upharpoonright_{M'} \vdash U_2 \rangle$. Hence by Remark 16, $M' : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \vdash U_1 \sqcap U_2 \rangle$ and, by lemma 39, $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \subseteq \Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2 \rangle$.
- If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2 \rangle$, then $M \rhd^*_{\beta\eta} N$, $N : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $N : \langle \Gamma \vdash U_1 \rangle$ and $N : \langle \Gamma \vdash U_2 \rangle$.
  Hence, $M : \langle \mathbb{H}^L \vdash^* U_1 \rangle$ and $M : \langle \mathbb{H}^L \vdash^* U_2 \rangle$.

We deduce that $\mathbb{I}_{\beta\eta}(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2 \rangle\}$.

– $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V \rangle\}$ and $\mathbb{I}_{\beta\eta}(T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* T \rangle\}$. Note that $\mathbb{I}_{\beta\eta}(V \to T) = \mathbb{I}_{\beta\eta}(V) \rightsquigarrow \mathbb{I}_{\beta\eta}(T)$.

- Let $M \in \mathbb{I}_{\beta\eta} V) \rightsquigarrow \mathbb{I}_{\beta\eta} T)$ and, by lemma 37, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin \mathrm{fv}(M)$. Then $M \diamond y^K$. By remark 16, $y^K : \langle (y^K : V) \vdash^* V \rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash^* V \rangle$. Thus, $y^K \in \mathbb{I}_{\beta\eta}(V)$ and $My^K \in \mathbb{I}_{\beta\eta}(T)$.
  - If $My^K \in \mathcal{O}^\oslash$, then since $y \in \mathcal{V}_2$, by lemma 41, $M \in \mathcal{O}^\oslash$.
  - If $My^K \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* T \rangle\}$ then $My^K \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash T \rangle$, hence, $\lambda y^K.My^K \rhd^*_{\beta\eta} \lambda y^K.N$. We have two cases:
    - If $y^K \in \mathrm{dom}(\Gamma)$, then $\Gamma = \Delta, (y^K : V)$ and by $\to_I$, $\lambda y^K.N : \langle \Delta \vdash V \to T \rangle$.
    - If $y^K \notin \mathrm{dom}(\Gamma)$, let $\Delta = \Gamma$. By $\to'_I$, $\lambda y^K.N : \langle \Delta \vdash \omega^K \to T \rangle$. By $\sqsubseteq$, since $\langle \Delta \vdash \omega^K \to T \rangle \sqsubseteq \langle \Delta \vdash V \to T \rangle$, we have $\lambda y^K.N : \langle \Delta \vdash V \to T \rangle$.

    Note that $\Delta \subset \mathbb{G}$. Since $\lambda y^K.My^K \rhd^*_{\beta\eta} M$ and $\lambda y^K.My^K \rhd^*_{\beta\eta} \lambda y^K.N$, by theorem 7 and theorem 20, there is M' such that $M \rhd^*_{\beta\eta} M'$, $\lambda y^K.N \rhd^*_{\beta\eta} M'$, $M' : \langle \Delta \upharpoonright_{M'} \vdash V \to T \rangle$. Since $\Delta \upharpoonright_{M'} \subseteq \Delta \subset \mathbb{H}^\oslash$, $M : \langle \mathbb{H}^\oslash \vdash^* V \to T \rangle$.
- Let $M \in \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* V \to T \rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V \rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K$.
  - If $M \in \mathcal{O}^\oslash$, then, by lemma 41, $MN \in \mathcal{O}^\oslash$.
  - If $M \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* V \to T \rangle\}$, then
    - If $N \in \mathcal{O}^K$, then, by lemma 41, $MN \in \mathcal{O}^\oslash$.
    - If $N \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash^* V \rangle\}$ then $M \rhd^*_{\beta\eta} M_1$, $N \rhd^*_{\beta\eta} N_1$, $M_1 : \langle \Gamma_1 \vdash V \to T \rangle$ and $N_1 : \langle \Gamma_2 \vdash V \rangle$ where $\Gamma_1 \subset \mathbb{H}^\oslash$ and $\Gamma_2 \subset \mathbb{H}^K$. By lemma 46, $MN \rhd^*_{\beta\eta} M_1 N_1$ and, by $\to_E$, $M_1 N_1 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle$. By lemma 39, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^\oslash$. Therefore $MN : \langle \mathbb{H}^\oslash \vdash^* T \rangle$.

We deduce that $\mathbb{I}_{\beta\eta}(V \to T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash^* V \to T \rangle\}$.

3. We only do the case $r = \beta$. By induction on U.

– $U = a$: By definition of $\mathbb{I}_\beta$.

– $U = \omega^L$: By definition, $\mathbb{I}_\beta(\omega^L) = \mathcal{M}^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash \omega^L\rangle\} \subseteq \mathbb{I}_\beta(\omega^L)$.
Let $M \in \mathbb{I}_\beta(\omega^L)$ where $\mathrm{fv}(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$. We have $M : \langle (x_i^{L_i} : \omega^{L_i})_n \vdash \omega^L\rangle$ and $M \in \mathcal{M}^L$. $\forall\ 1 \leq i \leq n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 14, $M : \langle \Gamma \vdash \omega^L\rangle$. Hence $M : \langle \mathbb{H}^L \vdash \omega^L\rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash \omega^L\rangle\}$.
We deduce $\mathbb{I}_\beta(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash \omega^L\rangle\}$.

– $U = e_i V$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 41, $\mathbb{I}_\beta(e_i V) = (\mathbb{I}_\beta(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i} =$
$\mathcal{O}^{i::L} \cup (\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i}$.
  - If $M \in \mathcal{M}^K$ and $M : \langle \mathbb{H}^K \vdash V\rangle$, then $M : \langle \Gamma \vdash V\rangle$ where $\Gamma \subset \mathbb{H}^K$. By e and 39, $M^{+i} : \langle e_i\Gamma \vdash e_i V\rangle$ and $e_i\Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash U\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash U\rangle$, then $M : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas 13, and 39, $M^{-i} : \langle \Gamma^{-i} \vdash V\rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by lemma 46, $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\}$.

  Hence $(\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i} = \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U\rangle\}$ and $\mathbb{I}_\beta(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U\rangle\}$.

– $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_\beta(U_1 \sqcap U_2) = \mathbb{I}_\beta(U_1) \cap \mathbb{I}_\beta(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_1\rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_2\rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_1\rangle\} \cap \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_2\rangle\})$.
  - If $M \in \mathcal{M}^L$, $M : \langle \mathbb{H}^L \vdash U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash U_2\rangle$, then $M : \langle \Gamma_1 \vdash U_1\rangle$ and $M : \langle \Gamma_2 \vdash U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. Hence by Remark 16, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2\rangle$ and, by lemma 39, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle$, then $M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $M : \langle \Gamma \vdash U_1\rangle$ and $M : \langle \Gamma \vdash U_2\rangle$. Hence, $M : \langle \mathbb{H}^L \vdash U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash U_2\rangle$.

  We deduce that $\mathbb{I}_\beta(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle\}$.

– $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_\beta(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\}$ and $\mathbb{I}_\beta(T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash T\rangle\}$. Note that $\mathbb{I}_\beta(V \to T) = \mathbb{I}_\beta(V) \rightsquigarrow \mathbb{I}_\beta(T)$.
  - Let $M \in \mathbb{I}_\beta V) \rightsquigarrow \mathbb{I}_\beta T)$ and, by lemma 37, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin \mathrm{fv}(M)$. Then $M \diamond y^K$. By remark 16, $y^K : \langle (y^K : V) \vdash^* V\rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash V\rangle$. Thus, $y^K \in \mathbb{I}_\beta(V)$ and $My^K \in \mathbb{I}_\beta(T)$.
    - If $My^K \in \mathcal{O}^\oslash$, then since $y \in \mathcal{V}_2$, by lemma 41, $M \in \mathcal{O}^\oslash$.
    - If $My^K \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash T\rangle\}$ then $My^K : \langle \Gamma \vdash T\rangle$. Since by lemma 14, $\mathrm{dom}(\Gamma) = \mathrm{fv}(My^K)$ and $y^K \in \mathrm{fv}(My^K)$, $\Gamma = \Delta, (y^K : V')$. Since $(y^K : V') \in \mathbb{H}^\oslash$, by lemma 37, $V = V'$. So $My^K : \langle \Delta, (y^K : V) \vdash T\rangle$ and by lemma 17 $M : \langle \Delta \vdash V \to T\rangle$. Note that $\Delta \subset \mathbb{H}^\oslash$, hence $M : \langle \mathbb{H}^\oslash \vdash V \to T\rangle$.
  - Let $M \in \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash V \to T\rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K$.
    - If $M \in \mathcal{O}^\oslash$, then, by lemma 41, $MN \in \mathcal{O}^\oslash$.
    - If $M \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash V \to T\rangle\}$, then
      - If $N \in \mathcal{O}^K$, then, by lemma 41, $MN \in \mathcal{O}^\oslash$.
      - If $N \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash V\rangle\}$ then $M : \langle \Gamma_1 \vdash V \to T\rangle$ and $N : \langle \Gamma_2 \vdash V\rangle$ where $\Gamma_1 \subset \mathbb{H}^\oslash$ and $\Gamma_2 \subset \mathbb{H}^K$. By $\to_E$, $MN : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$. By lemma 39, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^\oslash$. Therefore $MN : \langle \mathbb{H}^\oslash \vdash T\rangle$.

  We deduce that $\mathbb{I}_\beta(V \to T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash V \to T\rangle\}$.

$\square$

Now, we use this crucial $\mathbb{I}$ to establish completeness of our semantics.

Theorem 44 (Completeness of $\vdash$). *Let $U \in \mathbb{U}$ such that $d(U) = L$.*

1. $[U]_{\beta\eta} = \{M \in \mathcal{M}^L \ / \ M$ *closed,* $M \rhd^*_{\beta\eta} N$ *and* $N : \langle () \vdash U \rangle\}$.
2. $[U]_\beta = [U]_h = \{M \in \mathcal{M}^L \ / \ M : \langle () \vdash U \rangle\}$.
3. $[U]_{\beta\eta}$ *is stable by reduction. I.e., If* $M \in [U]_{\beta\eta}$ *and* $M \rhd^*_{\beta\eta} N$ *then* $N \in [U]_{\beta\eta}$.

Proof Let $r \in \{\beta, h, \beta\eta\}$.

1. Let $M \in [U]_{\beta\eta}$. Then M is a closed term and $M \in \mathbb{I}_{\beta\eta}(U)$. Hence, by Lemma 43, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U \rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^* U \rangle$ and so, $M \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash U \rangle$ where $\Gamma \subset \mathbb{H}^L$. By Theorem 4, N is closed and, by Lemma 14.2, $N : \langle () \vdash U \rangle$.
   Conversely, take M closed such that $M \rhd^*_\beta N$ and $N : \langle () \vdash U \rangle$. Let $\mathcal{I} \in \beta$-int. By Lemma 31, $N \in \mathcal{I}(U)$. By Lemma 30.1, $\mathcal{I}(U)$ is $\beta\eta$-saturated. Hence, $M \in \mathcal{I}(U)$. Thus $M \in [U]$.
2. Let $M \in [U]_\beta$. Then M is a closed term and $M \in \mathbb{I}_\beta(U)$. Hence, by Lemma 43, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U \rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash U \rangle$ and so, $M : \langle \Gamma \vdash U \rangle$ where $\Gamma \subset \mathbb{H}^L$. By Lemma 14.2, $N : \langle () \vdash U \rangle$.
   Conversely, take M such that $M : \langle () \vdash U \rangle$. By Lemma 14.2, M is closed. Let $\mathcal{I} \in \beta$-int. By Lemma 31, $M \in \mathcal{I}(U)$. Thus $M \in [U]_\beta$.
   It is easy to see that $[U]_\beta = [U]_h$.
3. Let $M \in [U]$ such that $M \rhd^*_{\beta\eta} N$. By 1, M is closed, $M \rhd^*_{\beta\eta} P$ and $P : \langle () \vdash U \rangle$. By confluence Theorem 7, there is Q such that $P \rhd^*_{\beta\eta} Q$ and $N \rhd^*_{\beta\eta} Q$. By subject reduction Theorem 20, $Q : \langle () \vdash U \rangle$. By Theorem 4, N is closed and, by 1, $N \in [U]$.

$\square$

8 Conclusion

Expansion may be viewed to work like a multi-layered simultaneous substitution. Moreover, expansion is a crucial part of a procedure for calculating principal typings and helps support compositional type inference. Because the early definitions of expansion were complicated, expansion variables (E-variables) were introduced to simplify and mechanise expansion. The aim of this paper is to give a complete semantics for intersection type systems with expansion variables.

The only earlier attempt (see Kamareddine, Nour, Rahli and Wells [12]) at giving a semantics for expansion variables could only handle the λI-calculus, did not allow a universal type, and was incomplete in the presence of more than one expansion variable. This paper overcomes these difficulties and gives a complete semantics for an intersection type system with an arbitrary (possibly infinite) number of expansion variables using a calculus indexed with finite sequences of natural numbers.

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A Proofs of Section 2

The next lemma is needed in the proofs.

Lemma 45. *Let $M, N, N_1, \dots, N_n \in \mathcal{M}$.*

1. *If $M \diamond N$ and M' is a subterm of M then $M' \diamond N$.*
2. *If $d(M) = L$ and x^K occurs in M, then $K \succeq L$.*
3. *Let $\mathcal{X} = \{M\} \cup \{N_i / 1 \leq i \leq n\}$. If $\forall 1 \leq i \leq n, d(N_i) = L_i$ and $\diamond\mathcal{X}$, then $M[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $d(M[(x_i^{L_i} := N_i)_n]) = d(M)$.*
4. *Let $\mathcal{X} = \{M, N\} \cup \{N_i / 1 \leq i \leq n\}$. If $\forall 1 \leq i \leq n, d(N_i) = L_i$ and $\diamond\mathcal{X}$ then $M[(x_i^{L_i} := N_i)_n] \diamond N[(x_i^{L_i} := N_i)_n]$*

Proof

1. By induction on M.
   - Case $M = x^L$ is trivial.
   - Case $M = \lambda x^L.P$ where $\forall K \in \mathcal{L}_\mathbb{N}, x^K \notin \mathrm{fv}(N)$. If $M' = M$ then nothing to prove. Else M' is a subterm of P. If we prove that $P \diamond N$ then we can use IH to get $M' \diamond N$. Hence, now we prove $P \diamond N$. Let $y \in \mathcal{V}$ such that $y^K \in \mathrm{fv}(P)$ and $y^{K'} \in \mathrm{fv}(N)$. Since $x^{K'} \notin \mathrm{fv}(N)$, then $x \neq y$ and $y^K \neq x^L$. Hence $y^K \in \mathrm{fv}(M)$ and since $M \diamond N$ then $K = K'$. Hence, $P \diamond N$.
   - Case $M = M_1 M_2$. Let $i \in \{1, 2\}$. First we prove that $M_i \diamond N$: let $x \in \mathcal{V}$, such that $x^L \in \mathrm{fv}(M_i)$ and $x^K \in \mathrm{fv}(N)$, then $x^L \in \mathrm{fv}(M)$ and so $L = K$. Now, if $M' = M$ then nothing to prove. Else
     - Either M' is a subterm of M_1 and so by IH, since $M_1 \diamond N$, $M' \diamond N$.
     - Or M' is a subterm of M_2 and so by IH, since $M_2 \diamond N$, $M' \diamond N$.
2. By induction on M.
   - If $M = x^K$ then $\mathrm{d}(M) = K$ and since $\succeq$ is an order relation, $K \succeq K$.
   - If $M = M_1 M_2$ then $\mathrm{d}(M) = \mathrm{d}(M_1)$. Let $L' = \mathrm{d}(M_2)$ so $L' \succeq L$. By IH, if x^K occurs in M_1 then $K \succeq L$ and if x^K occurs in M_2 then $K \succeq L'$. Since x^K occurs in M, $K \succeq L$.
   - If $M = \lambda x^{L_1}.M_1$ then $L_1 \succeq \mathrm{d}(M_1) = \mathrm{d}(\lambda x^{L_1}.M_1) = L$. If x^K occurs in M, then $x^K = x^{L_1}$ or x^K occurs in M_1. By IH, if x^K occurs in M_1 then $K \succeq L$.
3. By induction on M.
   - If $M = y^K$ then if $y^K = x_i^{L_i}$, for $1 \leq i \leq n$, then $M[(x_i^{L_i} := N_i)_n] = N_i \in \mathcal{M}$ and $\mathrm{d}(M[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(N_i) = L_i = K$. Else, $M[(x_i^{L_i} := N_i)_n] = y^K \in \mathcal{M}$ and $\mathrm{d}(M[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(y^K)$.
   - If $M = M_1 M_2$ then $\mathrm{d}(M) = \mathrm{d}(M_1)$ and $M[(x_i^{L_i} := N_i)_n] = M_1[(x_i^{L_i} := N_i)_n] M_2[(x_i^{L_i} := N_i)_n]$. Since $\forall N \in \mathcal{X}, M \diamond N$, by 1., $\forall N \in \mathcal{X}, M_1 \diamond N$ and $M_2 \diamond N$. Since $M_1, M_2 \in \mathcal{M}$, by IH, $M_1[(x_i^{L_i} := N_i)_n], M_2[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$, $\mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1)$ and $\mathrm{d}(M_2[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_2)$. Let $x^K \in \mathrm{fv}(M_1[(x_i^{L_i} := N_i)_n])$ and $x^{K'} \in \mathrm{fv}(M_2[(x_i^{L_i} := N_i)_n])$. If $x^K \in \mathrm{fv}(M_1)$ then by 1., $\diamond(\{M_1, M_2\} \cup \{N_i / 1 \leq i \leq n\})$ hence $K = K'$. Let $1 \leq i \leq n$. If $x^K \in \mathrm{fv}(N_i)$ then by 1., $\diamond(\{M_2\} \cup \{N_i / 1 \leq i \leq n\})$ hence $K = K'$. So $M_1[(x_i^{L_i} := N_i)_n] \diamond M_2[(x_i^{L_i} := N_i)_n]$. Furthermore, $\mathrm{d}(M_2[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_2) \succeq \mathrm{d}(M_1) = \mathrm{d}(M_1[(x_i^{L_i} := N_i)_n])$ hence $M_1[(x_i^{L_i} := N_i)_n] M_2[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $\mathrm{d}(M_1[(x_i^{L_i} := N_i)_n] M_2[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1) = \mathrm{d}(M)$.
   - If $M = \lambda y^K.M_1$ where $K \succeq \mathrm{d}(M_1)$ and $\forall 1 \leq i \leq n$, $y \neq x_i$ and $\forall K' \in \mathcal{L}_\mathbb{N}$, $y^{K'} \notin \mathrm{fv}(N_i)$ then $M[(x_i^{L_i} := N_i)_n] = \lambda y^K.M_1[(x_i^{L_i} := N_i)_n]$. Since $M_1 \in \mathcal{M}$, then by 1. and IH $M_1[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $\mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1)$. So $\lambda y^K.M_1[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$ and $\mathrm{d}(\lambda y^K.M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1[(x_i^{L_i} := N_i)_n]) = \mathrm{d}(M_1) = \mathrm{d}(M)$.

4. By 3., $M[(x_i^{L_i} := N_i)_n], N[(x_i^{L_i} := N_i)_n] \in \mathcal{M}$. Let $x^L \in \mathrm{fv}(M[(x_i^{L_i} := N_i)_n])$ and $x^K \in \mathrm{fv}(N[(x_i^{L_i} := N_i)_n])$. So $x^L \in \mathrm{fv}(M) \cup \mathrm{fv}(N_1) \cup ... \cup \mathrm{fv}(N_n)$ and $x^K \in \mathrm{fv}(N) \cup \mathrm{fv}(N_1) \cup ... \cup \mathrm{fv}(N_n)$. Since $\diamond\mathcal{X}$, then $K = L$. Hence, $M[(x_i^{L_i} := N_i)_n] \diamond N[(x_i^{L_i} := N_i)_n]$

□

Proof [Of Theorem 4]

1. By induction on $M \rhd_\eta^* N$, we only do the induction step:
   - $M = \lambda x^L.Nx^L \rhd_\eta N$ and $x^L \notin \mathrm{fv}(N)$. By definition $N \in \mathcal{M}$, $\mathrm{fv}(M) = \mathrm{fv}(Nx^L) \setminus \{x^L\} = \mathrm{fv}(N)$ and $\mathrm{d}(M) = \mathrm{d}(Nx^L) = \mathrm{d}(N)$.
   - $M = \lambda x^L.M_1 \rhd_\eta \lambda x^L.N_1 = N$ and $M_1 \rhd_\eta N_1$. By IH, $N_1 \in \mathcal{M}$, $\mathrm{fv}(N_1) = \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. By defintion $\mathrm{d}(M_1) \preceq L$, so $\mathrm{d}(N_1) \preceq L$ hence $N \in \mathcal{M}$. By defintion $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$ and $\mathrm{fv}(N) = \mathrm{fv}(N_1) \setminus \{x^L\} = \mathrm{fv}(M_1) \setminus \{x^L\} = \mathrm{fv}(M)$.
   - $M = M_1M_2 \rhd_\eta N_1M_2 = N$, $M_1 \diamond M_2$, $N_1 \diamond M_2$ and $M_1 \rhd_\eta N_1$. By IH, $N_1 \in \mathcal{M}$, $\mathrm{fv}(N_1) = \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. Since $\mathrm{d}(N_1) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, $N \in \mathcal{M}$. By defintion, $\mathrm{fv}(N) = \mathrm{fv}(N_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$.
   - $M = M_1M_2 \rhd_\eta M_1N_2 = N$, $M_1 \diamond M_2$, $M_1 \diamond N_2$ and $M_2 \rhd_\eta N_2$. By IH, $N_2 \in \mathcal{M}$, $\mathrm{fv}(N_2) = \mathrm{fv}(M_2)$ and $\mathrm{d}(M_2) = \mathrm{d}(N_2)$. Since $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2) = \mathrm{d}(N_2)$, $N \in \mathcal{M}$. By defintion, $\mathrm{fv}(N) = \mathrm{fv}(M_1) \cup \mathrm{fv}(N_2) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N)$.
2. Case $r = \beta$. By induction on $M \rhd_\beta^* N$, we only do the induction step:
   - $M = (\lambda x^L.M_1)M_2 \rhd_\beta M_1[x^L := M_2] = N$ and $\mathrm{d}(M_2) = L$. $(\lambda x^L.M_1) \diamond M_2$ by definition, so $M_1 \diamond M_2$ by lemma 45.1 and $N \in \mathcal{M}$ by lemma 45.3. If $x^L \in \mathrm{fv}(M_1)$ then $\mathrm{fv}(N) = (\mathrm{fv}(M_1) \setminus \{x^L\}) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$. If $x^L \notin \mathrm{fv}(M_1)$ then $\mathrm{fv}(N) = \mathrm{fv}(M_1) = \mathrm{fv}(M_1) \setminus \{x^L\} \subseteq \mathrm{fv}(M)$. By definition, $\mathrm{d}(M) = \mathrm{d}(\lambda x^L.M_1) = \mathrm{d}(M_1)$ and by lemma 45, $\mathrm{d}(N) = \mathrm{d}(M_1)$.
   - $M = \lambda x^L.M_1 \rhd_\beta \lambda x^L.N_1 = N$ and $M_1 \rhd_\beta N_1$. By IH, $N_1 \in \mathcal{M}$, $\mathrm{fv}(N_1) \subseteq \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. By defintion $\mathrm{d}(M_1) \preceq L$, so $\mathrm{d}(N_1) \preceq L$ hence $N \in \mathcal{M}$. By defintion $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$ and $\mathrm{fv}(N) = \mathrm{fv}(N_1) \setminus \{x^L\} \subseteq \mathrm{fv}(M_1) \setminus \{x^L\} = \mathrm{fv}(M)$.
   - $M = M_1M_2 \rhd_\beta N_1M_2 = N$, $M_1 \diamond M_2$, $N_1 \diamond M_2$ and $M_1 \rhd_\beta N_1$. By IH, $N_1 \in \mathcal{M}$, $\mathrm{fv}(N_1) \subseteq \mathrm{fv}(M_1)$ and $\mathrm{d}(M_1) = \mathrm{d}(N_1)$. Since $\mathrm{d}(N_1) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, $N \in \mathcal{M}$. By defintion, $\mathrm{fv}(N) = \mathrm{fv}(N_1) \cup \mathrm{fv}(M_2) \subseteq \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N_1) = \mathrm{d}(N)$.
   - $M = M_1M_2 \rhd_\beta M_1N_2 = N$, $M_1 \diamond M_2$, $M_1 \diamond N_2$ and $M_2 \rhd_\beta N_2$. By IH, $N_2 \in \mathcal{M}$, $\mathrm{fv}(N_2) \subseteq \mathrm{fv}(M_2)$ and $\mathrm{d}(M_2) = \mathrm{d}(N_2)$. Since $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2) = \mathrm{d}(N_2)$, $N \in \mathcal{M}$. By defintion, $\mathrm{fv}(N) = \mathrm{fv}(M_1) \cup \mathrm{fv}(N_2) \subseteq \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(M_1) = \mathrm{d}(N)$.

   Case $r = \beta\eta$, by the β and η cases. Case $r = h$, by the β case.

□

The next lemma is again needed in the proofs.

Lemma 46. *Let $M, N, N_1, N_2, \ldots, N_p \in \mathcal{M}$, $\blacktriangleright' \in \{\rhd_\beta, \rhd_\eta, \rhd_{\beta\eta}, \rhd_\beta^*, \rhd_\eta^*, \rhd_{\beta\eta}^*\}$, $\blacktriangleright \in \{\rhd_\beta, \rhd_\eta, \rhd_{\beta\eta}, \rhd_h, \rhd_\beta^*, \rhd_\eta^*, \rhd_{\beta\eta}^*, \rhd_h^*\}$, and $i, p \geq 0$. We have:*

1. *$M^{+i} \in \mathcal{M}$ and x^K occurs in M^{+i} iff $K = i :: L$ and x^L occurs in M.*
2. *If $M \diamond N$ then $M^{+i} \diamond N^{+i}$.*
3. *$d(M^{+i}) = i :: d(M)$ and $(M^{+i})^{-i} = M$.*
4. *$(M[(x_j^{L_j} := N_j)_p])^{+i} = M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.*
5. *If $M \blacktriangleright N$, then $M^{+i} \blacktriangleright N^{+i}$.*

6. *If* $d(M) = i :: L$, *then:*
   (a) $M = P^{+i}$ *for some* $P \in \mathcal{M}$, $d(M^{-i}) = L$ *and* $(M^{-i})^{+i} = M$.
   (b) *If* $\forall 1 \leq j \leq p, d(N_j) = i :: K_j$, *then* $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = M^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.
   (c) *If* $M \blacktriangleright N$ *then* $M^{-i} \blacktriangleright N^{-i}$.
7. *If* $M \blacktriangleright N$, $P \blacktriangleright Q$ *and* $M \diamond P$ *then* $N \diamond Q$
8. *If* $M \blacktriangleright N^{+i}$, *then there is* $P \in \mathcal{M}$ *such that* $M = P^{+i}$ *and* $P \blacktriangleright N$.
9. *If* $M^{+i} \blacktriangleright N$, *then there is* $P \in \mathcal{M}$ *such that* $N = P^{+i}$ *and* $M \blacktriangleright P$.
10. *If* $M \blacktriangleright N$ *and* $d(P) = L$, *then* $M[x^L := P] \blacktriangleright N[x^L := P]$.
11. *If* $N \blacktriangleright' P$ *and* $d(N) = L$, *then* $M[x^L := N] \blacktriangleright' M[x^L := P]$.
12. *If* $M \blacktriangleright' M'$, $P \blacktriangleright' P'$ *and* $d(P) = L$, *then* $M[x^L := P] \blacktriangleright' M'[x^L := P']$.

Proof

1. We only prove $M^{+i} \in \mathcal{M}$, by induction on M:
   – If $M = x^L$ then $M^{+i} = x^{i::L} \in \mathcal{M}$.
   – If $M = \lambda x^L.M_1$ then $M^{+i} = \lambda x^{i::L}.M_1^{+i}$. By IH, $M_1^{+i} \in \mathcal{M}$, so $\lambda x^{i::L}.M_1^{+i} \in \mathcal{M}$.
   – If $M = M_1 M_2$ then $M^{+i} = M_1^{+i} M_2^{+i}$. By IH, $M_1^{+i}, M_2^{+i} \in \mathcal{M}$. If $y^{K_1} \in \mathrm{fv}(M_1^{+i})$ and $y^{K_2} \in \mathrm{fv}(M_2^{+i})$, then $K_1 = i :: K_1'$, $K_2 = i :: K_2'$, $x^{K_1'} \in \mathrm{fv}(M_1)$ and $x^{K_2'} \in \mathrm{fv}(M_2)$. Thus $K_1' = K_2'$, so $K_1 = K_2$. Hence $M_1^{+i} \diamond M_2^{+i}$ and so, $M^{+i} \in \mathcal{M}$
2. Easy, using 1.
3. By induction on M.
4. By induction on M:
   – Let $M = y^K$. If $\forall 1 \leq j \leq p, y^K \neq x_j^{L_j}$ then $y^K[(x_j^{L_j} := N_j)_p] = y^K$. Hence $(y^K[(x_j^{L_j} := N_j)_p])^{+i} = y^{i::K} = y^{i::K}[(x_j^{i::L_j} := N_j^{+i})_p]$. If $\exists 1 \leq j \leq p, y^K = x_j^{L_j}$ then $y^K[(x_j^{L_j} := N_j)_p] = N_j$. Hence $(y^K[(x_j^{L_j} := N_j)_p])^{+i} = N_j^{+i} = y^{i::K}[(x_j^{i::L_j} := N_j^{+i})_p]$.
   – Let $M = \lambda y^K.M_1$. $M[(x_j^{L_j} := N_j)_p] = \lambda y^K.M_1[(x_j^{L_j} := N_j)_p]$ where $\forall 1 \leq j \leq p, y^K \notin N_j$. By IH, $(M_1[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
     Hence, $(M[(x_j^{L_j} := N_j)_p])^{+i} = \lambda y^{i::K}.(M_1[(x_j^{L_j} := N_j)_p])^{+i} =$
     $\lambda y^{i::K}.M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] = (\lambda y^K.M_1)^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
   – Let $M = M_1 M_2$. $M[(x_j^{L_j} := N_j)_p] = M_1[(x_j^{L_j} := N_j)_p] M_2[(x_j^{L_j} := N_j)_p]$. By IH, $(M_1[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$ and $(M_2[(x_j^{L_j} := N_j)_p])^{+i} = M_2^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
     Hence $(M[(x_j^{L_j} := N_j)_p])^{+i} = (M_1[(x_j^{L_j} := N_j)_p])^{+i}(M_2[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] M_2^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] = M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
5. – Let $\blacktriangleright$ be $\rhd_\beta$. By induction on $M \rhd_\beta N$.
     • Let $M = (\lambda x^L.M_1)M_2 \rhd_\beta M_1[x^L := M_2] = N$ where $\mathrm{d}(M_2) = L$, then $M^{+i} = (\lambda x^{i::L}.M_1^{+i})M_2^{+i} \rhd_\beta M_1^{+i}[x^{i::L} := M_2^{+i}] = (M_1[x^L := M_2])^{+i}$.
     • Let $M = \lambda x^L.M_1 \rhd_\beta \lambda x^L N_1 = N$ where $M_1 \rhd_\beta N_1$. By IH, $M_1^{+i} \rhd_\beta N_1^{+i}$, hence $M^{+i} = \lambda x^{i::L}.M_1^{+i} \rhd_\beta \lambda x^{i::L} N_1^{+i} = N^{+i}$.
     • Let $M = M_1 M_2 \rhd_\beta N_1 M_2 = N$ where $M_1 \diamond M_2$, $N_1 \diamond M_2$ and $M_1 \rhd_\beta N_1$. By IH, $M_1^{+i} \rhd_\beta N_1^{+i}$, hence $M^{+i} = M_1^{+i} M_2^{+i} \rhd_\beta N_1^{+i} M_2^{+i} = N^{+i}$.
     • Let $M = M_1 M_2 \rhd_\beta M_1 N_2 = N$ where $M_1 \diamond M_2$, $M_1 \diamond N_2$ and $M_2 \rhd_\beta N_2$. By IH, $M_2^{+i} \rhd_\beta N_2^{+i}$, hence $M^{+i} = M_1^{+i} M_2^{+i} \rhd_\beta N_1^{+i} M_2^{+i} = N^{+i}$.
   – Let $\blacktriangleright$ be $\rhd_\beta^*$. By induction on $\rhd_\beta^*$. using $\rhd_\beta$.
   – Let $\blacktriangleright$ be $\rhd_\eta$. We only do the basic case. The inductive cases are as for $\rhd_\beta$. Let $M = \lambda x^L.N x^L \rhd_\eta N$ where $x^L \notin \mathrm{fv}(N)$. Then $M^{+i} = \lambda x^{i::L}.N^{+i} x^{i::L} \rhd_\eta N^{+i}$.

– Let ► be $\rhd_\eta^*$. By induction on $\rhd_\eta^*$ using $\rhd_\eta$.
– Let ► be $\rhd_{\beta\eta}$, $\rhd_{\beta\eta}^*$, $\rhd_h$ or $\rhd_h^*$. By the previous items.

6. (a) By induction on M:
   – Let $M = y^{i::L}$. Let $N = y^L \in \mathcal{M}$, then $N^{+i} = M$.
   – Let $M = \lambda y^K.M_1$. Since $\mathrm{d}(M_1) = \mathrm{d}(M) = i :: L$, by IH, $M_1 = P^{+i}$ for some $P \in \mathcal{M}$, $\mathrm{d}(M_1^{-i}) = L$ and $(M_1^{-i})^{+i} = M_1$. Moreover, $K \succeq i :: L$ hence $K = i :: L :: K'$ for some K'. Let $Q = \lambda y^{L::K'}.P$. Since $P = (P^{+i})^{-i} = M_1^{-i}$, $\mathrm{d}(P) = L$. Since $L \preceq L :: K'$, $Q \in \mathcal{M}$ and $Q^{+i} = M$. $\mathrm{d}(M^{-i}) = \mathrm{d}(\lambda y^{L::K'}.P) = \mathrm{d}(P) = L$ and $(M^{-i})^{+i} = P^{+i} = M$.
   – Let $M = M_1M_2$. Then $\mathrm{d}(M) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, so $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH $M_1 = P_1^{+i}$ for some $P_1 \in \mathcal{M}$, $\mathrm{d}(M_1^{-i}) = L$ and $(M_1^{-i})^{+i} = M_1$. Again by IH, $M_2 = P_2^{+i}$ for some $P_2 \in \mathcal{M}$, $\mathrm{d}(M_2^{-i}) = L :: L'$ and $(M_2^{-i})^{+i} = M_2$. If $y^{K_1} \in \mathrm{fv}(P_1)$ and $y^{K_2} \in \mathrm{fv}(P_2)$, then $K_1' = i :: K_1$, $K_2' = i :: K_2$, $x^{K_1'} \in \mathrm{fv}(M_1)$ and $x^{K_2'} \in \mathrm{fv}(M_2)$. Thus $K_1' = K_2'$, so $K_1 = K_2$ and $P_1 \diamond P_2$. Hence $M = P_1^{+i}P_2^{+i} = (P_1P_2)^{+i}$. Let $Q = P_1P_2 \in \mathcal{M}$. $\mathrm{d}(P_1) = \mathrm{d}(M_1^{-i}) = L \preceq L :: L' = \mathrm{d}(M_2^{-i}) = \mathrm{d}(P_2)$, so $Q \in \mathcal{M}$ and $Q^{+i} = M$. $\mathrm{d}(M^{-i}) = \mathrm{d}(Q) = \mathrm{d}(P_1) = L$ and $(M^{-i})^{+i} = Q^{+i} = M$.

   (b) By induction on M:
   – Let $M = y^{i::L}$. If $\forall 1 \leq j \leq p, y^{i::L} \neq x_j^{i::K_j}$ then $y^{i::L}[(x_j^{i::K_j} := N_j)_p] = y^{i::L}$. Hence $(y^{i::L}[(x_j^{i::K_j} := N_j)_p])^{-i} = y^L = y^L[(x_j^{K_j} := N_j^{-i})_p]$. If $\exists 1 \leq j \leq p, y^{i::L} = x_j^{i::K_j}$ then $y^{i::L}[(x_j^{i::K_j} := N_j)_p] = N_j$. Hence $(y^{i::L}[(x_j^{i::K_j} := N_j)_p])^{-i} = N_j^{-i} = y^L[(x_j^{K_j} := N_j^{-i})_p]$.
   – Let $M = \lambda y^K.M_1$. $M[(x_j^{i::k_j} := N_j)_p] = \lambda y^K.M_1[(x_j^{i::K_j} := N_j)_p]$ where $\forall 1 \leq j \leq p, y^K \notin N_j$. By IH, $(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]$. Since $\mathrm{d}(i :: L) \preceq K$, $K = i :: L :: K'$ for some K'.
   Hence, $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = \lambda y^{L::K'}.(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = \lambda y^{L::K'}.M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p] = (\lambda y^K.M_1)^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.
   – Let $M = M_1M_2$. $M[(x_j^{i::K_j} := N_j)_p] = M_1[(x_j^{i::K_j} := N_j)_p]M_2[(x_j^{i::K_j} := N_j)_p]$. By IH, $(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]$ and $(M_2[(x_j^{i::K_j} := N_j)_p])^{-i} = M_2^{-i}[(x_j^{K_j} := N_j^{-i})_p]$. Hence
   $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = (M_1[(x_j^{i::K_j} := N_j)_p])^{-i}(M_2[(x_j^{i::K_j} := N_j)_p])^{-i}$
   $= M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]M_2^{-i}[(x_j^{K_j} := N_j^{-i})_p] = M^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.

   (c) – Let ► be $\rhd_\beta$. By induction on $M \rhd_\beta N$.
   • Let $M = (\lambda x^K.M_1)M_2 \rhd_\beta M_1[x^K := M_2] = N$ where $\mathrm{d}(M_2) = K$. Since $i :: L = \mathrm{d}(M) = \mathrm{d}(M_1) \preceq K$, $K = i :: L :: K'$. Then $M^{-i} = (\lambda x^{L::K'}.M_1^{-i})M_2^{-i} \rhd_\beta M_1^{-i}[x^{L::K'} := M_2^{-i}] = (M_1[x^K := M_2])^{-i}$.
   • Let $M = \lambda x^K.M_1 \rhd_\beta \lambda x^L N_1 = N$ where $M_1 \rhd_\beta N_1$. Since $i :: L = \mathrm{d}(M) = \mathrm{d}(M_1) \preceq K$, $K = i :: L :: K'$ for some K'. By IH, $M_1^{-i} \rhd_\beta N_1^{-i}$, hence $M^{-i} = \lambda x^{L::K'}.M_1^{-i} \rhd_\beta \lambda x^{L::K'} N_1^{-i} = N^{-i}$.
   • Let $M = M_1M_2 \rhd_\beta N_1M_2 = N$ where $M_1 \diamond M_2$, $N_1 \diamond M_2$ and $M_1 \rhd_\beta N_1$. Since $i :: L = \mathrm{d}(M) = \mathrm{d}(M_1)$, by IH, $M_1^{-i} \rhd_\beta N_1^{-i}$, hence $M^{-i} = M_1^{-i}M_2^{-i} \rhd_\beta N_1^{-i}M_2^{-i} = N^{-i}$.
   • Let $M = M_1M_2 \rhd_\beta M_1N_2 = N$ where $M_1 \diamond M_2$, $M_1 \diamond N_2$ and $M_2 \rhd_\beta N_2$. Since $i :: L = \mathrm{d}(M) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, by IH, $M_2^{-i} \rhd_\beta N_2^{-i}$, hence $M^{-i} = M_1^{-i}M_2^{-i} \rhd_\beta N_1^{-i}M_2^{-i} = N^{-i}$.
   – Let ► be $\rhd_\beta^*$. By induction on $\rhd_\beta^*$. using $\rhd_\beta$.
   – Let ► be $\rhd_\eta$. We only do the basic case. The inductive cases are as for $\rhd_\beta$. Let $M = \lambda x^K.Nx^K \rhd_\eta N$ where $x^K \notin \mathrm{fv}(N)$. Since $i :: L = \mathrm{d}(M) = \mathrm{d}(N) \preceq K$, $K = i :: L :: K'$ for some K'.Then $M^{-i} = \lambda x^{L::K'}.N^{-i}x^{L::K'} \rhd_\eta N^{-i}$.

– Let $\blacktriangleright$ be $\rhd^*_\eta$. By induction on $\rhd^*_\eta$ using $\rhd_\eta$.
– Let $\blacktriangleright$ be $\rhd_{\beta\eta}$, $\rhd^*_{\beta\eta}$, $\rhd_h$ or $\rhd^*_h$.. By the previous items.

7. Let $x^L \in \mathrm{fv}(N) \subseteq \mathrm{fv}(M)$ and $X^K \in \mathrm{fv}(Q) \subseteq \mathrm{fv}(P)$, since $M \diamond P$, $L = K$. Hence $N \diamond Q$.

□

Next we give a lemma that will be used in the rest of the article.

Lemma 47. *1. If $M[y^L := x^L] \rhd_\beta N$ then $M \rhd_\beta N'$ where $N = N'[y^L := x^L]$.*
2. If $M[y^L := x^L]$ is β-normalising then M is β-normalising.
3. Let $k \geq 1$. If $Mx_1^{L_1}...x_k^{L_k}$ is β-normalising, then M is β-normalising.
4. Let $k \geq 1$, $1 \leq i \leq k$, $l \geq 0$, $x_i^{L_i} N_1...N_l$ be in normal form and M be closed. If $Mx_1^{L_1}...x_k^{L_k} \rhd^*_\beta x_i^{L_i} N_1...N_l$, then for some $m \geq i$ and $n \leq l$, $M \rhd^*_\beta \lambda x_1^{L_1}....\lambda x_m^{L_m}.x_i^{L_i} M_1...M_n$ where $n + k = m + l$, $M_j \simeq_\beta N_j$ for every $1 \leq j \leq n$ and $N_{n+j} \simeq_\beta x_{m+j}^{L_{m+j}}$ for every $1 \leq j \leq k - m$.

Proof

1. By induction on $M[y^L := x^L] \rhd_\beta N$.
2. Immediate by 1.
3. By induction on $k \geq 1$. We only prove the basic case. The proof is by cases.
   - If $M\, x_1^{L_1} \rhd^*_\beta M'\, x_1^{L_1}$ where $M'\, x_1^{L_1}$ is in β-normal form and $M \rhd^*_\beta M'$ then M' is in β-normal form and M is β-normalising.
   - If $M\, x_1^{L_1} \rhd^*_\beta (\lambda y^{L_1}.N)\, x_1^{L_1} \rhd_\beta N[y^{L_1} := x_1^{L_1}] \rhd^*_\beta P$ where P is in β-normal form and $M \rhd^*_\beta \lambda y^{L_1}.N$ then by 2, N has a β-normal form and so, $\lambda y^{L_1}.N$ has a β-normal form. Hence, M has a β-normal form.
4. By 3, M is β-normalising and, since M is closed, its β-normal form is $\lambda x_1^{L_1}....\lambda x_m^{L_m}.x_p^{L_p} M_1...M_n$ for $n, m \geq 0$ and $1 \leq p \leq m$.
   Since by theorem 7, $x_i^{L_i} N_1...N_l \simeq_\beta (\lambda x_1^{L_1}....\lambda x_m^{L_m}.x_p^{L_p} M_1...M_n) x_1^{L_1}...x_k^{L_k}$ then $m \leq k$, $x_i^{L_i}\ N_1...N_l \simeq_\beta x_p^{L_p} M_1...M_n x_{m+1}^{L_{m+1}}...x_k^{L_k}$. Hence, $n \leq l$, $i = p \leq m$, $l = n + k - m$, for every $1 \leq j \leq n$, $M_j \simeq_\beta N_j$ and for every $1 \leq j \leq k - m$, $N_{n+j} \simeq_\beta x_{m+j}^{n_{m+j}}$.

□

A.1 Confluence of $\rhd^*_\beta$, $\rhd^*_h$ and $\rhd^*_{\beta\eta}$

In this section we establish the confluence of $\rhd^*_\beta$, $\rhd^*_h$ and $\rhd^*_{\beta\eta}$ using the standard parallel reduction method for $\rhd^*_\beta$ and $\rhd^*_{\beta\eta}$.

Definition 48. *Let $r \in \{\beta, \beta\eta\}$. We define on $\mathcal{M}$ the binary relation $\stackrel{\rho_r}{\to}$ by:*

– $M \stackrel{\rho_r}{\to} M$
– *If $M \stackrel{\rho_r}{\to} M'$ then $\lambda x^L.M \stackrel{\rho_r}{\to} \lambda x^L.M'$.*
– *If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_r}{\to} N'$ and $M \diamond N$ then $MN \stackrel{\rho_r}{\to} M'N'$*
– *If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_r}{\to} N'$, $d(N) = L$ and $M \diamond N$, then $(\lambda x^L.M)N \stackrel{\rho_r}{\to} M'[x^n := N']$*
– *If $M \stackrel{\rho_{\beta\eta}}{\to} M'$, $\forall L \in \mathcal{L}_\mathbb{N}, x^L \notin \mathrm{fv}(M)$ and $L \succeq d(M)$ then $\lambda x^L.Mx^L \stackrel{\rho_{\beta\eta}}{\to} M'$*

We denote the transitive closure of $\stackrel{\rho_r}{\to}$ by $\stackrel{\rho_r}{\twoheadrightarrow}$. When $M \stackrel{\rho_r}{\to} N$ (resp. $M \stackrel{\rho_r}{\twoheadrightarrow} N$), we can also write $N \stackrel{\rho_r}{\leftarrow} M$ (resp. $N \stackrel{\rho_r}{\twoheadleftarrow} M$). If $R, R' \in \{\stackrel{\rho_r}{\to}, \stackrel{\rho_r}{\twoheadrightarrow}, \stackrel{\rho_r}{\leftarrow}, \stackrel{\rho_r}{\twoheadleftarrow}\}$, we write $M_1RM_2R'M_3$ instead of M_1RM_2 and $M_2R'M_3$.

Lemma 49. *Let $M \in \mathcal{M}$.*

1. If $M \rhd_r M'$, then $M \stackrel{\rho_r}{\to} M'$.

2. *If* $M \overset{\rho_r}{\to} M'$*, then* $M' \in \mathcal{M}$*,* $M \rhd_r^* M'$*,* $\mathrm{fv}(M') \subseteq \mathrm{fv}(M)$ *and* $d(M) = d(M')$*.*
3. *If* $M \overset{\rho_r}{\to} M'$*,* $N \overset{\rho_r}{\to} N'$ *and* $M \diamond N$ *then* $M' \diamond N'$

Proof 1. By induction on the derivation $M \rhd_r M'$. 2. By induction on the derivation of $M \overset{\rho_r}{\to} M'$ using theorem 4 and lemma 46. 3. Let $x^L \in \mathrm{fv}(M')$ and $x^K \in \mathrm{fv}(N')$. By 2., $\mathrm{fv}(M') \subseteq \mathrm{fv}(M)$ and $\mathrm{fv}(N') \subseteq \mathrm{fv}(N)$. Hence, since $M \diamond N$, $L = K$, so $M' \diamond N'$. □

Lemma 50. *Let* $M, N \in \mathcal{M}$*,* $M \diamond N$ *and* $N \overset{\rho_r}{\to} N'$*. We have:*

1. $M[x^L := N] \overset{\rho_r}{\to} M[x^L := N']$*.*
2. *If* $M \overset{\rho_r}{\to} M'$ *and* $d(N) = L$*, then* $M[x^L := N] \overset{\rho_r}{\to} M'[x^L := N']$*.*

Proof 1. By induction on M:

- Let $M = y^K$. If $y^K = x^L$, then $M[x^L := N] = N$, $M[x^L := N'] = N'$ and by hypothesis, $N \overset{\rho_r}{\to} N'$. If $y^K \neq x^L$, then $M[x^L := N] = M$, $M[x^L := N'] = M$ and by definition, $M \overset{\rho_r}{\to} M$.
- Let $M = \lambda y^K.M_1$. $M[x^L := N] = \lambda y^K.M_1[x^L := N]$ and since $M_1 \diamond N$, by IH, $M_1[x^L := N] \overset{\rho_r}{\to} M_1[x^L := N']$ and so $\lambda y^K.M_1[x^L := N] \overset{\rho_r}{\to} \lambda y^K.M_1[x^L := N']$
- Let $M = M_1M_2$. $M[x^L := N] = M_1[x^L := N]M_2[x^L := N]$ and since $M_1 \diamond N$ and $M_2 \diamond N$, by IH, $M_1[x^L := N] \overset{\rho_r}{\to} M_1[x^L := N']$ and $M_2[x^L := N] \overset{\rho_r}{\to} M_2[x^L := N']$. By lemma 45.4, $M_1[x^L := N] \diamond M_2[x^L := N]$, so $M_1[x^L := N]M_2[x^L := N] \overset{\rho_r}{\to} M_1[x^L := N']M_2[x^L := N']$.

2. By induction on $M \overset{\rho_r}{\to} M'$.

- If $M = M'$, then 1..
- If $\lambda y^K.M \overset{\rho_r}{\to} \lambda y^K.M'$ where $M \overset{\rho_r}{\to} M'$, then by IH, $M[x^L := N] \overset{\rho_r}{\to} M'[x^L := N']$. Hence $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N] \overset{\rho_r}{\to} \lambda y^K.M'[x^L := N'] = (\lambda y^K.M')[x^L := N']$ where $y^K \notin \mathrm{fv}(N') \subseteq \mathrm{fv}(N)$.
- If $PQ \overset{\rho_r}{\to} P'Q'$ where $P \overset{\rho_r}{\to} P'$, $Q \overset{\rho_r}{\to} Q'$ and $P \diamond Q$, then by IH, $P[x^L := N] \overset{\rho_r}{\to} P'[x^L := N']$ and $Q[x^L := N] \overset{\rho_r}{\to} Q'[x^L := N']$. By lemma 45.4, $P[x^L := N] \diamond Q[x^L := N]$, so $P[x^L := N]Q[x^L := N] \overset{\rho_r}{\to} P'[x^L := N']Q'[x^L := N']$.
- $(\lambda y^K.P)Q \overset{\rho_r}{\to} P'[y^K := Q']$ where $P \overset{\rho_r}{\to} P'$, $Q \overset{\rho_r}{\to} Q'$, $P \diamond Q$ and $\mathrm{d}(Q) = K$, then by IH, $P[x^L := N] \overset{\rho_r}{\to} P'[x^L := N']$, $Q[x^L := N] \overset{\rho_r}{\to} Q'[x^L := N']$. Moreover, $((\lambda y^K.P)Q)[x^L := N] = (\lambda y^K.P)[x^L := N]Q[x^L := N] = \lambda y^K.P[x^L := N]Q[x^L := N]$ where $y^K \notin \mathrm{fv}(N') \subseteq \mathrm{fv}(N)$. By lemma 45.4, $P[x^L := N] \diamond Q[x^L := N]$ and by lemma 45.3 $\mathrm{d}(Q) = \mathrm{d}(Q[x^L := N])$ so $\lambda y^K.P[x^L := N]Q[x^L := N] \overset{\rho_r}{\to} P'[x^L := N'][y^K := Q'[x^L := N']] = P'[y^K := Q'][x^L := N']$.
- If $\lambda y^K.My^K \overset{\rho_{\beta\eta}}{\to} M'$ where $M \overset{\rho_{\beta\eta}}{\to} M'$, $K \succeq \mathrm{d}(M)$ and $\forall K \in \mathcal{L}_{\mathbb{N}}, y^K \notin \mathrm{fv}(M)$, then by IH $M[x^L := N] \overset{\rho_{\beta\eta}}{\to} M'[x^L := N']$. Moreover, $(\lambda y^K.My^K)[x^L := N] = \lambda y^K.M[x^L := N]y^K[x^L := N] = \lambda y^K.M[x^L := N]y^K$ where $\forall K \in \mathcal{L}_{\mathbb{N}}, y^K \notin \mathrm{fv}(N') \subseteq \mathrm{fv}(N)$. Since by lemma 45.3 $\mathrm{d}(M) = \mathrm{d}(M[x^L := N])$, $\lambda y^K.M[x^L := N]y^K \overset{\rho_{\beta\eta}}{\to} M'[x^L := N']$.

□

Lemma 51. *1. If* $x^L \overset{\rho_r}{\to} N$*, then* $N = x^L$*.*

2. *If* $\lambda x^L.P \overset{\rho_{\beta\eta}}{\to} N$ *then one of the following holds:*
   - $N = \lambda x^L.P'$ *where* $P \overset{\rho_{\beta\eta}}{\to} P'$*.*
   - $P = P'x^L$ *where* $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \notin \mathrm{fv}(P')$*,* $L \succeq d(P')$ *and* $P' \overset{\rho_{\beta\eta}}{\to} N$*.*
3. *If* $\lambda x^L.P \overset{\rho_\beta}{\to} N$ *then* $N = \lambda x^L.P'$ *where* $P \overset{\rho_\beta}{\to} P'$*.*
4. *If* $PQ \overset{\rho_r}{\to} N$*, then one of the following holds:*

– $N = P'Q'$, $P \stackrel{\rho_r}{\to} P'$, $Q \stackrel{\rho_r}{\to} Q'$ *and* $P \diamond Q$.
– $P = \lambda x^L.P'$, $N = P''[x^L := Q']$, $P' \stackrel{\rho_r}{\to} P''$, $Q \stackrel{\rho_r}{\to} Q'$, $P' \diamond Q$ *and* $d(Q) = L$.

Proof 1. By induction on the derivation $x^L \stackrel{\rho_r}{\to} N$.
2. By induction on the derivation $\lambda x^L.P \stackrel{\rho_{\beta\eta}}{\to} N$.
3. By induction on the derivation $\lambda x^L.P \stackrel{\rho_\beta}{\to} N$.
4. By induction on the derivation $PQ \stackrel{\rho_r}{\to} N$. □

Lemma 52. *Let* $M, M_1, M_2 \in \mathcal{M}$.

1. *If* $M_2 \stackrel{\rho_r}{\leftarrow} M \stackrel{\rho_r}{\to} M_1$, *then there is* $M' \in \mathcal{M}$ *such that* $M_2 \stackrel{\rho_r}{\to} M' \stackrel{\rho_r}{\leftarrow} M_1$.
2. *If* $M_2 \stackrel{\rho_r}{\twoheadleftarrow} M \stackrel{\rho_r}{\twoheadrightarrow} M_1$, *then there is* $M' \in \mathcal{M}$ *such that* $M_2 \stackrel{\rho_r}{\twoheadrightarrow} M' \stackrel{\rho_r}{\twoheadleftarrow} M_1$.

Proof 1. By induction on M:

– Let $r = \beta\eta$:
  - If $M = x^L$, by lemma 51, $M_1 = M_2 = x^L$. Take $M' = x^L$.
  - If $N_2P_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} NP \stackrel{\rho_{\beta\eta}}{\to} N_1P_1$ where $N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} N \stackrel{\rho_{\beta\eta}}{\to} N_1$, $P_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} P \stackrel{\rho_{\beta\eta}}{\to} P_1$ and $N \diamond P$ then, by IH, $\exists N', P'$ such that $N_2 \stackrel{\rho_{\beta\eta}}{\to} N' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1$ and $P_2 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_1$. By lemma 49.3, $N_1 \diamond P_1$ and $N_2 \diamond P_2$, hence $N_2P_2 \stackrel{\rho_{\beta\eta}}{\to} N'P' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1P_1$.
  - If $(\lambda x^L.P_1)Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} (\lambda x^L.P)Q \stackrel{\rho_{\beta\eta}}{\to} P_2[x^L := Q_2]$ where $\lambda x^L.P \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.P_1$, $P \stackrel{\rho_{\beta\eta}}{\to} P_2$, $Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} Q \stackrel{\rho_{\beta\eta}}{\to} Q_2$, $\mathrm{d}(Q) = L$, $(\lambda x^L.P) \diamond Q$ and $P \diamond Q$ then, by lemma 51, $P \stackrel{\rho_{\beta\eta}}{\to} P_1$. By IH, $\exists P', Q'$ such that $P_1 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho_{\beta\eta}}{\to} Q' \stackrel{\rho_{\beta\eta}}{\leftarrow} Q_2$. By lemma 49.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 49.3, $P_1 \diamond Q_1$. Hence, $(\lambda x^L.P_1)Q_1 \stackrel{\rho_{\beta\eta}}{\to} P'[x^L := Q']$.
    Moreover, since $P_2 \stackrel{\rho_{\beta\eta}}{\to} P'$, $Q_2 \stackrel{\rho_{\beta\eta}}{\to} Q'$, $\mathrm{d}(Q_2) = L$ and by lemma 49.3, $P_2 \diamond Q_2$, then, by lemma 50.2, $P_2[x^L := Q_2] \stackrel{\rho_{\beta\eta}}{\to} P'[x^L := Q']$.
  - If $P_1[x^L := Q_1] \stackrel{\rho_{\beta\eta}}{\leftarrow} (\lambda x^L.P)Q \stackrel{\rho_{\beta\eta}}{\to} P_2[x^L := Q_2]$ where $P_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} P \stackrel{\rho_{\beta\eta}}{\to} P_2$, $Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} Q \stackrel{\rho_{\beta\eta}}{\to} Q_2$, $\mathrm{d}(Q) = L$ and $P \diamond Q$, then, by IH, $\exists P', Q'$ where $P_1 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho_{\beta\eta}}{\to} Q' \stackrel{\rho_{\beta\eta}}{\leftarrow} Q_2$. By lemma 49.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 49.3, $P_1 \diamond Q_1$ and $P_2 \diamond Q_2$. Hence, by lemma 50.2, $P_1[x^L := Q_1] \stackrel{\rho_{\beta\eta}}{\to} P'[x^L := Q'] \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2[x^L := Q_2]$.
  - If $\lambda x^L.N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^L.N \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.N_1$ where $N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} N \stackrel{\rho_{\beta\eta}}{\to} N_1$, by IH, there is N' such that $N_2 \stackrel{\rho_{\beta\eta}}{\to} N' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1$. Hence, $\lambda x^L.N_2 \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.N' \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^L.N_1$.
  - If $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^L.Px^L \stackrel{\rho_{\beta\eta}}{\to} M_2$ where $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \not\in \mathrm{fv}(P)$, $L \succeq \mathrm{d}(P)$ and $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} P \stackrel{\rho_{\beta\eta}}{\to} M_2$, then, by IH, there is M' such that $M_2 \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} M_1$.
  - If $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^L.Px^L \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.P'$, where $P \stackrel{\rho_{\beta\eta}}{\to} M_1$, $Px^L \stackrel{\rho_{\beta\eta}}{\to} P'$ and $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \not\in \mathrm{fv}(P)$ and $L \succeq \mathrm{d}(P)$. By lemma 51 there are two cases:
    * $P' = P''x^L$ and $P \stackrel{\rho_{\beta\eta}}{\to} P''$. By IH, there is M' such that $P'' \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} M_1$. By lemma 49.2, $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \not\in \mathrm{fv}(P'')$ and $L \succeq \mathrm{d}(P'')$, hence, $\lambda x^L.P' = \lambda x^L.P''x^L \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} M_1$.
    * $P = \lambda y^L.Q$, $Q \stackrel{\rho_{\beta\eta}}{\to} Q'$, $Q \diamond x^L$ and $P' = Q'[y^L := x^L]$. So we have $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^L.(\lambda y^L.Q)x^L \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.Q'[y^L := x^L]$ where $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda y^L.Q = \lambda x^L.Q[y^L := x^L]$ since $\forall L \in \mathcal{L}_{\mathbb{N}}, x^L \not\in \mathrm{fv}(P)$.
      By lemma 50.2, $\lambda x^L.Q[y^L := x^L] \stackrel{\rho_{\beta\eta}}{\to} \lambda x^L.Q'[y^L := x^L]$. Hence by IH, there is M' such that $M_1 \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^L.Q'[y^L := x^L]$.

– Let $r = \beta$:
  - If $M = x^L$, by lemma 51, $M_1 = M_2 = x^L$. Take $M' = x^L$.
  - If $N_2P_2 \stackrel{\rho_\beta}{\leftarrow} NP \stackrel{\rho_\beta}{\to} N_1P_1$ where $N_2 \stackrel{\rho_\beta}{\leftarrow} N \stackrel{\rho_\beta}{\to} N_1$, $P_2 \stackrel{\rho_\beta}{\leftarrow} P \stackrel{\rho_\beta}{\to} P_1$ and $N \diamond P$, then, by IH, $\exists N', P'$ such that $N_2 \stackrel{\rho_\beta}{\to} N' \stackrel{\rho_\beta}{\leftarrow} N_1$ and $P_2 \stackrel{\rho_\beta}{\to} P' \stackrel{\rho_\beta}{\leftarrow} P_1$. By lemma 49.3, $N_1 \diamond P_1$ and $N_2 \diamond P_2$. Hence, $N_2P_2 \stackrel{\rho_\beta}{\to} N'P' \stackrel{\rho_\beta}{\leftarrow} N_1P_1$.

- If $(\lambda x^L.P_1)Q_1 \overset{\rho_\beta}{\leftarrow} (\lambda x^L.P)Q \overset{\rho_\beta}{\to} P_2[x^L := Q_2]$ where $\lambda x^L.P \overset{\rho_\beta}{\to} \lambda x^L.P_1$, $P \overset{\rho_\beta}{\to} P_2$, $Q_1 \overset{\rho_\beta}{\leftarrow} Q \overset{\rho_\beta}{\to} Q_2$, $\mathrm{d}(Q) = L$, $P \diamond Q$ and $(\lambda x^L.P) \diamond Q$, then, by lemma 51, $P \overset{\rho_\beta}{\to} P_1$. By IH, $\exists P', Q'$ such that $P_1 \overset{\rho_\beta}{\to} P' \overset{\rho_\beta}{\leftarrow} P_2$ and $Q_1 \overset{\rho_\beta}{\to} Q' \overset{\rho_\beta}{\leftarrow} Q_2$. By lemma 49.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 49.3, $P_1 \diamond Q_1$. Hence, $(\lambda x^L.P_1)Q_1 \overset{\rho_\beta}{\to} P'[x^L := Q']$.
  Moreover, since $P_2 \overset{\rho_\beta}{\to} P'$, $Q_2 \overset{\rho_\beta}{\to} Q'$, $\mathrm{d}(Q_2) = L$ and by lemma 49.3, $P_2 \diamond Q_2$., then, by lemma 50.2, $P_2[x^L := Q_2] \overset{\rho_\beta}{\to} P'[x^L := Q']$.
- If $P_1[x^L := Q_1] \overset{\rho_\beta}{\leftarrow} (\lambda x^L.P)Q \overset{\rho_\beta}{\to} P_2[x^L := Q_2]$ where $P_1 \overset{\rho_\beta}{\leftarrow} P \overset{\rho_\beta}{\to} P_2$, $Q_1 \overset{\rho_\beta}{\leftarrow} Q \overset{\rho_\beta}{\to} Q_2$, $\mathrm{d}(Q) = L$ and $P \diamond Q$ then by IH, $\exists P', Q'$ where $P_1 \overset{\rho_\beta}{\to} P' \overset{\rho_\beta}{\leftarrow} P_2$ and $Q_1 \overset{\rho_\beta}{\to} Q' \overset{\rho_\beta}{\leftarrow} Q_2$. By lemma 49.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = \mathrm{d}(Q) = L$. By lemma 49.3, $P_1 \diamond Q_1$ and $P_2 \diamond Q_2$. Hence, by lemma 50.2, $P_1[x^L := Q_1] \overset{\rho_\beta}{\to} P'[x^L := Q'] \overset{\rho_\beta}{\leftarrow} P_2[x^L := Q_2]$.
- If $\lambda x^L.N_2 \overset{\rho_\beta}{\leftarrow} \lambda x^L.N \overset{\rho_\beta}{\to} \lambda x^L.N_1$ where $N_2 \overset{\rho_\beta}{\leftarrow} N \overset{\rho_\beta}{\to} N_1$, by IH, there is N' such that $N_2 \overset{\rho_\beta}{\to} N' \overset{\rho_\beta}{\leftarrow} N_1$. Hence, $\lambda x^L.N_2 \overset{\rho_\beta}{\to} \lambda x^L.N' \overset{\rho_\beta}{\leftarrow} \lambda x^L.N_1$.

2. First show by induction on $M \overset{\rho_r}{\twoheadrightarrow} M_1$ (and using 1) that if $M_2 \overset{\rho_r}{\leftarrow} M \overset{\rho_r}{\twoheadrightarrow} M_1$, then there is M' such that $M_2 \overset{\rho_r}{\twoheadrightarrow} M' \overset{\rho_r}{\leftarrow} M_1$. Then use this to show 2 by induction on $M \overset{\rho_r}{\twoheadrightarrow} M_2$. □

Proof [Of Theorem 7]

1. For $r \in \{\beta, \beta\eta\}$, by lemma 52.2, $\overset{\rho_r}{\twoheadrightarrow}$ is confluent. by lemma 49.1 and 49.2, $M \overset{\rho_r}{\twoheadrightarrow} N$ iff $M \rhd_r^* N$. Then $\rhd_r^*$ is confluent.
   For $r = h$, since if $M \rhd_r^* M_1$ and $M \rhd_r^* M_2$, $M_1 = M_2$, we take $M' = M_1$.
2. If) is by definition of $\simeq_r$. Only if) is by induction on $M_1 \simeq_r M_2$ using 1.

□

B Proofs of section 3

Proof [Of lemma 12]

1. By definition.
2. By induction on U.
   - If $U = a$ ($\mathrm{d}(U) = \oslash$), nothing to prove.
   - If $U = V \to T$ ($\mathrm{d}(U) = \oslash$), nothing to prove.
   - If $U = \omega^L$, nothing to prove.
   - If $U = U_1 \sqcap U_2$ ($\mathrm{d}(U) = \mathrm{d}(U_1) = \mathrm{d}(U_2) = L$), by IH we have four cases:
     - If $U_1 = U_2 = \omega^L$ then $U = \omega^L$.
     - If $U_1 = \omega^L$ and $U_2 = \boldsymbol{e}_L \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}$ then $U = U_2$ (since ω^L is a neutral).
     - If $U_2 = \omega^L$ and $U_1 = \boldsymbol{e}_L \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}$ then $U = U_1$ (since ω^L is a neutral).
     - If $U_1 = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ and $U_2 = \boldsymbol{e}_L \sqcap_{i=p+1}^{p+q} T_i$ where $p, q \geq 1$, $\forall 1 \leq i \leq p+q$, $T_i \in \mathbb{T}$ then $U = \boldsymbol{e}_L \sqcap_{i=1}^{p+q} T_i$.
   - If $U = e_{n_1} V$ ($L = \mathrm{d}(U) = n_1 :: \mathrm{d}(V) = n_1 :: K$), by IH we have two cases:
     - If $V = \omega^K$, $U = e_{n_1} \omega^K = \omega^L$.
     - If $V = \boldsymbol{e}_K \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $T_i \in \mathbb{T}$ then $U = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $T_i \in \mathbb{T}$.
3. (a) By induction on $U_1 \sqsubseteq U_2$.
   (b) By induction on $U_1 \sqsubseteq U_2$.
   (c) By induction on K. We do the induction step. Let $U_1 = e_i U$. By induction on $e_i U \sqsubseteq U_2$ we obtain $U_2 = e_i U'$ and $U \sqsubseteq U'$.

(d) same proof as in the previous item.

(e) By induction on $U_1 \sqsubseteq U_2$:

- By ref, $U_1 = U_2$.
- If $\dfrac{\sqcap_{i=1}^{p} \boldsymbol{e}_K(U_i \to T_i) \sqsubseteq U \qquad U \sqsubseteq U_2}{\sqcap_{i=1}^{p} \boldsymbol{e}_K(U_i \to T_i) \sqsubseteq U_2}$. If $U = \omega^K$ then by (b), $U_2 = \omega^K$. If $U = \sqcap_{j=1}^{q} \boldsymbol{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$ then by IH, $U_2 = \omega^K$ or $U_2 = \sqcap_{k=1}^{r} \boldsymbol{e}_K(U''_k \to T''_k)$ where $r \geq 1$ and $\forall 1 \leq k \leq r$, $\exists 1 \leq j \leq q$ such that $U''_k \sqsubseteq U'_j$ and $T'_j \sqsubseteq T''_k$. Hence, by tr, $\forall 1 \leq k \leq r$, $\exists 1 \leq i \leq p$ such that $U''_k \sqsubseteq U_i$ and $T_i \sqsubseteq T''_k$.
- By $\sqcap_E$, $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_K(U'_j \to T'_j)$ where $1 \leq q \leq p$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U_i = U'_j$ and $T_i = T'_j$.
- Case $\sqcap$ is by IH.
- Case $\to$ is trivial.
- If $\dfrac{\sqcap_{i=1}^{p} \boldsymbol{e}_L(U_i \to T_i) \sqsubseteq U_2}{\sqcap_{i=1}^{p} \boldsymbol{e}_K(U_i \to T_i) \sqsubseteq e_i U_2}$ where $K = i :: L$ then by IH, $U_2 = \omega^L$ and so $e_i U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_L(U'_j \to T'_j)$ so $e_i U_2 = \sqcap_{j=1}^{q} \boldsymbol{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.

4. By $\sqcap_E$ and since ω^L is a neutral.
5. By induction on $U \sqsubseteq U'_1 \sqcap U'_2$.
   - Let $\dfrac{}{U'_1 \sqcap U'_2 \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$.
   - Let $\dfrac{U \sqsubseteq U'' \qquad U'' \sqsubseteq U'_1 \sqcap U'_2}{U \sqsubseteq U'_1 \sqcap U'_2}$. By IH, $U'' = U''_1 \sqcap U''_2$ such that $U''_1 \sqsubseteq U'_1$ and $U''_2 \sqsubseteq U'_2$. Again by IH, $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U''_1$ and $U_2 \sqsubseteq U''_2$. So by tr, $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
   - Let $\dfrac{}{(U'_1 \sqcap U'_2) \sqcap U \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$. Moreover $\mathrm{d}(U) = \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U'_1)$ then by $\sqcap_E$, $U'_1 \sqcap U \sqsubseteq U'_1$.
   - If $\dfrac{U_1 \sqsubseteq U'_1 \quad \& \quad U_2 \sqsubseteq U'_2}{U_1 \sqcap U_2 \sqsubseteq U'_1 \sqcap U'_2}$ there is nothing to prove.
   - $\dfrac{V_2 \sqsubseteq V_1 \quad \& \quad T_1 \sqsubseteq T_2}{V_1 \to T_1 \sqsubseteq V_2 \to T_2}$ then $U'_1 = U'_2 = V_2 \to T_2$ and $U = U_1 \sqcap U_2$ such that $U_1 = U_2 = V_1 \to T_1$ and we are done.
   - If $\dfrac{U \sqsubseteq U'_1 \sqcap U'_2}{e_i U \sqsubseteq e_i U'_1 \sqcap e_i U'_2}$ then by IH $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$. So, $e_i U = e_i U_1 \sqcap e_i U_2$ and by $\sqsubseteq_e$, $e_i U_1 \sqsubseteq e_i U'_1$ and $e_i U_2 \sqsubseteq e_i U'_2$.
6. By induction on $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$.
   - Let $\dfrac{}{\Gamma'_1 \sqcap \Gamma'_2 \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}$. By ref, $\Gamma'_1 \sqsubseteq \Gamma'_1$ and $\Gamma'_2 \sqsubseteq \Gamma'_2$.
   - Let $\dfrac{\Gamma \sqsubseteq \Gamma'' \qquad \Gamma'' \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}{\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}$. By IH, $\Gamma'' = \Gamma''_1 \sqcap \Gamma''_2$ such that $\Gamma''_1 \sqsubseteq \Gamma'_1$ and $\Gamma''_2 \sqsubseteq \Gamma'_2$. Again by IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma''_1$ and $\Gamma_2 \sqsubseteq \Gamma''_2$. So by tr, $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.
   - Let $\dfrac{U_1 \sqsubseteq U_2}{\Gamma, (y^n : U_1) \sqsubseteq \Gamma, (y^n : U_2)}$ where $\Gamma, (y^n : U_2) = \Gamma'_1 \sqcap \Gamma'_2$.
     - If $\Gamma'_1 = \Gamma''_1, (y^n : U'_2)$ and $\Gamma'_2 = \Gamma''_2, (y^n : U''_2)$ such that $U_2 = U'_2 \sqcap U''_2$, then by 5, $U_1 = U'_1 \sqcap U''_1$ such that $U'_1 \sqsubseteq U'_2$ and $U''_1 \sqsubseteq U''_2$. Hence $\Gamma = \Gamma''_1 \sqcap \Gamma''_2$ and $\Gamma, (y^n : U_1) = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 = \Gamma''_1, (y^n : U'_1)$ and $\Gamma_2 = \Gamma''_2, (y^n : U''_1)$ such that $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$ by $\sqsubseteq_c$.
     - If $y^n \notin \mathrm{dom}(\Gamma'_1)$ then $\Gamma = \Gamma'_1 \sqcap \Gamma''_2$ where $\Gamma''_2, (y^n : U_2) = \Gamma'_2$. Hence, $\Gamma, (y^n : U_1) = \Gamma'_1 \sqcap \Gamma_2$ where $\Gamma_2 = \Gamma''_2, (y^n : U_1)$. By ref and $\sqsubseteq_c$, $\Gamma'_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.
     - If $y^n \notin \mathrm{dom}(\Gamma'_2)$ then similar to the above case.

□

Proof [Of lemma 13] 1. First show by induction on the derivation $\Gamma \sqsubseteq \Gamma'$ that if $\Gamma \sqsubseteq \Gamma'$ and $\Gamma, (x^L : U)$ is an environment, then $\Gamma, (x^L : U) \sqsubseteq \Gamma', (x^L : U)$. Then use tr.
2. Only if) By induction on the derivation $\Gamma \sqsubseteq \Gamma'$. If) By induction on n using 1.
3. Only if) By induction on the derivation $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$. If) By $\sqsubseteq_{\langle\rangle}$.
4. Let $\mathrm{fv}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ and $\Gamma = (x_i^{L_i} : U_i)_n$. By definition, $env_M^\omega = (x_i^{L_i}, \omega^{L_i})_n$. Hence, by lemma 12.4 and 2, $\Gamma \sqsubseteq env_M^\omega$.
5. Let $x^{L_1} \in \mathrm{dom}(\Gamma^{-K})$ and $x^{L_2} \in \mathrm{dom}(\Delta^{-K})$, then $x^{K::L_1} \in \mathrm{dom}(\Gamma)$ and $x^{K::L_2} \in \mathrm{dom}(\Delta)$, hence $K :: L_1 = K :: L_2$ and so $L_1 = L_2$.
6. Let $\mathrm{d}(U) = L = K :: K'$. By lemma 12:

- If $U = \omega^L$ then by lemma 12.3b, $U' = \omega^L$ and by ref, $U^{-K} = \omega^{K'} \sqsubseteq \omega^{K'} = U'^{-K}$.
- If $U = \boldsymbol{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$ then by lemma 12.3c, $U' = \boldsymbol{e}_L V$ and $\sqcap_{i=1}^p T_i \sqsubseteq V$. Hence, by $\sqsubseteq_e$, $U^{-K} = \boldsymbol{e}_{K'} \sqcap_{i=1}^p T_i \sqsubseteq \boldsymbol{e}_{K'} V = U'^{-K}$.

7 Let $\mathrm{d}(\Gamma) = L = K :: K'$. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so by lemma 13.2, $\Gamma' = (x_i^{L_i} : U_i')_n$ and $\forall 1 \leq i \leq n$, $U_i \sqsubseteq U_i'$. Since $\mathrm{d}(\Gamma) \succeq K$, $\forall 1 \leq i \leq n$, $\mathrm{d}(U_i) = L_i = \mathrm{d}(U_i') \succeq K$, so $\mathrm{d}(U_i) = \mathrm{d}(U_i') = K :: K'$. By 1., $\forall 1 \leq i \leq n$, $U_i^{-K} \sqsubseteq U'^{-K}_i$ and by lemma 13.2, $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$. □

Proof [Of theorem 15]

1. - If $\dfrac{}{x^\oslash : \langle (x^\oslash : T) \vdash T \rangle}$ then $\mathrm{d}(T) = \oslash = \mathrm{d}(x^\oslash)$.
   - If $\dfrac{}{M : \langle env_M^\omega \vdash \omega^{\mathrm{d}(M)} \rangle}$. Let $\mathrm{fv}(M) = \{x^{L_1}, \ldots, x^{L_n}\}$, so $env_M^\omega = (x_i^{L_i} : \omega^{L_i})_n$ and by lemma 45, $\forall 1 \leq i \leq n, L_i \succeq \mathrm{d}(M)$.
   - If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash T \rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T \rangle}$ then by IH, $\mathrm{d}(\Gamma, (x^L : U)) \succeq \mathrm{d}(T) = \mathrm{d}(M)$. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so $\forall 1 \leq i \leq n, \mathrm{d}(U_i) \succeq \mathrm{d}(T) = \mathrm{d}(U \to T)$ and $\mathrm{d}(\lambda x^L.M) = \mathrm{d}(M) = \mathrm{d}(T) = \mathrm{d}(U \to T)$.
   - If $\dfrac{M : \langle \Gamma \vdash T \rangle \quad x^L \not\in \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T \rangle}$ then by IH, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(T) = \mathrm{d}(M)$. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so $\forall 1 \leq i \leq n, \mathrm{d}(U_i) \succeq \mathrm{d}(T) = \mathrm{d}(\omega^L \to T)$ and $\mathrm{d}(\lambda x^L.M) = \mathrm{d}(M) = \mathrm{d}(T) = \mathrm{d}(\omega^L \to T)$.
   - If $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T \rangle \qquad M_2 : \langle \Gamma_2 \vdash U \rangle \qquad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}$ then by IH, $\mathrm{d}(\Gamma_1) \succeq \mathrm{d}(U \to T) = \mathrm{d}(M_1)$ and $\mathrm{d}(\Gamma_2) \succeq \mathrm{d}(U) = \mathrm{d}(M_2)$. Let $\Gamma_1 = (x_i^{L_i} : U_i)_n, (y_i^{K_i} : V_i)_m$ and $\Gamma_2 = (x_i^{L_i} : U_i')_n, (z_i^{K_i'} : W_i)_r$ so $\Gamma_1 \sqcap \Gamma_2 = (x_i^{L_i} : U_i \sqcap U_i')_n, (y_i^{K_i} : V_i)_m, (z_i^{K_i'} : W_i)_r$ and $\forall 1 \leq i \leq n, \mathrm{d}(U_i \sqcap U_i') = \mathrm{d}(U_i) \succeq \mathrm{d}(U \to T) = \mathrm{d}(T)$, $\forall 1 \leq i \leq m, \mathrm{d}(V_i) \succeq \mathrm{d}(U \to T) = \mathrm{d}(T)$ and $\forall 1 \leq i \leq r, \mathrm{d}(W_i) \succeq \mathrm{d}(U) \succeq \mathrm{d}(T)$. Moreover $\mathrm{d}(M_1 M_2) = \mathrm{d}(M_1) = \mathrm{d}(U \to T) = \mathrm{d}(T)$.
   - If $\dfrac{M : \langle \Gamma \vdash U_1 \rangle \qquad M : \langle \Gamma \vdash U_2 \rangle}{M : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$ then by IH, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U_1) = \mathrm{d}(M)$ and $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U_2) = \mathrm{d}(M)$, so $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U_1 \sqcap U_2) = \mathrm{d}(U_1) = \mathrm{d}(M)$.
   - If $\dfrac{M : \langle \Gamma \vdash U \rangle}{M^{+k} : \langle e_k \Gamma \vdash e_k U \rangle}$ then by IH, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = \mathrm{d}(M)$. Let $\Gamma = (x_j^{L_j} : U_j)_n$ so $e_k \Gamma = (x_j^{k::L_j} : e_k U_j)_n$ and since $\forall 1 \leq j \leq n, \mathrm{d}(U_j) \succeq \mathrm{d}(U)$ then $\forall 1 \leq j \leq n, \mathrm{d}(e_k U_j) = k :: \mathrm{d}(U_j) \succeq k :: \mathrm{d}(U) = \mathrm{d}(e_k U) = k :: \mathrm{d}(M) = \mathrm{d}(M^{+k})$.
   - If $\dfrac{M : \langle \Gamma \vdash U \rangle \qquad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{M : \langle \Gamma' \vdash U' \rangle}$ then by IH, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = \mathrm{d}(M)$. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so $\forall 1 \leq i \leq n, \mathrm{d}(U_i) \succeq \mathrm{d}(U)$. By lemma 13.2, $\Gamma' =$

$(x_i^{L_i} : U'_i)_n$ and $\forall 1 \le i \le n$, $U_i \sqsubseteq U'_i$ so by lemma 12.3a, $\mathrm{d}(U_i) = \mathrm{d}(U'_i)$. By lemma 13.3, $U \sqsubseteq U'$ so by lemma 12.3a, $\mathrm{d}(U) = \mathrm{d}(U')$. Hence $\forall 1 \le i \le n, \mathrm{d}(U'_i) \succeq \mathrm{d}(U') = \mathrm{d}(M)$.

2. By induction on $M : \langle \Gamma \vdash U \rangle$. Case $K = \oslash$ is trivial, consider $K = i :: K'$. Let $\mathrm{d}(U) = K :: L$. Since $\mathrm{d}(U) \succeq K$, U^{-K} is well defined. Since by 1. $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = \mathrm{d}(M)$, M^{-K} and Γ^{-K} are well defined too.
   - If $\dfrac{}{M : \langle env^{\omega}_{M} \vdash \omega^{\mathrm{d}(M)} \rangle}$. By ω, $M^{-K} : \langle env^{\omega}_{M^{-K}} \vdash \omega^{L} \rangle$.
   - $\sqcap_I$ is by IH.
   - If $\dfrac{M : \langle \Gamma \vdash U \rangle}{M^{+i} : \langle e_i\Gamma \vdash e_iU \rangle}$. Since $\mathrm{d}(e_iU) = i :: K' :: L$, $\mathrm{d}(U) = K' :: L$, so $\mathrm{d}(U) \succeq K'$ and by IH, $M^{-K'} : \langle \Gamma^{-K'} \vdash U^{-K'} \rangle$, so by e, $(M^{+i})^{-K} : \langle (e_i\Gamma)^{-K} \vdash (e_iU^{-K} \rangle$.
   - If $\dfrac{M : \langle \Gamma \vdash U \rangle \qquad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{M : \langle \Gamma' \vdash U' \rangle}$ then by lemma 13.3, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. By lemma 12.3a, $\mathrm{d}(U) = \mathrm{d}(U') \succeq K$. By IH, $M^{-K} : \langle \Gamma^{-K} \vdash U^{-K} \rangle$. Hence by lemma 13 and $\sqsubseteq$, $M^{-K} : \langle \Gamma'^{-K} \vdash U'^{-K} \rangle$.

□

Proof [Of remark 16]

1. Let $M : \langle \Gamma_1 \vdash U_1 \rangle$ and $M : \langle \Gamma_2 \vdash U_2 \rangle$. By lemma 14.2, $\mathrm{dom}(\Gamma_1) = \mathrm{fv}(M) = \mathrm{dom}(\Gamma_2)$. Let $\Gamma_1 = (x_i^{L_i} : V_i)_n$ and $\Gamma_2 = (x_i^{L_i} : V'_i)_n$. Then, $\forall 1 \le i \le n$, $\mathrm{d}(V_i) = \mathrm{d}(V'_i) = L_i$. By $\sqcap_E$, $V_i \sqcap V'_i \sqsubseteq V_i$ and $V_i \sqcap V'_i \sqsubseteq V'_i$. Hence, by lemma 13.2, $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_1$ and $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_2$ and by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \rangle$ and $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_2 \rangle$. Finally, by $\sqcap_I$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2 \rangle$.
2. By lemma 12, either $U = \omega^L$ so by ω, $x^L : \langle (x^L : \omega^L) \vdash \omega^L \rangle$.Or $U = \sqcap_{i=1}^{p} \boldsymbol{e}_L T_i$ where $p \ge 1$, and $\forall 1 \le i \le p$, $T_i \in \mathbb{T}$. Let $1 \le i \le p$.By ax, $x^{\oslash} : \langle (x^{\oslash} : T_i) \vdash T_i \rangle$, hence by e, $x^L : \langle (x^L : \boldsymbol{e}_L T_i) \vdash \boldsymbol{e}_L T_i \rangle$. Now, by $\sqcap'_I$, $x^L : \langle (x^L : U) \vdash U \rangle$.

□

C Proofs of section 4

Proof [Of lemma 17] 1. By induction on the derivation $x^L : \langle \Gamma \vdash U \rangle$. We have fives cases:

- If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T \rangle}$, nothing to prove.
- If $\dfrac{}{x^{L} : \langle (x^{L} : \omega^L) \vdash \omega^L \rangle}$, nothing to prove.
- If $\dfrac{x^L : \langle \Gamma \vdash U_1 \rangle \qquad x^L : \langle \Gamma \vdash U_2 \rangle}{x^L : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$. By IH, $\Gamma = (x^L : V)$, $V \sqsubseteq U_1$ and $V \sqsubseteq U_2$, then by rule $\sqcap$, $V \sqsubseteq U_1 \sqcap U_2$.
- If $\dfrac{x^L : \langle \Gamma \vdash U \rangle}{x^{i::L} : \langle e_i\Gamma \vdash e_iU \rangle}$. Then by IH, $\Gamma = (x^L : V)$ and $V \sqsubseteq U$, so $e_i\Gamma = (x^{i::L} : e_iV)$ and by $\sqsubseteq_e$, $e_iV \sqsubseteq e_iU$,
- If $\dfrac{x^L : \langle \Gamma' \vdash U' \rangle \quad \langle \Gamma' \vdash U' \rangle \sqsubseteq \langle \Gamma \vdash U \rangle}{x^L : \langle \Gamma \vdash U \rangle}$. By lemma 13.3, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$ and, by IH, $\Gamma' = (x^L : V')$ and $V' \sqsubseteq U'$. Then, by lemma 13.2, $\Gamma = (x^L : V)$, $V \sqsubseteq V'$ and, by rule tr, $V \sqsubseteq U$.

2. By induction on the derivation $\lambda x^L.M : \langle \Gamma \vdash U \rangle$. We have five cases:

- If $\dfrac{}{\lambda x^L.M : \langle env^{\omega}_{\lambda x^L.M} \vdash \omega^{\mathrm{d}(\lambda x^L.M)} \rangle}$, nothing to prove.

- If $\dfrac{M : \langle \Gamma, x^L : U \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}$ $(\mathrm{d}(U \to T) = \oslash)$, nothing to prove.
- If $\dfrac{\lambda x^L.M : \langle \Gamma \vdash U_1\rangle \quad \lambda x^L.M : \langle \Gamma \vdash U_2\rangle}{\lambda x^L.M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$ then $\mathrm{d}(U_1 \sqcap U_2) = \mathrm{d}(U_1) = \mathrm{d}(U_2) = K$.
  By IH, we have four cases:
  - If $U_1 = U_2 = \omega^K$, then $U_1 \sqcap U_2 = \omega^K$.
  - If $U_1 = \omega^K$, $U_2 = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$, then $U_1 \sqcap U_2 = U_2$ (ω^K is a neutral element).
  - If $U_2 = \omega^K$, $U_1 = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$, then $U_1 \sqcap U_2 = U_1$ (ω^K is a neutral element).
  - If $U_1 = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$, $U_2 = \sqcap_{i=p+1}^{p+q} \boldsymbol{e}_K(V_i \to T_i)$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{p+q} \boldsymbol{e}_K(V_i \to T_i)$) where $p, q \geq 1$, $\forall 1 \leq i \leq p+q$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$, we are done.
- If $\dfrac{\lambda x^L.M : \langle \Gamma \vdash U\rangle}{\lambda x^{i::L}.M^{+i} : \langle e_i\Gamma \vdash e_i U\rangle}$. $\mathrm{d}(e_i U) = i :: \mathrm{d}(U) = i :: K' = K$. By IH, we have two cases:
  - If $U = \omega^{K'}$ then $e_i U = \omega^K$.
  - If $U = \sqcap_{j=1}^p \boldsymbol{e}_{K'}(V_j \to T_j)$, where $p \geq 1$ and for all $1 \leq j \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_{K'} V_j \vdash \boldsymbol{e}_{K'} T_j\rangle$. So $e_i U = \sqcap_{j=1}^p \boldsymbol{e}_K(V_j \to T_j)$ and by e, for all $1 \leq j \leq p$, $M^{+i} : \langle e_i\Gamma, x^{i::L} : \boldsymbol{e}_K V_j \vdash \boldsymbol{e}_K T_j\rangle$.
- Let $\dfrac{\lambda x^L.M : \langle \Gamma \vdash U\rangle \quad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{\lambda x^L.M : \langle \Gamma' \vdash U'\rangle}$. By lemma 13.3, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$ and by lemma 12.3a $\mathrm{d}(U) = \mathrm{d}(U') = K$. By IH, we have two cases:
  - If $U = \omega^K$, then, by lemma 12.3b, $U' = \omega^K$.
  - If $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$, where $p \geq 1$ and for all $1 \leq i \leq p$ $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash \boldsymbol{e}_K T_i\rangle$. By lemma 12.3e:
    * Either $U' = \omega^K$.
    * Or $U' = \sqcap_{i=1}^q \boldsymbol{e}_K(V'_i \to T'_i)$, where $q \geq 1$ and $\forall 1 \leq i \leq q$, $\exists 1 \leq j_i \leq p$ such that $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. Let $1 \leq i \leq q$. Since, by lemma 13.3, $\langle \Gamma, x^L : \boldsymbol{e}_K V_{j_i} \vdash \boldsymbol{e}_K T_{j_i}\rangle \sqsubseteq \langle \Gamma', x^L : \boldsymbol{e}_K V'_i \vdash \boldsymbol{e}_K T'_i\rangle$, then $M : \langle \Gamma', x^L : \boldsymbol{e}_K V'_i \vdash \boldsymbol{e}_K T'_i\rangle$.

3. Same proof as that of 2.
4. By induction on the derivation $M\,x^L : \langle \Gamma, x^L : U \vdash T\rangle$. We have two cases:

- Let $\dfrac{M : \langle \Gamma \vdash V \to T\rangle \quad x^L : \langle (x^L : U) \vdash V\rangle \quad \Gamma \diamond (x^L : U)}{M\,x^L : \langle \Gamma, (x^L : U) \vdash T\rangle}$ (where, by 1. $U \sqsubseteq V$). Since $V \to T \sqsubseteq U \to T$, we have $M : \langle \Gamma \vdash U \to T\rangle$.
- Let $\dfrac{M\,x^L : \langle \Gamma', (x^L : U') \vdash V'\rangle \quad \langle \Gamma', (x^L : U') \vdash V'\rangle \sqsubseteq \langle \Gamma, (x^L : U) \vdash V\rangle}{M\,x^L : \langle \Gamma, (x^L : U) \vdash V\rangle}$ (by lemma 13).
  By lemma 13, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. By IH, $M : \langle \Gamma' \vdash U' \to V'\rangle$ and by $\sqsubseteq$, $M : \langle \Gamma \vdash U \to V\rangle$.

□

Proof [Of lemma 18] By induction on the derivation $M : \langle \Gamma, x^L : U \vdash V\rangle$.

- If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T\rangle}$ and $N : \langle \Delta \vdash T\rangle$, then $x^{\oslash}[x^{\oslash} := N] = N : \langle \Delta \vdash T\rangle$.
- If $\dfrac{}{M : \langle env^{\omega}_{\mathrm{fv}(M)\setminus\{x^L\}}, (x^L : \omega^L) \vdash \omega^{\mathrm{d}(M)}\rangle}$ and $N : \langle \Delta \vdash \omega^L\rangle$ then by ω, $M[x^L := N] : \langle env^{\omega}_{M[x^L:=N]} \vdash \omega^{\mathrm{d}(M[x^L:=N])}\rangle$. By lemma 45 $\mathrm{d}(M[x^L := N]) = \mathrm{d}(M)$. Since $x^L \in \mathrm{fv}(M)$ (and so $\mathrm{fv}(N) \subseteq \mathrm{fv}(M[x^L := N])$), by $\sqsubseteq$, $M[x^L := N] : \langle env^{\omega}_{\mathrm{fv}(M)\setminus\{x^L\}} \sqcap \Delta \vdash \omega^{\mathrm{d}(M)}\rangle$.

- Let $\dfrac{M : \langle \Gamma, x^L : U, y^K : U' \vdash T\rangle}{\lambda y^K.M : \langle \Gamma, x^L : U \vdash U' \to T\rangle}$ where $y^K \notin \mathrm{fv}(N)$. By IH, $M[x^L := N] : \langle \Gamma \sqcap \Delta, y^K : U' \vdash T\rangle$. By $\to_I$, $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash U' \to T\rangle$.
- Let $\dfrac{M : \langle \Gamma, x^L : U \vdash T\rangle \quad y^K \notin \mathrm{dom}(\Gamma, x^L : U)}{\lambda y^K.M : \langle \Gamma, x^L : U \vdash \omega^K \to T\rangle}$ where $y^K \notin \mathrm{fv}(N)$. By IH, $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash T\rangle$. By $\to'_I$, $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash \omega^K \to T\rangle$.
- Let $\dfrac{M_1 : \langle \Gamma_1, x^L : U_1 \vdash V \to T\rangle \quad M_2 : \langle \Gamma_2, x^L : U_2 \vdash V\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1M_2 : \langle \Gamma_1 \sqcap \Gamma_2, x^L : U_1 \sqcap U_2 \vdash T\rangle}$ where $x^L \in \mathrm{fv}(M_1) \cap \mathrm{fv}(M_2)$, $N : \langle \Delta \vdash U_1 \sqcap U_2\rangle$ and $(\Gamma_1 \sqcap \Gamma_2) \diamond \Delta$. It is easy to show that $\Gamma_1 \diamond \Delta$ and $\Gamma_2 \diamond \Delta$. By $\sqcap_E$ and $\sqsubseteq$, $N : \langle \Delta \vdash U_1\rangle$ and $N : \langle \Delta \vdash U_2\rangle$. Now use IH and $\to_E$.
  The cases $x^L \in \mathrm{fv}(M_1) \setminus \mathrm{fv}(M_2)$ or $x^L \in \mathrm{fv}(M_2) \setminus \mathrm{fv}(M_1)$ are easy.
- If $\dfrac{M : \langle \Gamma, x^L : U \vdash U_1\rangle \quad M : \langle \Gamma, x^L : U \vdash U_2\rangle}{M : \langle \Gamma, x^L : U \vdash U_1 \sqcap U_2\rangle}$ use IH and $\sqcap_I$.
- Let $\dfrac{M : \langle \Gamma, x^L : U \vdash V\rangle}{M^{+i} : \langle e_i\Gamma, x^{i::L} : e_iU \vdash e_iV\rangle}$ where $N : \langle \Delta \vdash e_iU\rangle$. By lemma 15, $N^{-i} : \langle \Delta^{-i} \vdash U\rangle$. By IH, $M[x^L := N^{-i}] : \langle \Gamma \sqcap \Delta^{-i} \vdash V\rangle$. By e and lemma 46.4, $M^{+i}[x^{i::L} := N] : \langle e_i\Gamma \sqcap \Delta \vdash e_iV\rangle$.
- Let $\dfrac{M : \langle \Gamma', x^L : U' \vdash V'\rangle \quad \langle \Gamma', x^L : U' \vdash V'\rangle \sqsubseteq \langle \Gamma, x^L : U \vdash V\rangle}{M : \langle \Gamma, x^L : U \vdash V\rangle}$ (lemma 13). By lemma 13, $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma')$, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. Hence $N : \langle \Delta \vdash U'\rangle$ and, by IH, $M[x^L := N] : \langle \Gamma' \sqcap \Delta \vdash V'\rangle$. It is easy to show that $\Gamma \sqcap \Delta \sqsubseteq \Gamma' \sqcap \Delta$. Hence, $\langle \Gamma' \sqcap \Delta \vdash V'\rangle \sqsubseteq \langle \Gamma \sqcap \Delta \vdash V\rangle$ and $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash V\rangle$.

□

The next lemma is needed in the proofs.

Lemma 53. *1. If* $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$, *then* $env^M_\omega \restriction_N = env^N_\omega$.
2. If $\mathrm{fv}(M) \subseteq \mathrm{dom}(\Gamma_1)$ *and* $\mathrm{fv}(N) \subseteq \mathrm{dom}(\Gamma_2)$, *then* $(\Gamma_1 \sqcap \Gamma_2) \restriction_{MN} \sqsubseteq (\Gamma_1 \restriction_M) \sqcap (\Gamma_2 \restriction_N)$.
3. $e_i(\Gamma \restriction_M) = (e_i\Gamma) \restriction_{M^{+i}}$

Proof 1. Easy. 2. First, note that $\mathrm{dom}((\Gamma_1 \sqcap \Gamma_2) \restriction_{MN}) = \mathrm{fv}(MN) = \mathrm{fv}(M) \cup \mathrm{fv}(N) = \mathrm{dom}(\Gamma_1 \restriction_M) \cup \mathrm{dom}(\Gamma_2 \restriction_N) = \mathrm{dom}((\Gamma_1 \restriction_M) \sqcap (\Gamma_2 \restriction_N))$. Now, we show by cases that if $(x^L : U_1) \in (\Gamma_1 \sqcap \Gamma_2) \restriction_{MN}$ and $(x^L : U_2) \in (\Gamma_1 \restriction_M) \sqcap (\Gamma_2 \restriction_N)$ then $U_1 \sqsubseteq U_2$:

- If $x^L \in \mathrm{fv}(M) \cap \mathrm{fv}(N)$ then $(x^L : U'_1) \in \Gamma_1$, $(x^L : U''_1) \in \Gamma_2$ and $U_1 = U'_1 \sqcap U''_1 = U_2$.
- If $x^L \in \mathrm{fv}(M) \setminus \mathrm{fv}(N)$ then
  - If $x^L \in \mathrm{dom}(\Gamma_2)$ then $(x^L : U_2) \in \Gamma_1$, $(x^L : U'_1) \in \Gamma_2$ and $U_1 = U'_1 \sqcap U_2 \sqsubseteq U_2$.
  - If $x^L \notin \mathrm{dom}(\Gamma_2)$ then $(x^L : U_2) \in \Gamma_1$ and $U_1 = U_2$.
- If $x^L \in \mathrm{fv}(N) \setminus \mathrm{fv}(M)$ then
  - If $x^L \in \mathrm{dom}(\Gamma_1)$ then $(x^L : U'_1) \in \Gamma_1$, $(x^L : U_2) \in \Gamma_2$ and $U_1 = U'_1 \sqcap U_2 \sqsubseteq U_2$.
  - If $x^L \notin \mathrm{dom}(\Gamma_1)$ then $x^L : U_2 \in \Gamma_2$ and $U_1 = U_2$.

  3. Let $\Gamma = (x_j^{L_j} : U_j)_n$ and let $\mathrm{fv}(M) = \{y_1^{K_1}, \ldots, y_m^{K_m}\}$ where $m \leq n$ and $\forall 1 \leq k \leq m\ \exists 1 \leq j \leq n$ such that $y_k^{K_k} = x_j^{L_j}$. So $\Gamma \restriction_M = (y_k^{K_k} : U_k)_m$ and $e_i(\Gamma \restriction_M) = (y_k^{i::K_k} : e_iU_k)_m$. Since $e_i\Gamma = (x_j^{i::L_j} : e_iU_j)_n$, $\mathrm{fv}(M^{+i}) = \{y_1^{i::K_1}, \ldots, y_m^{i::K_m}\}$ and $\forall 1 \leq k \leq m\ \exists 1 \leq j \leq n$ such that $y_k^{i::K_k} = x_j^{i::L_j}$ then $(e_i\Gamma) \restriction_{M^{+i}} = (y_k^{i::K_k} : U_k)_m$.

□

The next two theorems are needed in the proof of subject reduction.

Theorem 54. *If $M : \langle \Gamma \vdash U\rangle$ and $M \rhd_\beta N$, then $N : \langle \Gamma \upharpoonright_N \vdash U\rangle$.*

Proof By induction on the derivation $M : \langle \Gamma \vdash U\rangle$.

- Rule ω follows by theorem 4.2 and lemma 53.1.
- If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}$ then $N = \lambda x^L N'$ and $M \rhd_\beta N'$. By IH, $N' : \langle (\Gamma, (x^L : U)) \upharpoonright_{N'} \vdash T\rangle$. If $x^L \in \mathrm{fv}(N')$ then $N' : \langle \Gamma \upharpoonright_{\mathrm{fv}(N')\setminus\{x^L\}}, (x^L : U) \vdash T\rangle$ and by $\to_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash U \to T\rangle$. Else $N' : \langle \Gamma \upharpoonright_{\mathrm{fv}(N')\setminus\{x^L\}} \vdash T\rangle$ so by $\to'_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash \omega^L \to T\rangle$ and since by lemma 12.4, $U \sqsubseteq \omega^L$, by $\sqsubseteq$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash U \to T\rangle$.
- If $\dfrac{M : \langle \Gamma \vdash T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T\rangle}$ then $N = \lambda x^L N'$ and $M \rhd_\beta N'$. Since $x^L \notin \mathrm{fv}(M)$, by theorem 4.2, $x^L \notin \mathrm{fv}(N')$. By IH, $N' : \langle \Gamma \upharpoonright_{\mathrm{fv}(N')\setminus\{x^L\}} \vdash T\rangle$ so by $\to'_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash \omega^L \to T\rangle$.
- If $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$. Using lemma 53.2, case $M_1 \rhd_\beta N_1$ and $N = N_1 M_2$ and case $M_2 \rhd_\beta N_2$ and $N = M_1 N_2$ are easy. Let $M_1 = \lambda x^L.M'_1$ and $N = M'_1[x^L := M_2]$. If $x^L \in FV(M'_1)$ then by lemma 17.2, $M'_1 : \langle \Gamma_1, x^L : U \vdash T\rangle$. By lemma 18, $M'_1[x^L := M_2] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$. If $x^L \notin FV(M'_1)$ then by lemma 17.3, $M'_1[x^L := M_2] = M'_1 : \langle \Gamma_1 \vdash T\rangle$ and by $\sqsubseteq$, $N : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_N \vdash T\rangle$.
- Case $\sqcap_I$ is by IH.
- If $\dfrac{M : \langle \Gamma \vdash U\rangle}{M^{+i} : \langle e_i\Gamma \vdash e_i U\rangle}$ and $M^{+i} \rhd_\beta N$, then by lemma 46.9, there is $P \in \mathcal{M}$ such that $P^{+i} = N$ and $M \rhd_\beta P$. By IH, $P : \langle \Gamma \upharpoonright_P \vdash U\rangle$ and by e and lemma 53.3, $N : \langle (e_i\Gamma) \upharpoonright_N \vdash e_i U\rangle$.
- If $\dfrac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}$ then by IH, lemma 13.3 and $\sqsubseteq$, $N : \langle \Gamma' \upharpoonright_N \vdash U'\rangle$.

□

Theorem 55. *If $M : \langle \Gamma \vdash U\rangle$ and $M \rhd_\eta N$, then $N : \langle \Gamma \vdash U\rangle$.*

Proof By induction on the derivation $M : \langle \Gamma \vdash U\rangle$.

- If $\dfrac{}{M : \langle env^\omega_M \vdash \omega^{\mathrm{d}(M)}\rangle}$ then by lemma 4.1, $\mathrm{d}(M) = \mathrm{d}(N)$ and $\mathrm{fv}(M) = \mathrm{fv}(N)$ and by ω, $N : \langle env^\omega_M \vdash \omega^{\mathrm{d}(M)}\rangle$.
- If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash T\rangle}{\lambda x^L.M : \langle \Gamma \vdash U \to T\rangle}$ then we have two cases:
  - $M = N x^L$ and so by lemma 17.4, $N : \langle \Gamma \vdash U \to T\rangle$.
  - $N = \lambda x^L N'$ and $M \rhd_\eta N'$. By IH, $N' : \langle \Gamma, (x^L : U) \vdash T\rangle$ and by $\to_I$, $N : \langle \Gamma \vdash U \to T\rangle$.
- if $\dfrac{M : \langle \Gamma \vdash T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash \omega^L \to T\rangle}$ then $N = \lambda x^L N'$ and $M \rhd_\eta N'$. By IH, $N' : \langle \Gamma \vdash T\rangle$ and by $\to'_I$, $N : \langle \Gamma \vdash \omega^L \to T\rangle$.
- If $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$, then we have two cases:
  - $M_1 \rhd_\eta N_1$ and $N = N_1 M_2$. By IH $N_1 : \langle \Gamma_1 \vdash U \to T\rangle$ and by $\to_E$, $N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$.
  - $M_2 \rhd_\eta N_2$ and $N = M_1 N_2$. By IH $N_2 : \langle \Gamma_2 \vdash U\rangle$ and by $\to_E$, $N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$.

- Case $\sqcap_I$ is by IH and $\sqcap_I$.
- If $\dfrac{M : \langle \Gamma \vdash U\rangle}{M^{+i} : \langle e_i\Gamma \vdash e_i U\rangle}$ then by lemma 46.9, there is $P \in \mathcal{M}$ such that $P^{+i} = N$ and $M \rhd_\eta P$. By IH, $P : \langle \Gamma \vdash U\rangle$ and by e, $N : \langle e_i\Gamma \vdash e_i U\rangle$.
- If $\dfrac{M : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M : \langle \Gamma' \vdash U'\rangle}$ then by IH, lemma 13.3 and $\sqsubseteq$, $N : \langle \Gamma' \vdash U'\rangle$.

$\square$

The next auxiliary lemma is needed in proofs.

Lemma 56. *Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash U\rangle$. We have:*

1. *If $(x^L : U_1) \in \Gamma$ and $(y^K : U_2) \in \Gamma$, then:*
   (a) If $(x^L : U_1) \neq (y^K : U_2)$, then $x^L \neq y^K$.
   (b) If $x = y$, then $L = K$ and $U_1 = U_2$.
2. *If $(x^L : U_1) \in \Gamma$ and $(y^K : U_2) \in \Gamma$ and $(x^L : U_1) \neq (y^K : U_2)$, then $x \neq y$ and $x^L \neq y^K$.*

Proof 1. By induction on the derivation of $M : \langle \Gamma \vdash U\rangle$. 2. Corollary of 1. $\square$

Proof [Of theorem 20] Proofs are by induction on derivations using theorem 54 and theorem 55. $\square$

D Proofs for section 5

Proof [Of lemma 22] By induction on the derivation $M[x^L := N] : \langle \Gamma \vdash U\rangle$.

- If $\dfrac{}{y^\oslash : \langle (y^\oslash : T) \vdash T\rangle}$ then $M = x^\oslash$ and $N = y^\oslash$. By ax, $x^\oslash : \langle (x^\oslash : T) \vdash T\rangle$.
- If $\dfrac{}{M[x^L := N] : \langle env^\omega_{M[x^L:=N]} \vdash \omega^{\mathrm{d}(M[x^L:=N])}\rangle}$ then by lemma 45, $\mathrm{d}(M) = \mathrm{d}(M[x^L := N])$. By ω, $M : \langle env^\omega_{\mathrm{fv}(M)\setminus\{x^L\}}, (x^L : \omega^L) \vdash \omega^{\mathrm{d}(M)}\rangle$ and $N : \langle env^\omega_N \vdash \omega^L\rangle$ and it's easy to see that $env^\omega_{\mathrm{fv}(M)\setminus\{x^L\}} \sqcap env^\omega_N = env^\omega_{M[x^L:=N]}$.
- If $\dfrac{M[x^L := N] : \langle \Gamma, (y^K : W) \vdash T\rangle}{\lambda y^K.M[x^L := N] : \langle \Gamma \vdash W \to T\rangle}$ where $y^K \notin \mathrm{fv}(N)$. By IH, $\exists\ V$ type such that $\mathrm{d}(V) = L$ and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash T\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma, y^K : W = \Gamma_1 \sqcap \Gamma_2$. Since $y^K \in \mathrm{fv}(M)$ and $y^K \notin \mathrm{fv}(N)$, $\Gamma_1 = \Delta_1, y^K : W$. Hence $M : \langle \Delta_1, y^K : W, x^L : V \vdash T\rangle$. By rule $\to_I$, $\lambda y^K.M : \langle \Delta_1, x^L : V \vdash W \to T\rangle$. Finally $\Gamma = \Delta_1 \sqcap \Gamma_2$.
- If $\dfrac{M[x^L := N] : \langle \Gamma \vdash T\rangle \quad y^K \notin \mathrm{dom}(\Gamma)}{\lambda y^K.M[x^L := N] : \langle \Gamma \vdash \omega^K \to T\rangle}$. By IH, $\exists\ V$ type such that $\mathrm{d}(V) = L$ and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash T\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. Since $y^K \neq x^L$, $\lambda y^K.M : \langle \Gamma_1, x^L : V \vdash \omega^K \to T\rangle$.
- If $\dfrac{M_1[x^L := N] : \langle \Gamma_1 \vdash W \to T\rangle \quad M_2[x^L := N] : \langle \Gamma_2 \vdash W\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1[x^L := N]\ M_2[x^L := N] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ where $M = M_1 M_2$, then we have three cases:
  - If $x^L \in \mathrm{fv}(M_1) \cap \mathrm{fv}(M_2)$ then by IH, $\exists\ V_1, V_2$ types and $\exists\ \Delta_1, \Delta_2, \nabla_1, \nabla_2$ type environments such that $M_1 : \langle \Delta_1, (x^L : V_1) \vdash W \to T\rangle$, $M_2 : \langle \nabla_1, (x^L : V_2) \vdash W\rangle$, $N : \langle \Delta_2 \vdash V_1\rangle$, $N : \langle \nabla_2 \vdash V_2\rangle$, $\Gamma_1 = \Delta_1 \sqcap \Delta_2$ and $\Gamma_2 = \nabla_1 \sqcap \nabla_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Delta_1 \diamond \nabla_1$ and since $\Delta_1, (x^L : V_1)$ and $\nabla_1, (x^L : V_2)$ are type environments, by lemma 56, $(\Delta_1, (x^L : V_1)) \diamond (\nabla_1, (x^L : V_2))$. Then, by rules $\sqcap_I$ and $\to_E$, $M_1 M_2 : \langle \Delta_1 \sqcap \nabla_1, (x^L : V_1 \sqcap V_2) \vdash T\rangle$ and by $\sqsubseteq$ and $\sqcap_I$, $N : \langle \Delta_2 \sqcap \nabla_2 \vdash V_1 \sqcap V_2\rangle$. Finally, $\Gamma_1 \sqcap \Gamma_2 = (\Delta_1 \sqcap \Delta_2) \sqcap (\nabla_1 \sqcap \nabla_2)$.

- If $x^L \in \text{fv}(M_1) \setminus \text{fv}(M_2)$ then by IH, $\exists\ V$ types and $\exists\ \Delta_1, \Delta_1$ type environments such that $M_1 : \langle \Delta_1, (x^L : V) \vdash W \to T\rangle$, $N : \langle \Delta_2 \vdash V\rangle$ and $\Gamma_1 = \Delta_1 \sqcap \Delta_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Delta_1 \diamond \Gamma_2$ and since $\Gamma_1 \sqcap \Gamma_2$ is a type environment, by lemma 56, $(\Delta_1, (x^L : V)) \diamond \Gamma_2$. By $\to_E$, $M_1M_2 : \langle \Delta_1 \sqcap \Gamma_2, (x^L : V) \vdash T\rangle$ and $\Gamma_1 \sqcap \Gamma_2 = (\Delta_1 \sqcap \Delta_2) \sqcap \Gamma_2$.
- If $x^L \in \text{fv}(M_2) \setminus \text{fv}(M_1)$ then by IH, $\exists\ V$ types and $\exists\ \Delta_1, \Delta_2$ type environments such that $M_2 : \langle \Delta_1, (x^L : V) \vdash W\rangle$, $N : \langle \Delta_2 \vdash V\rangle$ and $\Gamma_2 = \Delta_1 \sqcap \Delta_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Gamma_1 \diamond \Delta_1$ and since $\Gamma_1 \sqcap \Gamma_2$ is a type environment, by lemma 56, $(\Delta_1, (x^L : V)) \diamond \Gamma_1$. By $\to_E$, $M_1M_2 : \langle \Gamma_1 \sqcap \Delta_1, (x^L : V) \vdash T\rangle$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \sqcap (\Delta_1 \sqcap \Delta_2)$.

– Let $\dfrac{M[x^L := N] : \langle \Gamma \vdash U_1\rangle \quad M[x^L := N] : \langle \Gamma \vdash U_2\rangle}{M[x^L := N] : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$. By IH, $\exists\ V_1, V_2$ types and $\exists\ \Delta_1, \Delta_2, \nabla_1, \nabla_2$ type environments such that $M : \langle \Delta_1, x^L : V_1 \vdash U_1\rangle$, $M : \langle \nabla_1, x^L : V_2 \vdash U_2\rangle$, $N : \langle \Delta_2 \vdash V_1\rangle$, $N : \langle \nabla_2 \vdash V_2\rangle$, $\Gamma = \Delta_1 \sqcap \Delta_2$ and $\Gamma = \nabla_1 \sqcap \nabla_2$. Then, by rule $\sqcap'_I$, $M : \langle \Delta_1 \sqcap \nabla_1, x^L : V_1 \sqcap V_2 \vdash U_1 \sqcap U_2\rangle$ and $N : \langle \Delta_2 \sqcap \nabla_2 \vdash V_1 \sqcap V_2\rangle$. Finally, $\Gamma = (\Delta_1 \sqcap \Delta_2) \sqcap (\nabla_1 \sqcap \nabla_2)$.

– If $\dfrac{M[x^L := N] : \langle \Gamma \vdash U\rangle}{M^{+j}[x^{j::L} := N^{+j}] : \langle e_j\Gamma \vdash e_jU\rangle}$ then by IH, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash U\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. So by e, $M^{+j} : \langle e_j\Gamma_1, x^{j::L} : e_jV \vdash e_jU\rangle$, $N : \langle e_j\Gamma_2 \vdash e_jV\rangle$ and $e_j\Gamma = e_j\Gamma_1 \sqcap e_j\Gamma_2$.

– If $\dfrac{M[x^L := N] : \langle \Gamma' \vdash U'\rangle \quad \langle \Gamma' \vdash U'\rangle \sqsubseteq \langle \Gamma \vdash U\rangle}{M[x^L := N] : \langle \Gamma \vdash U\rangle}$ then by lemma 13.2, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$. By IH, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma'_1, x^L : V \vdash U'\rangle$, $N : \langle \Gamma'_2 \vdash V\rangle$ and $\Gamma' = \Gamma'_1 \sqcap \Gamma'_2$. Then by lemma 12.6, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$. So by $\sqsubseteq$, $M : \langle \Gamma_1, x^L : V \vdash U\rangle$ and $N : \langle \Gamma_2 \vdash V\rangle$.

□

The next lemma is basic for the proof of subject expansion for β.

Lemma 57. *If $M[x^L := N] : \langle \Gamma \vdash U\rangle$, $d(N) = L$, $d(U) = K$, $x^L \notin \text{fv}(N)$ and $\mathcal{U} = \text{fv}((\lambda x^L.M)N)$, then $(\lambda x^L.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U\rangle$.*

Proof By lemma 45 and theorem 15.1, $K = \text{d}(M[x^L := N]) = \text{d}(M) = \text{d}((\lambda x^L.M)N)$. We have two cases:

– If $x^L \in \text{fv}(M)$, then, by lemma 22, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash U\rangle$, $N : \langle \Gamma_2 \vdash V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. By lemma 45, $L \succeq K$, so $L = K :: K'$. By lemma 12, we have two cases :
  - If $U = \omega^K$, then by lemma 14.1, $(\lambda x^L.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U\rangle$.
  - If $U = \boldsymbol{e}_K \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$, then by theorem 15.2, $M^{-K} : \langle \Gamma_1^{-K}, x^{K'} : V^{-K} \vdash \sqcap_{i=1}^p T_i\rangle$. By $\sqsubseteq$, $\forall\ 1 \leq i \leq p$, $M^{-K} : \langle \Gamma_1^{-K}, x^{K'} : V^{-K} \vdash T_i\rangle$, so by $\to_I$, $\lambda x^{K'}.M^{-K} : \langle \Gamma_1^{-K} \vdash V^{-K} \to T_i\rangle$. Again by theorem 15.2, $N^{-K} : \langle \Gamma_2^{-K} \vdash V^{-K}\rangle$ and since $\Gamma_1 \diamond \Gamma_2$, $\Gamma_1^{-K} \diamond \Gamma_2^{-K}$, so by $\to_E$, $\forall\ 1 \leq i \leq p$, $(\lambda x^{K'}.M^{-K})N^{-K} : \langle \Gamma_1^{-K} \sqcap \Gamma_2^{-K} \vdash T_i\rangle$. Finally by $\sqcap_I$ and e, $(\lambda x^L.M)N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U\rangle$, so $(\lambda x^L.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U\rangle$.

– If $x^L \notin \text{fv}(M)$, then $M : \langle \Gamma \vdash U\rangle$ and, by rule $\to'_I$, $\lambda x^L.M : \langle \Gamma \vdash \omega^L \to U\rangle$. By rule ω, $N : \langle env_N^\omega \vdash \omega^L\rangle$, then, since $M \diamond N$, by rule $\to_E$, $(\lambda x^L.M)N : \langle \Gamma \sqcap env_N^\omega \vdash U\rangle$. Since $\text{fv}((\lambda x^L.M)N) = \text{fv}(M[x^L := N]) \cup \text{fv}(N)$, then $\Gamma\uparrow^{\mathcal{U}} = \Gamma \sqcap env_N^\omega$.

□

Next, we give the main block for the proof of subject expansion for β.

Theorem 58. *If $N : \langle \Gamma \vdash U\rangle$ and $M \rhd_\beta N$, then $M : \langle \Gamma\uparrow^M \vdash U\rangle$.*

Proof By induction on the derivation $N : \langle \Gamma \vdash U\rangle$.

- If $\dfrac{}{x^\oslash : \langle (x^\oslash : T) \vdash T\rangle}$ and $M \rhd_\beta x^\oslash$, then $M = (\lambda y^K.M_1)M_2$ where $y^K \notin \mathrm{fv}(M_2)$ and $x^\oslash = M_1[y^K := M_2]$. By lemma 57, $M : \langle (x^\oslash : T)\uparrow^M \vdash T\rangle$.
- If $\dfrac{}{N : \langle env^\omega_N \vdash \omega^{\mathrm{d}(N)}\rangle}$ and $M \rhd_\beta N$, then since by theorem 4.2, $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, $(env^\omega_N)\uparrow^M = env^\omega_M$. By ω, $M : \langle env^\omega_M \vdash \omega^{\mathrm{d}(M)}\rangle$. Hence, $M : \langle (env^N_\omega)\uparrow^M \vdash \omega^{\mathrm{d}(N)}\rangle$.
- If $\dfrac{N : \langle \Gamma, x^L : U \vdash T\rangle}{\lambda x^L.N : \langle \Gamma \vdash U \to T\rangle}$ and $M \rhd_\beta \lambda x^L.N$, then we have two cases:
  - If $M = \lambda x.M'$ where $M' \rhd_\beta N$, then by IH, $M' : \langle (\Gamma, (x^L : U))\uparrow^{M'} \vdash T\rangle$. Since by theorem 4.2 and lemma 14.2, $x^L \in \mathrm{fv}(N) \subseteq \mathrm{fv}(M')$, then we have $(\Gamma, (x^L : U))\uparrow^{\mathrm{fv}(M')} = \Gamma\uparrow^{\mathrm{fv}(M')\setminus\{x^L\}}, (x^L : U)$ and $\Gamma\uparrow^{\mathrm{fv}(M')\setminus\{x^L\}} = \Gamma\uparrow^{\lambda x^L.M'}$. Hence, $M' : \langle \Gamma\uparrow^{\lambda x^L.M'}, (x^L : U) \vdash T\rangle$ and finally, by $\to_I$, $\lambda x^L.M' : \langle \Gamma\uparrow^{\lambda x^L.M'} \vdash U \to T\rangle$.
  - If $M = (\lambda y^K.M_1)M_2$ where $y^K \notin \mathrm{fv}(M_2)$ and $\lambda x^L.N = M_1[y^K := M_2]$, then, by lemma 57, since $y^K \notin \mathrm{fv}(M_2)$ and $M_1[y^K := M_2] : \langle \Gamma \vdash U \to T\rangle$, we have $(\lambda y^K.M_1)M_2 : \langle \Gamma\uparrow^{(\lambda y^K.M_1)M_2} \vdash U \to T\rangle$.
- If $\dfrac{N : \langle \Gamma \vdash T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.N : \langle \Gamma \vdash \omega^L \to T\rangle}$ and $M \rhd_\beta N$ then similar to the above case.
- If $\dfrac{N_1 : \langle \Gamma_1 \vdash U \to T\rangle \quad N_2 : \langle \Gamma_2 \vdash U\rangle \quad \Gamma_1 \diamond \Gamma_2}{N_1\ N_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ and $M \rhd_\beta N_1 N_2$, we have three cases:
  - $M = M_1 N_2$ where $M_1 \rhd_\beta N_1$ and $M_1 \diamond N_2$. By IH, $M_1 : \langle \Gamma_1\uparrow^{M_1} \vdash U \to T\rangle$. It is easy to show that $(\Gamma_1 \sqcap \Gamma_2)\uparrow^{M_1 N_2} = \Gamma_1\uparrow^{M_1} \sqcap \Gamma_2$. Since $M_1 \diamond N_2$, $\Gamma_1\uparrow^{M_1} \diamond \Gamma_2$, hence use $\to_E$.
  - $M = N_1 M_2$ where $M_2 \rhd_\beta N_2$. Similar to the above case.
  - $M = (\lambda x^L.M_1)M_2$ where $x^L \notin \mathrm{fv}(M_2)$ and $N_1 N_2 = M_1[x^L := M_2]$. By lemma 57, $(\lambda x^L.M_1)M_2 : \langle (\Gamma_1 \sqcap \Gamma_2)\uparrow^{(\lambda x^L.M_1)M_2} \vdash T\rangle$.
- If $\dfrac{N : \langle \Gamma \vdash U_1\rangle \qquad N : \langle \Gamma \vdash U_2\rangle}{N : \langle \Gamma \vdash U_1 \sqcap U_2\rangle}$ and $M \rhd_\beta N$ then use IH.
- If $\dfrac{N : \langle \Gamma \vdash U\rangle}{N^{+j} : \langle e_j\Gamma \vdash e_j U\rangle}$ then by lemma 46.8 then there is $P \in \mathcal{M}$ such that $M = P^{+j}$ and $P \rhd_\beta N$. By IH, $P : \langle \Gamma\uparrow^P \vdash U\rangle$ and by e, $M : \langle (e_j\Gamma)\uparrow^M \vdash e_j U\rangle$.
- If $\dfrac{N : \langle \Gamma \vdash U\rangle \qquad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{N : \langle \Gamma' \vdash U'\rangle}$ and $M \rhd_\beta N$. By lemma 13.3, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. It is easy to show that $\Gamma'\uparrow^M \sqsubseteq \Gamma\uparrow^M$ and hence by lemma 13.3, $\langle \Gamma\uparrow^M \vdash U\rangle \sqsubseteq \langle \Gamma'\uparrow^M \vdash U'\rangle$. By IH, $M\uparrow^M : \langle \Gamma \vdash U\rangle$. Hence, by $\sqsubseteq_{\langle\rangle}$, we have $M : \langle \Gamma'\uparrow^M \vdash U'\rangle$.

□

Proof [Of theorem 24] By induction on the length of the derivation $M \rhd^*_\beta N$ using theorem 58 and the fact that if $\mathrm{fv}(P) \subseteq \mathrm{fv}(Q)$, then $(\Gamma\uparrow^P)\uparrow^Q = \Gamma\uparrow^Q$. □

E Proofs of section 6

Proof [Of lemma 28] 1. and 2. are easy. 3. If $M \rhd^*_r N^{+i}$ where $N \in \mathcal{X}$, then, by lemma 46.8, $M = P^{+i}$ and $P \rhd_r N$. As $\mathcal{X}$ is r-saturated, $P \in \mathcal{X}$ and so $P^{+i} = M \in \mathcal{X}^{+i}$.

4. Let $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $N \rhd_r^* M$. If $P \in \mathcal{X}$ such that $P \diamond N$, then $P \diamond M$ and $NP \rhd_r^* MP$. Since $MP \in \mathcal{Y}$ and $\mathcal{Y}$ is r-saturated, $NP \in \mathcal{Y}$. Hence, $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$.
5. Let $M \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$, then $M = N^{+i}$ and $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$. If $P \in \mathcal{X}^{+i}$ such that $M\diamond$, then $P = Q^{+i}$, $Q \in \mathcal{X}$, $MP = N^{+i}Q^{+i} = (NQ)^{+i}$ and $N \diamond Q$. Hence $NQ \in \mathcal{Y}$ and $MP \in \mathcal{Y}^{+i}$. Thus $M \in \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.
6. Let $M \in \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$ such that $\mathcal{X}^{+} \wr \mathcal{Y}^{+}$. If $P \in \mathcal{X}^{+i}$ such that $M \diamond P$, then $MP \in \mathcal{Y}^{+i}$ hence $MP = Q^{+i}$ such that $Q \in \mathcal{Y}$. Hence, $M = M_1^{+}$. Let $N_1 \in \mathcal{X}$ such that $M_1 \diamond N_1$. By lemma 46, $M \diamond N_1^{+}$ and we have $(M_1N_1)^{+} = M_1^{+}N_1^{+} \in \mathcal{Y}^{+}$. Hence $M_1N_1 \in \mathcal{Y}$. Thus $M_1 \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $M = M_1^{+} \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+}$. □

Proof [Of lemma 30] 1.1a . By induction on T using lemma 28.
1.1b. We prove $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$ by induction on U. Case $U = a$: by definition. Case $U = \omega^L$: We have $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{M}^L \subseteq \mathcal{M}^L$. Case $U = U_1 \sqcap U_2$ (resp. $U = e_iV$): use IH since $\mathrm{d}(U_1) = \mathrm{d}(U_2)$ (resp. $\mathrm{d}(U) = i :: \mathrm{d}(V)$, $\forall x \in \mathcal{V}_1, (\mathcal{N}_x^K)^{+i} = \mathcal{N}_x^{i::K}$ and $(\mathcal{M}^K)^{+i} = \mathcal{M}^{i::K}$). Case $U = V \to T$: by definition, $K = \mathrm{d}(V) \succeq \mathrm{d}(T) = \oslash$.

- Let $x \in \mathcal{V}_1$, $N_1, ..., N_k$ such that $\forall\ 1 \leq i \leq k$, $\mathrm{d}(N_i) \succeq \oslash$ and let $N \in \mathcal{I}(V)$ such that $(x^{\oslash}N_1...N_k) \diamond N$. By IH, $\mathrm{d}(N) = K \succeq \oslash$. Again, by IH, $x^{\oslash}N_1...N_kN \in \mathcal{I}(T)$. Thus $x^{\oslash}N_1...N_k \in \mathcal{I}(V \to T)$.
- Let $M \in \mathcal{I}(V \to T)$. Let $x \in \mathcal{V}_1$ such that $\forall L, x^L \notin \mathrm{fv}(M)$. By IH, $x^K \in \mathcal{I}(V)$, then $Mx^K \in \mathcal{I}(T)$ and, by IH, $\mathrm{d}(Mx^K) = \oslash$. Thus $\mathrm{d}(M) = \oslash$.

2. By induction of the derivation $U \sqsubseteq V$. □

Proof [Of lemma 31] By induction on the derivation $M : \langle (x_j^{L_j} : U_j)_n \vdash U \rangle$.

- If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash T \rangle}$ and $N \in \mathcal{I}(T)$, then $x^{\oslash}[x^{\oslash} := N] = N \in \mathcal{I}(T)$.
- If $\dfrac{}{M : \langle env_M^{\omega} \vdash \omega^{\mathrm{d}(M)} \rangle}$. Let $env_M^{\omega} = (x_j^{L_j} : U_j)_n$ so $\mathrm{fv}(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$. Since $\forall 1 \leq j \leq n, \mathrm{d}(U_j) = L_j$ by lemma 30.1, $\mathcal{I}(U_j) \subseteq \mathcal{M}^{L_j}$, hence, $\mathrm{d}(N_j) = L_j$. Then, by lemma 45, $\mathrm{d}(M[(x_j^{L_j} := N_j)_n]) = \mathrm{d}(M)$ and $M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}^{\mathrm{d}(M)} = \mathcal{I}(\omega^{\mathrm{d}(M)})$.
- If $\dfrac{M : \langle (x_j^{L_j} : U_j)_n, (x^K : V) \vdash T \rangle}{\lambda x^K.M : \langle (x_j^{L_j} : U_j)_n \vdash V \to T \rangle}$, $\forall 1 \leq j \leq n$, $N_j \in \mathcal{I}(U_j)$ and $N \in \mathcal{I}(V)$ such that $(\lambda x^K.M) \diamond N$.
$(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] = \lambda x^K.M[(x_j^{L_j} := N_j)_n]$, where $\forall 1 \leq j \leq n, y^K \notin \mathrm{fv}(N_j)$. Since $N \in \mathcal{I}(V)$ and by lemma 30.1, $\mathcal{I}(V) \subseteq \mathcal{M}^K$, $\mathrm{d}(N) = K$. Hence, $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \rhd_r M[(x_j^{L_j} := N_j)_n, (x^K := N)]$. By IH, $M[(x_j^{L_j} := N_j)_n, (x^K := N)] \in \mathcal{I}(T)$. Since, by lemma 30.1 $\mathcal{I}(T)$ is r-saturated, then $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{I}(T)$ and so $\lambda x^K.M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(V \to T)$.
- If $\dfrac{M : \langle (x_j^{L_j} : U_j)_n \vdash T \rangle \quad x^K \notin \mathrm{dom}((x_j^{L_j} : U_j)_n)}{\lambda x^K.M : \langle (x_j^{L_j} : U_j)_n \vdash \omega^K \to T \rangle}$, $\forall 1 \leq j \leq n$, $x^K \neq x_i^{L_j}$, $N_j \in \mathcal{I}(U_j)$ and $N \in \mathcal{I}(\omega^K)$ such that $(\lambda x^K.M) \diamond N$.
$(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] = \lambda x^K.M[(x_j^{L_j} := N_j)_n]$, where $\forall 1 \leq j \leq n, y^K \notin \mathrm{fv}(N_j)$. Since $N \in \mathcal{I}(\omega^K)$ and by lemma 30.1, $\mathcal{I}(\omega^K) = \mathcal{M}^K$, then $\mathrm{d}(N) = K$. Hence, $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \rhd_r M[(x_j^{L_j} := N_j)_n]$. By IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(T)$. Since, by lemma 30.1 $\mathcal{I}(T)$ is r-saturated, then $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{I}(T)$ and so $\lambda x^K.M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(\omega^K) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(\omega^K \to T)$.

– Let $\dfrac{M_1 : \langle \Gamma_1 \vdash V \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash V\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ where $\Gamma_1 = (x_j^{L_j} : U_j)_n, (y_j^{K_j} : V_j)_m$, $\Gamma_2 = (x_j^{L_j} : U'_j)_n, (z_j^{S_j} : W_j)_p$ and $\Gamma_1 \sqcap \Gamma_2 = (x_j^{L_j} : U_j \sqcap U'_j)_n, (y_j^{K_j} : V_j)_m, (z_j^{S_j} : W_j)_p$.
Let $\forall\ 1 \leq j \leq n, P_j \in \mathcal{I}(U_j \sqcap U'_j)$, $\forall\ 1 \leq j \leq m, Q_j \in \mathcal{I}(V_j)$ and $\forall\ 1 \leq j \leq p, R_j \in \mathcal{I}(W_j)$. Let $A = M_1[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m]$ and $B = M_2[(x_j^{L_j} := P_j)_n, (z_j^{S_j} := R_j)_p]$.
By lemma 14, $\mathrm{fv}(M_1) = \mathrm{dom}(\Gamma_1)$ and $\mathrm{fv}(M_2) = \mathrm{dom}(\Gamma_2)$. Hence, $(M_1 M_2)[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m, (z_j^{S_j} := R_j)_p] = AB$.
By IH, $A \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T)$ and $B \in \mathcal{I}(V)$. Hence, $AB = (M_1 M_2)[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m, (z_j^{S_j} := R_j)_p] \in \mathcal{I}(T)$.
– Let $\dfrac{M : \langle (x_j^{L_j} : U_j)_n \vdash V_1\rangle \quad M : \langle (x_j^{L_j} : U_j)_n \vdash V_2\rangle}{M : \langle (x_j^{L_j} : U_j)_n \vdash V_1 \sqcap V_2\rangle}$. By IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_1)$ and $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_2)$. Hence, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_1 \sqcap V_2)$.
– Let $\dfrac{M : \langle (x_k^{L_k} : U_k)_n \vdash U\rangle}{M^{+j} : \langle (x_k^{j::L_k} : e_j U_k)_n \vdash e_j U\rangle}$ and $\forall\ 1 \leq k \leq n$, $N_k \in \mathcal{I}(e_j T_k) = \mathcal{I}(T_k)^{+j}$.
Then $\forall\ 1 \leq k \leq n$, $N_k = P_k^{+j}$ where $P_k \in \mathcal{I}(U_k)$. By IH, $M[(x_k^{L_k} := P_k)_n] \in \mathcal{I}(T)$. Hence, by lemma 46, $M^{+j}[(x_k^{j::L_k} := N_k)_n] = (M[(x_k^{L_k} := P_k)_n])^{+j} \in \mathcal{I}(U)^{+j} = \mathcal{I}(e_j U)$.
– Let $\dfrac{M : \Phi \quad \Phi \sqsubseteq \Phi'}{M : \Phi'}$ where $\Phi' = \langle (x_j^{L_j} : U_j)_n \vdash U\rangle$. By lemma 13, we have $\Phi = \langle (x_j^{L_j} : U'_j)_n \vdash U'\rangle$, where for every $1 \leq j \leq n$, $U_j \sqsubseteq U'_j$ and $U' \sqsubseteq U$. By lemma 30.2, $N_j \in \mathcal{I}(U'_j)$, then, by IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U')$ and, by lemma 30.2, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$.

□

Proof [Of lemma 35]

1. Let $y \in \mathcal{V}_2$ and $\mathcal{X} = \{M \in \mathbb{M}^{\oslash}\ /\ M \rhd_\beta^* x^{\oslash} N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1$ or $M \rhd_\beta^* y^{\oslash}\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{X} \subseteq \mathcal{M}^{\oslash}$. Take an β-interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Id_0]_\beta$, then M is closed and $M \in \mathcal{X} \rightsquigarrow \mathcal{X}$. Since $y^{\oslash} \in \mathcal{X}$ and $m \diamond Y^{\oslash}$ then $My^{\oslash} \in \mathcal{X}$ and $My^{\oslash} \rhd_\beta^* x^{\oslash} N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1$ or $My^{\oslash} \rhd_\beta^* y^{\oslash}$. Since M is closed and $x^{\oslash} \neq y^{\oslash}$, by lemma 4.2, $My^{\oslash} \rhd_\beta^* y^{\oslash}$. Hence, by lemma 47.4, $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash}$ and, by lemma 4, $M \in \mathcal{M}^{\oslash}$.
Conversely, let $M \in \mathcal{M}^{\oslash}$ such that $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash}$. Let $\mathcal{I}$ be an β-interpretation and $N \in \mathcal{I}(a)$. Since $\mathcal{I}(a)$ is β-saturated and $MN \rhd_\beta^* N$, $MN \in \mathcal{I}(a)$ and hence $M \in \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$. Hence, $M \in [Id_0]_\beta$.
2. By lemma 33, $[Id'_1]_\beta = [e_1 a \to e_1 a]_\beta = [e_1(a \to a)]_\beta = [Id_1] = [a \to a]_\beta^{+1} = [Id_0]_\beta^{+1}$. By 1., $[Id_0]_\beta^{+1} = \{M \in \mathcal{M}^{(1)}\ /\ M \rhd_\beta^* \lambda y^{(1)}.y^{(1)}\}$.
3. Let $y \in \mathcal{V}_2$, $\mathcal{X} = \{M \in \mathcal{M}^{\oslash}\ /\ M \rhd_\beta^* y^{\oslash}$ or $M \rhd_\beta^* x^{\oslash} N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$ and $\mathcal{Y} = \{M \in \mathcal{M}^{\oslash}\ /\ M \rhd_\beta^* y^{\oslash} y^{\oslash}$ or $M \rhd_\beta^* x^{\oslash} N_1 ... N_k$ or $M \rhd_\beta^* y^{\oslash}(x^{\oslash} N_1 ... N_k)$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$, $\mathcal{Y}$ are β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}^{\oslash}$. Let $\mathcal{I}$ be an β-interpretation such that $\mathcal{I}(a) = \mathcal{X}$ and $\mathcal{I}(b) = \mathcal{Y}$. If $M \in [D]_\beta$, then M is closed (hence $M \diamond y^{\oslash}$) and $M \in (\mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})) \rightsquigarrow \mathcal{Y}$. Since $y^{\oslash} \in \mathcal{X}$ and $y^{\oslash} \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, $y^{\oslash} \in \mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})$ and $My^{\oslash} \in \mathcal{Y}$. Since $x^{\oslash} \neq y^{\oslash}$, by lemma 4.2, $My^{\oslash} \rhd_\beta^* y^{\oslash} y^{\oslash}$. Hence, by lemma 47.4, $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash} y^{\oslash}$ and, by lemma 4, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}^{\oslash}$.
Conversely, let $M \in \mathcal{M}^{\oslash}$ such that $M \rhd_\beta^* \lambda y^{\oslash}.y^{\oslash} y^{\oslash}$. Let $\mathcal{I}$ be an β-interpretation and $N \in \mathcal{I}(a \sqcap (a \to b)) = \mathcal{I}(a) \cap (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(b))$. Since $\mathcal{I}(b)$ is β-saturated,

$NN \in \mathcal{I}(b)$ and $MN \rhd_\beta^* NN$, we have $MN \in \mathcal{I}(b)$ and hence $M \in \mathcal{I}(a \sqcap (a \to b)) \rightsquigarrow \mathcal{I}(b)$. Therefore, $M \in [D]_\beta$.

4. Let $f, y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathcal{M}^\oslash \ / \ M \rhd_\beta^* (f^\oslash)^n(x^\oslash N_1...N_k)$ or $M \rhd_\beta^* (f^\oslash)^n y^\oslash$ where $k, n \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{X} \subseteq \mathcal{M}^\oslash$. Let $\mathcal{I}$ be an β-interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat_0]_\beta$, then M is closed and $M \in (\mathcal{X} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X} \rightsquigarrow \mathcal{X})$. We have $f^\oslash \in \mathcal{X} \rightsquigarrow \mathcal{X}$, $y^\oslash \in \mathcal{X}$ and $\diamond\{M, f^\oslash, y^\oslash\}$ then $Mf^\oslash y^\oslash \in \mathcal{X}$ and $Mf^\oslash y^\oslash \rhd_\beta^* (f^\oslash)^n(x^\oslash N_1...N_k)$ or $Mf^\oslash y^\oslash \rhd_\beta^* (f^\oslash)^n y^\oslash$ where $n \geq 0$ and $x \in \mathcal{V}_1$. Since M is closed and $\{x^\oslash\} \cap \{y^\oslash, f^\oslash\} = \emptyset$, by lemma 4.2, $Mf^\oslash y^\oslash \rhd_\beta^* (f^\oslash)^n y^\oslash$ where $n \geq 1$. Hence, by lemma 47.4, $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ where $n \geq 1$. Moreover, by lemma 4, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}^\oslash$.
   Conversely, let $M \in \mathcal{M}^\oslash$ such that $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ where $n \geq 1$. Let $\mathcal{I}$ be an β-interpretation, $N \in \mathcal{I}(a \to a) = \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)$. We show, by induction on $m \geq 0$, that $(N)^m N' \in \mathcal{I}(a)$. Since $MNN' \rhd_\beta^* (N)^m N'$ where $m \geq 0$ and $(N)^m N' \in \mathcal{I}(a)$ which is β-saturated, then $MNN' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat_0]_\beta$.
5. By lemma 33, $[Nat_1] = [eNat_0] = [Nat_0]^+$. Let $\mathcal{I}$ be an β-interpretation. By lemma 33, $\mathcal{I}(e_1(a \to a) \to (e_1 a \to e_1 a)) = \mathcal{I}((a \to a) \to (a \to a))^{+1}$ and hence $[Nat_1'] = [Nat_0]^{+1}$. By 4., $[Nat_1] = [Nat_1'] = [Nat_0]^{+1} = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd_\beta^* \lambda f^{(1)}.f^{(1)}$ or $M \rhd_\beta^* \lambda f^{(1)}.\lambda y^{(1)}.(f^{(1)})^n y^{(1)}$ where $n \geq 1\}$.
6. Let $f, y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathcal{M}^\oslash \ / \ M \rhd_\beta^* x^\oslash P_1...P_l$ or $M \rhd_\beta^* f^\oslash(x^\oslash Q_1...Q_n)$ or $M \rhd_\beta^* y^\oslash$ or $M \rhd_\beta^* f^\oslash y^{(1)}$ where $l, n \geq 0$ and $\mathrm{d}(Q_i) \succeq (1)\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{X} \subseteq \mathcal{M}^\oslash$. Let $\mathcal{I}$ be an β-interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat_0']_\beta$, then M is closed and $M \in (\mathcal{X}^{+1} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X}^{+1} \rightsquigarrow \mathcal{X})$. Let $N \in \mathcal{X}^{+1}$ such that $N \diamond f^\oslash$. We have $N \rhd_\beta^* x^{(1)} P_1^{+1}...P_k^{+1}$ or $N \rhd_\beta^* y^{(1)}$, then $f^\oslash N \rhd_\beta^* f^\oslash(\varkappa^{(1)} P_1^{+1}...P_k^{+1}) \in \mathcal{X}$ or $N \rhd_\beta^* f^\oslash y^{(1)} \in \mathcal{X}$, thus $f^\oslash \in \mathcal{X}^{+1} \rightsquigarrow \mathcal{X}$. We have $f^\oslash \in \mathcal{X}^{+1} \rightsquigarrow \mathcal{X}$, $y^{(1)} \in \mathcal{X}^{+1}$ and $\diamond\{M, f^\oslash, y^{(1)}\}$, then $Mf^\oslash y^{(1)} \in \mathcal{X}$. Since M is closed and $\{x^\oslash, x^{(1)}\} \cap \{y^{(1)}, f^\oslash\} = \emptyset$, by lemma 4.2, $Mf^\oslash y^{(1)} \rhd_\beta^* f^\oslash y^{(1)}$. Hence, by lemma 47.4, $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}$. Moreover, by lemma 4, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}^\oslash$.
   Conversely, let $M \in \mathcal{M}^\oslash$ and $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}$. Let $\mathcal{I}$ be an β-interpretation, $N \in \mathcal{I}(e_1 a \to a) = \mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)^{+1}$ where $\diamond\{M, N, N'\}$. Since $MNN' \rhd_\beta^* NN'$, $NN' \in \mathcal{I}(a)$ and $\mathcal{I}(a)$ is β-saturated, then $MNN' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat_0']$.

$\square$