A completeness result for a realisability semantics for an intersection type system

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Abstract

In this paper we consider a type system with a universal type ω where any term (whether open or closed, β-normalising or not) has type ω. We provide this type system with a realisability semantics where an atomic type is interpreted as the set of λ-terms saturated by a certain relation. The variation of the saturation relation gives a number of interpretations to each type. We show the soundness and completeness of our semantics and that for different notions of saturation (based on weak head reduction and normal β-reduction) we obtain the same interpretation for types. Since the presence of ω prevents typability and realisability from coinciding and creates extra difficulties in characterizing the interpretation of a type, we define a class $\mathbb{U}^+$ of the so-called positive types (where ω can only occur at specific positions). We show that if a term inhabits a positive type, then this term is β-normalisable and reduces to a closed term. In other words, positive types can be used to represent abstract data types. The completeness theorem for $\mathbb{U}^+$ becomes interesting indeed since it establishes a perfect equivalence between typable terms and terms that inhabit a type. In other words, typability and realisability coincide on $\mathbb{U}^+$. We give a number of examples to explain the intuition behind the definition of $\mathbb{U}^+$ and to show that this class cannot be extended while keeping its desired properties.

1 Introduction

The ground work for intersection types and related notions was developed in the seventies [5, 6, 18] and have since proved to be a valuable tool in the theoretical studies and applications of the lambda calculus. Intersection types incorporate type polymorphism in a finitary way (where the usage of types is listed rather than quantified over). Since the late seventies, numerous intersection type systems have been developed or used for a multitude of purposes (the list is huge; for a very brief list we simply refer the reader to the recent articles [1, 4] and the references there, for a longer list we refer the reader to the bibliography of intersection types and related systems available (while that URL address is active) at `http://www.macs.hw.ac.uk/~jbw/itrs/bibliography.html`). In this paper, we are interested in the interpretation of an intersection type. We study this interpretation in the context of the so-called realisability semantics.

The idea of realisability semantics is to associate to each type a set of terms which realise this type. Under this semantics, an atomic type is interpreted as the set of

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λ-terms saturated by a certain relation. Then, arrow and intersection types receive their intuitive interpretation of functional space and set intersection. For example, a term which realises the type $\mathbb{N} \to \mathbb{N}$ is a function from $\mathbb{N}$ to $\mathbb{N}$. Realisability semantics has been a powerful method for establishing the strong normalisation of type systems à la Tait and Girard. The realisability of a type system enables one to also show the soundness of the system in the sense that the interpretation of a type contains all the terms that have this type. Soundness has been an important method for characterising the algorithmic behaviour of typed terms through their types as has been illuminative in the work of Krivine.

It is also interesting to find the class of types for which the converse of soundness holds. I.e., to find the types A for which the realisability interpretation contains exactly (in a certain sense) the terms typable by A. This property is called completeness and has not yet been studied for every type system.

In addition to the questions of soundness and completeness for a realisability semantics, one is interested in the additional three questions:

1. Can different interpretations of a type given by different saturation relations be compared?

2. For a particular saturation relation, what are the types uniquely realised by the λ-terms which are typable by these types?

3. Is there a class of types for which typability and realisability coincide?

In this paper we establish the soundness and completeness as well as give answers to questions 1, 2 and 3 for a strict non linear intersection type system with a universal type. We show that for different notions of saturation (based on weak head reduction and normal β-reduction) we obtain the same interpretation for types answering question 1 partially. Questions 2 and 3 are affected by the presence of ω which prevents typability and realisability from coinciding and creates extra difficulties in characterizing the interpretation of a type. We define a class $\mathbb{U}^+$ of the so-called positive types (where ω can only occur at specific positions). We show that if a term inhabits a positive type, then this term is β-normalisable and reduces to a closed term. In other words, positive types can be used to represent abstract data types. This result answers question 2 and depends on the full power of soundness. The completeness theorem for $\mathbb{U}^+$ becomes interesting indeed since it establishes a perfect equivalence between typable terms and terms that inhabit a type. In other words, typability and realisability coincide on $\mathbb{U}^+$ answering question 3. We give a number of examples to explain the intuition behind the definition of $\mathbb{U}^+$ and to show that this class cannot be extended while keeping its desired properties.

Hindley [12, 13, 14] was the first to study the completeness of a simple type system and he showed that all the types of that system have the completeness property. Then, he generalised his completeness proof for an intersection type system [11]. Using his completeness theorem for the realisability semantics based on the sets of λ-terms saturated by $\beta\eta$-equivalence, Hindley has shown that simple types have property 2 above. However, his completeness theorem for intersection types does not allow him to establish property 2 for the intersection type system. Moreover, Hindley's completeness theorems were established with the sets of λ-terms saturated by $\beta\eta$-equivalence, and hence they don't permit a comparison between the different possible interpretations. In our method, saturation is not by $\beta\eta$-equivalence. Rather, it is by the weaker requirement of weak head normal forms. Hence, all of Hindley's saturated models are also saturated in our framework and moreover, there are saturated models based on weak head normal form which cannot be models in Hindley's framework.

[16] has established completeness for a class of types in Girard's system F (also independently discovered by Reynolds as the second order typed λ-calculus) known

as the strictly positive types. [9, 10] generalised the result of [16] for the larger class which includes all the positive types and also for second order functional arithmetic. [7] established recently by a different method using Kripke models, the completeness for the simply typed λ-calculus. Finally [17] introduced a realizability semantics for the simply typed $\lambda\mu$-calculus and proved a completeness result.

The paper is structured as follows: In section 2, we introduce the intersection type system that will be studied in this paper. In section 3 we study both the subject reduction and subject expansion properties for β. In section 4 we establish the soundness and completeness of the realisability semantics based on two notions of saturated sets (one using weak head reduction and the other using β-reduction). In section 5 we show that the meaning of a type does not depend on the chosen notion of saturation (based on either weak head reduction or β-reduction). We also define a subset of types which we show to satisfy the (weak) normalisation property and for which typability and realisability coincide.

2 The typing system

A number of intersection type systems have been given in the literature (for a very brief list see [1, 4] and the references there; for a longer list (and while that URL address is active) see `http://www.macs.hw.ac.uk/~jbw/itrs/bibliography.html`). In this paper we introduce an interesection type system due to J.B. Wells and inspired by his work with Sébastien Carlier on expansion [4]. We follow [4] and write the type judgements $\Gamma \vdash M : U$ as $M : \langle \Gamma \vdash U \rangle$. There are many reasons why this latter notation is to be prefered over the former (see [4]). In particular, this typing notation allowed J.B. Wells in [20] to give a very simple yet general definition of principal typings.

Before presenting the type system, we give a number of its characteristics:

- The type system is *relevant*: this means that the type environments contain all and only the necessary assumptions as is shown in lemma 7.1.
- The type system is *strict* and *non-linear*. Following the terminology of [19] (who advocated the use of of linear systems of intersection types only with strict intersection types), types are strict if ω and $\sqcap$ do not occur immediately to the right of arrows. Our type system is non-linear since $\sqcap$ is idempotant. We guarantee strictness by using two sets of types $\mathbb{T}$ and $\mathbb{U}$ such that $\mathbb{T} \subset \mathbb{U}$ and $\mathbb{T}$ is only formed by either basic types or using the arrow constructor (without permitting ω and $\sqcap$ to occur immediately to the right of arrows). This means that one does not need to state laws relating $A \to (B_1 \sqcap B_2)$ to $(A \to B_1) \sqcap (A \to B_2)$, yet one can still establish a number of type inclusion properties as is shown in lemma 5.

Definition 1 *1. Let $\mathcal{V}$ be a denumerably infinite set of variables. The set of terms $\mathcal{M}$, of the λ-calculus is defined as usual by the following grammar:*

$$\mathcal{M} ::= \mathcal{V} \mid (\lambda \mathcal{V}.\mathcal{M}) \mid (\mathcal{M}\mathcal{M})$$

We let x, y, z, etc. range over $\mathcal{V}$ and $M, N, P, Q, M_1, M_2, \ldots$ range over $\mathcal{M}$. We assume the the usual definition of subterms and the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $M\ N_1...N_n$ instead of $(...(M\ N_1)\ N_2...N_{n-1})\ N_n$.

We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $FV(M)$ for the set of the free variables of term M.

2. *We define as usual the substitution* $M[x := N]$ *of the term* N *for all free occurrences of* x *in the term* M *and similarly,* $M[(x_i := N_i)_1^n]$*, the simultaneous substitution of* N_i *for all free occurrences of* x_i *in* M *for* $1 \leq i \leq n$*.*

3. *We assume the usual definition of compatibility.*

   - *The weak head reduction* $\rhd_f$ *on* $\mathcal{M}$ *is defined by:* $M \rhd_f N$ *if* $M = (\lambda x.P)Q\ Q_1...Q_n$ *and* $N = P[x := Q]\ Q_1...Q_n$ *where* $n \geq 0$*.*
   - *The reduction relation* $\rhd_\beta$ *on* $\mathcal{M}$ *is defined as the least compatible relation closed under the rule:* $(\lambda x.M)N \rhd_\beta M[x := N]$*.*
   - *For* $r \in \{f, \beta\}$*,* $\rhd_r^*$ *denotes the reflexive transitive closure of* $\rhd_r$*.*
   - $\simeq_\beta$ *denotes the equivalence relation induced by* $\rhd_\beta^*$*.*

The next theorem is standard and is needed for the rest of the paper.

Theorem 2 1. *Let* $r \in \{f, \beta\}$*. If* $M \rhd_r^* N$*, then* $FV(N) \subseteq FV(M)$*.*

2. *If* $M \rhd_f^* N$*, then, for all* $P \in \mathcal{M}$*,* $MP \rhd_f^* NP$*.*

3. *If* $M \rhd_\beta^* M_1$ *and* $M \rhd_\beta^* M_2$*, then there is* M' *such that* $M_1 \rhd_\beta^* M'$ *and* $M_2 \rhd_\beta^* M'$*.*

4. $M_1 \simeq_\beta M_2$ *iff there is a term* M *such that* $M_1 \rhd_\beta^* M$ *and* $M_2 \rhd_\beta^* M$*.*

5. *Let* $n \geq 1$ *and assume* $x_i \notin FV(M)$ *for every* $1 \leq i \leq n$*. If* $Mx_1...x_n \rhd_\beta^* x_j\ N_1...N_m$ *for some* $1 \leq j \leq n$ *and* $m \geq 0$*, then for some* $k \geq j$ *and* $s \leq m$*,* $M \rhd_\beta^* \lambda x_1....\lambda x_k.x_j\ M_1...M_s$ *where* $s + n = k + m$*,* $M_i \simeq_\beta N_i$ *for every* $1 \leq i \leq s$ *and* $N_{s+i} \simeq_\beta x_{k+i}$ *for every* $1 \leq i \leq n - k$*.*

6. *If* $M\,x$ *is weakly* β*-normalising and* $x \notin FV(M)$*, then* M *is also weakly* β*-normalising.*

Proof See [3] for more detail. Here, we sketch the proofs. 1 (resp. 2) is by induction on $M \rhd_r^* N$ (resp. $M \rhd_f^* N$). 3 is the Church-Rosser. 4 if) is by definition of $\simeq_\beta$ whereas only if) is by induction on $M_1 \simeq_\beta M_2$ using 3.

5. is as follows: Since $Mx_1...x_n \rhd_\beta^* x_j N_1...N_m$, then by page 23 of [15], $Mx_1...x_n$ is solvable and hence, M is also solvable and its head reduction terminates. Therefore, $M \rhd_\beta^* \lambda x_1...\lambda x_k.zM_1...M_s$ for $s, k \geq 0$. Since $x_j\ N_1...N_m \simeq_\beta (\lambda x_k.zM_1...M_s)x_1...x_n$ then $k \leq n$, $x_j\ N_1...N_m \simeq_\beta zM_1...M_s x_{k+1}...x_n$. Hence, $z = x_j$, $s \leq m$, $j \leq k$ (since $x_j \notin FV(M)$), $m = s+(n-(k+1))+1 = s+n-k$, $M_i \simeq_\beta N_i$ for every $1 \leq i \leq s$ and $N_{s+i} \simeq_\beta x_{k+i}$ for every $1 \leq i \leq n - k$.

6. is by cases:

   – If $M\,x \rhd_\beta^* M'\,x$ where $M'\,x$ is in β-normal form and $M \rhd_\beta^* M'$ then M' is in β-normal form and M is β-normalising.

   – If $M\,x \rhd_\beta^* (\lambda y.N)\,x \rhd_\beta N[y := x] \rhd_\beta^* P$ where P is in β-normal form and $M \rhd_\beta^* \lambda y.N$ then by 1, $x \notin FV(N)$ and so, $M \rhd_\beta^* \lambda y.N = \lambda x.N[y := x] \rhd_\beta^* \lambda x.P$. Since $\lambda x.P$ is in β-normal form, M is β-normalising.

□

Definition 3 1. *Let $\mathcal{A}$ be a denumerably infinite set of atomic types. The types are defined by the following grammars:*

$$\mathbb{T} ::= \mathcal{A} \mid \mathbb{U} \to \mathbb{T}$$

$$\mathbb{U} ::= \omega \mid \mathbb{U} \sqcap \mathbb{U} \mid \mathbb{T}$$

We let $a, b, c, a_1, a_2, \ldots$ range over $\mathcal{A}$, $T, T_1, T_2, T', \ldots$ range over $\mathbb{T}$ and U, V, W, U_1, V_1, $U', \ldots$ range over $\mathbb{U}$.

We quotient types by taking $\sqcap$ to be commutative (i.e. $U_1 \sqcap U_2 = U_2 \sqcap U_1$), associative (i.e. $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$), idempotent (i.e. $U \sqcap U = U$) and to have ω as neutral (i.e. $\omega \sqcap U = U$).
We denote $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ by $\sqcap_{i=n}^{m} U_i$ (when $n \leq m$).

2. *A type environment is a set $\{x_i : U_i \ / \ 1 \leq i \leq n, n \geq 0,$ and $\forall 1 \leq i \leq n,\ x_i \in \mathcal{V},\ U_i \in \mathbb{U}$ and $\forall 1 \leq i, j \leq n,$ if $i \neq j$ then $x_i \neq x_j\}$. We denote such environment (call it Γ) by $x_1 : U_1, \ldots, x_n : U_n$ or simply by $(x_i : U_i)_n$ and define $dom(\Gamma) = \{x_i \ / \ 1 \leq i \leq n\}$. We use $\Gamma, \Delta, \Gamma_1, \ldots$ to range over environments and write $()$ for the empty environment.*

   If M is a term and $FV(M) = \{x_1, ..., x_n\}$, we denote $env_\omega^M = (x_i : \omega)_n$.

   If $\Gamma = (x_i : U_i)_n$, $x \notin dom(\Gamma)$ and $U \in \mathbb{U}$, we denote $\Gamma, x : U$ the type environment $x_1 : U_1, \ldots, x_n : U_n, x : U$.

   Let $\Gamma_1 = (x_i : U_i)_n, (y_j : V_j)_m$ and $\Gamma_2 = (x_i : U'_i)_n, (z_k : W_k)_l$. We denote $\Gamma_1 \sqcap \Gamma_2$ the type environment $(x_i : U_i \sqcap U'_i)_n, (y_j : V_j)_m, (z_k : W_k)_l$. Note that $dom(\Gamma_1 \sqcap \Gamma_2) = dom(\Gamma_1) \cup dom(\Gamma_2)$ and that $\sqcap$ is commutative, associative and idempotent on environments.

3. *The typing rules are the following:*

$$\frac{}{x : \langle x : T \vdash T \rangle}\ ax$$

$$\frac{}{M : \langle env_\omega^M \vdash \omega \rangle}\ \omega$$

$$\frac{M : \langle \Gamma, x : U \vdash T \rangle}{\lambda x.M : \langle \Gamma \vdash U \to T \rangle}\ \to_i$$

$$\frac{M : \langle \Gamma \vdash T \rangle \quad x \notin dom(\Gamma)}{\lambda x.M : \langle \Gamma \vdash \omega \to T \rangle}\ \to'_i$$

$$\frac{M_1 : \langle \Gamma_1 \vdash U \to T \rangle \qquad M_2 : \langle \Gamma_2 \vdash U \rangle}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}\ \to_e$$

$$\frac{M : \langle \Gamma \vdash U_1 \rangle \qquad M : \langle \Gamma \vdash U_2 \rangle}{M : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}\ \sqcap_i$$

$$\frac{M : \langle \Gamma \vdash U \rangle \qquad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{M : \langle \Gamma' \vdash U' \rangle}\ \sqsubseteq$$

   In the last clause, the binary relation $\sqsubseteq$ is defined by the following rules:

$$\frac{}{\Phi \sqsubseteq \Phi}\ ref$$

$$\frac{\Phi_1 \sqsubseteq \Phi_2 \qquad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ tr$$

$$\frac{}{U_1 \sqcap U_2 \sqsubseteq U_1}\ \sqcap_e$$

$$\frac{U_1 \sqsubseteq V_1 \qquad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ \sqcap$$

$$\frac{U_2 \sqsubseteq U_1 \qquad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ \to$$

$$\frac{U_1 \sqsubseteq U_2 \qquad x \notin dom(\Gamma)}{\Gamma, x : U_1 \sqsubseteq \Gamma, x : U_2}\ \sqsubseteq_c$$

$$\frac{U_1 \sqsubseteq U_2 \qquad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash U_1 \rangle \sqsubseteq \langle \Gamma_2 \vdash U_2 \rangle}\ \sqsubseteq_{\langle\rangle}$$

Throughout, we use $\Phi, \Phi', \Phi_1, \ldots$ to denote $U \in \mathbb{U}$, or environments Γ or typings $\langle \Gamma \vdash U \rangle$. Note that when $\Phi \sqsubseteq \Phi'$, then Φ and Φ' belong to the same set (either $\mathbb{U}$ or environments or typings).

The next lemma gives the shape of a type in $\mathbb{U}$.

Lemma 4 1. *If $U \in \mathbb{U}$, then $U = \omega$ or $U = \sqcap_{i=1}^{n} T_i$ where $n \geq 1$ and $\forall\ 1 \leq i \leq n$, $T_i \in \mathbb{T}$.*

2. *$U \sqsubseteq \omega$.*

3. *If $\omega \sqsubseteq U$, then $U = \omega$.*

Proof

1. By induction on $U \in \mathbb{U}$.

2. By rule $\sqcap_e$, $U = \omega \sqcap U \sqsubseteq \omega$.

3. By induction on the derivation $\omega \sqsubseteq U$.

□

The next lemma studies the relation $\sqsubseteq$ on $\mathbb{U}$.

Lemma 5 *Let $V \neq \omega$.*

1. *If $U \sqsubseteq V$, then $U = \sqcap_{j=1}^{k} T_j$, $V = \sqcap_{i=1}^{p} T'_i$ where $p, k \geq 1$, $\forall 1 \leq j \leq k$, $1 \leq i \leq p$, $T_j, T'_i \in \mathbb{T}$, and $\forall\ 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ such that $T_j \sqsubseteq T'_i$.*

2. *If $U \sqsubseteq V' \sqcap a$, then $U = U' \sqcap a$ and $U' \sqsubseteq V'$.*

3. *Let $p, k \geq 1$. If $\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq \sqcap_{i=1}^{p}(U'_i \to T'_i)$, then $\forall 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ such that $U'_i \sqsubseteq U_j$ and $T_j \sqsubseteq T'_i$.*

4. *If $U \to T \sqsubseteq V$, then $V = \sqcap_{i=1}^{p}(U_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $U_i \sqsubseteq U$ and $T \sqsubseteq T_i$.*

5. *If $\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq V$ where $k \geq 1$, then $V = \sqcap_{i=1}^{p}(U'_i \to T'_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ $U'_i \sqsubseteq U_j$ and $T_j \sqsubseteq T'_i$.*

Proof

1. By induction on the derivation $U \sqsubseteq V$ using lemma 4.1.

2. By induction on $U \sqsubseteq V' \sqcap a$.

3. By induction on $\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq \sqcap_{i=1}^{p}(U'_i \to T'_i)$. We only do the *tr* case. If $\dfrac{\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq V \quad V \sqsubseteq \sqcap_{i=1}^{p}(U'_i \to T'_i)}{\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq \sqcap_{i=1}^{p}(U'_i \to T'_i)}$, then, by 1, $V = \sqcap_{l=1}^{q} T''_l$ where $q \geq 1$ and $\forall 1 \leq l \leq q$, $\exists 1 \leq j \leq k$, such that $U_j \to T_j \sqsubseteq T''_l$. If $T''_l = a$, then, by 2, $U_j \to T_j = U' \sqcap a$. Absurd. Hence, $\forall 1 \leq l \leq q$, $T''_l = V_l \to T'''_l$ and $V = \sqcap_{l=1}^{q}(V_l \to T'''_l)$. Let $1 \leq i \leq p$. By IH, $\exists 1 \leq l \leq q$, $U'_i \sqsubseteq V_l$ and $T'''_l \sqsubseteq T'_i$. Also, by IH, $\exists 1 \leq j \leq k$, $V_l \sqsubseteq U_j$ and $T_j \sqsubseteq T'''_l$. Hence, $\forall 1 \leq i \leq p$, $\exists 1 \leq j \leq k$, such that $U'_i \sqsubseteq U_j$ and $T_j \sqsubseteq T'_i$.

4. By 1, $V = \sqcap_{i=1}^{p} T'_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $U \to T \sqsubseteq T'_i$. If $T'_i = a$, then, by 2, $U \to T = U' \sqcap a$. Absurd. Hence, $T'_i = U_i \to T_i$. Hence, by 3, $\forall 1 \leq i \leq p$, $U_i \sqsubseteq U$ and $T \sqsubseteq T_i$.

5. Since $V \neq \omega$, then, by lemma 4.1, $V = \sqcap_{i=1}^{p} T'_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $T'_i \in \mathbb{T}$. Let $1 \leq i \leq p$. By 1, $\exists 1 \leq j_i \leq k$ such that $U_{j_i} \to T_{j_i} \sqsubseteq T'_i$. By 4, and since $T'_i \in \mathbb{T}$, $T'_i = U'_i \to T''_i$ where $U'_i \sqsubseteq U_{j_i}$ and $T_{j_i} \sqsubseteq T''_i$. Hence, $V = \sqcap_{i=1}^{p}(U'_i \to T''_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ $U'_i \sqsubseteq U_{j_i}$ and $T_{j_i} \sqsubseteq T''_i$.

□

The next lemma studies the relation $\sqsubseteq$ on environments and typings.

Lemma 6 *1. If $\Gamma \sqsubseteq \Gamma'$, then $dom(\Gamma) = dom(\Gamma')$.*

2. If $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $x \notin dom(\Gamma)$, then $\Gamma, x : U \sqsubseteq \Gamma', x : U'$.

3. $\Gamma \sqsubseteq \Gamma'$ iff $\Gamma = (x_i : U_i)_n$, $\Gamma' = (x_i : U'_i)_n$ and for every $1 \leq i \leq n$, $U_i \sqsubseteq U'_i$.

4. If $dom(\Gamma) = FV(M)$, then $\Gamma \sqsubseteq env^M_\omega$

5. If $env^M_\omega \sqsubseteq \Gamma$, then $\Gamma = env^M_\omega$.

6. $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$ iff $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$.

7. If $\Gamma \sqsubseteq \Gamma'$ and $\Delta \sqsubseteq \Delta'$, then $\Gamma \sqcap \Delta \sqsubseteq \Gamma' \sqcap \Delta'$.

Proof

1. By induction on the derivation $\Gamma \sqsubseteq \Gamma'$.

2. First show, by induction on the derivation $\Gamma \sqsubseteq \Gamma'$ (using 1), that if $\Gamma \sqsubseteq \Gamma'$, $V \in \mathbb{U}$ and $y \notin dom(\Gamma)$ then $\Gamma, y : V \sqsubseteq \Gamma', y : V$. Then use tr.

3. Only if) By 1, $\Gamma = (x_i : U_i)_n$ and $\Gamma' = (x_i : U_i)_n$. The proof is by induction on the derivation $(x_i : U_i)_n \sqsubseteq (x_i : U'_i)_n$. If) By induction on n using 2.

4. Let $FV(M) = \{x_1, \ldots, x_n\}$ and $\Gamma = (x_i : U_i)_n$. By definition, $env^M_\omega = (x_i, \omega)_n$. Hence, by lemma 4.2 and 3, $\Gamma \sqsubseteq env^M_\omega$.

5. Let $FV(M) = \{x_1, \ldots, x_n\}$. By definition, $env^M_\omega = (x_i, \omega)_n$. By 3, $\Gamma = (x_i : U_i)_n$ and $\forall 1 \leq i \leq n$, $\omega \sqsubseteq U_i$. Hence by lemma 4.3, $\forall 1 \leq i \leq n$, $\omega = U_i$.

6. Only if) By induction on the derivation $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$. If) By $\sqsubseteq_{\langle\rangle}$.

7. This is a corollary of 3.

□

The next lemma shows that we do not allow weakening in our type system.

Lemma 7 *1. If $M : \langle \Gamma \vdash U \rangle$, then $dom(\Gamma) = FV(M)$.*

2. For every Γ and M such that $dom(\Gamma) = FV(M)$, we have $M : \langle \Gamma \vdash \omega \rangle$.

Proof

1. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$.

2. By ω, $M : \langle env_\omega^M \vdash \omega \rangle$. By lemma 6.4, $\Gamma \sqsubseteq env_\omega^M$. Hence, by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma \vdash \omega \rangle$.

□

Finally, it may come as a surprise that the rule *ax* uses types in $\mathbb{T}$ instead of $\mathbb{U}$ and that in the rule $\sqcap$ we take the same environment. The lemma below shows that this is not restrictive.

Lemma 8 *1. The rule* $\dfrac{M : \langle \Gamma_1 \vdash U_1 \rangle \qquad M : \langle \Gamma_2 \vdash U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2 \rangle}$ $\sqcap_i'$ *is derivable.*

2. The rule $\dfrac{}{x : \langle (x : U) \vdash U \rangle}$ ax' *is derivable.*

Proof

1. Let $M : \langle \Gamma_1 \vdash U_1 \rangle$ and $M : \langle \Gamma_2 \vdash U_2 \rangle$. By lemma 7, $dom(\Gamma_1) = dom(\Gamma_2) = FV(M)$. Let $\Gamma_1 = (x_i : V_i)_n$ and $\Gamma_2 = (x_i : V_i')_n$. Hence, $\Gamma_1 \sqcap \Gamma_2 = (x_i : V_i \sqcap V_i')_n$. By $V_i \sqcap V_i' \sqsubseteq V_i$ and $V_i \sqcap V_i' \sqsubseteq V_i'$ for all $1 \leq i \leq n$. Hence, by lemma 6.3, $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_1$ and $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_2$, and, by rules $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma_1 \sqcap \Gamma_2, U_1 \rangle$ and $M : \langle \Gamma_1 \sqcap \Gamma_2, U_2 \rangle$. Finally, by rule $\sqcap_i$, $M : \langle \Gamma_1 \sqcap \Gamma_2, U_1 \sqcap U_2 \rangle$.

2. By lemma 4.1:

   – Either $U = \omega$, then, by rule ω, we have $x : \langle (x : \omega) \vdash \omega \rangle$.

   – Or $U = \sqcap_{i=1}^k T_i$ where $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}$, then, by rule *ax*, $x : \langle (x : T_i) \vdash T_i \rangle$ and, by $k - 1$ applications of rule $\sqcap_i'$, $x : \langle (x : U) \vdash U \rangle$.

□

3 Subject reduction and expansion properties

In this section we establish the subject reduction and subject expansion properties for β.

3.1 Subject reduction for β

We start with a form of the generation lemma.

Lemma 9 (Generation) *1. If $x : \langle \Gamma \vdash U \rangle$, then $\Gamma = (x : V)$ and $V \sqsubseteq U$.*

2. If $M\ x : \langle \Gamma, x : U \vdash V \rangle$ and $x \notin FV(M)$, then $V = \omega$ or $V = \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash U \to T_i \rangle$.

3. If $\lambda x.M : \langle \Gamma \vdash U \rangle$ and $x \in FV(M)$, then $U = \omega$ or $U = \sqcap_{i=1}^k (V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma, x : V_i \vdash T_i \rangle$.

4. If $\lambda x.M : \langle \Gamma \vdash U \rangle$ and $x \notin FV(M)$, then $U = \omega$ or $U = \sqcap_{i=1}^k (V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash T_i \rangle$.

Proof 1. By induction on the derivation $x : \langle \Gamma \vdash U \rangle$. We have four cases:

- If $\dfrac{}{x : \langle (x : T) \vdash T \rangle}$, nothing to prove.
- If $\dfrac{}{x : \langle (x : \omega) \vdash \omega \rangle}$, nothing to prove.
- Let $\dfrac{x : \langle \Gamma \vdash U_1 \rangle \quad x : \langle \Gamma \vdash U_2 \rangle}{x : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$. By IH, $\Gamma = (x : V)$, $V \sqsubseteq U_1$ and $V \sqsubseteq U_2$, then, by rule $\sqcap$, $V \sqsubseteq U_1 \sqcap U_2$.
- Let $\dfrac{x : \langle \Gamma' \vdash U' \rangle \quad \langle \Gamma' \vdash U' \rangle \sqsubseteq \langle \Gamma \vdash U \rangle}{x : \langle \Gamma \vdash U \rangle}$. By lemma 6.6, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$ and, by IH, $\Gamma' = (x : V')$ and $V' \sqsubseteq U'$. Then, by lemma 6.3, $\Gamma = (x : V)$, $V \sqsubseteq V'$ and, by rule tr, $V \sqsubseteq U$.

2. By induction on the derivation $M\,x : \langle \Gamma, x : U \vdash V \rangle$. We have four cases:

- If $\dfrac{}{M\ x : \langle env_{\omega}^{M\ x} \vdash \omega \rangle}$, nothing to prove.
- Let $\dfrac{M : \langle \Gamma \vdash U \to T \rangle \quad x : \langle (x : V) \vdash U \rangle}{M\,x : \langle \Gamma, x : V \vdash T \rangle}$ (where, by 1. $V \sqsubseteq U$).

  Since $U \to T \sqsubseteq V \to T$, we have $M : \langle \Gamma \vdash V \to T \rangle$.
- Let $\dfrac{M\ x : \langle \Gamma, x : U \vdash U_1 \rangle \quad M\ x : \langle \Gamma, x : U \vdash U_2 \rangle}{M\ x : \langle \Gamma, x : U \vdash U_1 \sqcap U_2 \rangle}$. By IH, we have four cases:
  - If $U_1 = U_2 = \omega$, then $U_1 \sqcap U_2 = \omega$.
  - If $U_1 = \omega$, $U_2 = \sqcap_{i=1}^{k} T_i$, $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash U \to T_i \rangle$, then $U_1 \sqcap U_2 = U_2$ (ω is a neutral element).
  - If $U_2 = \omega$, $U_1 = \sqcap_{i=1}^{k} T_i$, $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash U \to T_i \rangle$, then $U_1 \sqcap U_2 = U_1$ (ω is a neutral element).
  - If $U_1 = \sqcap_{i=1}^{k} T_i$ and $U_2 = \sqcap_{i=1}^{l} T_{k+i}$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{k+l} T_i$) , where $k, l \geq 1$ and $\forall 1 \leq i \leq k + l$, $M : \langle \Gamma \vdash U \to T_i \rangle$.
- Let $\dfrac{M\ x : \langle \Gamma', x : U' \vdash V' \rangle \quad \langle \Gamma', x : U' \vdash V' \rangle \sqsubseteq \langle \Gamma, x : U \vdash V \rangle}{M\ x : \langle \Gamma, x : U \vdash V \rangle}$ (by lemma 6).

  By lemma 6, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. By IH, we have two cases:
  - If $V' = \omega$, then, by lemma 4.3, $V = \omega$.
  - If $V' = \sqcap_{i=1}^{k} T'_i$, where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash U \to T'_i \rangle$. By lemma 5.1, $V = \omega$ (nothing to prove) or $V = \sqcap_{i=1}^{p} T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $T'_{j_i} \sqsubseteq T_i$. Since, by lemma 6.6, $\langle \Gamma' \vdash U' \to T'_{j_i} \rangle \sqsubseteq \langle \Gamma \vdash U \to T_i \rangle$ for any $1 \leq i \leq p$, then $\forall 1 \leq i \leq p$, $M : \langle \Gamma \vdash U \to T_i \rangle$.

3. By induction on the derivation $\lambda x.M : \langle \Gamma \vdash U \rangle$. We have four cases:

- If $\dfrac{}{\lambda x.M : \langle env_{\omega}^{\lambda x.M} \vdash \omega \rangle}$, nothing to prove.
- If $\dfrac{M : \langle \Gamma, x : U \vdash T \rangle}{\lambda x.M : \langle \Gamma \vdash U \to T \rangle}$, nothing to prove.
- Let $\dfrac{\lambda x.M : \langle \Gamma \vdash U_1 \rangle \quad \lambda x.M : \langle \Gamma \vdash U_2 \rangle}{\lambda x.M : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$. By IH, we have four cases:
  - If $U_1 = U_2 = \omega$, then $U_1 \sqcap U_2 = \omega$.

– If $U_1 = \omega$, $U_2 = \sqcap_{i=1}^{k}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma_2, x : V_i \vdash T_i \rangle$, then $U_1 \sqcap U_2 = U_2$ (ω is a neutral element).

– If $U_2 = \omega$, $U_1 = \sqcap_{i=1}^{k}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma_1, x : V_i \vdash T_i \rangle$, then $U_1 \sqcap U_2 = U_1$ (ω is a neutral element).

– If $U_1 = \sqcap_{i=1}^{k}(V_i \to T_i)$, $U_2 = \sqcap_{i=k+1}^{k+l}(V_i \to T_i)$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{k+l}(V_i \to T_i)$) where $k, l \geq 1$, $\forall 1 \leq i \leq k+l$, $M : \langle \Gamma, x : V_i \vdash T_i \rangle$, we are done.

- Let $\dfrac{\lambda x.M : \langle \Gamma \vdash U \rangle \quad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{\lambda x.M : \langle \Gamma' \vdash U' \rangle}$. By lemma 6.6, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. By IH, we have two cases:
  - If $U = \omega$, then, by lemma 4.3, $U' = \omega$.
  - Assume $U = \sqcap_{i=1}^{k}(V_i \to T_i)$, where $k \geq 1$ and $M : \langle \Gamma, x : V_i \vdash T_i \rangle$ for all $1 \leq i \leq k$. By lemma 4.1:
    * Either $U' = \omega$, and hence nothing to prove.
    * Or, by lemma 5.5, $U' = \sqcap_{i=1}^{p}(V'_i \to T'_i)$, where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. Let $1 \leq i \leq p$. Since, by lemma 6.6, $\langle \Gamma, x : V_{j_i} \vdash T_{j_i} \rangle \sqsubseteq \langle \Gamma', x : V'_i \vdash T'_i \rangle$, then $M : \langle \Gamma', x : V'_i \vdash T'_i \rangle$.

4. Same proof as that of 3. □

Now, we establish the substitution lemma.

Lemma 10 (Substitution) *If $M : \langle \Gamma, x : U \vdash V \rangle$ and $N : \langle \Delta \vdash U \rangle$, then $M[x := N] : \langle \Gamma \sqcap \Delta \vdash V \rangle$.*

Proof By induction on the derivation $M : \langle \Gamma, x : U \vdash V \rangle$.

- If $\dfrac{}{x : \langle (x : T) \vdash T \rangle}$ and $N : \langle \Delta \vdash T \rangle$, then $N = x[x := N] : \langle \Delta \vdash T \rangle$.

- If $\dfrac{}{M : \langle (x_i : \omega)_n, x : \omega \vdash \omega \rangle}$ where $FV(M) = \{x_1, \ldots, x_n, x\}$ and if $N : \langle \Delta \vdash \omega \rangle$, then since $FV(M[x := N]) = \{x_1, \ldots, x_n\} \cup FV(N)$, we have by ω, $M[x := N] : \langle (x_i : \omega)_n \sqcap env_\omega^N \vdash \omega \rangle$. By lemmas 6.4 and 7, $\Delta \sqsubseteq env_\omega^N$ and by lemma 6.7, $(x_i : \omega)_n \sqcap \Delta \sqsubseteq (x_i : \omega)_n \sqcap env_\omega^N$. Hence, by $\sqsubseteq_{\langle\rangle}$, $M[x := N] : \langle (x_i : \omega)_n \sqcap \Delta \vdash \omega \rangle$.

- Let $\dfrac{M : \langle \Gamma, x : U, y : U' \vdash T \rangle}{\lambda y.M : \langle \Gamma, x : U \vdash U' \to T \rangle}$. By IH, $M[x := N] : \langle \Gamma \sqcap \Delta, y : U' \vdash T \rangle$. By rule $\to_i$, $(\lambda y.M)[x := N] = \lambda y.M[x := N] : \langle \Gamma \sqcap \Delta \vdash U' \to T \rangle$.

- Let $\dfrac{M : \langle \Gamma, x : U \vdash T \rangle \quad y \notin dom(\Gamma) \cup \{x\}}{\lambda y.M : \langle \Gamma, x : U \vdash \omega \to T \rangle}$. By IH, $M[x := N] : \langle \Gamma \sqcap \Delta \vdash T \rangle$. By rule $\to'_i$, $(\lambda y.M)[x := N] = \lambda y.M[x := N] : \langle \Gamma \sqcap \Delta \vdash \omega \to T \rangle$.

- Let $\dfrac{M_1 : \langle \Gamma_1, x : U_1 \vdash V \to T \rangle \quad M_2 : \langle \Gamma_2, x : U_2 \vdash V \rangle}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2, x : U_1 \sqcap U_2 \vdash T \rangle}$ where $x \in FV(M_1) \cap FV(M_2)$ and $N : \langle \Delta \vdash U_1 \sqcap U_2 \rangle$. By rules $\sqcap_e$ and $\sqsubseteq$, $N : \langle \Delta \vdash U_1 \rangle$ and $N : \langle \Delta \vdash U_2 \rangle$. Now use IH and rule $\to_e$.
  The cases $x \in FV(M_1) \setminus FV(M_2)$ or $x \in FV(M_2) \setminus FV(M_1)$ are easy.

- If $\dfrac{M : \langle \Gamma, x : U \vdash U_1 \rangle \quad M : \langle \Gamma, x : U \vdash U_2 \rangle}{M : \langle \Gamma, x : U \vdash U_1 \sqcap U_2 \rangle}$ use IH and $\sqcap_i$.

- Let $\dfrac{M : \langle \Gamma', x : U' \vdash V' \rangle \quad \langle \Gamma', x : U' \vdash V' \rangle \sqsubseteq \langle \Gamma, x : U \vdash V \rangle}{M : \langle \Gamma, x : U \vdash V \rangle}$ (by lemma 6).

  By lemma 6, $dom(\Gamma) = dom(\Gamma')$, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. Hence by $\sqsubseteq$, $N : \langle \Delta \vdash U' \rangle$ and, by IH, $M[x := N] : \langle \Gamma' \sqcap \Delta \vdash V' \rangle$. It is easy to show $\Gamma \sqcap \Delta \subseteq \Gamma' \sqcap \Delta$. Hence, $\langle \Gamma' \sqcap \Delta \vdash V' \rangle \sqsubseteq \langle \Gamma \sqcap \Delta \vdash V \rangle$ and by $\sqsubseteq$, $M[x := N] : \langle \Gamma \sqcap \Delta \vdash V \rangle$.

□

Since our system does not allow weakening, we need the next definition (and the related lemma below it) since when a term is reduced, it may lose some of its free variables and hence will need to be typed in a smaller environment.

Definition 11 *If Γ is a type environment and $\mathcal{U} \subseteq dom(\Gamma)$, then we write $\Gamma \upharpoonright_{\mathcal{U}}$ for the restriction of Γ on the variables of $\mathcal{U}$. If $\mathcal{U} = FV(M)$ for a term M, we write $\Gamma \upharpoonright_M$ instead of $\Gamma \upharpoonright_{FV(M)}$.*

Lemma 12 *1. If $FV(N) \subseteq FV(M)$, then $env^M_\omega \upharpoonright_N = env^N_\omega$.*

2. If $FV(M) \subseteq dom(\Gamma_1)$ and $FV(N) \subseteq dom(\Gamma_2)$, then $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN} \sqsubseteq (\Gamma_1 \upharpoonright_M) \sqcap \Gamma_2$.

Proof 1. Easy. 2. First, note that $dom((\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN}) = FV(MN) = FV(M) \cup FV(N) = dom(\Gamma_1 \upharpoonright_M) \cup dom(\Gamma_2) = dom((\Gamma_1 \upharpoonright_M) \sqcap \Gamma_2)$. Now, we show by cases that if $x : U_1 \in (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN}$ and $x : U_2 \in (\Gamma_1 \upharpoonright_M) \sqcap \Gamma_2$ then $U_1 \sqsubseteq U_2$:

- If $x \in FV(M) \cap FV(N)$ then $x : U_1' \in \Gamma_1$, $x : U_1'' \in \Gamma_2$ and $U_1 = U_1' \sqcap U_1'' = U_2$.
- If $x \in FV(M) \setminus FV(N)$ then $x \notin dom(\Gamma_2)$, $x : U_1 \in \Gamma_1$ and $U_1 = U_2$.
- If $x \in FV(N) \setminus FV(M)$ then
  - If $x \in dom(\Gamma_1)$ then $x : U_1' \in \Gamma_1$, $x : U_2 \in \Gamma_2$ and $U_1 = U_1' \sqcap U_2 \sqsubseteq U_2$.
  - If $x \notin dom(\Gamma_1)$ then $x : U_2 \in \Gamma_2$ and $U_1 = U_2$.

□

Now we give the basic block in the subject reduction for β.

Theorem 13 *If $M : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.*

Proof By induction on the derivation $M : \langle \Gamma \vdash U \rangle$. Rule ω follows by theorem 2.1 and lemma 12.1. Rules $\to_i$, $\to_i'$, $\sqcap_i$ and $\sqsubseteq$ are by IH. We do $\to_e$

Let $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T \rangle \quad Q : \langle \Gamma_2 \vdash U \rangle}{M_1\ Q : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}$.

- If $M = M_1 Q \rhd_\beta PQ = N$ where $M_1 \rhd_\beta P$ then by IH, $P : \langle \Gamma_1 \upharpoonright_P \vdash U \to T \rangle$. By $\to_e$, $P\ Q : \langle (\Gamma_1 \upharpoonright_P) \sqcap \Gamma_2 \vdash T \rangle$. By lemma 12.2, $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{PQ} \sqsubseteq (\Gamma_1 \upharpoonright_P) \sqcap \Gamma_2$. Finally, by $\sqsubseteq_{\langle\rangle}$, $P\ Q : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{PQ} \vdash T \rangle$.
- The case $M = M_1 Q \rhd_\beta M_1 P = N$ where $Q \rhd_\beta P$ is similar to the above.
- Assume $M_1 = \lambda x.P$ and $M_1\ M_2 = (\lambda x.P) M_2 \rhd_\beta P[x := M_2] = N$. Since $\lambda x.P : \langle \Gamma_1 \vdash U \to T \rangle$, we have two cases:
  - If $x \in FV(P)$, then, by lemma 9.3, $P : \langle \Gamma_1, x : U \vdash T \rangle$. By lemma 10, $P[x := M_2] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle$. Moreover, $FV(M_1 M_2) = FV(N) = dom(\Gamma_1 \sqcap \Gamma_2)$. Hence $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_N = \Gamma_1 \sqcap \Gamma_2$ and $N : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_N \vdash T \rangle$.
  - If $x \notin FV(P)$, then, by lemma 9.4, $P : \langle \Gamma_1 \vdash T \rangle$. Moreover, by lemma 7.1, $FV(P) = FV(M_1) = dom(\Gamma_1)$. Hence, $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_P = \Gamma_1 \upharpoonright_P = \Gamma_1$ and $P[x := M_2] = P : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_P \vdash T \rangle$.

□

Corollary 14 (Subject reduction for β)
If $M : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$.
Proof By induction on the length of the derivation $M \rhd_\beta^* N$ using theorem 13. $\square$

Remark 15 *Note that using lemma 9.(2 and 3), we can also prove the subject reduction property for η-reduction.*

3.2 Subject expansion for β

Subject reduction for β was shown using generation, substitution and environment restriction. Subject expansion for β needs something like the converse of the substitution lemma and environment enlargement.

The next lemma can be seen as the converse of the substitution lemma.

Lemma 16 *If $M[x := N] : \langle \Gamma \vdash U \rangle$, $x \in FV(M)$ and $x \notin FV(N)$, then $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that:*

- $M : \langle \Gamma_1, x : V \vdash U \rangle$
- $N : \langle \Gamma_2 \vdash V \rangle$
- $\Gamma \sqsubseteq \Gamma_1 \sqcap \Gamma_2$

Proof By induction on the derivation $M[x := N] : \langle \Gamma \vdash U \rangle$.

If $M = x$, then $x : \langle x : U \vdash U \rangle$, $N : \langle \Gamma \vdash U \rangle$ and $\Gamma = \Gamma \sqcap ()$. Then we can assume that $M \neq x$.

- The last typing rule can not be ax.
- Let $\dfrac{M[x := N] : \langle \Gamma, y : W \vdash T \rangle}{\lambda y.M[x := N] : \langle \Gamma \vdash W \to T \rangle}$ where $y \notin FV(N)$.

  By IH, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x : V \vdash T \rangle$, $N : \langle \Gamma_2 \vdash V \rangle$ and $\Gamma, y : W \sqsubseteq \Gamma_1 \sqcap \Gamma_2$. Since $y \in FV(M)$ and $y \notin FV(N)$, by lemma 6.3, $\Gamma_1 = \Delta_1, y : W'$ and $W \sqsubseteq W'$. Hence $M : \langle \Delta_1, y : W', x : V \vdash T \rangle$. By rule $\to_i$, $\lambda y.M : \langle \Delta_1, x : V \vdash W' \to T \rangle$ and since $W' \to T \sqsubseteq W \to T$, then by rule $\sqsubseteq$, $\lambda y.M : \langle \Delta_1, x : V \vdash W \to T \rangle$. Finally by lemma 6.3, $\Gamma \sqsubseteq \Delta_1 \sqcap \Gamma_2$.

- Let $\dfrac{M[x := N] : \langle \Gamma \vdash T \rangle \quad y \notin dom(\Gamma)}{\lambda y.M[x := N] : \langle \Gamma \vdash \omega \to T \rangle}$.

  By IH, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x : V \vdash T \rangle$, $N : \langle \Gamma_2 \vdash V \rangle$ and $\Gamma \sqsubseteq \Gamma_1 \sqcap \Gamma_2$. Since $y \neq x$, $\lambda y.M : \langle \Gamma_1, x : V \vdash \omega \to T \rangle$.

- Let $\dfrac{M_1[x := N] : \langle \Gamma_1 \vdash W \to T \rangle \quad M_2[x := N] : \langle \Gamma_2 \vdash W \rangle}{M_1[x := N]\ M_2[x := N] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}$

  where $M = M_1 M_2$ and $x \in FV(M_1) \cap FV(M_2)$.

  By IH, $\exists\ V_1, V_2$ types and $\exists\ \Delta_1, \Delta_2; \nabla_1, \nabla_2$ type environments such that $M_1 : \langle \Delta_1, x : V_1 \vdash W \to T \rangle$, $M_2 : \langle \nabla_1, x : V_2 \vdash W \rangle$, $N : \langle \Delta_2 \vdash V_1 \rangle$, $N : \langle \nabla_2 \vdash V_2 \rangle$, $\Gamma_1 \sqsubseteq \Delta_1 \sqcap \Delta_2$ and $\Gamma_2 \sqsubseteq \nabla_1 \sqcap \nabla_2$. Then, by rules $\sqcap'$ and $\to_e$, $M_1 M_2 : \langle \Delta_1 \sqcap \nabla_1, x : V_1 \sqcap V_2 \vdash T \rangle$ and $N : \langle \Delta_2 \sqcap \nabla_2 \vdash V_1 \sqcap V_2 \rangle$. Finally, by lemma 6.7, $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq (\Delta_1 \sqcap \Delta_2) \sqcap (\nabla_1 \sqcap \nabla_2)$.

  The cases $x \in FV(M_1) \setminus FV(M_2)$ or $x \in FV(M_2) \setminus FV(M_1)$ are easy.

- Let $\dfrac{M[x := N] : \langle \Gamma \vdash U_1 \rangle \quad M[x := N] : \langle \Gamma \vdash U_2 \rangle}{M[x := N] : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$.

  By IH, $\exists\ V_1, V_2$ types and $\exists\ \Gamma_1, \Gamma_2; \Delta_1, \Delta_2$ type environments such that $M : \langle \Gamma_1, x : V_1 \vdash U_1 \rangle$, $M : \langle \Delta_1, x : V_2 \vdash U_2 \rangle$, $N : \langle \Gamma_2 \vdash V_1 \rangle$, $N : \langle \Delta_2 \vdash V_2 \rangle$, $\Gamma \sqsubseteq \Gamma_1 \sqcap \Gamma_2$ and $\Gamma \sqsubseteq \Delta_1 \sqcap \Delta_2$. Then, by rule $\sqcap'$, $M : \langle \Gamma_1 \sqcap \Delta_1, x : V_1 \sqcap V_2 \vdash U_1 \sqcap U_2 \rangle$ and $N : \langle \Gamma_2 \sqcap \Delta_2 \vdash V_1 \sqcap V_2 \rangle$. Finally, by lemma 6.7, $\Gamma \sqsubseteq (\Gamma_1 \sqcap \Gamma_2) \sqcap (\Delta_1 \sqcap \Delta_2)$.

- Let $\dfrac{M[x := N] : \langle \Gamma' \vdash U' \rangle \quad \langle \Gamma' \vdash U' \rangle \sqsubseteq \langle \Gamma \vdash U \rangle}{M : \langle \Gamma \vdash U \rangle}$.

  By lemma 6.3, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$. By IH, $\exists\ V$ type and $\exists\ \Gamma'_1, \Gamma'_2$ type environments such that $M : \langle \Gamma_1, x : V \vdash U' \rangle$, $N : \langle \Gamma_2 \vdash V \rangle$ and $\Gamma' \sqsubseteq \Gamma_1 \sqcap \Gamma_2$. Then by rules $\sqsubseteq_{\langle\rangle}$, $\sqsubseteq$ and *tr*, $M : \langle \Gamma_1, x : V \vdash U \rangle$ and $\Gamma \sqsubseteq \Gamma_1 \sqcap \Gamma_2$.

□

Since more free variables might appear in the β-expansion of a term, the next definition gives a possible enlargement of an environment.

Definition 17 *Let $m \geq n$, $\Gamma = (x_i : U_i)_n$ and $\mathcal{U} = \{x_1, ..., x_m\}$. We write $\Gamma\uparrow^{\mathcal{U}}$ for $x_1 : U_1, ..., x_n : U_n, x_{n+1} : \omega, ..., x_m : \omega$. If $dom(\Gamma) \subseteq FV(M)$, we write $\Gamma\uparrow^{M}$ instead of $\Gamma\uparrow^{FV(M)}$.*

The next lemma is basic for the proof of subject expansion for β.

Lemma 18 *If $M[x := N] : \langle \Gamma \vdash U \rangle$, $x \notin FV(N)$ and $\mathcal{U} = FV((\lambda x.M)N)$, then $(\lambda x.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U \rangle$.*

Proof We have three cases:

- If $U = \omega$: By lemma 7.2, we have $(\lambda x.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash \omega \rangle$.
- If $U \in \mathbb{T}$: We have two cases:
  - If $x \in FV(M)$, then, by lemma 16, $\exists\ V$ type and $\exists\ \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x : V \vdash U \rangle$, $N : \langle \Gamma_2 \vdash V \rangle$ and $\Gamma \sqsubseteq \Gamma_1 \sqcap \Gamma_2$. Hence, by rules $\to_i$ and $\to_e$, $\lambda x.M : \langle \Gamma_1 \vdash V \to U \rangle$ and $(\lambda x.M)N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U \rangle$. Since $FV((\lambda x.M)N) = FV(M[x := N])$, then $\Gamma\uparrow^{\mathcal{U}} = \Gamma$, and, by rule $\sqsubseteq$, $(\lambda x.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U \rangle$.
  - If $x \notin FV(M)$, then $M : \langle \Gamma \vdash U \rangle$ and, by rule $\to'_i$, $\lambda y.M : \langle \Gamma \vdash \omega \to U \rangle$. By rule ω, $N : \langle env^N_\omega \vdash \omega \rangle$, then, by rule $\to_e$, $(\lambda x.M)N : \langle \Gamma \sqcap env^N_\omega \vdash U \rangle$. Since $FV((\lambda x.M)N) = FV(M[x := N]) \cup FV(N)$, then $\Gamma\uparrow^{\mathcal{U}} = \Gamma \sqcap env^N_\omega$.
- If $U = \sqcap_{i=1}^{k} T_i$ where $\forall\ 1 \leq i \leq k$, $T_i \in \mathbb{T}$: By rule $\sqsubseteq$, we have $\forall\ 1 \leq i \leq k$, $M[x := N] : \langle \Gamma \vdash T_i \rangle$, then, by the previous case, $\forall\ 1 \leq i \leq k$, $(\lambda x.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash T_i \rangle$, then, by $k - 1$ applications of rule $\sqcap_i$, $(\lambda x.M)N : \langle \Gamma\uparrow^{\mathcal{U}} \vdash U \rangle$.

□

Next, we give the main block for the proof of subject expansion for β.

Theorem 19 *If $N : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta N$, then $M : \langle \Gamma\uparrow^{M} \vdash U \rangle$.*

Proof By induction on the derivation $N : \langle \Gamma \vdash U \rangle$.

- If $\dfrac{}{x : \langle x : T \vdash T \rangle}$ and $M \rhd_\beta x$, then $M = (\lambda y.M_1)M_2$ where $y \notin FV(M_2)$ and $x = M_1[y := M_2]$. By lemma 18, $M : \langle (x : T)\uparrow^{M} \vdash T \rangle$.
- If $\dfrac{}{N : \langle env^N_\omega \vdash \omega \rangle}$ and $M \rhd_\beta N$, then since by theorem 2.1, $FV(N) \subseteq FV(M)$, $(env^N_\omega)\uparrow^{M} = env^M_\omega$. By ω, $M : \langle env^M_\omega \vdash \omega \rangle$. Hence, $M : \langle (env^N_\omega)\uparrow^{M} \vdash \omega \rangle$.
- If $\dfrac{N : \langle \Gamma, x : U \vdash T \rangle}{\lambda x.N : \langle \Gamma \vdash U \to T \rangle}$ and $M \rhd_\beta \lambda x.N$, then we have two cases:
  - If $M = \lambda x.M'$ where $M' \rhd_\beta N$, then by IH, $M' : \langle (\Gamma, x : U)\uparrow^{M'} \vdash T \rangle$. Since by theorem 2.1 and lemma 7.1, $x \in FV(N) \subseteq FV(M')$, then we have $(\Gamma, x : U)\uparrow^{FV(M')} = \Gamma\uparrow^{FV(M')\setminus\{x\}}, x : U$ and $\Gamma\uparrow^{FV(M')\setminus\{x\}} = \Gamma\uparrow^{\lambda x.M'}$. Hence, $M' : \langle \Gamma\uparrow^{\lambda x.M'}, x : U \vdash T \rangle$ and finally, by $\to_i$, $\lambda x.M' : \langle \Gamma\uparrow^{\lambda x.M'} \vdash U \to T \rangle$.

– If $M = (\lambda y.M_1)M_2$ where $y \notin FV(M_2)$ and $\lambda x.N = M_1[y := M_2]$, then, by lemma 18, since $y \notin FV(M_2)$ and $M_1[y := M_2] : \langle \Gamma \vdash U \to T \rangle$, we have $(\lambda y.M_1)M_2 : \langle \Gamma \uparrow^{(\lambda y.M_1)M_2} \vdash U \to T \rangle$.

- If $\dfrac{N : \langle \Gamma \vdash T \rangle \quad x \notin dom(\Gamma)}{\lambda x.N : \langle \Gamma \vdash \omega \to T \rangle}$ and $M \rhd_\beta N$ then similar to the above case.

- If $\dfrac{N_1 : \langle \Gamma_1 \vdash U \to T \rangle \qquad N_2 : \langle \Gamma_2 \vdash U \rangle}{N_1\ N_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle}$ and $M \rhd_\beta N_1 N_2$, we have three cases:

  – $M = M_1 N_2$ where $M_1 \rhd_\beta N_1$. By IH, $M_1 : \langle \Gamma_1 \uparrow^{M_1} \vdash U \to T \rangle$. It is easy to show that $(\Gamma_1 \sqcap \Gamma_2)\uparrow^{M_1 N_2} = \Gamma_1 \uparrow^{M_1} \sqcap \Gamma_2$. Now use $\to_e$.

  – $M = N_1 M_2$ where $M_2 \rhd_\beta N_2$. Similar to the above case.

  – $M = (\lambda x.M_1)M_2$ where $x \notin FV(M_2)$ and $N_1 N_2 = M_1[x := M_2]$. By lemma 18, $(\lambda x.M_1)M_2 : \langle (\Gamma_1 \sqcap \Gamma_2)\uparrow^{(\lambda x.M_1)M_2} \vdash T \rangle$.

- If $\dfrac{N : \langle \Gamma \vdash U_1 \rangle \qquad N : \langle \Gamma \vdash U_2 \rangle}{N : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$ and $M \rhd_\beta N$ then use IH.

- Let $\dfrac{N : \langle \Gamma \vdash U \rangle \qquad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{N : \langle \Gamma' \vdash U' \rangle}$ and $M \rhd_\beta N$. By lemma 6.6, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. It is easy to show that $\Gamma' \uparrow^M \sqsubseteq \Gamma \uparrow^M$ and hence by lemma 6.6, $\langle \Gamma \uparrow^M \vdash U \rangle \sqsubseteq \langle \Gamma' \uparrow^M \vdash U' \rangle$. By IH, $M \uparrow^M : \langle \Gamma \vdash U \rangle$. Hence, by $\sqsubseteq_{\langle\rangle}$, we have $M : \langle \Gamma' \uparrow^M \vdash U' \rangle$.

□

Corollary 20 (Subject expansion for β)
If $N : \langle \Gamma \vdash U \rangle$ and $M \rhd_\beta^* N$, then $M : \langle \Gamma \uparrow^M \vdash U \rangle$.

Proof By induction on the length of the derivation $M \rhd_\beta^* N$ using theorem 19 and the fact that if $FV(P) \subseteq FV(Q)$, then $(\Gamma \uparrow^P)\uparrow^Q = \Gamma \uparrow^Q$. □

4 The realisability semantics, its soundness and completeness

In this section we give a realisability semantics for our type system and establish both the soundness and completeness of this semantics.

We start with the definition of the function space and saturated sets.

Definition 21 *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$.*

1. *We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of $\mathcal{X}$, i.e. $\{\mathcal{Y} \;/\; \mathcal{Y} \subseteq \mathcal{X}\}$.*

2. *We define $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M} \;/\; M\ N \in \mathcal{Y}$ for all $N \in \mathcal{X}\}$.*

3. *Let $r \in \{f, \beta\}$. We say that $\mathcal{X}$ is r-saturated if whenever $M \rhd_r^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.*

Lemma 22 *Let $r \in \{f, \beta\}$.*

1. *If $\mathcal{X}$ is β-saturated, then $\mathcal{X}$ is f-saturated.*

2. *If $\mathcal{X}, \mathcal{Y}$ are r-saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is r-saturated.*

3. *If $\mathcal{Y}$ is r-saturated, then, for every set $\mathcal{X} \subseteq \mathcal{M}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is r-saturated.*

Proof 1. Note that $\rhd_f^* \subset \rhd_\beta^*$. 2. is easy. 3. Let $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, $M \rhd_r^* N$ and $P \in \mathcal{X}$. Then, by theorem 2.2, $M\ P \rhd_r^* N\ P$ and $N\ P \in \mathcal{Y}$. Since $\mathcal{Y}$ is r-saturated, then $M\ P \in \mathcal{Y}$. Thus, $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$. □

We interpret basic types as saturated sets. The interpretation of complex types is built up from smaller types in the obvious way.

Definition 23 *Let $r \in \{f, \beta\}$.*

1. *An r-interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M})$ is a function such that:*
   $\forall\ a \in \mathcal{A}$, $\mathcal{I}(a)$ is r-saturated.

2. *An r-interpretation $\mathcal{I}$ can be extended to $\mathbb{U}$ as follows:*
   $\mathcal{I}(\omega) = \mathcal{M} \qquad \mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2) \qquad \mathcal{I}(U \to T) = \mathcal{I}(U) \leadsto \mathcal{I}(T)$

Lemma 24 *If $\mathcal{I}$ is a β-interpretation then $\mathcal{I}$ is an f-interpretation.*

Proof Use lemma 22.1. □

The next lemma shows that the interpretation of any type (basic or complex) is saturated, that the interpretation function respects the relation $\sqsubseteq$ and that we can in some sense expand the terms in the interpretation.

Lemma 25 *Let $r \in \{f, \beta\}$ and let $\mathcal{I}$ be an r-interpretation.*

1. *For any $U \in \mathbb{U}$, we have $\mathcal{I}(U)$ is r-saturated.*

2. *If $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.*

3. *Let $n \geq 0$ and $\forall 1 \leq i \neq j \leq n$, $x_i \neq x_j$. If $\forall\ N_i \in \mathcal{I}(U_i)$ $(1 \leq i \leq n)$, $M[(x_i := N_i)_1^n] \in \mathcal{I}(U)$, then*
   $\lambda x_1 .... \lambda x_n.M \in \mathcal{I}(U_1 \to (U_2 \to (... \to (U_n \to U)...)))$.

Proof 1. By induction on U using lemma 22.
2. By induction of the derivation $U \sqsubseteq V$. 3. By induction on $n \geq 0$ using 1. □

We now show the soundness of our sematics.

Theorem 26 (Soundness) *Let $r \in \{f, \beta\}$. If $M : \langle (x_i : U_i)_n \vdash U \rangle$, $\mathcal{I}$ is an r-interpretation and $\forall 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i)$, then $M[(x_i := N_i)_1^n] \in \mathcal{I}(U)$.*

Proof By induction on the derivation $M : \langle (x_i : U_i)_n \vdash U \rangle$.

- Let $\dfrac{}{x : \langle (x : T) \vdash T \rangle}$. If $N \in \mathcal{I}(T)$ then $x[x := N] = N \in \mathcal{I}(T)$.

- Let $\dfrac{}{M : \langle env_\omega^M \vdash \omega \rangle}$ where $env_\omega^M = (x_i : \omega)_n$.
  We have $M[(x_i := N_i)_1^n] \in \mathcal{M} = \mathcal{I}(\omega)$.

- Let $\dfrac{P : \langle (x_i : U_i)_1^n, x : U \vdash T \rangle}{\lambda x.P : \langle (x_i : U_i)_n \vdash U \to T \rangle}$.
  If $\mathcal{I}(U) = \emptyset$ then $(\lambda x.P)[(x_i := N_i)_1^n] \in \mathcal{I}(U) \leadsto \mathcal{I}(T) = \mathcal{M}$.
  If $\mathcal{I}(U) \neq \emptyset$ then let $N \in \mathcal{I}(U)$. By IH, $P[(x_i := N_i)_1^n, x := N] \in \mathcal{I}(T)$. By lemma 25.1, $\mathcal{I}(T)$ is r-saturated.
  Moreover, $(\lambda x.P)[(x_i := N_i)_1^n]\ N \rhd_r^* P[(x_i := N_i)_1^n, x := N]$. Hence,
  $(\lambda x.P)[(x_i := N_i)_1^n] N \in \mathcal{I}(T)$ and $(\lambda x.P)[(x_i := N_i)_1^n] \in \mathcal{I}(U) \leadsto \mathcal{I}(T)$.

- Let $\dfrac{P : \langle (x_i : U_i)_n \vdash T \rangle \quad x \neq x_i}{\lambda x.P : \langle (x_i : U_i)_n \vdash \omega \to T \rangle}$ and $N \in \mathcal{M}$. Note that $x \notin FV(P)$.
  By IH, $P[(x_i := N_i)_1^n] \in \mathcal{I}(T)$. By lemma 25.1, $\mathcal{I}(T)$ is r-saturated.
  Moreover, $(\lambda x.P)[(x_i := N_i)_1^n]\ N \rhd_r^* P[(x_i := N_i)_1^n]$. Hence
  $(\lambda x.P)[(x_i := N_i)_1^n]\ N \in \mathcal{I}(T)$ and $(\lambda x.P)[(x_i := N_i)_1^n] \in \mathcal{I}(\omega) \leadsto \mathcal{I}(T)$.

- Let $\dfrac{M_1 : \langle \Gamma_1 \vdash U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash U\rangle}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle}$ where $\Gamma_1 = (x_i : U_i)_n, (y_j : V_j)_m$, $\Gamma_2 = (x_i : U'_i)_n, (z_k : W_k)_l$ and $\Gamma_1 \sqcap \Gamma_2 = (x_i : U_i \sqcap U'_i)_n, (y_j : V_j)_m, (z_k : W_k)_l$. Let $\forall 1 \leq i \leq n, P_i \in \mathcal{I}(U_i \sqcap U'_i)$, $\forall 1 \leq j \leq m, Q_j \in \mathcal{I}(V_j)$ and $\forall 1 \leq k \leq l, R_k \in \mathcal{I}(W_k)$. By IH, $M_1[(x_i := P_i)_1^n, (y_j := Q_j)_1^m] \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and
  $M_2[(x_i := P_i)_1^n, (z_k := R_k)_1^l] \in \mathcal{I}(U)$,
  then $(M_1 M_2)[(x_i := P_i)_1^n, (y_j := Q_j)_1^m, (z_k := R_k)_1^l] =$
  $M_1[(x_i := P_i)_1^n, (y_j := Q_j)_1^m]\ M_2[(x_i := P_i)_1^n, (z_k := R_k)_1^l] \in \mathcal{I}(T)$.
- Let $\dfrac{M : \langle (x_i : U_i)_n \vdash V_1\rangle \quad M : \langle (x_i : U_i)_n \vdash V_2\rangle}{M : \langle (x_i : U_i)_n \vdash V_1 \sqcap V_2\rangle}$. By IH, $M[(x_i := N_i)_1^n] \in \mathcal{I}(V_1)$ and $M[(x_i := N_i)_1^n] \in \mathcal{I}(V_2)$. Hence, $M[(x_i := N_i)_1^n] \in \mathcal{I}(V_1 \sqcap V_2)$.
- Let $\dfrac{M : \Phi \quad \Phi \sqsubseteq \Phi'}{M : \Phi'}$ where $\phi' = \langle (x_i : U_i)_n \vdash U\rangle$.
  By lemma 6.6 and 6.3, $\Phi = \langle (x_i : U'_i)_n \vdash U'\rangle$, $\forall\ 1 \leq i \leq n$, $U_i \sqsubseteq U'_i$ and $U' \sqsubseteq U$. By lemma 25.2, $N_i \in \mathcal{I}(U'_i)$, then, by IH, $M[(x_i := N_i)_1^n] \in \mathcal{I}(U')$ and, by lemma 25.2, $M[(x_i := N_i)_1^n] \in \mathcal{I}(U)$.

□

Roughly speaking, completeness of the semantics amounts to saying that if M is in the meaning of type U (i.e., M is in $\mathcal{I}(U)$ for any interpretation $\mathcal{I}$) then M has type U. In order to show completeness, we define a special interpretation function $\mathbb{I}$ through the typing relation $\vdash$ in such a way that, if $M \in \mathbb{I}(U)$ then M can be shown to have type U. This is done in the next definition and lemma.

Definition 27 *1. For every $U \in \mathbb{U}$, let an infinite subset $\mathbb{V}_U$ of $\mathcal{V}$ such that:*
- *If $U \neq V$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.*
- *$\bigcup_{U \in \mathbb{U}} \mathbb{V}_U = \mathcal{V}$.*

2. *We denote $\mathbb{G} = \{(x : U)\ /\ U$ is a type and $x \in \mathbb{V}_U\}$. Note that since $\mathbb{G}$ is infinite, $\mathbb{G}$ is not a type environment.*
3. *Let $M \in \mathcal{M}$ and $U \in \mathbb{U}$. We write $M : \langle \mathbb{G} \vdash U\rangle$ if there is a type environment $\Gamma \subset \mathbb{G}$ such that $M : \langle \Gamma \vdash U\rangle$.*
4. *Let $\mathbb{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M})$ be the function defined by:*
   $\forall\ a \in \mathcal{A}$, $\mathbb{I}(a) = \{M \in \mathcal{M}\ /\ M : \langle \mathbb{G} \vdash a\rangle\}$.

Remark 28 *Note that in Definition 27, we have associated to each $U \in \mathbb{U}$, an infinite set of variables $\mathbb{V}_U$ in such a way that no variable is used in two different types, and each variable of $\mathcal{V}$ is associated to a type. Obviously, as long as these conditions are satisfied, we have the liberty of dividing the set $\mathcal{V}$ as we wish. We will practice this liberty in the proof of theorem 32.*

Lemma 29 *1. If $\Gamma, \Gamma' \subset \mathbb{G}$ and $dom(\Gamma) = dom(\Gamma')$, then $\Gamma = \Gamma'$.*

2. *If $\Gamma, \Gamma' \subset \mathbb{G}$, then $\Gamma \sqcap \Gamma' = \Gamma \cup \Gamma' \subset \mathbb{G}$.*
3. *$\mathbb{I}$ is a β-interpretation. I.e., $\forall\ a \in \mathcal{A}$, $\mathbb{I}(a)$ is β-saturated.*
   Hence, $\mathbb{I}$ is an f-interpretation.
   Furthermore, we extend $\mathbb{I}$ to $\mathbb{U}$ as in Definition 23.2.
4. *If $U \in \mathbb{U}$, then $\mathbb{I}(U) \neq \emptyset$ and $\mathbb{I}(U) = \{M \in \mathcal{M}\ /\ M : \langle \mathbb{G} \vdash U\rangle\}$.*

Proof

1. Let $(x : U) \in \Gamma$ and $(x : U') \in \Gamma'$. Hence, $x \in \mathbb{V}_U$ and $x \in \mathbb{V}_{U'}$ and so, $U = U'$ (otherwise, $\mathbb{V}_U \cap \mathbb{V}_{U'} = \emptyset$).

2. Let $\Gamma = (x_i : U_i)_n, (y_j : V_j)_m$ and $\Gamma' = (x_i : U'_i)_n, (z_k : W_k)_l$ where $y_j \neq z_k$ for all $1 \leq j \leq m$ and $1 \leq k \leq l$. Since $(x_i : U_i)_n \subset \mathbb{G}$ and $(x_i : U'_i)_n \subset \mathbb{G}$, by 1, $U_i = U'_i$ for all $1 \leq i \leq n$. Hence, $\Gamma \sqcap \Gamma' = \Gamma \cup \Gamma' \subset \mathbb{G}$.

3. Let $a \in \mathcal{A}$, $M \in \mathcal{M}$, $M \rhd^*_\beta N$ and $N \in \mathbb{I}(a)$. Then $N : \langle \Gamma \vdash a \rangle$ where $\Gamma \subset \mathbb{G}$. Let $FV(M) \setminus dom(\Gamma) = \{x_1, ..., x_n\}$ and $\forall\ 1 \leq i \leq n$, take U_i such that $x_i \in \mathbb{V}_{U_i}$. Then $\Delta = \Gamma, (x_i : U_i)_n \subset \mathbb{G}$ and $\Gamma{\uparrow}^M = \Gamma, (x_i : \omega)_n$. By corollary 20, $M : \langle \Gamma{\uparrow}^M \vdash a \rangle$ and, by lemma 6.3, $\Delta \sqsubseteq \Gamma{\uparrow}^M$. Hence, by rule $\sqsubseteq$, $M : \langle \Delta \vdash a \rangle$. Thus, $M \in \mathbb{I}(a)$. Hence $\mathbb{I}(a)$ is β-saturated and so, $\mathbb{I}$ is a β-interpretation. Finally, by lemma 24, $\mathbb{I}$ is an f-interpretation.

4. The proof of $\mathbb{I}(U) \neq \emptyset$ is as follows: let $x \in \mathbb{V}_U \neq \emptyset$. Then, $x : U \in \mathbb{G}$ and since $x : \langle (x : U) \vdash U \rangle$ then $x \in \mathbb{I}(U)$.

   Now we do the second part by induction on U.

   - $U = a$: By definition of $\mathbb{I}$.
   - $U = \omega$: By definition, $\mathbb{I}(\omega) = \mathcal{M}$. So, $\{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash \omega \rangle\} \subseteq \mathbb{I}(\omega)$. Conversely, let $M \in \mathbb{I}(\omega)$ where $FV(M) = \{x_1, ..., x_n\}$. We have $M : \langle (x_i : \omega)_n \vdash \omega \rangle$. $\forall\ 1 \leq i \leq n$, take U_i such that $x_i \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i : U_i)_n \subset \mathbb{G}$. By lemma 7.2, $M : \langle \Gamma \vdash \omega \rangle$. Hence $M : \langle \mathbb{G} \vdash \omega \rangle$. Thus, $\mathbb{I}(\omega) \subseteq \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash \omega \rangle\}$.

     We deduce $\mathbb{I}(\omega) = \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash \omega \rangle\}$.
   - $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}(U_1 \sqcap U_2) = \mathbb{I}(U_1) \cap \mathbb{I}(U_2) =$ $\{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash U_1 \rangle\} \cap \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash U_2 \rangle\}$.
     * If $M : \langle \mathbb{G} \vdash U_1 \rangle$ and $M : \langle \mathbb{G} \vdash U_2 \rangle$, then $M : \langle \Gamma_1 \vdash U_1 \rangle$ and $M : \langle \Gamma_2 \vdash U_1 \rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{G}$. By lemma 7.1, $dom(\Gamma_1) = dom(\Gamma_2) = FV(M)$. By lemma 8.1, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2 \rangle$. Since $\Gamma_1, \Gamma_2 \subset \mathbb{G}$, then, by 1, $\Gamma_1 = \Gamma_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \subset \mathbb{G}$. Thus $M : \langle \mathbb{G} \vdash U_1 \sqcap U_2 \rangle$.
     * If $M : \langle \mathbb{G} \vdash U_1 \sqcap U_2 \rangle$, then $M : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle$ where $\Gamma \subset \mathbb{G}$. By $\sqsubseteq$, $M : \langle \Gamma \vdash U_1 \rangle$ and $M : \langle \Gamma \vdash U_2 \rangle$, then $M : \langle \mathbb{G} \vdash U_1 \rangle$ and $M : \langle \mathbb{G} \vdash U_2 \rangle$.

     We deduce $\mathbb{I}(U_1 \sqcap U_2) = \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash U_1 \sqcap U_2 \rangle\}$.
   - $U = V \to T$: Then $\mathbb{I}(V \to T) = \mathbb{I}(V) \rightsquigarrow \mathbb{I}(T)$. By IH, $\mathbb{I}(V) = \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash V \rangle\}$ and $\mathbb{I}(T) = \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash T \rangle\}$.
     * Let $M \in \mathbb{I}(V) \rightsquigarrow \mathbb{I}(T)$ and $x \in \mathbb{V}_V$ such that $x \notin FV(M)$. By rule ax' (see lemma 8.2), $x : \langle (x : V) \vdash V \rangle$. Since $(x : V) \subset \mathbb{G}$, then $x : \langle \mathbb{G} \vdash V \rangle$. By IH, $x \in \mathbb{I}(V)$. Hence $Mx \in \mathbb{I}(T)$ and so $Mx : \langle \Gamma \vdash T \rangle$ where $\Gamma \subset \mathbb{G}$. Since $x \notin FV(M)$, then $\Gamma = \Delta, x : V$ and $\Delta \subset \mathbb{G}$. By lemma 9.2, we deduce that $M : \langle \Delta \vdash V \to T \rangle$.
     * Let $M, N \in \mathcal{M}$ such that $M : \langle \mathbb{G} \vdash V \to T \rangle$ and $N : \langle \mathbb{G} \vdash V \rangle$. We have $M : \langle \Gamma_1 \vdash V \to T \rangle$ and $N : \langle \Gamma_2 \vdash V \rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{G}$. Thus $M\ N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle$. Since, by lemma 29.2, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{G}$. Therefore $MN : \langle \mathbb{G} \vdash T \rangle$.

     We deduce $\mathbb{I}(V \to T) = \{M \in \mathcal{M} \ / \ M : \langle \mathbb{G} \vdash V \to T \rangle\}$.

□

Now, the $\mathbb{I}$ of definition 27 will be used to show the completeness of the semantics.

Theorem 30 (Completeness) *Let $r \in \{f, \beta\}$. Let $U_1, ..., U_n, U \in \mathbb{U}$ and $M \in \mathcal{M}$ such that $FV(M) = \{x_1, ..., x_n\}$. If $\forall$ r-interpretation $\mathcal{I}$ and $\forall$ $N_i \in \mathcal{I}(U_i)$ $(1 \leq i \leq n)$, $M[(x_i := N_i)_1^n] \in \mathcal{I}(U)$, then $M : \langle (x_i : U_i)_n \vdash U \rangle$.*

Proof We distinguish three cases:

- If $U = \omega$, then $M : \langle (x_i : \omega)_n \vdash \omega \rangle$. Thus, by lemma 7.2, $M : \langle (x_i : U_i)_n \vdash \omega \rangle$.

- If $U \in \mathbb{T}$, then, let $V = U_1 \to (U_2 \to (\ldots \to (U_n \to U)\ldots))$. By hypothesis and lemma 25.3, $\forall$ r-interpretation $\mathcal{I}$, $\lambda x_1 \ldots . \lambda x_n . M \in \mathcal{I}(V)$. Hence, $\lambda x_1 \ldots . \lambda x_n . M \in \mathbb{I}(V)$ where $\mathbb{I}$ is the interpretation of definition 27.4. By lemma 29.4, $\lambda x_1 \ldots . \lambda x_n . M : \langle \Gamma \vdash V \rangle$ where $\Gamma \subset \mathbb{G}$ and, since $\lambda x_1 \ldots . \lambda x_n . M$ is closed, $\Gamma = ()$. By rule ax', $\forall\ 1 \leq i \leq n$, $x_i : \langle x_i : U_i \vdash U_i \rangle$, by n applications of $\to_e$ we deduce $(\lambda x_1 \ldots . \lambda x_n . M) x_1 \ldots x_n : \langle (x_i : U_i)_n \vdash U \rangle$. Since $(\lambda x_1 \ldots . \lambda x_n . M) x_1 \ldots x_n \rhd_\beta^* M$, then by corollary 14, $M : \langle (x_i : U_i)_n \vdash U \rangle$.

- If $U = \sqcap_{j=1}^m T_j$, then, by hypothesis, $\forall$ r-interpretation $\mathcal{I}$, $\forall$ $N_i \in \mathcal{I}(U_i)$ $(1 \leq i \leq n)$, and $\forall\ 1 \leq j \leq m$, $M[(x_i := N_i)_1^n] \in \mathcal{I}(T_j)$. By the previous case, $\forall\ 1 \leq j \leq m$, $M : \langle (x_i : U_i)_n \vdash T_j \rangle$. By $m-1$ applications of $\sqcap_i$ we deduce $M : \langle (x_i : U_i)_n \vdash U \rangle$.

□

5 The meaning of types

Obviously the meaning of a type U should be based on the intersection of all the interpretations of U. However, since we have been using two different kinds of interpretations (β- and f-interpretations), we give two definitions for the meaning of a type. We will show that these two definitions are equivalent.

Definition 31 *Let $r \in \{f, \beta\}$. We define the meaning $[U]_r$ of $U \in \mathbb{U}$ by:*

$$[U]_r = \bigcap_{\mathcal{I}\ \ r-interpretation} \mathcal{I}(U)$$

The next theorem shows that the meaning $[U]$ of U is the set of terms typable by U in a special environment and that $[U]$ is stable by β-reduction and β-expansion.

Theorem 32 *Let $r \in \{f, \beta\}$ and $U \in \mathbb{U}$.*

1. *$[U]_r = \{M \in \mathcal{M}\ /\ M : \langle env_\omega^M \vdash U \rangle\}$.*
2. *$[U]_r$ is stable by β-reduction. I.e., if $M \in [U]_r$ and $M \rhd_\beta^* N$, then $N \in [U]_r$.*
3. *$[U]_r$ is stable by β-expansion. I.e., if $M \in [U]_r$, $N \rhd_\beta^* M$, then $N \in [U]_r$.*
4. *$[U]_r = \{M \in \mathcal{M}\ /\ M \rhd_\beta^* N$ and $N : \langle env_\omega^N \vdash U \rangle\}$.*

Proof

1. Let $M \in \mathcal{M}$ such that $M : \langle env_\omega^M \vdash U \rangle$. Let $\mathcal{I}$ be an r-interpretation and take $FV(M) = dom(env_\omega^M) = \{x_1, x_2, \ldots, x_n\}$. By theorem 26, since $\forall 1 \leq i \leq n$, $x_i \in \mathcal{I}(\omega) = \mathcal{M}$, then $M = M[(x := x_i)_1^n] \in \mathcal{I}(U)$. Hence, $M \in [U]_r$.
   Conversely, let $M \in [U]_r$. Take the interpretation $\mathbb{I}$ given in Definition 27 such that (recall remark 28) $FV(M) \subset \mathbb{V}_\omega$. Since $M \in \mathbb{I}(U)$ then $M : \langle \Gamma \vdash U \rangle$ where $\Gamma \subseteq \mathbb{G}$. But $FV(M) \subset \mathbb{V}_\omega$ and by lemma 7.1, $FV(M) = dom(\Gamma)$. Hence $\Gamma = env_\omega^M$.
   We conclude that $[U]_r = \{M \in \mathcal{M}\ /\ M : \langle env_\omega^M \vdash U \rangle\}$.

2. Let $M \in [U]_r$ such that $M \rhd_\beta^* N$. By 1, $M : \langle env_\omega^M \vdash U \rangle$. By subject reduction for β corollary 14, $N : \langle (env_\omega^M) \upharpoonright_N \vdash U \rangle$. Since by theorem 2.1, $FV(N) \subseteq FV(M)$ then $(env_\omega^M) \upharpoonright_N = env_\omega^N$. Thus by 1, $N \in [U]_r$.

3. Let $M \in [U]_r$ such that $N \rhd_\beta^* M$. By 1, $M : \langle env_\omega^M \vdash U \rangle$. By subject expansion for β corollary 20, $N : \langle (env_\omega^M) \uparrow^N \vdash U \rangle$. Since by theorem 2.1, $FV(M) \subseteq FV(N)$ then $(env_\omega^M) \uparrow^N = env_\omega^N$. Thus by 1, $N \in [U]_r$.

4. By 1, $[U]_r \subseteq \{M \in \mathcal{M} \ / \ M \rhd_\beta^* N$ and $N : \langle env_\omega^N \vdash U \rangle\}$. Conversely, let $M \rhd_\beta^* N$ and $N : \langle env_\omega^N \vdash U \rangle$. By 1, $N \in [U]_r$. Hence, by 3, $M \in [U]_r$.

□

Corollary 33 *Let $U \in \mathbb{U}$. We have that $[U]_f = [U]_\beta$.*

Proof By theorem 32.1, $[U]_f = [U]_\beta = \{M \in \mathcal{M} \ / \ M : \langle env_\omega^M \vdash U \rangle\}$. □

Hence, we write $[U]$ instead of either $[U]_f$ or $[U]_\beta$.

Remark 34 *The reader may ask here why we introduced the two notions of saturation if the meaning of a type does not depend on whether this meaning was made using β-interpretations or f-interpretations. The answer to this question is that up to here, we could equally use β-interpretations or f-interpretations. However, to establish further results related to the meaning of types, especially for those types whose meaning consists of terms that reduce to closed terms, then we need β-saturation. For this reason, in the rest of paper, we only consider β-saturation.*

Let us now reflect further on the meaning of types as given in definition 31. The next lemma gives three examples.

Lemma 35 *Let $a \in \mathcal{A}$, $U = \omega \to (a \to a)$, $V = a \to (\omega \to a)$ and $W = (\omega \to a) \to a$. We have:*

1. $[U] = \{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.\lambda y.y\}$. *Note that* $\lambda x.\lambda y.y : \langle () \vdash U \rangle$.
2. $[V] = \{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.\lambda y.x\}$. *Note that* $\lambda x.\lambda y.x : \langle () \vdash V \rangle$.
3. $[W] = \{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.xP$ *where* $P \in \mathcal{M}\}$.
   Note that $\lambda x.xP : \langle env_\omega^{\lambda x.xP} \vdash W \rangle$.

Proof

1. It is easy to show that $\lambda x.\lambda y.y : \langle () \vdash U \rangle$. Note that $env_\omega^{\lambda x.\lambda y.y} = ()$. Hence, $\{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.\lambda y.y\} = \{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.\lambda y.y$ and $\lambda x.\lambda y.y : \langle env_\omega^{\lambda x.\lambda y.y} \vdash U \rangle\} \subseteq [U]$ by theorem 32.4.

   Conversely, let $M \in [U]$ and $y \notin FV(M)$. Take the β-interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X} = \{M \in \mathcal{M}/M \rhd_\beta^* y\}$. Since $M \in [U]$ then $M \in \mathcal{I}(U) = \mathcal{M} \rightsquigarrow (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)) = \mathcal{M} \rightsquigarrow (\mathcal{X} \rightsquigarrow \mathcal{X})$. Let $x \neq y$ such that $x \notin FV(M)$. Since $x \in \mathcal{M}$ and $y \in \mathcal{X}$, then $Mxy \in \mathcal{X}$, $Mxy \rhd_\beta^* y$ and by theorem 2.5, $M \rhd_\beta^* \lambda x.\lambda y.y$.

2. It is easy to show that $\lambda x.\lambda y.x : \langle () \vdash V \rangle$. Let $\mathcal{I}$ be a β-interpretation. By theorem 26, $\lambda x.\lambda y.x \in \mathcal{I}(V)$. By lemma 25.1, $\mathcal{I}(V)$ is β-saturated. Hence, $\{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.\lambda y.x\} \subseteq \mathcal{I}(V)$. Thus, $\{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.\lambda y.x\} \subseteq [V]$.
   Conversely, let $M \in [V]$ and $x \notin FV(M)$. Take the β-interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X} = \{M \in \mathcal{M}/M \rhd_\beta^* x\}$. Since $M \in [V]$ then $M \in \mathcal{I}(V) = \mathcal{I}(a) \rightsquigarrow (\mathcal{M} \rightsquigarrow \mathcal{I}(a)) = \mathcal{X} \rightsquigarrow (\mathcal{M} \rightsquigarrow \mathcal{X})$. Let $y \neq x$ such that $y \notin FV(M)$. We have $x \in \mathcal{X}$ and $y \in \mathcal{M}$, then $Mxy \in \mathcal{X}$ and $Mxy \rhd_\beta^* x$. Thus, by theorem 2.5, $M \rhd_\beta^* \lambda x.\lambda y.x$.

3. Let $P \in \mathcal{M}$. Using lemma 7.2, we can show that $\lambda x.xP : \langle env_\omega^{\lambda x.xP} \vdash W \rangle$ (irrespectively of whether $x \in FV(P)$ or not). Now, $\{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.xP\} = \{M \in \mathcal{M}/M \rhd_\beta^* \lambda x.xP$ and $\lambda x.xP : \langle env_\omega^{\lambda x.xP} \vdash W \rangle\} \subseteq [W]$ by theorem 32.4.

Conversely, let $M \in [W]$ and $x \notin FV(M)$. Take the β-interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X} = \{M \in \mathcal{M} / M \rhd_\beta^* xP$ where $P \in \mathcal{M}\}$. Then $M \in \mathcal{I}(W) = (\mathcal{M} \rightsquigarrow \mathcal{X}) \rightsquigarrow \mathcal{X}$. Since $x \in \mathcal{M} \rightsquigarrow \mathcal{X}$, then $M\ x \in \mathcal{X}$ and $M\ x \rhd_\beta^* xP$ where $P \in \mathcal{M}$. Thus, by theorem 2.5, $M \rhd_\beta^* \lambda x.xQ$ where $Q \in \mathcal{M}$.

□

The meanings of the types U and V (of lemma 35) contain only terms which are reduced to closed terms. Due to the position of ω in W, the meaning of W does not solely contain terms which are reduced to closed terms. In U and V, ω has a negative occurence, but in W, ω has a positive one. We will generalize this result.

Definition 36 *1. We define two subsets $\mathbb{U}^+$ and $\mathbb{U}^-$ of $\mathbb{U}$ as follows:*

- *$\forall\ a \in \mathcal{A}$, $a \in \mathbb{U}^+$ and $a \in \mathbb{U}^-$.*
- *$\omega \in \mathbb{U}^-$.*
- *If $U \in \mathbb{U}^+$, then $U \sqcap V \in \mathbb{U}^+$.*
- *If $U, V \in \mathbb{U}^-$, then $U \sqcap V \in \mathbb{U}^-$.*
- *If $U \in \mathbb{U}^-$ and $T \in \mathbb{U}^+$, then $U \to T \in \mathbb{U}^+$.*
- *If $U \in \mathbb{U}^+$ and $T \in \mathbb{U}^-$, then $U \to T \in \mathbb{U}^-$.*

2. Let $\mathcal{S} \subseteq \mathcal{V}$ where $\mathcal{S} \neq \emptyset$.

(a) We say that a term M is $\mathcal{S}$-almost closed if $M \rhd_\beta^* N$ and $FV(N) \subseteq \mathcal{S}$. We denote $\mathcal{M}^{\mathcal{S}}$ the set of $\mathcal{S}$-almost closed terms.

(b) We define the function $\mathcal{I}_{\mathcal{S}} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M})$ by: $\forall\ a \in \mathcal{A}$, $\mathcal{I}_{\mathcal{S}}(a) = \mathcal{M}^{\mathcal{S}}$.

The next lemma shows that $\mathcal{I}_{\mathcal{S}}$ is a β-interpretation and relates $\mathcal{I}_{\mathcal{S}}(U)$ and $\mathcal{M}^{\mathcal{S}}$ according to whether $U \in \mathbb{U}^+$ or $U \in \mathbb{U}^-$.

Lemma 37 *Let $\mathcal{S} \subseteq \mathcal{V}$ where $\mathcal{S} \neq \emptyset$.*

1. *$\mathcal{I}_{\mathcal{S}}$ is a β-interpretation. I.e., $\forall\ a \in \mathcal{A}$, $\mathcal{I}_{\mathcal{S}}(a)$ is β-saturated. Hence, we extend $\mathcal{I}_{\mathcal{S}}$ to $\mathbb{U}$ as in Definition 23.2.*
2. *If $U \in \mathbb{U}^+$, then $\mathcal{I}_{\mathcal{S}}(U) \subseteq \mathcal{M}^{\mathcal{S}}$.*
3. *If $U \in \mathbb{U}^-$, then $\mathcal{M}^{\mathcal{S}} \subseteq \mathcal{I}_{\mathcal{S}}(U)$.*

Proof 1. Easy since $\mathcal{I}_{\mathcal{S}}(a) = \mathcal{M}^{\mathcal{S}}$ which is β-saturated (use theorem 2.1).
We show 2 and 3 by simultaneous induction on U.

2. Let $U \in \mathbb{U}^+$ and $M \in \mathcal{I}_{\mathcal{S}}(U)$.
   - If $U = a$, the result comes by definition of $\mathcal{I}_{\mathcal{S}}$.
   - If $U = U_1 \sqcap U_2$ and $U_1 \in \mathbb{U}^+$, then $M \in \mathcal{I}_{\mathcal{S}}(U_1)$ and, by IH, $M \in \mathcal{M}^{\mathcal{S}}$.
   - If $U = V \to T$, $V \in \mathbb{U}^-$ and $T \in \mathbb{U}^+$, then let $x \in \mathcal{S}$. We have $x \in \mathcal{M}^{\mathcal{S}}$, then, by IH, $x \in \mathcal{I}_{\mathcal{S}}(V)$ and $Mx \in \mathcal{I}_{\mathcal{S}}(T)$. By IH, $Mx \in \mathcal{M}^{\mathcal{S}}$, then $Mx \rhd_\beta^* N$ and $FV(N) \subseteq \mathcal{S}$. We examine the reduction $Mx \rhd_\beta^* N$.
     - If $M \rhd_\beta^* P$ and $N = Px$, then $FV(P) \subseteq FV(N) \subseteq \mathcal{S}$.
     - If $M \rhd_\beta^* \lambda y.Q$ and $Q[y := x] \rhd_\beta^* N$, then $M \rhd_\beta^* \lambda y.Q = \lambda x.Q[y := x] \rhd_\beta^* \lambda x.N$ and $FV(\lambda x.N) \subseteq FV(N) \subseteq \mathcal{S}$.

     Then $M \rhd_\beta^* M'$ and $FV(M') \subseteq \mathcal{S}$. Thus $M \in \mathcal{M}^{\mathcal{S}}$.
3. Let $U \in \mathbb{U}^-$ and $M \in \mathcal{M}^{\mathcal{S}}$.

– If $U = a$, the result comes by definition of $\mathcal{I}_\mathcal{S}$.
– If $U = \omega$, then $M \in \mathcal{I}_\mathcal{S}(U) = \mathcal{M}$.
– If $U = U_1 \sqcap U_2$ and $U_1, U_2 \in \mathbb{U}^-$, then, by IH, $M \in \mathcal{I}_\mathcal{S}(U_1)$ and $M \in \mathcal{I}_\mathcal{S}(U_2)$, then $M \in \mathcal{I}_\mathcal{S}(U_1 \sqcap U_2)$.
– If $U = V \to T$, $V \in \mathbb{U}^+$ and $T \in \mathbb{U}^-$, then let $P \in \mathcal{I}_\mathcal{S}(V)$. We have $M \rhd_\beta^* N$ and $FV(N) \subseteq \mathcal{S}$. By IH, $P \in \mathcal{M}^\mathcal{S}$, then $P \rhd_\beta^* Q$ and $FV(Q) \subseteq \mathcal{S}$. We have $MP \rhd_\beta^* NQ$ and $FV(NQ) = FV(N) \cup FV(Q) \subseteq \mathcal{S}$, then $MP \in \mathcal{M}^\mathcal{S}$, and, by IH, $MP \in \mathcal{I}_\mathcal{S}(T)$. Thus $M \in \mathcal{I}_\mathcal{S}(V \to T)$.

□

The next corollary shows that if $U \in \mathbb{U}^+$ then $[U]$ contains only elements which β-reduce to closed terms and $[U]$ is the set of all terms that β-reduce to closed terms typable by U. Note that in the proof of 2 below, we need β-saturation and that this is the reason why we adopted exclusively β-saturation since remark 34.

Corollary 38 *Let $U \in \mathbb{U}^+$.*

1. *If $M \in [U]$, then $M \rhd_\beta^* N$ and N is closed.*
2. *$[U] = \{M \in \mathcal{M} \ / \ M \rhd_\beta^* N$ and $N : \langle () \vdash U \rangle\}$.*

Proof

1. Let $\mathcal{S} \subseteq \mathcal{V}$ such that $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \cap FV(M) = \emptyset$. Since $M \in [U]$, then $M \in \mathcal{I}_\mathcal{S}(U)$, and, by lemma 37, $M \rhd_\beta^* N$ and $FV(N) \subseteq \mathcal{S}$. But, by theorem 2.1, $F(N) \subseteq FV(M)$, then $FV(N) = \emptyset$.
2. Let $M \in [U]$. By lemma 29.4, $M : \langle \Gamma \vdash U \rangle$. By 1, $M \rhd_\beta^* N$ and N is closed. Hence by subject reduction for β corollary 14, $N : \langle \Gamma \upharpoonright_N \vdash U \rangle$. Since N is closed $N : \langle () \vdash U \rangle$.

   Conversely, let M such that $M \rhd_\beta^* N$ and $N : \langle () \vdash U \rangle$, and take a β-interpretation $\mathcal{I}$. By theorem 26, $N \in \mathcal{I}(U)$ and, since $\mathcal{I}(U)$ is β-saturated, $M \in \mathcal{I}(U)$. Then $M \in \bigcap_{\mathcal{I}\ \beta-interpretation} \mathcal{I}(U)$ and so, $M \in [U]$.

□

Remark 39 *Note that neither strong nor weak normalisation holds in general for typable terms. For example, $(\lambda x.xx)(\lambda x.xx) : \langle () \vdash \omega \rangle$. As another example, take $\lambda y.y((\lambda x.xx)(\lambda x.xx)) : \langle () \vdash (\omega \to a) \to a \rangle$ by lemma 35.*

We cannot even establish a strong normalisation result for positive types. For example, $(\lambda y.\lambda x.x)((\lambda x.xx)(\lambda x.xx)) : \langle () \vdash a \to a \rangle$. In what follows however, we will establish a weak normalisation result for positive types.

Definition 40 *We define the function $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M})$ by: $\forall\ a \in \mathcal{A}$, $\mathcal{I}(a) = \mathcal{N}$ where $\mathcal{N}$ is the set of β-normalising terms.*

Lemma 41 1. *$\mathcal{I}$ is a β-interpretation. I.e., $\forall\ a \in \mathcal{A}$, $\mathcal{I}(a)$ is β-saturated. Hence, we extend $\mathcal{I}$ to $\mathbb{U}$ as in Definition 23.2.*

2. *If $U \in \mathbb{U}^+$, then $\mathcal{I}(U) \subseteq \mathcal{N}$.*
3. *Let $\mathcal{N}' = \{xM_1 \dots M_n \in \mathcal{M} / x \in \mathcal{V}$ and $M_1 \dots M_n \in \mathcal{N}\}$. Note, $\mathcal{N}' \subseteq \mathcal{N}$. If $U \in \mathbb{U}^-$, then $\mathcal{N}' \subseteq \mathcal{I}(U)$.*

Proof 1 is obvious. We show 2 and 3 by simultaneous induction on U.

2. Let $U \in \mathbb{U}^+$ and $M \in \mathcal{I}(U)$.

– If $U = a$, the result comes by definition of $\mathcal{I}$.
– If $U = U_1 \sqcap U_2$ and $U_1 \in \mathbb{U}^+$, then $M \in \mathcal{I}(U_1)$ and, by IH, $M \in \mathcal{N}$.
– If $U = V \to T$, $V \in \mathbb{U}^-$ and $T \in \mathbb{U}^+$, then let $x \in \mathcal{V} \subseteq \mathcal{N}'$ such that $x \notin FV(M)$. By IH, $x \in \mathcal{I}(V)$ and $Mx \in \mathcal{I}(T)$. By IH, $Mx \in \mathcal{N}$. Hence, by theorem 2.6, $M \in \mathcal{N}$.

3. Let $U \in \mathbb{U}^-$ and $M \in \mathcal{N}'$.

– If $U = a$, the result comes by definition of $\mathcal{I}$.
– If $U = \omega$, then $M \in \mathcal{I}(U) = \mathcal{M}$.
– If $U = U_1 \sqcap U_2$ and $U_1, U_2 \in \mathbb{U}^-$, then, by IH, $M \in \mathcal{I}(U_1)$ and $M \in \mathcal{I}(U_2)$, then $M \in \mathcal{I}(U_1 \sqcap U_2)$.
– If $U = V \to T$, $V \in \mathbb{U}^+$ and $T \in \mathbb{U}^-$, then let $P \in \mathcal{I}(V)$. We have $M = xM_1 \dots M_n$ where $M_i \in \mathcal{N}$ for $1 \leq i \leq n$. By IH, $P \in \mathcal{N}$. Hence, $MP \in \mathcal{N}'$ and by IH, $MP \in \mathcal{I}(T)$. Thus $M \in \mathcal{I}(V \to T)$.

□

The next corollary shows that if $U \in \mathbb{U}^+$ then $[U]$ contains only elements which are normalisable.

Corollary 42 *Let $U \in \mathbb{U}^+$.*

1. *If $M \in [U]$, then M is normalisable.*
2. *If $M : \langle () \vdash U \rangle$ then M is normalisable.*
3. *$[U] = \{M \in \mathcal{M} \ / \ M \rhd_\beta^* N,\ N$ is in normal form and $N : \langle () \vdash U \rangle\}$.*

Proof

1. By lemma 41, $M \in [U] \subseteq \mathcal{I}(U) \subseteq \mathcal{N}$.

2 By Theorem 26, $M \in \mathcal{I}(U)$. By lemma 41, $M \in \mathcal{N}$.

3. Let $M \in [U]$. By Corollary 38.2, $M \rhd_\beta^* P$ and $P : \langle () \vdash U \rangle$. Since by 1, M is normalisable then by Church-Rosser P is normalising. Let N be the normal form of P. By Subject reduction corollary 14, $N : \langle () \vdash U \rangle$.
The inverse inclusion is obvious by corollary 38.2.

□

Remark 43 *It should be noted that positive types are not exlusively the types which satisfy the properties proved about them (e.g., corollary 38). For example, let us take the non-positive type $U' = (\omega \to b) \to (a \to a)$ where a and b are different. We can show that $[U']$ only contains terms which reduce to the closed term $\lambda x.\lambda y.y$ (and that $\lambda x.\lambda y.y : \langle () \vdash U' \rangle$). Hence, U' is a type which is not positive, yet for which corollary 38 holds. Note that, since a and b are different, then $(\omega \to b)$ cannot be used in type derivations.*

6 Conclusion

In this article, we considered an elegant intersection type system for which we established basic properties which include the subject reduction and expansion properties for β. We gave this system a realisability semantics and we showed its soundness and completeness using a method comparable to (yet more detailed than) Hindley's completeness semantics for an earlier intersection type system. The basic difference between both proofs is that Hindley's notion of saturation is based on equivalence classes whereas ours is based on a weaker requirement of weak head normal forms. Hence, all of Hindley's saturated models are also saturated in our framework yet on the other hand, there are saturated models based on weak head normal form which cannot be models in Hindley's framework. This means that our method provides a larger set of possible models and this leaves the choice open for better models or counter-models for particular applications. We have even proved that for different notions of saturation (based on weak head reduction and normal β-reduction) we obtain the same interpretation for types. Another difference between our approach and that of Hindley is that he constructs his models modulo the convertibility relation, whereas we establish that the interpretation of types is stable by both β-reduction and β-expansion.

Furthermore, we reflected on the meaning of types, especially on the so-called abstract data types where typability and realisability coincide. The presence of ω in intersection type systems prevents typability and realisability from coinciding as one sees for example in $\lambda x.xP$ (where P may contain free variable and may not be normalisable) whose type is $(\omega \to a) \to a$. We found a set of types $\mathbb{U}^+$ for which we showed that typability and realisability coincide. We have also shown that this set satisfies the weak normalisation property.

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