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Reducibility proofs in the λ-calculus

Fairouz Kamareddine, Vincent Rahli and J. B. Wells*

Abstract. Reducibility, despite being quite mysterious and inflexible, has been used to prove a number of properties of the λ-calculus and is well known to offer general proofs which can be applied to a number of instantiations. In this paper, we look at two related but different results in λ-calculi with intersection types.

1. We show that one such result (which aims at giving reducibility proofs of Church-Rosser, standardisation and weak normalisation for the untyped λ-calculus) faces serious problems which break the reducibility method. We provide a proposal to partially repair the method.
2. We consider a second result whose purpose is to use reducibility for typed terms in order to show the Church-Rosser of β-developments for the untyped terms (and hence the Church-Rosser of β-reduction). In this second result, strong normalisation is not needed. We extend the second result to encompass both βI- and $\beta\eta$-reduction rather than simply β-reduction.

Keywords: Lambda-Calculus, Reducibility, Church-Rosser, Developments

1. Introduction

Based on realisability semantics [Kle45], the reducibility method has been developed by Tait [Tai67] in order to prove the normalisation of some functional theories. The basic idea of reducibility is to interpret types by sets of λ-terms which are closed under some properties. Girard [Gir72] developed the reducibility method further and used it to prove the strong normalisation of a typed λ-calculus by introducing the candidates of reducibility [Gal90]. Statman [Sta85], Koletsos [Kol85], and Mitchell [Mit90, Mit96] also used reducibility to prove the Church-Rosser property (also called confluence) of the simply typed λ-calculus. Furthermore, Krivine [Kri90] uses reducibility to prove the strong normalisation of system $\mathcal{D}$, an intersection type system [CDC80, CDCV80, CDCV81]. Moreover, Gallier [Gal97, Gal98] uses some aspects of Koletsos's method to prove a number of results such as the strong normalisation of the λ-terms

Address for correspondence: ULTRA Group (Useful Logics, Types, Rewriting, and their Automation), Heriot-Watt University, School of Mathematical and Computer Sciences, Mountbatten building, Edinburgh EH14 4AS, UK. Email: http://www.macs.hw.ac.uk/ultra/
*School of Mathematical and Computer Sciences, Heriot-Watt Univ., Riccarton, Edinburgh EH14 4AS, Scotland

that are typable in systems like $\mathcal{D}$ or $\mathcal{D}\Omega$ [Kri90]. In particular, Gallier states some conditions a property needs to satisfy in order to be enjoyed by some typable terms under some restrictions.

Similarly, Ghilezan and Likavec [GL02] state some conditions a property has to satisfy in order to hold for all λ-terms typable under some type restrictions in a type system close to $\mathcal{D}\Omega$. Furthermore, they state a condition that a property has to satisfy in order to step from the statement "a λ-term *typable under some restrictions on types* has the property" to the statement "a λ-term *of the untyped λ-calculus* has the property". If successful, the method of [GL02] would provide an attractive way for establishing properties such as Church-Rosser for all the untyped λ-terms, by simply showing easier conditions on typed terms. However, we show in this paper that Ghilezan and Likavec's method fails in both the typed and the untyped settings. We outline the obstacle we faced when trying to repair the result for the typed setting and explain how far we have been able to to repair it. However, the result for the untyped setting seems unrepairable. Ghilezan and Likavec also present a weaker version of their method for a type system similar to system $\mathcal{D}$, which allows one to use reducibility to prove properties of the terms typable by this system, namely the strongly normalisable terms. As far as we know, this portion of their result is correct. (They do not actually apply this weaker method to any sets of terms.)

In addition to the method proposed by Ghilezan and Likavec (which does not actually work for the full untyped λ-calculus), other steps of establishing properties like Church-Rosser for typed λ-terms and concluding the properties for all the untyped λ-terms have been successfully exploited in the literature. Koletsos and Stavrinos [KS08] use reducibility to state that the λ-terms that are typable in system $\mathcal{D}$ satisfies the Church-Rosser property. Using this result together with a method based on β-developments [Klo80, Kri90], they show that β-developments are Church-Rosser and this in turn will imply the confluence of the untyped λ-calculus. Although Klop [Klo80] proves the confluence of β-developments [BBKV76], his proof is based on strong normalisation whereas the Koletsos and Stavrinos's proof only uses an embedding of β-developments in the reduction of typable λ-terms. In this paper, we apply Koletsos and Stavrinos's method to βI-reduction and then generalise it to $\beta\eta$-reduction.

In section 2 we introduce the formal machinery and establish some needed lemmas. In section 3 we present the reducibility method used by Ghilezan and Likavec and show that it fails at a number of important propositions which makes it inapplicable to the full untyped λ-calculus, although a version of their method works for the strongly normalisable terms. We give counterexamples where all the conditions stated in Ghilezan and Likavec's paper are satisfied, yet the claimed property does not hold. In section 4 we indicate the limits of the method, show how these limits affect its salvation and then we partially salvage it so that it can be correctly used to establish confluence, standardisation and weak head normal forms but only for restricted sets of lambda terms and types (that we believe to be equal to the set of strongly normalisable terms). We point out some links between the work of [GL02] and that of Gallier [Gal98]. In section 5, we give a precise formalisation of β-developments where we formally deal with occurrences of redexes using paths and we adapt definitions from [Kri90] to allow βI- and $\beta\eta$-reduction. In section 6, we introduce the reducibility semantics for both βI- and $\beta\eta$-reduction and establish its soundness. Then, we show that all typable terms satisfy the Church-Rosser property. In section 7 we adapt the Church-Rosser proof of Koletsos and Stavrinos [KS08] to βI-reduction. In section 8 we non-trivially generalise Koletsos and Stavrinos's method to handle $\beta\eta$-reduction. We formalise $\beta\eta$-residuals and $\beta\eta$-developments in section 8.1. Then, we compare our notion of $\beta\eta$-residuals with those of Curry and Feys [CF58] and Klop [Klo80] in section 8.2, establishing that we allow less residuals than Klop but we believe more residuals than Curry and Feys. Finally, we establish in section 8.3 the confluence of $\beta\eta$-developments and hence of $\beta\eta$-reduction. We conclude in section 9.

2. The Formal Machinery

This section provides some known formal machinery and introduces new definitions and lemmas that are necessary for the paper. Let n, m be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We take as convention that if a metavariable v ranges over a set s then the metavariables v_i such that $i \geq 0$ and the metavariables $v', v'', etc.$ also range over s.

A binary relation is a set of pairs. Let rel range over binary relations. Let $\mathrm{dom}(rel) = \{x \mid \langle x, y\rangle \in rel\}$ and $\mathrm{ran}(rel) = \{y \mid \langle x, y\rangle \in rel\}$. A function is a binary relation fun such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq fun$ then $y = z$. Let fun range over functions. Let $s \to s' = \{fun \mid \mathrm{dom}(fun) \subseteq s \wedge \mathrm{ran}(fun) \subseteq s'\}$.

Given n sets $s_1, \dots, s_n$, where $n \geq 2$, $s_1 \times \cdots \times s_n$ stands for the set of all the tuples built on the sets $s_1, \dots, s_n$. If $x \in s_1 \times \cdots \times s_n$, then $x = \langle x_1, \dots, x_n\rangle$ such that $x_i \in s_i$ for all $i \in \{1, \dots, n\}$.

2.1. Familiar background on λ-calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the λ-calculus and one lemma which deals with the shape of reductions.

Definition 2.1. 1. let $x, y, z, etc.$ range over $\mathcal{V}$, a countable infinite set of λ-term variables. The set of terms of the λ-calculus is defined by:

$$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$$

We let $M, N, P, Q, etc.$ range over Λ. We assume the usual definition of subterms: we write $N \subseteq M$ if N is a subterm of M. We also assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $M\ N_1 ... N_n$ instead of $(...(M\ N_1)\ N_2 ... N_{n-1})\ N_n$.

We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of the bound variables differ from the names of the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $\mathrm{fv}(M)$ for the set of the free variables of term M.

2. For $n \geq 0$, define $M^n(N)$, by induction on n by: $M^0(N) = N$ and $M^{n+1}(N) = M(M^n(N))$.

3. A path in a term M is a pointer to a subterm of M. The set of paths is defined as follows:

$$p \in \mathsf{Path} ::= 0 \mid 1.p \mid 2.p$$

We define $M|_p$ by: $M|_0 = M$, $(\lambda x.M)|_{1.p} = M|_p$, $(MN)|_{1.p} = M|_p$, and $(MN)|_{2.p} = N|_p$. We define $2^n.p$ by induction on $n \geq 0$: $2^0.p = p$ and $2^{n+1}.p = 2^n.2.p$.

4. The set $\Lambda\mathrm{I} \subset \Lambda$, of terms of the λI-calculus is defined by:

   - If $x \in \mathcal{V}$ then $x \in \Lambda\mathrm{I}$.
   - If $M \in \Lambda\mathrm{I}$ and $x \in \mathrm{fv}(M)$ then $\lambda x.M \in \Lambda\mathrm{I}$.
   - If $M, N \in \Lambda\mathrm{I}$ then $MN \in \Lambda\mathrm{I}$.

5. The substitution $M[x := N]$ of N for all free occurrences of x in M and the simultaneous substitution $M[x_i := N_i, \dots, x_n := N_n]$ for $1 \leq i \leq n$, of N_i for all free occurrences of x_i in M are defined as usual.

6. We define the following four common relations:

   - $\mathsf{Beta} ::= \langle(\lambda x.M)N, M[x := N]\rangle$.
   - $\mathsf{BetaI} ::= \langle(\lambda x.M)N, M[x := N]\rangle$, where $x \in \mathrm{fv}(M)$.
   - $\mathsf{Eta} ::= \langle\lambda x.Mx, M\rangle$, where $x \notin \mathrm{fv}(M)$.
   - $\mathsf{BetaEta} = \mathsf{Beta} \cup \mathsf{Eta}$.

   Let $\langle s, r\rangle \in \{\langle\mathsf{Beta}, \beta\rangle, \langle\mathsf{BetaI}, \beta I\rangle, \langle\mathsf{Eta}, \eta\rangle, \langle\mathsf{BetaEta}, \beta\eta\rangle\}$.

   We define $\mathcal{R}^r$ to be $\{L \mid \langle L, R\rangle \in s\}$. If $\langle L, R\rangle \in s$ then we call L a r-redex and R a r-contractum of L (or a L r-contractum). We define the ternary relation $\to_r$ as follows:

   - $M \xrightarrow{0}_r M'$ if $\langle M, M'\rangle \in s$
   - $\lambda x.M \xrightarrow{1.p}_r \lambda x.M'$ if $M \xrightarrow{p}_r M'$
   - $MN \xrightarrow{1.p}_r M'N$ if $M \xrightarrow{p}_r M'$
   - $NM \xrightarrow{2.p}_r NM'$ if $M \xrightarrow{p}_r M'$

   We define the binary relation $\to_r$ (for simplicity we use the same name as for the ternary relation) as follows: $M \to_r M'$ if there exists p such that $M \xrightarrow{p}_r M'$. We define $\mathcal{R}^r_M = \{p \mid M|_p \in \mathcal{R}^r\}$.

7. Let $M \in \Lambda$ and $\mathcal{F} \subseteq \Lambda$. $\mathcal{F} \upharpoonright M = \{N \mid N \in \mathcal{F} \wedge N \subseteq M\}$.

8. Let $\to_{h\beta}$ be the set of pairs of the form $\langle\lambda x_1.\dots x_n.(\lambda x.M_0)M_1 \dots M_m, \lambda x_1.\dots x_n.M_0[x := M_1]M_2 \dots M_m\rangle$ where $n \geq 0$ and $m \geq 1$.

   If $\langle L, R\rangle \in \to_{h\beta}$ then $L = \lambda x_1.\dots x_n.(\lambda x.M_0)M_1 \dots M_m$ where $n \geq 0$ and $m \geq 1$ and $(\lambda x.M_0)M_1$ is called the β-head redex of L. We define the binary relation $\to_{i\beta}$ as $\to_\beta \setminus \to_{h\beta}$.

9. Let $r \in \{\to_\beta, \to_\eta, \to_{\beta\eta}, \to_{\beta I}, \to_{h\beta}, \to_{i\beta}\}$. We use $\to_r^*$ to denote the reflexive transitive closure of $\to_r$. We let $\simeq_r$ denote the equivalence relation induced by $\to_r$. If the r-reduction from M to N is in k steps, we write $M \to_r^k N$.

10. Let $r \in \{\beta I, \beta\eta\}$ and $n \geq 0$. A term $(\lambda x.M')N_0'N_1' \dots N_n'$ is a direct r-reduct of a term $(\lambda x.M)N_0N_1 \dots N_n$ iff $M \to_r^* M'$ and $\forall i \in \{0, \dots, n\}$. $N_i \to_r^* N_i'$.

11. The set NF (of β-normal forms) and WN (of weakly β-normalisable terms) are defined by:

    - $\mathsf{NF} = \{\lambda x_1.\dots\lambda x_n.x_0N_1 \dots N_m \mid n, m \geq 0, N_1, \dots, N_m \in \mathsf{NF}\}$.
    - $\mathsf{WN} = \{M \in \Lambda \mid \exists N \in \mathsf{NF}, M \to_\beta^* N\}$.

12. Let $r \in \{\beta, \beta I, \beta\eta\}$. We say that M has the Church-Rosser property for r (has r-CR) if whenever $M \to_r^* M_1$ and $M \to_r^* M_2$ then there is an M_3 such that $M_1 \to_r^* M_3$ and $M_2 \to_r^* M_3$. We define:

    - $\mathsf{CR}^r = \{M \mid M \text{ has } r\text{-}\mathsf{CR}\}$.
    - $\mathsf{CR}_0^r = \{xM_1 \dots M_n \mid n \geq 0 \wedge x \in \mathcal{V} \wedge (\forall i \in \{1, \dots, n\}, M_i \in \mathsf{CR}^r)\}$.
    - We use CR to denote CR^β and CR_0 to denote CR_0^β.
    - A term is a weak head normal form if it is a λ-abstraction (a term of the form $\lambda x.M$) or if it starts with a variable (a term of the form $xM_1 \cdots M_n$). A term is weakly head normalising if it reduces to a weak head normal form. Let $\mathsf{W}^r = \{M \in \Lambda \mid \exists n \geq 0, \exists x \in \mathcal{V}, \exists P, P_1, \dots, P_n \in \Lambda, M \to_r^* \lambda x.P \text{ or } M \to_r^* xP_1 \dots P_n\}$. We use W to denote W^β.

13. We say that M has the standardisation property if whenever $M \to_\beta^* N$ then there is an M' such that $M \to_h^* M'$ and $M' \to_i^* N$. Let $\mathsf{S} = \{M \in \Lambda \mid M$ has the standardisation property$\}$.

The next lemma deals with the shape of reductions.

Lemma 2.2. 1. $M \xrightarrow{p}_{\beta\eta} M'$ iff ($M \xrightarrow{p}_\beta M'$ or $M \xrightarrow{p}_\eta M'$).

2. If $x \in \mathrm{fv}(M_1)$ then $\mathrm{fv}((\lambda x.M_1)M_2) = \mathrm{fv}(M_1[x := M_2])$.
   If $(\lambda x.M_1)M_2 \in \Lambda\mathrm{I}$ then $M_1[x := M_2] \in \Lambda\mathrm{I}$.
3. If $M \to_{\beta\eta}^* M'$ then $\mathrm{fv}(M') \subseteq \mathrm{fv}(M)$.
4. If $M \to_{\beta I}^* M'$ then $\mathrm{fv}(M) = \mathrm{fv}(M')$ and if $M \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.
5. $\lambda x.M \xrightarrow{p}_{\beta\eta} P$ iff ($p = 1.p'$, $P = \lambda x.M'$ and $M \xrightarrow{p'}_{\beta\eta} M'$) or ($p = 0$, $M = Px$ and $x \notin \mathrm{fv}(P)$).
6. If $r \in \{\beta I, \beta\eta\}$, $n \geq 0$, P is not a direct r-reduct of $N = (\lambda x.M)N_0 \ldots N_n$ and $N \to_r^k P$, then:
   (a) $k \geq 1$, and if $k = 1$ then $P = M[x := N_0]N_1 \ldots N_n$.
   (b) There exists a direct r-reduct $(\lambda x.M')N_0'N_1' \ldots N_n'$ of $(\lambda x.M)N_0 \ldots N_n$ such that $M'[x := N_0']N_1' \ldots N_n' \to_r^* P$.
7. Let $r \in \{\beta I, \beta\eta\}$, $n \geq 0$ and $(\lambda x.M)N_0N_1 \ldots N_n \to_r^* P$. There exists P' such that $P \to_r^* P'$ and if ($r = \beta I$ and $x \in \mathrm{fv}(M)$) or $r = \beta\eta$ then $M[x := N_0]N_1 \ldots N_n \to_r^* P'$.
8. There exists M' such that $M \xrightarrow{p}_r M'$ iff $p \in \mathcal{R}_M^r$.
9. If $M \xrightarrow{p}_r M_1$ and $M \xrightarrow{p}_r M_2$ then $M_1 = M_2$.

Proof: 1) By induction on p.
2) By induction on the structure of M_1.
3) (resp. 4)) By induction on the length of the reduction $M \to_{\beta\eta}^* M'$ (resp. $M \to_{\beta I}^* M'$).
5) $\Rightarrow$) Let $\lambda x.M \xrightarrow{p}_{\beta\eta} P$. We prove the result by case on p. Either $p = 0$ and $M = Px$ such that $x \notin \mathrm{fv}(P)$. Or $p = 1.p'$, $P = \lambda x.M'$ and $M \xrightarrow{p'}_{\beta\eta} M'$.
$\Leftarrow$) If $P = \lambda x.M'$ and $M \to_{\beta\eta} pM'$. So, $\lambda x.M \xrightarrow{1.p}_{\beta\eta} P$ and $\lambda x.M \to_{\beta\eta} P$. If $M = Px$ and $x \notin fvP$ then $\lambda x.M = \lambda x.Px \xrightarrow{0}_{\beta\eta} P$, so $\lambda x.M \to_{\beta\eta} P$.
6a) If $k = 0$ then $P = (\lambda x.M)N_1N_1 \ldots N_n$ is a direct r-reduct of $(\lambda x.M)N_0N_1 \ldots N_n$, absurd. So $k \geq 1$. Assume $k = 1$, we prove $P = M[x := N_0]N_1 \ldots N_n$ by induction on $n \geq 0$.
6b) By 6a, $k \geq 1$. We prove the statement by induction on $k \geq 1$.
7) If P is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$ then $P = (\lambda x.M')N_0' \ldots N_n'$ such that $M \to_r^* M'$ and $\forall i \in \{0, \ldots, n\}, N_i \to_r^* N_i'$. So $P \to_r M'[x := N_0']N_1' \ldots N_n'$ (if $r = \beta I$, note that $x \in \mathrm{fv}(M')$ by lemma 2.2.4) and $M[x := N_0]N_1 \ldots N_n \to_r^* M'[x := N_0']N_1' \ldots N_n'$. If P is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$ then by lemma 6.6b, there exists a direct r-reduct, $(\lambda x.M')N_0' \ldots N_n'$, such that $M \to_r^* M'$ and $\forall i \in \{0, \ldots, n\}, N_i \to_r^* N_i'$, of $(\lambda x.M)N_0 \ldots N_n$. We have $M[x := N_0]N_1 \ldots N_n \to_r^* M'[x := N_0']N_1' \ldots N_n' \to_r^* P$.
8) and 9) By induction on the structure of p. □

(ref)	$\tau \leq \tau$
(tr)	$(\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3) \Rightarrow \tau_1 \leq \tau_3$
(in_L)	$\tau_1 \cap \tau_2 \leq \tau_1$
(in_R)	$\tau_1 \cap \tau_2 \leq \tau_2$
$(\rightarrow\text{-}\cap)$	$(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \leq \tau_1 \rightarrow (\tau_2 \cap \tau_3)$
(mon')	$(\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3) \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$
(mon)	$(\tau_1 \leq \tau_1' \wedge \tau_2 \leq \tau_2') \Rightarrow \tau_1 \cap \tau_2 \leq \tau_1' \cap \tau_2'$
$(\rightarrow\text{-}\eta)$	$(\tau_1 \leq \tau_1' \wedge \tau_2' \leq \tau_2) \Rightarrow \tau_1' \rightarrow \tau_2' \leq \tau_1 \rightarrow \tau_2$
(Ω)	$\tau \leq \Omega$
$(\Omega'\text{-}lazy)$	$\tau \rightarrow \Omega \leq \Omega \rightarrow \Omega$
$(idem)$	$\tau \leq \tau \cap \tau$

Figure 1. The ordering axioms on types

2.2. Background on Types and Type Systems

This section provides the necessary background for the type systems used in this paper. The type systems $\lambda\cap^1$ and $\lambda\cap^2$ are used in section 3, and the type systems $\mathcal{D}$ and $\mathcal{D}_I$ are used in section 6.

Definition 2.3. Let $i \in \{1, 2\}$.

1. Let $\mathcal{A}$ be a countably infinite set of type variables, let α range over $\mathcal{A}$ and let $\Omega \notin \mathcal{A}$ be a constant type. The sets of types $\mathsf{Type}^1 \subset \mathsf{Type}^2$ are defined as follows:

$$\sigma \in \mathsf{Type}^1 ::= \alpha \mid \sigma_1 \rightarrow \sigma_2 \mid \sigma_1 \cap \sigma_2$$

$$\tau \in \mathsf{Type}^2 ::= \alpha \mid \tau_1 \rightarrow \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega$$

2. Let $\Gamma \in \mathcal{B}^1 = \{\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\} \mid \forall i, j \in \{1, \ldots, n\}.\ x_i = x_j \Rightarrow \sigma_i = \sigma_j\}$ and $\Gamma, \Delta \in \mathcal{B}^2 = \{\{x_1 : \tau_1, \ldots, x_n : \tau_n\} \mid \forall i, j \in \{1, \ldots, n\}.\ x_i = x_j \Rightarrow \tau_i = \tau_j\}$.

   Let $\mathrm{dom}(\Gamma) = \{x \mid x : \sigma \in \Gamma\}$.

   When $\mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) = \varnothing$, we write Γ_1, Γ_2 for $\Gamma_1 \cup \Gamma_2$. We write $\Gamma, x : \sigma$ for $\Gamma, \{x : \sigma\}$ and $x : \sigma$ for $\{x : \sigma\}$. We denote $\Gamma = x_m : \sigma_m, \ldots, x_n : \sigma_n$ where $n \geq m \geq 0$, by $(x_i : \sigma_i)_n^m$. If $m = 1$, we simply denote Γ by $(x_i : \sigma_i)_n$.

   If $\Gamma_1 = (x_i : \tau_i)_n, (y_i : \tau_i'')_p$ and $\Gamma_2 = (x_i : \tau_i')_n, (z_i : \tau_i''')_q$ where $x_1, \ldots, x_n$ are the only shared variables, then let $\Gamma_1 \sqcap \Gamma_2 = (x_i : \tau_i \cap \tau_i')_n, (y_i : \tau_i'')_p, (z_i : \tau_i''')_q$.

   Let $X \subseteq \mathcal{V}$. We define $\Gamma \upharpoonright X = \Gamma' \subseteq \Gamma$ where $\mathrm{dom}(\Gamma') = \mathrm{dom}(\Gamma) \cap X$.

   Let $\sqsubseteq$ be the reflexive transitive closure of the axioms $\tau_1 \cap \tau_2 \sqsubseteq \tau_1$ and $\tau_1 \cap \tau_2 \sqsubseteq \tau_2$. If $\Gamma = (x_i : \tau_i)_n$ and $\Gamma' = (x_i : \tau_i')_n$ then $\Gamma \sqsubseteq \Gamma'$ iff for all $i \in \{1, \ldots, n\}$, $\tau_i \sqsubseteq \tau_i'$.

3. • – Let $\nabla_1 = \{(ref), (tr), (in_L), (in_R), (\rightarrow\text{-}\cap), (mon'), (mon), (\rightarrow\text{-}\eta)\}$.

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ (ax) \qquad \frac{}{x : \tau \vdash x : \tau}\ (ax^I)$$

$$\frac{\Gamma \vdash M : \tau_1 \to \tau_2 \quad \Gamma \vdash N : \tau_1}{\Gamma \vdash MN : \tau_2}\ (\to_E) \qquad \frac{\Gamma_1 \vdash M : \tau_1 \to \tau_2 \quad \Gamma_2 \vdash N : \tau_1}{\Gamma_1 \sqcap \Gamma_2 \vdash MN : \tau_2}\ (\to_{E^I})$$

$$\frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x.M : \tau_1 \to \tau_2}\ (\to_I) \qquad \frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2}\ (\cap_I)$$

$$\frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_1}\ (\cap_{E1}) \qquad \frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_2}\ (\cap_{E2})$$

$$\frac{\Gamma \vdash M : \tau_1 \quad \tau_1 \leq^{\nabla} \tau_2}{\Gamma \vdash M : \tau_2}\ (\leq^{\nabla}) \qquad \frac{}{\Gamma \vdash M : \Omega}\ (\Omega)$$

Figure 2. The typing rules

- Let $\nabla_2 = \nabla_1 \cup \{(\Omega), (\Omega' - lazy)\}$.
- Let $\nabla_D = \{(in_L), (in_R)\}$.
- Let $\nabla_{D_I} = \nabla_D \cup \{(idem)\}$.

- – Let Type^{∇_1}, Type^{∇_D}, and $\mathsf{Type}^{\nabla_{D_I}}$ be Type^1.
  – Let Type^{∇_2} be Type^2.

- – Let ∇ be a set of axioms from Figure 1. The relation $\leq^{\nabla}$ is defined on types Type^{∇} and axioms ∇. We use $\leq^1$ instead of $\leq^{\nabla_1}$ and $\leq^2$ instead of $\leq^{\nabla_2}$.
  – The equivalence relation is defined by: $\tau_1 \sim^{\nabla} \tau_2 \iff \tau_1 \leq^{\nabla} \tau_2 \wedge \tau_2 \leq^{\nabla} \tau_1$. We use $\sim^1$ instead of $\sim^{\nabla_1}$ and $\sim^2$ instead of $\sim^{\nabla_2}$.

- – Let the type system $\lambda\cap^1$ be the type derivability relation $\vdash^1$ between the elements of $\mathcal{B}^1$, Λ, and Type^1 generated using the following typing rules of Figure 2: (ax), $(\to_E)$, $(\to_I)$, $(\cap_I)$ and $(\leq^1)$).
  – Let the type system $\lambda\cap^2$ be the type derivability relation $\vdash^2$ between the elements of $\mathcal{B}^2$, Λ, and Type^2 generated using the following typing rules of Figure 2: (ax), $(\to_E)$, $(\to_I)$, $(\cap_I)$, $(\leq^2)$ and (Ω).
  – Let the type system $\mathcal{D}$ be the type derivability relation $\vdash^{\beta\eta}$ between the elements of $\mathcal{B}^1$, Λ, and Type^1 generated using the following typing rules of Figure 2: (ax), $(\to_E)$, $(\to_I)$, $(\cap_I)$, $(\cap_{E1})$ and $(\cap_{E2})$. Note that system $\mathcal{D}$ does not use subtyping.
  – Let the type system $\mathcal{D}_I$ be the type derivability relation $\vdash^{\beta I}$ between the elements of $\mathcal{B}^1$, Λ, and Type^1 generated using the following typing rule of Figure 2: (ax^I), $(\to_{E^I})$, $(\to_I)$, $(\cap_I)$, $(\cap_{E1})$ and $(\cap_{E2})$. Moreover, in this type system, we assume that $\sigma \cap \sigma = \sigma$. Note that system $\mathcal{D}_I$ does not use subtyping.

3. Problems of Ghilezan and Likavec's reducibility method [GL02]

This section introduces the reducibility method of [GL02] and shows exactly where it fails. Throughout, we let $\circledast = \lambda x.xx$.

Definition 3.1. (Type interpretations and the reducibility method of [GL02])
Let $i \in \{1, 2\}$ and $\mathcal{P}$ range over 2^Λ.

1. The type interpretation $[\![-]\!]^i_- \in \mathsf{Type}^i \to 2^\Lambda \to 2^\Lambda$ is defined by:
   - $[\![\alpha]\!]^i_{\mathcal{P}} = \mathcal{P}$.
   - $[\![\tau_1 \cap \tau_2]\!]^i_{\mathcal{P}} = [\![\tau_1]\!]^i_{\mathcal{P}} \cap [\![\tau_2]\!]^i_{\mathcal{P}}$.
   - $[\![\Omega]\!]^2_{\mathcal{P}} = \Lambda$.
   - $[\![\sigma_1 \to \sigma_2]\!]^1_{\mathcal{P}} = \{M \mid \forall N \in [\![\sigma_1]\!]^1_{\mathcal{P}}.MN \in [\![\sigma_2]\!]^1_{\mathcal{P}}\}$.
   - $[\![\tau_1 \to \tau_2]\!]^2_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^2_{\mathcal{P}}, MN \in [\![\tau_2]\!]^2_{\mathcal{P}}\}$.
2. A valuation of term variables in Λ is a function $\nu \in \mathcal{V} \to \Lambda$. We write $v(x := M)$ for the function v' where $v'(x) = M$ and $v'(y) = v(y)$ if $y \neq x$.
3. let ν be a valuation of term variables in Λ. Then the term interpretation $[\![-]\!]_\nu \in \Lambda \to \Lambda$ is defined as follows: $[\![M]\!]_\nu = M[x_1 := \nu(x_1), \ldots, x_n := \nu(x_n)]$, where $\mathrm{fv}(()M) = \{x_1, \ldots, x_n\}$.
4.
   - $\nu \models^i_{\mathcal{P}} M : \tau$ iff $[\![M]\!]_\nu \in [\![\tau]\!]^i_{\mathcal{P}}$.
   - $\nu \models^i_{\mathcal{P}} \Gamma$ iff $\forall (x : \tau) \in \Gamma.\ \nu(x) \in [\![\tau]\!]^i_{\mathcal{P}}$.
   - $\Gamma \models^i_{\mathcal{P}} M : \tau$ iff $\forall \nu \in \mathcal{V} \to \Lambda.\ \nu \models^i_{\mathcal{P}} \Gamma \Rightarrow \nu \models^i_{\mathcal{P}} M : \tau$.
5. Let $\mathcal{X} \subseteq \Lambda$. We recall here the variable, saturation, closure, and invariance under abstraction predicates defined by Ghilezan and Likavec (see Definitions 3.6 and 3.15 of [GL02]):
   - $\mathrm{VAR}^1(\mathcal{P}, \mathcal{X}) \iff \mathrm{VAR}^2(\mathcal{P}, \mathcal{X}) \iff \mathcal{V} \subseteq \mathcal{X}$.
   - $\mathrm{SAT}^1(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall N \in \mathcal{P}.\ M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X})$.
   - $\mathrm{SAT}^2(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X})$.
   - $\mathrm{CLO}^1(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ Mx \in \mathcal{X} \Rightarrow M \in \mathcal{P})$.
   - $\mathrm{CLO}^2(\mathcal{P}, \mathcal{X}) \iff \mathrm{CLO}(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P})$.
   - $\mathrm{VAR}(\mathcal{P}, \mathcal{X}) \iff (\forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \mathcal{P}.\ xN_1 \ldots N_n \in \mathcal{X})$.
   - $\mathrm{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \mathcal{P}.$
     $M[x := N]N_1 \ldots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \ldots N_n \in \mathcal{X})$.
   - $\mathrm{INV}(\mathcal{P}) \iff (\forall M \in \Lambda.\ \forall x \in \mathcal{V}.\ M \in \mathcal{P} \iff \lambda x.M \in \mathcal{P})$.

   For $\mathcal{R} \in \{\mathrm{VAR}^i, \mathrm{SAT}^i, \mathrm{CLO}^i\}$, let $\mathcal{R}(\mathcal{P}) \iff \forall \tau \in \mathsf{Type}^i.\ \mathcal{R}(\mathcal{P}, [\![\tau]\!]^i_{\mathcal{P}})$.

Lemma 3.2. (Basic lemmas proved in [GL02] and needed for this section)

1. (a) $[\![M]\!]_{\nu(x:=N)} \equiv [\![M]\!]_{\nu(x:=x)}[x := N]$.
   (b) $[\![MN]\!]_\nu \equiv [\![M]\!]_\nu [\![N]\!]_\nu$.

(c) $[\![\lambda x.M]\!]_\nu \equiv \lambda x.[\![M]\!]_{\nu(x:=x)}$.

2. If $\mathrm{VAR}^1(\mathcal{P})$ and $\mathrm{CLO}^1(\mathcal{P})$ then for all $\sigma \in \mathsf{Type}^1$, $[\![\sigma]\!]^1_{\mathcal{P}} \subseteq \mathcal{P}$.
3. If $\mathrm{VAR}^1(\mathcal{P})$, $\mathrm{CLO}^1(\mathcal{P})$, $\mathrm{SAT}^1(\mathcal{P})$, and $\Gamma \vdash^1 M : \sigma$ then $\Gamma \models^1_{\mathcal{P}} M : \sigma$.
4. If $\mathrm{VAR}^1(\mathcal{P})$, $\mathrm{CLO}^1(\mathcal{P})$, $\mathrm{SAT}^1(\mathcal{P})$, and $\Gamma \vdash^1 M : \sigma$ then $M \in \mathcal{P}$.
5. For all $\tau \in \mathsf{Type}^2$, if $\tau \not\sim^2 \Omega$ then $[\![\tau]\!]^2_{\mathcal{P}} \subseteq \mathcal{P}$.
6. If $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]^2_{\mathcal{P}} \subseteq [\![\tau_2]\!]^2_{\mathcal{P}}$.
7. If $\mathrm{VAR}^2(\mathcal{P})$, $\mathrm{SAT}^2(\mathcal{P})$ and $\mathrm{CLO}^2(\mathcal{P})$ then $\Gamma \vdash^2 M : \tau$ implies $\Gamma \models^2_{\mathcal{P}} M : \tau$.
8. If $\mathrm{VAR}^2(\mathcal{P})$, $\mathrm{SAT}^2(\mathcal{P})$ and $\mathrm{CLO}^2(\mathcal{P})$ then for all $\tau \in \mathsf{Type}^2$, if $\tau \not\sim^2 \Omega$ and $\Gamma \vdash^2 M : \tau$ then $M \in \mathcal{P}$.
9. $\mathrm{CLO}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ \tau \not\sim^2 \Omega \Rightarrow \mathrm{CLO}^2(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$.

Note that lemma 3.2.3 states that $\lambda\cap^1$ is sound w.r.t. the $\models^1_{\mathcal{P}}$ interpretation, and lemma 3.2.7 states that $\lambda\cap^2$ is sound w.r.t. the $\models^2_{\mathcal{P}}$ interpretation. Based on these soundness lemmas, Ghilezan and Likavec prove lemmas 3.2.4 and 3.2.8 which are key results in their reducibility method.

Ghilezan and Likavec (see Remark 3.9 of [GL02]) note that if $\mathrm{CLO}^1(\mathcal{P})$, $\mathrm{VAR}^1(\mathcal{P})$ and $\mathrm{SAT}^1(\mathcal{P})$ are true then $\mathsf{SN}_\beta \subseteq \mathcal{P}$ (note that this result does not make any use of the type system $\lambda\cap^1$).

Furthermore, given the notions and statements of definition 3.1 and lemma 3.2, [GL02] states that the predicates $\mathrm{VAR}^i(\mathcal{P})$, $\mathrm{SAT}^i(\mathcal{P})$ and $\mathrm{CLO}^i(\mathcal{P})$ for $i \in \{1, 2\}$ are sufficient to develop the reducibility method. However, in order to prove these predicates (for various instances of $\mathcal{P}$), [GL02] states that one needs stronger and easier to prove induction hypotheses. Therefore, Ghilezan and Likavec introduce the following conditions: $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ and $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ (see Definition 3.1 above or Definition 3.15 of [GL02]). These conditions imply restrictions of $\mathrm{VAR}^2(\mathcal{P}, \mathcal{X})$, $\mathrm{SAT}^2(\mathcal{P}, \mathcal{X})$, and $\mathrm{CLO}^2(\mathcal{P}, \mathcal{X})$. However, as we show below, this attempt fails. (They do not develop the necessary stronger induction hypotheses for the case when $i = 1$, and $\lambda\cap^1$ can only type strongly normalisable terms, so we will not consider the case $i = 1$ further.)

Our definition 3.4 and lemma 3.5 given below are necessary to establish the results of this section (the failure of the method of [GL02]). In definition 3.4, we use the following fact that the defined preorder relation is commutative, associative and idempotent:

Remark 3.3. Commutativity, associativity and idempotence w.r.t. the preorder relation are given by the axioms (in_L), (in_R), (mon'), (tr) and (ref) listed in figure 1.

Proof: • Commutativity: by (in_R), $\tau_1 \cap \tau_2 \leq^2 \tau_2$ and by (in_L), $\tau_1 \cap \tau_2 \leq^2 \tau_1$ so by (mon'), $\tau_1 \cap \tau_2 \leq^2 \tau_2 \cap \tau_1$. By (in_L), $\tau_2 \cap \tau_1 \leq^2 \tau_2$ and by (in_R), $\tau_2 \cap \tau_1 \leq^2 \tau_1$ so by (mon'), $\tau_2 \cap \tau_1 \leq^2 \tau_1 \cap \tau_2$. Hence, $\tau_1 \cap \tau_2 \sim^2 \tau_2 \cap \tau_1$.

• Associativity: by (in_R), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_3$, by (in_L), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_1 \cap \tau_2$, by (in_R), $\tau_1 \cap \tau_2 \leq^2 \tau_2$, by (in_L), $\tau_1 \cap \tau_2 \leq^2 \tau_1$, so by (tr), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_1$ and $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_2$. By (mon'), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_2 \cap \tau_3$ and again by (mon'), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_1 \cap (\tau_2 \cap \tau_3)$. By (in_L), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_1$, by (in_R), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_2 \cap \tau_3$, by (in_L), $\tau_2 \cap \tau_3 \leq^2 \tau_2$, by (in_R), $\tau_2 \cap \tau_3 \leq^2 \tau_3$, so by (tr), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_2$ and $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_3$. By (mon'), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_1 \cap \tau_2$ and again by (mon'), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 (\tau_1 \cap \tau_2) \cap \tau_3$. Hence, $(\tau_1 \cap \tau_2) \cap \tau_3 \sim^2 \tau_1 \cap (\tau_2 \cap \tau_3)$.

• Idempotence: by (in_L), $\tau \cap \tau \leq^2 \tau$ and by (ref) and (mon'), $\tau \leq^2 \tau \cap \tau$, hence, $\tau \sim^2 \tau \cap \tau$. □

Definition 3.4. Let $to \in \mathsf{TypeOmega} ::= \Omega \mid to_1 \cap to_2$.

Let $\mathrm{inInter}(\tau, \tau')$ be true iff $\tau = \tau'$ or $\tau' = \tau_1 \cap \tau_2$ and ($\mathrm{inInter}(\tau, \tau_1)$ or $\mathrm{inInter}(\tau, \tau_2)$).

By commutativity, associativity, and reflexivity we write $\tau_1 \cap \cdots \cap \tau_n$, where $n \geq 1$, instead of τ iff the following condition holds: $\mathrm{inInter}(\tau', \tau)$ iff there exists $i \in \{1, \ldots, n\}$ such that $\tau' = \tau_i$.

Lemma 3.5.

1. If $\tau_1 \leq^2 \tau_2$ and $\tau_1 \in \mathsf{TypeOmega}$ then $\tau_2 \in \mathsf{TypeOmega}$.
2. If $\tau \leq^2 \tau'$ and $\tau' \not\sim^2 \Omega$ then $\tau \not\sim^2 \Omega$.
3. If $\tau \cap \tau' \not\sim^2 \Omega$ then $\tau \not\sim^2 \Omega$ or $\tau' \not\sim^2 \Omega$.
4. If $\tau' \sim^2 \Omega$ then $\tau \leq^2 \tau \cap \tau'$.
5. If $\tau \leq^2 \tau'$ and $\mathrm{inInter}(\tau_1 \to \tau_2, \tau')$ and $\tau_2 \not\sim^2 \Omega$ then there exist $n \geq 1$ and $\tau'_1, \tau''_1, \ldots, \tau'_n, \tau''_n$ such that for all $i \in \{1, \ldots, n\}$, $\mathrm{inInter}(\tau'_i \to \tau''_i, \tau)$ and $\tau''_i \not\sim^2 \Omega$ and $\tau''_1 \cap \cdots \cap \tau''_n \leq^2 \tau_2$. Moreover, if $\tau_1 \sim^2 \Omega$ then for all $i \in \{1, \ldots, n\}$, $\tau'_i \sim^2 \Omega$.
6. For all $\tau, \tau' \in \mathsf{Type}^2$, $\alpha \to \Omega \to \tau' \not\sim^2 \Omega \to \tau$.

Proof: 1) By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last derivation rule.

2) Let $\tau \leq^2 \tau'$. Assume $\tau \sim^2 \Omega$. Then $\Omega \leq^2 \tau$ and by transitivity $\Omega \leq^2 \tau'$. Moreover, by (Ω), $\tau' \leq^2 \Omega$. So $\tau' \sim^2 \Omega$.

3) By (Ω), $\tau \cap \tau' \leq^2 \Omega$. Let $\tau \sim^2 \Omega$ and $\tau' \sim^2 \Omega$, so $\Omega \leq^2 \tau$ and $\Omega \leq^2 \tau'$ and by (mon'), $\Omega \leq^2 \tau \cap \tau'$.

4) By (Ω), $\tau \leq^2 \Omega$ and by transitivity, $\tau \leq^2 \tau'$ because $\Omega \leq^2 \tau'$. By (ref), $\tau \leq^2 \tau$ and by (mon'), $\tau \leq^2 \tau \cap \tau'$.

5) By induction on the size of the derivation of $\tau \leq^2 \tau'$ and then by case on the last derivation rule.

6) Let $\tau' \in \mathsf{Type}^2$. First we prove that $\Omega \to \tau' \not\sim^2 \Omega$. Assume $\Omega \to \tau' \sim^2 \Omega$ then $\Omega \leq^2 \Omega \to \tau'$. By lemma 3.5.1, $\Omega \to \tau' \in \mathsf{TypeOmega}$ which is false. We distinguish the following two cases:

- Let $\tau \sim^2 \Omega$. Assume $\alpha \to \Omega \to \tau' \sim^2 \Omega \to \tau$ then $\Omega \to \tau \leq^2 \alpha \to \Omega \to \tau'$. By lemma 3.5.5, $\tau \leq^2 \Omega \to \tau'$ which is false.
- Let $\tau \not\sim^2 \Omega$. Assume $\alpha \to \Omega \to \tau' \sim^2 \Omega \to \tau$ then $\alpha \to \Omega \to \tau' \leq^2 \Omega \to \tau$. By lemma 3.5.5, $\alpha \sim^2 \Omega$ because $\Omega \sim^2 \Omega$, which is false.

□

The next lemma establishes the failure of a basic lemma of [GL02].

Lemma 3.6. (Lemma 3.16 of [GL02] does not hold)
The following lemma of [GL02] does not hold:
$\mathrm{VAR}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2. (\forall \tau' \in \mathsf{Type}^2. (\tau \not\sim^2 \Omega \to \tau') \Rightarrow \mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$.

Proof: To show that the above statement is false, we provide a counterexample. First, note that $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$ implies that $\mathcal{V} \subseteq [\![\tau]\!]^2_{\mathcal{P}}$. Let $x \in \mathcal{V}$, τ be $\alpha \to \Omega \to \alpha$ and $\mathcal{P}$ be WN. By lemma 3.5.6, for all $\tau' \in \mathsf{Type}^2$, $\tau \not\sim^2 \Omega \to \tau'$. Also $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ is trivially true. Now, assume $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$. By definition, $x \in [\![\tau]\!]^2_{\mathcal{P}}$. Then, $x \in [\![\alpha \to \Omega \to \alpha]\!]^2_{\mathcal{P}} = [\![\tau]\!]^2_{\mathcal{P}}$. Because $x \in \mathcal{P} = [\![\alpha]\!]^2_{\mathcal{P}}$ and $\circledast\circledast \in \Lambda = [\![\Omega]\!]^2_{\mathcal{P}}$ then $xx(\circledast\circledast) \in [\![\alpha]\!]^2_{\mathcal{P}} = \mathcal{P}$. But $xx(\circledast\circledast) \in \mathcal{P}$ is false, so $\mathrm{VAR}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}})$ is false. $\square$

The proof for Lemma 3.18 of [GL02] does not work (because of a wrong use of an induction hypothesis) but we have not yet proved or disproved that lemma:

Remark 3.7. (It is not clear that lemma 3.18 of [GL02] holds)
It is not clear whether the following lemma of [GL02] holds:
$\mathrm{SAT}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathsf{Type}^2.\ (\forall \tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \to \tau') \Rightarrow \mathrm{SAT}(\mathcal{P}, [\![\tau]\!]^2_{\mathcal{P}}))$.

The proof given in [GL02] does not go through and we have neither been able to prove nor disprove this lemma. It remains that this lemma is not yet proved and hence cannot be used in further proofs.

Furthermore, Ghilezan and Likavec state a proposition (Proposition 3.21) which is the reducibility method for typable terms. However, the proof of that proposition depends on two problematic lemmas (lemma 3.16 which we showed to fail in our lemma 3.6, and lemma 3.18 which by remark 3.7 has not been proved). The following lemma is needed to prove that Proposition 3.21 of [GL02] does not hold:

Lemma 3.8. $\mathrm{VAR}(\mathsf{WN}, \mathsf{WN})$, $\mathrm{CLO}(\mathsf{WN}, \mathsf{WN})$, $\mathrm{INV}(\mathsf{WN})$ and $\mathrm{SAT}(\mathsf{WN}, \mathsf{WN})$ hold.

Proof: • $\mathrm{VAR}(\mathsf{WN}, \mathsf{WN})$ holds because $\forall x \in \mathcal{V}$, $\forall n \geq 0$, $\forall N_1, \ldots, N_n \in \mathsf{WN}$, $xN_1 \ldots N_n \in \mathsf{WN}$.
• $\mathrm{CLO}(\mathsf{WN}, \mathsf{WN})$ holds because if $\exists n, m \geq 0$, $\exists x_0 \in \mathcal{V}$, $\exists N_1, \ldots, N_m \in \mathsf{NF}$ such that $M \to^*_\beta \lambda x_1. \ldots \lambda x_n.x_0 N_1 \ldots N_m$ then $\forall y \in \mathcal{V}$, $\lambda y.M \to^*_\beta \lambda y.\lambda x_1. \ldots \lambda x_n.x_0 N_1 \ldots N_m \in \mathsf{NF}$.
• $\mathrm{INV}(\mathsf{WN})$ holds because if $\exists n, m \geq 0$, $\exists x_0 \in \mathcal{V}$, $\exists N_1, \ldots, N_m \in \mathsf{NF}$ such that $\lambda x.M \to^*_\beta \lambda x_1. \ldots \lambda x_n.x_0 N_1 \ldots N_m$ then $x_1 = x$ and $M \to^*_\beta \lambda x_2. \ldots \lambda x_n.x_0 N_1 \ldots N_m$.
• $\mathrm{SAT}(\mathsf{WN}, \mathsf{WN})$ holds because since if $M[x := N]N_1 \ldots N_n \in \mathsf{WN}$ where $n \geq 0$ and $N_1, \ldots, N_n \in \mathsf{WN}$ then $\exists P \in \mathsf{NF}$ such that $M[x := N]N_1 \ldots N_n \to^*_\beta P$. Hence, $(\lambda x.M)NN_1 \ldots N_n \to_\beta M[x := N]N_1 \ldots N_n \to^*_\beta P$. $\square$

Lemma 3.9. (Proposition 3.21 of [GL02] does not hold)
Assume $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ and $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$. The following proposition of [GL02] does not hold:
$\forall \tau \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \wedge \forall \tau' \in \mathsf{Type}^2.\ (\tau \not\sim^2 \Omega \to \tau') \wedge \Gamma \vdash^2 M : \tau \Rightarrow M \in \mathcal{P})$.

Proof: Let $\mathcal{P}$ be WN. Note that $\lambda y.\lambda z.\circledast\circledast \notin \mathsf{WN}$ and $\varnothing \vdash^2 \lambda y.\lambda z.\circledast\circledast : \alpha \to \Omega \to \Omega$ is derivable, where $\alpha \to \Omega \to \Omega \not\sim^2 \Omega$ and by lemma 3.5.6, $\alpha \to \Omega \to \Omega \not\sim^2 \Omega \to \tau'$, for all $\tau' \in \mathsf{Type}^2$. Since $\mathrm{VAR}(\mathsf{WN}, \mathsf{WN})$, $\mathrm{CLO}(\mathsf{WN}, \mathsf{WN})$ and $\mathrm{SAT}(\mathsf{WN}, \mathsf{WN})$ hold by lemma 3.8, we get a counterexample for Proposition 3.21 of [GL02]. $\square$

Finally, Ghilezan and Likavec's proof method for untyped terms fails too.

Lemma 3.10. (Proposition 3.23 of [GL02] does not hold)
The following proposition of [GL02] does not hold:
If $\mathcal{P} \subseteq \Lambda$ is invariant under abstraction (i.e., $\mathrm{INV}(\mathcal{P})$), $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ and $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ then $\mathcal{P} = \Lambda$.

Proof: As by lemma 3.8, $\mathrm{VAR}(\mathsf{WN}, \mathsf{WN})$, $\mathrm{SAT}(\mathsf{WN}, \mathsf{WN})$, and $\mathrm{INV}(\mathsf{WN})$ hold, we get a counterexample for Proposition 3.23. Note that the proof in [GL02] depends on Proposition 3.21 which fails. $\square$

4. How much of the reducibility method of [GL02] can we salvage?

This section provides some indications on the limits of the method. We show how these limits affect the salvation of the method, we partially salvage it, and we show that the obtained method can correctly be used to establish confluence, standardisation, and weak head normal forms but only for restricted sets of lambda terms and types (that we believe to be equal to the set of strongly normalisable terms). We also point out some links between the work done by Ghilezan and Likavec and that of Gallier [Gal98].

Because we proved that Proposition 3.23 of [GL02] is false, we know that the set of properties that a set of terms $\mathcal{P}$ has to satisfy in order to be equal to the set of terms of the untyped λ-calculus cannot be $\{\mathrm{INV}(\mathcal{P}), \mathrm{VAR}(\mathcal{P}, \mathcal{P}), \mathrm{SAT}(\mathcal{P}, \mathcal{P})\}$. Therefore, even if one changes the soundness result or the type interpretation (the set of realisers) in order to obtain the same result as the one claimed by Ghilezan and Likavec, one also has to come up with a new set of properties.

Proposition 3.23 of [GL02] states a set of properties characterising the set of terms of the untyped λ-calculus. The predicate $\mathrm{VAR}(\Lambda, \Lambda)$ states that the variables (more generally, the terms of the form $xNM_1 \cdots M_n$) belong to the untyped λ-calculus. The predicate $\mathrm{INV}(\Lambda)$ states among other things that given a λ-term M, the abstraction of a variable over M is a λ-term too. Therefore, to get a full characterisation of the set of terms of the untyped λ-calculus, we need predicates that cover the application case, i.e., a predicate, say $\mathrm{APP}(\mathcal{P})$, stating that $(\lambda x.M)NM_1 \cdots M_n \in \mathcal{P}$ if $M, N, M_1, \ldots, M_n \in \mathcal{P}$, needs to hold. Note that this predicate cannot be equivalent to the sum of properties $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ and $\mathrm{INV}(\mathcal{P})$ since we saw that the set WN satisfies these properties but is not equal to the λ-calculus. Hence, these properties are not enough to characterise the λ-calculus.

The problem with these properties is that if one tries to salvage Ghilezan and Likavec's reducibility method, the properties $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ and $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ impose a restriction on the arrow types for which the interpretation is in $\mathcal{P}$ (the realisers of arrow types) as we can see below in the arrow type case of the proofs of lemmas 4.4.5 and 4.5. We show at the end of this section that even if the obtained result when considering these restrictions is an improvement of that of Ghilezan and Likavec using the type system $\lambda\cap^1$, it is not possible to salvage their method. (Note that this section does not introduce a new set of predicates. Instead it constrains further the type system used in the method.)

The non-trivial types introduced by Gallier [Gal98] (see below) are not much help in this case, because of the precise restriction imposed by $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$. One might also want to consider the sets of properties stated by Gallier [Gal98], but they are unfortunately not easy to prove for CR (Church-Rosser), because they require a proof of $xM \in \mathsf{CR}$ for all $M \in \Lambda$. Moreover, if one succeeds in proving that the variables are included in the interpretation of a defined set of types containing $\Omega \to \alpha$, where Ω is interpreted as Λ and α as $\mathcal{P}$, then one has proved that $xM \in \mathcal{P}$, which in the case $\mathcal{P} = \mathsf{CR}$ means $M \in \mathsf{CR}$ (this gives the intuition as why the arrow types in OType^3 defined below are of the form $\rho \to \varphi$, where ρ cannot be the Ω type).

It is worth pointing out that part of the work done by Gallier [Gal98] could be adapted to the type system $\lambda\cap^2$. Gallier defines the non-trivial types as follows (where $\tau \in \mathsf{Type}^2$):

$$\psi \in \mathsf{NonTrivial} ::= \alpha \mid \tau \to \psi \mid \tau \cap \psi \mid \psi \cap \tau$$

Note that $\mathsf{NonTrivial} \subset \mathsf{Type}^2$. Types in Type^2 are then interpreted as follows: $[\![\alpha]\!]_{\mathcal{P}} = \mathcal{P}$, $[\![\psi \cap \tau]\!]_{\mathcal{P}} = [\![\tau \cap \psi]\!]_{\mathcal{P}} = [\![\tau]\!]_{\mathcal{P}} \cap [\![\psi]\!]_{\mathcal{P}}$, $[\![\tau]\!]_{\mathcal{P}} = \Lambda$ if $\tau \notin \mathsf{NonTrivial}$ and $[\![\tau \to \psi]\!]_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau]\!]_{\mathcal{P}}.\, MN \in [\![\psi]\!]_{\mathcal{P}}\}$. One can easily prove that if $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]_{\mathcal{P}} \subseteq [\![\tau_2]\!]_{\mathcal{P}}$. Hence, considering the type system $\lambda\cap^2$ instead of $\mathcal{D}\Omega$, Gallier's method provides a set of predicates which when satisfied by a set of terms $\mathcal{P}$ implies that the set of terms typable in the system $\lambda\cap^2$ by a non-trivial type is a subset of $\mathcal{P}$. Gallier proved that the set of head-normalising λ-terms satisfies each of the given predicates.

Using a method similar to Ghilezan and Likavec's method, Gallier also proved that the set of weakly head-normalising terms (W) is equal to the set of terms typable by a weakly non-trivial type in the type system $\mathcal{D}\Omega$. The set of weakly non-trivial types is defined as follows:

$$\psi \in \mathsf{WeaklyNonTrivial} ::= \alpha \mid \tau \to \psi \mid \Omega \to \Omega \mid \tau \cap \psi \mid \psi \cap \tau$$

As explained above and inspired by Gallier's method, we can now try to salvage Ghilezan and Likavec's method by first restricting the set of realisers when defining the interpretation of the set of types in Type^2. The different restrictions lead us to the definition of NTType^3 (where "NT" stands for *non trivial* since $\mathsf{NTType}^3 = \mathsf{NonTrivial}$) and the following type interpretation:

Definition 4.1. We define NTType^3 by:

$$\rho \in \mathsf{NTType}^3 ::= \alpha \mid \tau \to \rho \mid \rho \cap \tau \mid \tau \cap \rho$$

Note that $\mathsf{NTType}^3 \subset \mathsf{Type}^2$. We define a new interpretation of the types in Type^2 as follows:

- $[\![\alpha]\!]^3_{\mathcal{P}} = \mathcal{P}$.
- $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$, if $\tau_1 \cap \tau_2 \in \mathsf{NTType}^3$.
- $[\![\tau]\!]^3_{\mathcal{P}} = \Lambda$, if $\tau \notin \mathsf{NTType}^3$.
- $[\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\}$, if $\tau_1 \to \tau_2 \in \mathsf{NTType}^3$.

In order to prove the relation between the stronger induction hypotheses (VAR, SAT, and CLO) and those depending on type interpretations (VAR^2, SAT^2, and CLO^2), and in order to be able to use these stronger induction hypotheses in the soundness lemma, we have to impose other restrictions (we especially need these restrictions to prove lemma 4.4.5 below which itself uses lemma 4.4.2 and the fact that arrow OType^3 types defined below are of the restricted form $\rho \to \varphi$).

Definition 4.2. We define the set OType^3 (where "O" stands for *omega*) as follows:

$$\varphi \in \mathsf{OType}^3 ::= \alpha \mid \Omega \mid \rho \to \varphi \mid \varphi \cap \tau \mid \tau \cap \varphi$$

Note that $\mathsf{OType}^3 \subset \mathsf{Type}^2$.

Let $\Gamma \in \mathcal{B}^3 = \{\{x_1 : \varphi_1, \ldots, x_n : \varphi_n\} \mid \forall i, j \in \{1, \ldots, n\}.\ x_i = x_j \Rightarrow \varphi_i = \varphi_j\}$, i.e., environments in $\mathcal{B}^3$ are built from types in OType^3.

Let $\vdash^3$ be $\vdash^2$ where $\mathcal{B}^2$ is replaced by $\mathcal{B}^3$, and let $\lambda\cap^3$ be the type system based on $\vdash^3$.

Let $\models^3_{\mathcal{P}}$ be the relation $\models^2_{\mathcal{P}}$ where $[\![\tau]\!]^2_{\mathcal{P}}$ is replaced by $[\![\tau]\!]^3_{\mathcal{P}}$.

Note that $\vdash^3$, $\lambda\cap^3$, and $\models^3_{\mathcal{P}}$ are still built on Type^2.

Due to the saturation predicate and its uses, we could impose further restrictions on the type system. Alternatively, we slightly modify this predicate (for simplicity of notation, we keep the same name):

Definition 4.3. $\mathrm{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda.\ \forall x \in \mathcal{V}.\ \forall n \in \mathbb{N}.\ \forall N_1, \ldots, N_n \in \Lambda.$
$M[x := N]N_1 \ldots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \ldots N_n \in \mathcal{X})$.

We can prove that if $\mathcal{P} \in \{\mathsf{CR}, \mathsf{S}, \mathsf{W}\}$, where CR is the Church-Rosser property, S is the standardisation property, and W is the weak head normalisation property, then $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ holds.

The next lemma (and the relation between the old/new induction hypothesis) is useful for soundness.

Lemma 4.4.
1. $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$.
2. $[\![\rho]\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
3. If $\tau_1 \leq^2 \tau_2$ and $\tau_2 \in \mathsf{NTType}^3$ then $\tau_1 \in \mathsf{NTType}^3$.
4. If $\tau_1 \leq^2 \tau_2$ then $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$.
5. If $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ then for all $\varphi \in \mathsf{OType}^3$, $\mathrm{VAR}(\mathcal{P}, [\![\varphi]\!]^3_{\mathcal{P}})$.
6. If $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ then for all $\tau \in \mathsf{Type}^2$, $\mathrm{SAT}(\mathcal{P}, [\![\tau]\!]^3_{\mathcal{P}})$.

Proof: 1) If $\tau_1 \cap \tau_2 \in \mathsf{NTType}^3$ then it is done by definition. Otherwise $\tau_1, \tau_2 \notin \mathsf{NTType}^3$. Hence $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = \Lambda = \Lambda \cap \Lambda = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$.
2) By induction on the structure of ρ.
3) By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.
4) By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.
5) By induction on the structure of φ.
6) By induction on the structure of τ. □

We now state the following soundness lemma:

Lemma 4.5. If $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$, $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ and $\Gamma \vdash^3 M : \tau$ then $\Gamma \models^3_{\mathcal{P}} M : \tau$.

Proof: By induction on the size of the derivation of $\Gamma \vdash^3 M : \tau$ and then by case on the last rule used in the derivation. Cases dealing with $\tau \notin \mathsf{NTType}^3$ are trivial since $[\![\tau]\!]^3_{\mathcal{P}} = \Lambda$. The intersection case is also trivial by IH. So we only consider $\tau \in \mathsf{NTType}^3$ where τ is not an intersection type.

- (ax): Let $\nu \models^3_{\mathcal{P}} \Gamma, x : \varphi$ then $\nu(x) \in [\![\varphi]\!]^3_{\mathcal{P}}$.
- $(\rightarrow_E)$: By IH, $\Gamma \models^3 M : \tau_1 \rightarrow \tau_2$ and $\Gamma \models^3 N : \tau_1$, so by lemma 3.2.1b, $\Gamma \models^3_{\mathcal{P}} MN : \tau_2$ (because if $\tau_2 \in \mathsf{NTType}^3$ then $\tau_1 \rightarrow \tau_2 \in \mathsf{NTType}^3$).
- $(\rightarrow_I)$: By IH, $\Gamma, x : \tau_1 \models^3_{\mathcal{P}} M : \tau_2$. Let $\nu \models^3_{\mathcal{P}} \Gamma$ and $N \in [\![\tau_1]\!]^3_{\mathcal{P}}$. Then $\nu(x := N) \models^3_{\mathcal{P}} \Gamma$ since $x \notin \mathrm{dom}(\Gamma)$ and $\nu(x := N) \models^3_{\mathcal{P}} x : \tau_1$ since $N \in [\![\tau_1]\!]^3_{\mathcal{P}}$. Therefore $\nu(x := N) \models^3_{\mathcal{P}} M : \tau_2$, i.e. $[\![M]\!]_{\nu(x:=N)} \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Hence, by lemma 3.2.1a, $[\![M]\!]_{\nu(x:=x)}[x := N] \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Since $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$ holds, we can apply lemma 4.4.6 to obtain $(\lambda x.[\![M]\!]_{\nu(x:=x)})N \in [\![\tau_2]\!]^3_{\mathcal{P}}$. By lemma 3.2.1c, $([\![\lambda x.M]\!]_{\nu})N \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Hence $[\![\lambda x.M]\!]_{\nu} \in \{M \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\}$.

  Since $\tau_1 \in \mathsf{OType}^3$ and because $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$ holds, then by lemma 4.4.5, $x \in [\![\tau_1]\!]^3_{\mathcal{P}}$. Hence, by the same argument as above we obtain $[\![M]\!]_{\nu(x:=x)} \in [\![\tau_2]\!]^3_{\mathcal{P}}$. Since $\tau_1 \rightarrow \tau_2 \in \mathsf{NTType}^3$ then $\tau_2 \in \mathsf{NTType}^3$. Because $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$ holds, then by lemma 4.4.2, $\lambda x.[\![M]\!]_{\nu(x:=x)} \in \mathcal{P}$, and by lemma 3.2.1c, $[\![\lambda x.M]\!]_{\nu} \in \mathcal{P}$. Hence, we conclude that $[\![\lambda x.M]\!]_{\nu} \in [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}}$.

- $(\leq^3)$: We conclude by IH and lemma 4.4.4.
- (Ω): This case is trivial because $\Omega \notin \mathsf{NTType}^3$.

□

The next lemma states that a set of terms satisfying the Church-Rosser, the standardisation, or the weak head normalisation properties, also satisfies the variable, saturation and closure predicates.

Lemma 4.6. Let $\mathcal{P} \in \{\mathsf{CR}, \mathsf{S}, \mathsf{W}\}$. Then $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$, $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$, and $\mathrm{CLO}(\mathcal{P}, \mathcal{P})$.

Proof: Straightforward using the relevant property and predicate conditions. □

We obtain the following proof method which is our attempt at salvaging the method of [GL02].

Proposition 4.7. If $\Gamma \vdash^3 M : \rho$ then $M \in \mathsf{CR}$, $M \in \mathsf{S}$, and $M \in \mathsf{W}$.

Proof: By lemma 4.6, lemma 4.4.2 and lemma 4.5 □

We conjecture that the set of terms typable in our type system $\vdash^3$ is no more than the set of strongly normalisable terms.

5. Formalising the background on developments

In this section we go through some needed background from [Kri90] on developments and we precisely formalise and establish all the necessary properties. Throughout the paper, we take c to be a metavariable ranging over $\mathcal{V}$. As far as we know, this is the first precise formalisation of developments. Our definition of developments is similar to Koletsos and Stavrinos's [KS08]. A major difference is that Koletsos and Stavrinos [KS08] deal informally with occurrences of redexes while the current paper deal with them formally using paths (see definition 2.1.3 above).

The next definition adapts Λ_c of [Kri90] to deal with βI- and $\beta\eta$-reduction. $\Lambda\mathrm{I}_c$ is Λ_c where in the abstraction construction rule (R1).2, we restrict abstraction to $\Lambda\mathrm{I}$. In $\Lambda\eta_c$ we introduce the new rule (R4) and replace the abstraction rule of Λ_c by (R1).3 and (R1).4.

Definition 5.1. ($\Lambda\eta_c$, $\Lambda\mathrm{I}_c$)

1. We let $\mathcal{M}_c$ range over $\Lambda\eta_c, \Lambda\mathrm{I}_c$ defined as follows (note that $\Lambda\mathrm{I}_c \subset \Lambda\mathrm{I}$):

   (R1) If x is a variable distinct from c then

   1. $x \in \mathcal{M}_c$.
   2. If $M \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(M)$ then $\lambda x.M \in \Lambda\mathrm{I}_c$.
   3. If $M \in \Lambda\eta_c$ then $\lambda x.M[x := c(cx)] \in \Lambda\eta_c$.
   4. If $Nx \in \Lambda\eta_c$ such that $x \notin \mathrm{fv}(N)$ and $N \neq c$ then $\lambda x.Nx \in \Lambda\eta_c$.

   (R2) If $M, N \in \mathcal{M}_c$ then $cMN \in \mathcal{M}_c$.

   (R3) If $M, N \in \mathcal{M}_c$ and M is a λ-abstraction then $MN \in \mathcal{M}_c$.

   (R4) If $M \in \Lambda\eta_c$ then $cM \in \Lambda\eta_c$.

As standard in lambda calculi, the next lemma gives necessary information on terms of $\mathcal{M}_c$.

Lemma 5.2. (Generation)

1. $M[x := c(cx)] \neq x$ and for any N, $M[x := c(cx)] \neq Nx$.
2. Let $x \notin \text{fv}(M)$. Then, $M[y := c(cx)] \neq x$ and for any N, $M[y := c(cx)] \neq Nx$.
3. If $M \in \mathcal{M}_c$ then $M \neq c$.
4. If $M, N \in \mathcal{M}_c$ then $M[x := N] \neq c$.
5. Let $MN \in \mathcal{M}_c$. Then $N \in \mathcal{M}_c$ and either:
   - $M = cM'$ where $M' \in \mathcal{M}_c$ or
   - $M = c$ and $\mathcal{M}_c = \Lambda\eta_c$ or
   - $M = \lambda x.P$ is in $\mathcal{M}_c$.
6. If $c^n(M) \in \mathcal{M}_c$ then $M \in \mathcal{M}_c$.
7. If $M \in \Lambda\eta_c$ and $n \geq 0$ then $c^n(M) \in \Lambda\eta_c$.
8. If $\lambda x.P \in \Lambda\eta_c$ then $x \neq c$ and either:
   - $P = Nx$ where $N, Nx \in \Lambda\eta_c$, $x \notin \text{fv}(N)$ and $N \neq c$ or
   - $P = N[x := c(cx))]$ where $N \in \Lambda\eta_c$.
9. If $\lambda x.P \in \Lambda\text{I}_c$ then $x \neq c$, $x \in \text{fv}(P)$ and $P \in \Lambda\text{I}_c$.
10. If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $M[x := N] \in \mathcal{M}_c$.
11. Let $y \notin \{x, c\}$. Then:
    - If $M[x := c(cx)] = y$ then $M = y$.
    - If $M[x := c(cx)] = Py$ then $M = Ny$ and $P = N[x := c(cx)]$.
    - If $M[x := c(cx)] = \lambda y.P$ then $M = \lambda y.N$ and $P = N[x := c(cx)]$.
    - If $M[x := c(cx)] = PQ$ then either $M = x$, $P = c$ and $Q = cx$ or $M = P'Q'$ and $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$.
    - If $M[x := c(cx)] = (\lambda y.P)Q$ then $M = (\lambda y.P')Q'$ and $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$.
12. Let $M \in \Lambda\eta_c$.
    (a) If $M = \lambda x.P$ then $P \in \Lambda\eta_c$.
    (b) If $M = \lambda x.Px$ then $Px, P \in \Lambda\eta_c$, $x \notin \text{fv}(P) \cup \{c\}$ and $P \neq c$.
13. (a) Let $x \neq c$. $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'$ iff $M' = N[x := c(cx)]$ and $M \xrightarrow{p}_{\beta\eta} N$.

(b) Let $n \geq 0$. If $c^n(M) \xrightarrow{p}_{\beta\eta} M'$ then $p = 2^n.p'$ and there exists $N \in \Lambda\eta_c$ such that $M' = c^n(N)$ and $M \xrightarrow{p'}_{\beta\eta} N$.

Proof: 1) and 2) By induction on the structure of M.

3) By cases on the derivation of $M \in \mathcal{M}_c$.

4) By cases on the structure of M using 3).

5) By cases on the derivation of $MN \in \mathcal{M}_c$.

6) By induction on n.

7) Easy.

8) By cases on the derivation of $\lambda x.P \in \Lambda\eta_c$.

9) By cases on the derivation of $\lambda x.P \in \Lambda\mathrm{I}_c$.

10) By induction on the structure of $M \in \mathcal{M}_c$.

11) By case on the structure of M.

12a) By definition, $x \neq c$. By 8), $P = Nx$ where $Nx \in \Lambda\eta_c$ or $P = N[x := c(cx)]$ where $N \in \Lambda\eta_c$. In the second case since by (R4) $c(cx) \in \Lambda\eta_c$, we get by 10) that $N[x := c(cx)] \in \Lambda\eta_c$.

12b) By 1) and 8).

13a) Both $\Rightarrow$) and $\Leftarrow$) are by induction on the structure of p.

13b) By induction on n. □

As the formalisation of developments is basic to our work, the next lemma is about sets/paths of redexes.

Lemma 5.3. Let $r \in \{\beta I, \beta\eta\}$ and $\mathcal{F} \subseteq \mathcal{R}^r_M$.

- If $M \in \mathcal{V}$ then $\mathcal{R}^r_M = \varnothing$ and $\mathcal{F} = \varnothing$.
- If $M = \lambda x.N$ then $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^r_N$ and:
  - if $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_N\}$ and $\mathcal{F} \setminus \{0\} = \{1.p \mid p \in \mathcal{F}'\}$.
  - else $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_N\}$ and $\mathcal{F} = \{1.p \mid p \in \mathcal{F}'\}$.
- If $M = PQ$ then $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^r_P$, $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^r_Q$ and:
  - if $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$ and $\mathcal{F} \setminus \{0\} = \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\}$.
  - else $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$ and $\mathcal{F} = \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\}$.

Proof: The part related to $\mathcal{R}^r_M$ is by case on the structure of M. The part related to $\mathcal{F}$ is also by case on the structure of M and uses the first part. □

The next lemma shows the role of redexes w.r.t. substitutions involving c.

Lemma 5.4. Let $r \in \{\beta\eta, \beta I\}$ and $x \neq c$.

1. $M \in \mathcal{R}^{\beta\eta}$ iff $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
2. If $p \in \mathcal{R}^{\beta\eta}_M$ then $M[x := c(cx)]|_p = M|_p[x := c(cx)]$.
3. $p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$ iff $p = 1.p'$ and $p' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.
4. $\mathcal{R}^{\beta\eta}_{M[x:=c(cx)]} = \mathcal{R}^{\beta\eta}_M$.
5. $\mathcal{R}^{\beta\eta}_{c^n(M)} = \{2^n.p \mid p \in \mathcal{R}^{\beta\eta}_M\}$.

Proof: 1) and 2) By induction on the structure of M.
3 $\Rightarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$. By lemma 5.2.1, $\lambda x.M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$ so by lemma 5.3, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.
$\Leftarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. By lemma 5.3, $1.p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$.
4) $\Rightarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. We prove the statement by induction on the structure of M.
$\Leftarrow$) Let $p \in \mathcal{R}^r_M$. Then by definition $M|_p \in \mathcal{R}^{\beta\eta}$. By 1), $M|_p[x := c(cx)] \in \mathcal{R}^{\beta\eta}$. By 2), $M[x := c(cx)]|_p \in \mathcal{R}^{\beta\eta}$. So $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.
5) By induction on $n \geq 0$. $\square$

The next lemma shows that any element $(\lambda x.P)Q$ of $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) is a βI- (resp. $\beta\eta$-) redex, that $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) contains the βI-redexes (resp. $\beta\eta$-redexes) of all its terms and generalises a lemma given in [Kri90] (and used in [KS08]) stating that $\Lambda\eta_c$ (resp. $\Lambda\mathrm{I}_c$) is closed under $\rightarrow_{\beta\eta}$- (resp. $\rightarrow_{\beta I}$-) reduction.

Lemma 5.5. 1. Let $(\mathcal{M}_c, r) \in \{(\Lambda\mathrm{I}_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$.

   (a) If $M = (\lambda x.P)Q$ then $M \in \mathcal{R}^r$.
   (b) If $p \in \mathcal{R}^r_M$ then $M|_p \in \mathcal{M}_c$.

2. (a) If $M \in \Lambda\eta_c$ and $M \rightarrow_{\beta\eta} M'$ then $M' \in \Lambda\eta_c$.
   (b) If $M \in \Lambda\mathrm{I}_c$ and $M \rightarrow_{\beta I} M'$ then $M' \in \Lambda\mathrm{I}_c$.

Proof: 1a) By case on r.
1b) By induction on the structure of M.
2a) Let $M \in \Lambda\eta_c$ and $M \rightarrow_{\beta\eta} M'$. Then there exists p such that $M \xrightarrow{p}_{\beta\eta} M'$. We prove that $M' \in \Lambda\eta_c$ by induction on the structure of p.
2b) By induction on $M \rightarrow_{\beta I} M'$. $\square$

The next definition, taken from [Kri90], erases all the c's from an $\mathcal{M}_c$-term. We extend it to paths.

Definition 5.6. ($|-|^c$)
We define $|M|^c$ and $|\langle M, p\rangle|^c$ inductively as follows:

- $|x|^c = x$
- $|cP|^c = |P|^c$
- $|\langle M, 0\rangle|^c = 0$
- $|\langle cM, 2.p\rangle|^c = |\langle M, p\rangle|^c$
- $|\langle MN, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$
- $|\lambda x.N|^c = \lambda x.|N|^c$, if $x \neq c$
- $|NP|^c = |N|^c|P|^c$ if $N \neq c$
- $|\langle \lambda x.M, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$, if $x \neq c$
- $|\langle NM, 2.p\rangle|^c = 2.|\langle M, p\rangle|^c$, if $N \neq c$

Let $\mathcal{F} \subseteq \mathsf{Path}$ then we define $|\langle M, \mathcal{F}\rangle|^c = \{|\langle M, p\rangle|^c \mid p \in \mathcal{F}\}$.

Now, c^n is indeed erased from $|c^n(M)|^c$ and from $|c^n(N)|^c$ for any $c^n(N)$ subterm of M.

Lemma 5.7. 1. Let $n \geq 0$ then $|c^n(M)|^c = |M|^c$.

2. $|\langle c^n(M), \mathcal{R}^{\beta\eta}_{c^n(M)}\rangle|^c = |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$.

3. $|\langle c^n(M), 2^n.p\rangle|^c = |\langle M, p\rangle|^c$.

4. Let $|M|^c = P$.

   - If $P \in \mathcal{V}$ then $\exists n \geq 0$ such that $M = c^n(P)$.
   - If $P = \lambda x.Q$ then $\exists n \geq 0$ such that $M = c^n(\lambda x.N)$ and $|N|^c = Q$.
   - If $P = P_1P_2$ then $\exists n \geq 0$ such that $M = c^n(M_1M_2)$, $M_1 \neq c$, $|M_1|^c = P_1$ and $|M_2|^c = P_2$.

Proof: 1), 2) and 3) By induction on n.
4) Each case is by induction on the structure of M. □

The next lemma shows that: if the c-erasures of two paths of M are equal, then these paths are also equal and inside a term; substituting x by $c(cx)$ is undone by c-erasure; c is definitely erased from the free variables of $|M|^c$; erasure propagates through substitutions; and c-erasing a $\Lambda\mathrm{I}_c$-term returns a $\Lambda\mathrm{I}$-term.

Lemma 5.8. 1. Let $r \in \{\beta I, \beta\eta\}$. If $p, p' \in \mathcal{R}^r_M$ and $|\langle M, p\rangle|^c = |\langle M, p'\rangle|^c$ then $p = p'$.

2. Let $x \neq c$. Then, $|M[x := c(cx)]|^c = |M|^c$.

3. Let $x \neq c$ and $p \in \mathcal{R}^{\beta\eta}_M$. Then, $|\langle M[x := c(cx)], p\rangle|^c = |\langle M, p\rangle|^c$.

4. If $M \in \mathcal{M}_c$ then $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(|M|^c)$.

5. If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $|M[x := N]|^c = |M|^c[x := |N|^c]$.

6. If $M \in \Lambda\mathrm{I}_c$ then $|M|^c \in \Lambda\mathrm{I}$.

7. Let $(\mathcal{M}_c, r) \in \{(\Lambda\mathrm{I}_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M, M_1, N_1, M_2, N_2 \in \mathcal{M}_c$.

   (a) If $p \in \mathcal{R}^r_M$ and $M \xrightarrow{p}_r M'$ then $|M|^c \xrightarrow{p'}_r |M'|^c$ such that $p' = |\langle M, p\rangle|^c$.

   (b) Let $x \neq c$, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$, $|\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$, $|M_1|^c = |M_2|^c$ and $|N_1|^c = |N_2|^c$. Then, $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

(c) Let $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$ and $|M_1|^c = |M_2|^c$. If $M_1 \xrightarrow{p_1}_r M_1'$, $M_2 \xrightarrow{p_2}_r M_2'$ such that $|\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$ then $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

Proof: 1) ... 6) By induction on the structure of M.

7a) By induction on the structure of p.

7b) and 7c) By induction on the structure of M_1. □

6. Reducibility method for the CR proofs w.r.t. βI- and $\beta\eta$-reductions

In this section, we introduce the reducibility semantics for both βI- and $\beta\eta$-reductions and establish its soundness (lemma 6.4). Then, we show that all terms typable in either $\mathcal{D}_I$ or $\mathcal{D}$ satisfy the Church-Rosser property, and that all terms of $\Lambda \mathrm{I}_c$ (resp. $\Lambda\eta_c$) are typable in system $\mathcal{D}_I$ (resp. $\mathcal{D}$).

The next definition introduces a reducibility semantics for Type^1 types.

Definition 6.1. 1. Let $r \in \{\beta I, \beta\eta\}$. We define the type interpretation $[\![-]\!]^r : \mathsf{Type}^1 \to 2^\Lambda$ by:

- $[\![\alpha]\!]^r = \mathsf{CR}^r$, where $\alpha \in \mathcal{A}$.
- $[\![\sigma \cap \tau]\!]^r = [\![\sigma]\!]^r \cap [\![\tau]\!]^r$.
- $[\![\sigma \to \tau]\!]^r = \{M \in \mathsf{CR}^r \mid \forall N \in [\![\sigma]\!]^r.\ MN \in [\![\tau]\!]^r\}$.

2. A set $\mathcal{X} \subseteq \Lambda$ is saturated iff $\forall n \geq 0.\ \forall M, N, M_1, \ldots, M_n \in \Lambda.\ \forall x \in \mathcal{V}.$
$M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$.

3. A set $\mathcal{X} \subseteq \Lambda\mathrm{I}$ is I-saturated iff $\forall n \geq 0.\ \forall M, N, M_1, \ldots, M_n \in \Lambda.\ \forall x \in \mathcal{V}.$
$x \in \mathrm{fv}(M) \Rightarrow M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$.

The next background lemma is familiar to many type systems.

Lemma 6.2. 1. If $\Gamma \vdash^{\beta I} M : \sigma$ then $M \in \Lambda\mathrm{I}$ and $\mathrm{fv}(M) = \mathrm{dom}(\Gamma)$.

2. Let $\Gamma \vdash^{\beta\eta} M : \sigma$. Then $\mathrm{fv}(M) \subseteq \mathrm{dom}(\Gamma)$ and if $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash^{\beta\eta} M : \sigma$.

3. Let $r \in \{\beta I, \beta\eta\}$. If $\Gamma \vdash^r M : \sigma$, $\sigma \sqsubseteq \sigma'$ and $\Gamma' \sqsubseteq \Gamma$ then $\Gamma' \vdash^r M : \sigma'$.

Proof: 1) By induction on $\Gamma \vdash^{\beta I} M : \sigma$.

2) By induction on $\Gamma \vdash^{\beta\eta} M : \sigma$.

3) First prove: if $\Gamma \vdash^r M : \sigma$, and $\sigma \sqsubseteq \sigma'$ then $\Gamma \vdash^r M : \sigma'$ by induction on $\sigma \sqsubseteq \sigma'$. Then, do the proof of 3. by induction on $\Gamma \vdash^r M : \sigma$. □

The next lemma states that the interpretations of types are saturated and only contain terms that are Church-Rosser. Krivine [Kri90] proved a similar result for $r = \beta$ and where CR^r_0 and CR^r were replaced by the corresponding sets of strongly normalising terms. Koletsos and Stavrinos [KS08] adapted Krivine's lemma for Church-Rosser w.r.t. β-reduction instead of strong normalisation. Here, we adapt the result to βI and $\beta\eta$.

Lemma 6.3. Let $r \in \{\beta I, \beta\eta\}$.

1. $\forall \sigma \in \mathsf{Type}^1.\ \mathsf{CR}_0^r \subseteq [\![\sigma]\!]^r \subseteq \mathsf{CR}^r$.
2. $\mathsf{CR}^{\beta I}$ is I-saturated.
3. $\mathsf{CR}^{\beta\eta}$ is saturated.
4. $\forall \sigma \in \mathsf{Type}^1.\ [\![\sigma]\!]^{\beta I}$ is I-saturated.
5. $\forall \sigma \in \mathsf{Type}^1.\ [\![\sigma]\!]^{\beta\eta}$ is saturated.

Proof: When $M \rightarrow_r^* N$ and $M \rightarrow_r^* P$, we write $M \rightarrow_r^* \{N, P\}$.
1) By induction on $\sigma \in \mathsf{Type}^1$.
2) Let $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta I}$ where $n \geq 0$, $x \in \mathrm{fv}(M)$ and $(\lambda x.M)NN_1 \ldots N_n \rightarrow_{\beta I}^* \{M_1, M_2\}$. By lemma 2.2.7, there exist M_1' and M_2' such that $M_1 \rightarrow_{\beta I}^* M_1'$, $M[x := N]N_1 \ldots N_n \rightarrow_{\beta I}^* M_1'$, $M_2 \rightarrow_{\beta I}^* M_2'$ and $M[x := N]N_1 \ldots N_n \rightarrow_{\beta I}^* M_2'$. Then, using $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta I}$.
3) Let $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta\eta}$ where $n \geq 0$ and $(\lambda x.M)NN_1 \ldots N_n \rightarrow_{\beta\eta}^* \{M_1, M_2\}$. By lemma 2.2.7, there exist M_1' and M_2' such that $M_1 \rightarrow_{\beta\eta}^* M_1'$, $M[x := N]N_1 \ldots N_n \rightarrow_{\beta\eta}^* M_1'$, $M_2 \rightarrow_{\beta\eta}^* M_2'$ and $M[x := N]N_1 \ldots N_n \rightarrow_{\beta\eta}^* M_2'$. Then we conclude using $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta\eta}$.
4) and 5) By induction on σ. □

Next, it is straightforward to adapt (and prove) the soundness lemma of [Kri90] to both $\vdash^{\beta I}$ and $\vdash^{\beta\eta}$.

Lemma 6.4. Let $r \in \{\beta I, \beta\eta\}$. If $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$ and $\forall i \in \{1, \ldots, n\}$, $N_i \in [\![\sigma_i]\!]^r$ then $M[(x_i := N_i)_1^n] \in [\![\sigma]\!]^r$.

Proof: By induction on $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$. □

Finally, we adapt a corollary from [KS08] to show that every term of Λ typable in system $\mathcal{D}_I$ (resp. $\mathcal{D}$) has the βI (resp. $\beta\eta$) Church-Rosser property.

Corollary 6.5. Let $r \in \{\beta I, \beta\eta\}$. If $\Gamma \vdash^r M : \sigma$ then $M \in \mathsf{CR}^r$.

Proof: Let $\Gamma = (x_i : \sigma_i)_n$. By lemma 6.3, $\forall i \in \{1, \ldots, n\}, x_i \in [\![\sigma_i]\!]^r$, so by lemma 6.4 and again by lemma 6.3, $M \in [\![\sigma]\!]^r \subseteq \mathsf{CR}^r$. □

To accommodate βI- and $\beta\eta$-reduction, the next lemma generalises a lemma given in [Kri90] (and used in [KS08]). This lemma states that every term of $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) is typable in system $\mathcal{D}_I$ (resp. $\mathcal{D}$).

Lemma 6.6. Let $\mathrm{fv}(M) \setminus \{c\} = \{x_1, \ldots, x_n\} \subseteq \mathrm{dom}(\Gamma)$ where $c \notin \mathrm{dom}(\Gamma)$.

1. If $M \in \Lambda\mathrm{I}_c$ then for $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(M)$, $\exists \sigma, \tau \in \mathsf{Type}^1$ such that if $c \in \mathrm{fv}(M)$ then $\Gamma', c : \sigma \vdash^{\beta I} M : \tau$, and if $c \notin \mathrm{fv}(M)$ then $\Gamma' \vdash^{\beta I} M : \tau$.
2. If $M \in \Lambda\eta_c$ then $\exists \sigma, \tau \in \mathsf{Type}^1$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} M : \tau$.

Proof: By induction on M. Note that by Lemma 5.2, $M \neq c$. □

7. Adapting Koletsos and Stavrinos's method [KS08] to βI-developments

Koletsos and Stavrinos [KS08] gave a proof of Church-Rosser for β-reduction for the intersection type system $\mathcal{D}$ of Definition 2.3 (studied in detail by Krivine in [Kri90]) and showed that this can be used to establish confluence of β-developments without using strong normalisation. In this section, we adapt their proof to βI. First, we adapt and formalise a number of definitions and lemmas given by Krivine in [Kri90] in order to make them applicable to βI-developments. Then, we adapt [KS08] to establish the confluence of βI-developments and hence of βI-reduction.

7.1. Formalising βI-developments

The next definition, taken from [Kri90] (and used in [KS08]) uses the variable c to "freeze" the βI-redexes of M which are not in the set $\mathcal{F}$ of βI-redex occurrences in M, and to neutralise applications so that they cannot be transformed into redexes after βI-reduction. For example, in $c(\lambda x.x)y$, c is used to freeze the βI-redex $(\lambda x.x)y$.

Definition 7.1. ($\Phi^c(-,-)$)
Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$.

1. If $M = x$ then $\mathcal{F} = \varnothing$ and $\Phi^c(x, \mathcal{F}) = x$
2. If $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ then $\Phi^c(\lambda x.N, \mathcal{F}) = \lambda x.\Phi^c(N, \mathcal{F}')$.
3. If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta I}$ then

$$\Phi^c(NP, \mathcal{F}) = \begin{cases} c\Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{if } 0 \notin \mathcal{F} \\ \Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{otherwise.} \end{cases}$$

The next lemma is an adapted version of a lemma which appears in [KS08] and which in turns adapts a lemma from [Kri90].

Lemma 7.2.

1. If $M \in \Lambda\mathrm{I}$, $c \notin \mathrm{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$ then
   (a) $\mathrm{fv}(M) = \mathrm{fv}(\Phi^c(M, \mathcal{F})) \setminus \{c\}$.
   (b) $\Phi^c(M, \mathcal{F}) \in \Lambda\mathrm{I}_c$.
   (c) $|\Phi^c(M, \mathcal{F})|^c = M$.
   (d) $|\langle \Phi^c(M, \mathcal{F}), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})} \rangle|^c = \mathcal{F}$.
2. Let $M \in \Lambda\mathrm{I}_c$.
   (a) $|\langle M, \mathcal{R}_M^{\beta I} \rangle|^c \subseteq \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta I} \rangle|^c)$.
   (b) $\langle |M|^c, |\langle M, \mathcal{R}_M^{\beta I} \rangle|^c \rangle$ is the one and only pair $\langle N, \mathcal{F} \rangle$ such that $N \in \Lambda\mathrm{I}$, $c \notin \mathrm{fv}(N)$, $\mathcal{F} \subseteq \mathcal{R}_N^{\beta I}$ and $\Phi^c(N, \mathcal{F}) = M$.

Proof: All items of 1) are by induction on the structure of $M \in \Lambda\mathrm{I}$. Note that 1b) uses 1a) and that 1d) uses 1b).

2a) By induction on the construction of $M \in \Lambda\mathrm{I}_c$. Note that by lemma 6, $|M|^c \in \Lambda\mathrm{I}$.

2b) By lemma 6, $|M|^c \in \Lambda\mathrm{I}$. By lemma 4, $c \notin \mathrm{fv}(|M|^c)$. By 2a, $|\langle M, \mathcal{R}_M^{\beta I}\rangle|^c \subseteq \mathcal{R}_{|M|^c}^{\beta I}$ and $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta I}\rangle|^c)$. To show unicity, let $\langle N', \mathcal{F}'\rangle$ be another such pair. We have $\mathcal{F}' \subseteq \mathcal{R}_{N'}^{\beta I}$ and $M = \Phi^c(N', \mathcal{F}')$. Then, $|M|^c = |\Phi^c(N', \mathcal{F}')|^c =^{1c} N'$ and $\mathcal{F}' =^{1d} |\langle \Phi^c(N', \mathcal{F}'), \mathcal{R}_{\Phi^c(N', \mathcal{F}')}^{\beta I}\rangle|^c = |\langle M, \mathcal{R}_M^{\beta I}\rangle|^c$. □

The next lemma is needed to define βI-developments.

Lemma 7.3. Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. Then, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}), p'\rangle|^c = p$.

Proof: By lemma 7.2.1c and lemma 5.8.5.8.1, there exists a unique $p' \in \mathcal{R}_{\Phi^c(M,\mathcal{F})}^{\beta I}$, such that $|\langle \mathcal{R}_{\Phi^c(M,\mathcal{F})}^{\beta I}, p'\rangle|^c = p$. By lemma 2.2.8, there exists P such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} P$. By lemma 5.8.7a, $M =^{7.2.1c} |\Phi^c(M, \mathcal{F})|^c \xrightarrow{p_0}_{\beta I} |P|^c$, such that $|\langle \mathcal{R}_{\Phi^c(M,\mathcal{F})}^{\beta I}, p'\rangle|^c = p_0$. So $p = p_0$ and by lemma 2.2.9, $M' = |P|^c$. Let $\mathcal{F}' = |\langle P, \mathcal{R}_P^{\beta I}\rangle|^c$. Because, $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} P$, by lemma 2 and lemma 7.2.1b, $P \in \Lambda\mathrm{I}_c$. By lemma 7.2.2a, $P = \Phi^c(M', \mathcal{F}')$ and $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$. By lemma 7.2.2b, $\mathcal{F}'$ is unique. □

We follow [Kri90] and define the set of βI-residuals of a set of βI-redexes $\mathcal{F}$ relative to a sequence of βI-redexes. First, we give the definition relative to one redex.

Definition 7.4. Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. By lemma 7.3, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}), p'\rangle|^c = p$. We call $\mathcal{F}'$ the set of **βI-residuals in M' of the set of βI-redexes $\mathcal{F}$ in M relative to p**.

Definition 7.5. (βI-development)
Let $M \in \Lambda\mathrm{I}$ where $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$. A one-step βI-development of $\langle M, \mathcal{F}\rangle$, denoted $\langle M, \mathcal{F}\rangle \to_{\beta Id} \langle M', \mathcal{F}'\rangle$, is a βI-reduction $M \xrightarrow{p}_{\beta I} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of βI-residuals in M' of the set of βI-redexes $\mathcal{F}$ in M relative to p. A βI-**development** is the transitive closure of a one-step βI-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta Id} M_n$ for the βI-development $\langle M, \mathcal{F}\rangle \to_{\beta Id}^* \langle M_n, \mathcal{F}_n\rangle$.

7.2. Confluence of βI-developments hence of βI-reduction

The next lemma is informative about βI-developments. It relates βI-reductions of frozen terms to βI-developments, and it states that given a βI-development, one can always define a new development that allows at least the same reductions.

Lemma 7.6. 1. Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$. Then: $\langle M, \mathcal{F}\rangle \to_{\beta Id}^* \langle M', \mathcal{F}'\rangle \iff \Phi^c(M, \mathcal{F}) \to_{\beta I}^* \Phi^c(M', \mathcal{F}')$.

2. Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}^{\beta I}_M$. If $\langle M, \mathcal{F}_1\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_1\rangle$ then there exists $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M'}$ such that $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and $\langle M, \mathcal{F}_2\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_2\rangle$.

Proof: 1) It sufficient to prove: $\langle M, \mathcal{F}\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'\rangle \iff \Phi^c(M, \mathcal{F}) \rightarrow_{\beta I} \Phi^c(M', \mathcal{F}')$.

- $\Rightarrow$) Let $\langle M, \mathcal{F}\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'\rangle$. By definition 7.5, $\exists p \in \mathcal{F}$ where $M \xrightarrow{p}_{\beta I} M'$ and $\mathcal{F}'$ is the set of βI-residuals in M' of the set of redexes $\mathcal{F}$ in M relative to p. By definition 7.4, $\Phi^c(M, \mathcal{F}) \rightarrow_{\beta I} \Phi^c(M', \mathcal{F}')$.

- $\Leftarrow$) Let $\Phi^c(M, \mathcal{F}) \rightarrow_{\beta I} \Phi^c(M', \mathcal{F}')$. By lemma 2.2.8, $\exists p \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}$ such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p}_{\beta I} \Phi^c(M', \mathcal{F}')$. Because, by lemma 7.2.1b, $\Phi^c(M, \mathcal{F}) \in \Lambda\mathrm{I}_c$, by lemma 5.8.7a and lemma 7.2.1c, $M = |\Phi^c(M, \mathcal{F})|^c \xrightarrow{p_0}_{\beta I} |\Phi^c(M', \mathcal{F}')|^c = M'$ such that $|\langle \Phi^c(M, \mathcal{F}), p_0\rangle|^c = p$. By definition 7.4, $\mathcal{F}'$ is the set of βI-residuals in M' of the set of redexes $\mathcal{F}$ in M relative to p_0. By definition 7.5, $\langle M, \mathcal{F}\rangle \rightarrow_{\beta d} \langle M', \mathcal{F}'\rangle$.

2) By lemma 7.2.1b, $\Phi^c(M, \mathcal{F}_1), \Phi^c(M, \mathcal{F}_2) \in \Lambda\mathrm{I}_c$. By lemma 7.2.1c, $|\Phi^c(M, \mathcal{F}_1)|^c = |\Phi^c(M, \mathcal{F}_2)|^c$. By lemma 7.2.1d, $|\langle \Phi^c(M, \mathcal{F}_1), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}\rangle|^c = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\langle \Phi^c(M, \mathcal{F}_2), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_2)}\rangle|^c$.

If $\langle M, \mathcal{F}_1\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_1\rangle$ then by lemma 1, $\Phi^c(M, \mathcal{F}_1) \rightarrow_{\beta I} \Phi^c(M', \mathcal{F}'_1)$. By lemma 2.2.8, there exists $p_1 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$ such that $\Phi^c(M, \mathcal{F}_1) \xrightarrow{p_1}_{\beta I} \Phi^c(M', \mathcal{F}'_1)$. Let $p_0 = |\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}, p_1\rangle|^c$, so by lemma 7.2.1d, $p_0 \in \mathcal{F}_1$. By lemma 5.8.7a and lemma 7.2.1c, $M \xrightarrow{p_0}_{\beta I} M'$.

By lemma 7.3 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M, \mathcal{F}_1) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}_1), p'\rangle|^c = p_0$. By lemma 2.2.8, $p' \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$. Since $p', p_1 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$, by lemma 5.8.1, $p' = p_1$. So, by lemma 2.2.9, $\Phi^c(M', \mathcal{F}') = \Phi^c(M', \mathcal{F}'_1)$. By lemma 7.2.1d, $\mathcal{F}' = \mathcal{F}'_1$ and $\mathcal{F}'_1 = |\langle \Phi^c(M', \mathcal{F}'_1), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}'_1)}\rangle|^c$.

By lemma 7.3 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M, \mathcal{F}_2) \xrightarrow{p_2}_{\beta I} \Phi^c(M', \mathcal{F}'_2)$ and $|\langle \Phi^c(M, \mathcal{F}_2), p_2\rangle|^c = p_0$.
By lemma 2.2.8, $p_2 \in \Phi^c(M, \mathcal{F}_2)$. By lemma 7.2.1d, $\mathcal{F}'_2 = |\langle \Phi^c(M', \mathcal{F}'_2), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}'_2)}\rangle|^c$.

Hence, by lemma 5.8.7c, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 1, $\langle M, \mathcal{F}_2\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_2\rangle$. □

The next lemma adapts the main theorem in [KS08] where as far as we know it first appeared.

Lemma 7.7. (Confluence of the βI-developments)
Let $M \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exist $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$, $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ and $M_3 \in \Lambda\mathrm{I}$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta Id} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta Id} M_3$.

Proof: If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exists $\mathcal{F}''_1, \mathcal{F}''_2$ such that $\langle M, \mathcal{F}_1\rangle \rightarrow^*_{\beta Id} \langle M_1, \mathcal{F}''_1\rangle$ and $\langle M, \mathcal{F}_2\rangle \rightarrow^*_{\beta Id} \langle M_2, \mathcal{F}''_2\rangle$. By definitions 7.4 and 7.5, $\mathcal{F}''_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}''_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$. Note that by definition 7.5 and lemma 2.2.4, $M_1, M_2 \in \Lambda\mathrm{I}$. By lemma 8.6.2, there exist $\mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ such that $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta Id} \langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle$ and $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta Id} \langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle$. By lemma 7.6.1, $T \rightarrow^*_{\beta I} T_1$ and $T \rightarrow^*_{\beta I} T_2$ where $T = \Phi^c(M, \mathcal{F}_1 \cup \mathcal{F}_2)$, $T_1 = \Phi^c(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $T_2 = \Phi^c(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$. Since by lemma 7.2.1b, $T \in \Lambda\mathrm{I}_c$ and by lemma 6.6.1, T is typable in the type system $\mathcal{D}_I$, so $T \in \mathsf{CR}^{\beta I}$ by corollary 6.5. So, by lemma 2.2b, there exists $T_3 \in \Lambda\mathrm{I}_c$, such that $T_1 \rightarrow^*_{\beta I} T_3$ and $T_2 \rightarrow^*_{\beta I} T_3$. Let $\mathcal{F}_3 = |\langle T_3, \mathcal{R}^{\beta I}_{T_3}\rangle|^c$ and $M_3 = |T_3|^{\beta I}$, then by lemma 7.2.2b, $T_3 = \Phi^c(M_3, \mathcal{F}_3)$. Hence, by lemma 7.6.1, $\langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle \rightarrow^*_{\beta Id} \langle M_3, \mathcal{F}_3\rangle$ and $\langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle \rightarrow^*_{\beta Id} \langle M_3, \mathcal{F}_3\rangle$, i.e. $M_1 \stackrel{\mathcal{F}''_1 \cup \mathcal{F}'''_1}{\rightarrow}_{\beta Id} M_3$ and $M_2 \stackrel{\mathcal{F}''_2 \cup \mathcal{F}'''_2}{\rightarrow}_{\beta Id} M_3$. □

We follow [Bar84] and [KS08] and define the following reduction relation:

Definition 7.8. Let $M, M' \in \Lambda\mathrm{I}$, such that $c \notin \mathrm{fv}(M)$. We define the following one step reduction: $M \rightarrow_{1I} M' \iff \exists \mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \rightarrow^*_{\beta Id} (M', \mathcal{F}')$.

Before establishing the main result of this section we need the following lemma that, among other things, relates βI-developments to βI-reductions (lemma 7.9.5).

Lemma 7.9.

1. Let $c \notin \mathrm{fv}(M)$. Then, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)} = \varnothing$.
2. Let $c \notin \mathrm{fv}(MN)$ and $x \neq c$. Then, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$.
3. Let $c \notin \mathrm{fv}(M)$. If $p \in \mathcal{R}^{\beta I}_M$ and $\Phi^c(M, \{p\}) \rightarrow_{\beta I} M'$ then $\mathcal{R}^{\beta I}_{M'} = \varnothing$.
4. Let $M \in \Lambda\mathrm{I}$ such that $c \notin \mathrm{fv}(M)$. If $M \stackrel{p}{\rightarrow}_{\beta I} M'$ then $\langle M, \{p\}\rangle \rightarrow_{\beta Id} \langle M', \varnothing\rangle$.
5. $\rightarrow^*_{\beta I} = \rightarrow^*_{1I}$.

Proof: 1), 2) and 3) By induction on the structure of M.
4) By lemma 2.2.8, $p \in \mathcal{R}^{\beta I}_M$. By lemma 7.3, there is a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M, \{p\}) \rightarrow_{\beta I} \Phi^c(M', \mathcal{F}')$. By lemma 7.9.3, $\mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}')} = \varnothing$, so $|\langle \Phi^c(M', \mathcal{F}'), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}')}\rangle|^c = \varnothing$ and $\mathcal{F}' = \varnothing$ by lemma 7.2.1d. Finally, by lemma 7.6.1, $\langle M, \{p\}\rangle \rightarrow_{\beta Id} \langle M', \varnothing\rangle$.
5) It is obvious that $\rightarrow^*_{1I} \subseteq \rightarrow^*_{\beta I}$. We prove $\rightarrow^*_{\beta I} \subseteq \rightarrow^*_{1I}$ by induction on the length of $M \rightarrow^*_{\beta I} M'$. □

Finally, we achieve what we started to do: the confluence of βI-reduction on $\Lambda\mathrm{I}$.

Lemma 7.10. $\Lambda\mathrm{I} \subseteq \mathsf{CR}^{\beta I}$.

Proof: Let $M \in \Lambda\mathrm{I}$ and c be a variable such that $c \notin \mathrm{fv}(M)$. Let $M \rightarrow^*_{\beta I} M_1$ and $M \rightarrow^*_{\beta I} M_2$. By lemma 5, $M \rightarrow^*_{1I} M_1$ and $M \rightarrow^*_{1I} M_2$. We prove the statement by induction on the length of $M \rightarrow^*_{1I} M_1$. □

8. Generalising Koletsos and Stavrinos's method [KS08] to $\beta\eta$-developments

In this section, we generalise the method of [KS08] to handle $\beta\eta$-reduction. This generalisation is not trivial since we needed to define developments involving η-reduction and to establish the important result of the closure under η-reduction of a defined set of frozen terms. These were the main reasons that led us to extend the various definitions related to developments. For example, clause (R4) of the definition of $\Lambda\eta_c$ in definition 5.1 aims to ensure closure under η-reduction. The definition of Λ_c in [Kri90] excluded such a rule and hence we lose closure under η-reduction as can be seen by the following example: Let $M = \lambda x.cNx \in \Lambda_c$ where $x \notin \mathrm{fv}(N)$ and $N \in \Lambda_c$, then $M \rightarrow_\eta cN \notin \Lambda_c$.

First, we formalise $\beta\eta$-residuals and $\beta\eta$-developments in section 8.1. Then, we compare our notion of $\beta\eta$-residuals with those of Curry and Feys [CF58] and Klop [Klo80] in section 8.2, establishing that we allow less residuals than Klop but we believe more residuals than Curry and Feys. Finally, we establish in section 8.3 the confluence of $\beta\eta$-developments and hence of $\beta\eta$-reduction.

8.1. Formalising $\beta\eta$-developments

The next definition adapts definition 7.1 to deal with $\beta\eta$-reduction. The variable c is used to 1) freeze the $\beta\eta$-redexes of M which are not in the set $\mathcal{F}$ of $\beta\eta$-redex occurrences in M; 2) neutralise applications so that they cannot be transformed into redexes after $\beta\eta$-reduction; and 3) neutralise bound variables so λ-abstraction cannot be transformed into redexes after $\beta\eta$-reduction. For example, in $\lambda x.y(c(cx))$ $(x \neq y)$, c is used to freeze the η-redex $\lambda x.yx$.

Definition 8.1. ($\Psi^c(-,-), \Psi^c_0(-,-)$)
Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$.

(P1) If $M \in \mathcal{V} \setminus \{c\}$ and $\mathcal{F} =^{\text{lem. 5.3}} \varnothing$ then:

$$\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n > 0\} \qquad \Psi^c_0(M, \mathcal{F}) = \{M\}$$

(P2) If $M = \lambda x.N$, $x \neq c$, and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{\text{lem. 5.3}} \mathcal{R}^{\beta\eta}_N$ then:

$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

$$\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

(P3) If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{\text{lem. 5.3}} \mathcal{R}^{\beta\eta}_N$, and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq^{\text{lem. 5.3}} \mathcal{R}^{\beta\eta}_P$ then:

$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$$

$$\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{N'P' \mid N' \in \Psi^c_0(N, \mathcal{F}_1) \wedge P' \in \Psi^c_0(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

The next lemma is needed to define $\beta\eta$-developments and relates the freezing and erasure operations.

Lemma 8.2. 1. Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$. We have:

(a) $\Psi^c_0(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.

(b) $\forall N \in \Psi^c(M, \mathcal{F}).\ \mathrm{fv}(M) = \mathrm{fv}(N) \setminus \{c\}$.

(c) $\Psi^c(M, \mathcal{F}) \subseteq \Lambda\eta_c$.

(d) Let $M = Nx$ where $x \notin \text{fv}(N) \cup \{c\}$ and $P \in \Psi^c_0(M, \mathcal{F})$. Then, $\mathcal{R}^{\beta\eta}_{\lambda x.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_P\}$.

(e) Let $M = Nx$. If $Px \in \Psi^c(Nx, \mathcal{F})$ then $Px \in \Psi^c_0(Nx, \mathcal{F})$.

(f) $\forall N \in \Psi^c(M, \mathcal{F}).\ \forall n \geq 0.\ c^n(N) \in \Psi^c(M, \mathcal{F})$.

(g) $\forall N \in \Psi^c(M, \mathcal{F}).\ |N|^c = M$.

(h) $\forall N \in \Psi^c(M, \mathcal{F}).\ \mathcal{F} = |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c$.

2. Let $M \in \Lambda\eta_c$. We have:

   (a) $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c \subseteq \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.

   (b) $\langle |M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\rangle$ is the unique $\langle N, \mathcal{F}\rangle$ where $c \notin \text{fv}(N)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_N$ and $M \in \Psi^c(N, \mathcal{F})$.

3. Let $M \in \Lambda$, where $c \notin \text{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. Then, $\exists$ a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ where $\forall N \in \Psi^c(M, \mathcal{F})$ there are $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p'\rangle|^c = p$.

Proof: 1a), 1b.), 1c), 1g) and 1h) By induction on the structure of M.

1d) and 1e) By case on the belonging of 0 in $\mathcal{F}$.

1f) By case on the structure of M and induction on n.

2a) By induction on the construction of M.

2b) By lemmas 5.8.4 and 8.2.2a, $c \notin \text{fv}(|M|^c)$, $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c \subseteq \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$. If $\langle N', \mathcal{F}'\rangle$ is another such pair then $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{N'}$ and $M \in \Psi^c(N', \mathcal{F}')$ and by lemmas 8.2.1g and 8.2.1h, $|M|^c = N'$ and $\mathcal{F}' = |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$. □

Definition 8.3. ($\beta\eta$-development)

1. Let $M \in \Lambda$, $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. By lemma 8.2.3, $\exists$ a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that $\forall N \in \Psi^c(M, \mathcal{F})$, there are $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ where $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p'\rangle|^c = p$. We call $\mathcal{F}'$ the set of $\beta\eta$-**residuals in** M' **of the set of** $\beta\eta$-**redexes** $\mathcal{F}$ **in** M **relative to** p.

2. Let $M \in \Lambda$, where $c \notin \text{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_M$. A one-step $\beta\eta$-development of $\langle M, \mathcal{F}\rangle$, denoted $\langle M, \mathcal{F}\rangle \to_{\beta\eta d} \langle M', \mathcal{F}'\rangle$, is a $\beta\eta$-reduction $M \xrightarrow{p}_{\beta\eta} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals in M' of the set of $\beta\eta$-redexes $\mathcal{F}$ in M relative to p. A $\beta\eta$-**development** is the transitive closure of a one-step $\beta\eta$-development. We write $M \xrightarrow{\mathcal{F}}_{\beta\eta d} M'$ for the $\beta\eta$-development $\langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$.

8.2. Comparison with Curry and Feys [CF58] and Klop [Klo80]

A common definition of a $\beta\eta$-residual is given by Curry and Feys [CF58] (p. 117, 118). Another definition of $\beta\eta$-residual (called λ-residual) is presented by Klop [Klo80] (definition 2.4, p. 254). Klop shows that these definitions allow one to prove different properties of developments. Following the definition of a $\beta\eta$-residual given by Curry and Feys [CF58] (and as pointed out in [CF58, Klo80, BBKV76]), if the η-redex $\lambda x.(\lambda y.M)x$, where $x \notin \mathrm{fv}(\lambda y.M)$, is reduced in the term $P = (\lambda x.(\lambda y.M)x)N$ to give the term $Q = (\lambda y.M)N$, then Q is not a $\beta\eta$-residual of P in P (note that following the definition of a λ-residual given by [Klo80], Q is a λ-residual of the redex $(\lambda y.M)x$ in P since the λ of the redex Q is the same as the λ of the redex $(\lambda y.M)x$ in P). Moreover, if the β-redex $(\lambda y.My)x$, where $y \notin \mathrm{fv}(M)$, is reduced in the term $P = \lambda x.(\lambda y.My)x$ to give the term $Q = \lambda x.Mx$, then Q is not a $\beta\eta$-residual of P in P (note that following the definition of a λ-residual given by [Klo80], Q is a λ-residual of the redex P in P since the λ of the redex Q is the same as the λ of the redex P in P). Our definition 8.3.1 differs from the common one stated by Curry and Feys [CF58] by the cases illustrated in the following example: $\Psi^c((\lambda x.(\lambda y.M)x)N, \{0, 1.0, 1.1.0\}) = \{c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \mid n \geq 0 \wedge P \in \Psi^c(M, \varnothing) \wedge Q \in \Psi^c(N, \varnothing)\}$, where $x \notin \mathrm{fv}(\lambda y.M)$. Let $p = 1.0$ then $(\lambda x.(\lambda y.M)x)N \xrightarrow{p}_{\beta\eta} (\lambda y.M)N$. Moreover, $P_0 = c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \xrightarrow{p'}_{\beta\eta} c^n((\lambda y.P[y := c(cy)])Q)$ such that $n \geq 0$, $P \in \Psi^c(M, \varnothing)$, $Q \in \Psi^c(N, \varnothing)$, and $|\langle P_0, p'\rangle|^c = |\langle P_0, 2^n.1.0\rangle|^c = p$, and $c^n((\lambda y.P[y := c(cy)])Q) \in \Psi^c((\lambda y.M)N, \{0\})$.

Let us now compare our definition of $\beta\eta$-residuals to the λ-residuals given by Klop [Klo80]. We believe that we accept more redexes as residuals of a set of redexes than Curry and Feys [CF58] (as shown by the examples of this section) and less than Klop.

We introduce the two calculi $\bar{\Lambda}$ and $\bar{\Lambda\eta_c}$ which are labelled versions of the calculi Λ and $\Lambda\eta_c$:

$$
\begin{array}{llll}
t & \in & \bar{\Lambda} & ::= \; x \mid \lambda_n x.t \mid t_1 t_2 \\
v & \in & \mathsf{ABS}_c & ::= \; \lambda_n \bar{x}.w\bar{x} \mid \lambda_n \bar{x}.u[\bar{x} := c(c\bar{x})], \text{ where } \bar{x} \notin \mathrm{fv}(w) \\
w & \in & \mathsf{APP}_c & ::= \; v \mid cu \\
u & \in & \bar{\Lambda\eta_c} & ::= \; \bar{x} \mid v \mid wu \mid cu
\end{array}
$$

where $\bar{x}, \bar{y} \in \mathcal{V} \setminus \{c\}$. Note that $\mathsf{ABS}_c \subseteq \mathsf{APP}_c \subseteq \bar{\Lambda\eta_c} \subseteq \bar{\Lambda}$.

The labels enable to distinguish two different occurrences of a λ.

Since these two calculi are only labelled versions of Λ and $\Lambda\eta_c$, let us assume in this section that the work done so far holds when Λ ans $\Lambda\eta_c$ are replaced by $\bar{\Lambda}$ and $\bar{\Lambda\eta_c}$.

Klop [Klo80] defines his λ-residuals as follows:

> "Let $\mathcal{R} = M_0 \to M_1 \to \ldots \to M_k \to \ldots$ be a $\beta\eta$-reduction, R_0 a redex in M_0 and R_k a redex in M_k such that the head-λ of R_k descends from that of R_0.
> Regardless whether R_0, R_k are β- or η-redexes, R_k is called a λ-residual of R_0 via $\mathcal{R}$."

We define the head-λ of a $\beta\eta$-redex by: $\mathrm{headlam}((\lambda_n x.t_1)t_2) = \langle 1, n\rangle$ and $\mathrm{headlam}(\lambda_n x.t_0 x) = \langle 2, n\rangle$, if $x \notin \mathrm{fv}(t_0)$. If $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$ we define $\mathrm{headlamred}(t, \mathcal{F})$ to be $\{\langle i, n\rangle \mid \exists p \in \mathcal{F}.\ \mathrm{headlam}(t|_p) = \langle i, n\rangle\}$. We define $\mathrm{hlr}(t)$ to be $\mathrm{headlamred}(t, \mathcal{R}_t^{\beta\eta})$.

The following lemma states the equality between the head-λ's of a set $\mathcal{F}$ of $\beta\eta$-redexes of a term t and the head-λ's of the $\beta\eta$-redexes of any term u in the application of the function Ψ^c to t and $\mathcal{F}$:

Lemma 8.4. Let $c \notin \mathrm{fv}(t)$ and $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$. If $u \in \Psi^c(t, \mathcal{F})$ then $\mathrm{hlr}(u) = \mathrm{headlamred}(t, \mathcal{F})$.

Proof: By induction on the structure of t. □

The following lemma states that if a term u_1 in $\Lambda\bar{\eta}_c$ reduces to a term u' then the set of head-λ's of the $\beta\eta$-redexes of u' is included in the set of head-λ's of the $\beta\eta$-redexes of u_1.

Lemma 8.5. If $u_1 \in \Lambda\bar{\eta}_c$ and $u_1 \xrightarrow{p}_{\beta\eta} u'$ then $\mathrm{hlr}(u') \subseteq \mathrm{hlr}(u_1)$.

Proof: By induction on the size of u_1 and then by case on the structure of u_1. □

Let us now prove that, following our definition, the set of head-λ's of the $\beta\eta$-residuals of a set of $\beta\eta$-redexes in a term is included in the set of head-λ's of the considered set of $\beta\eta$-redexes.

Let $c \notin \mathrm{fv}(t)$, $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$ and $t \xrightarrow{p}_{\beta\eta} t'$ then by definition 8.3.1, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{t'}^{\beta\eta}$, such that for all $u \in \Psi^c(t,\mathcal{F})$ (by lemma 8.2.1c, $u \in \Lambda\bar{\eta}_c$), there exist $u' \in \Psi^c(t',\mathcal{F}')$ and $p' \in \mathcal{R}_u^{\beta\eta}$ such that $u \xrightarrow{p'}_{\beta\eta} u'$ and $|\langle u, p'\rangle|^c = p$. The set $\mathcal{F}'$ is the set of $\beta\eta$-residuals in t' of the set of redexes $\mathcal{F}$ in t relative to p. By lemma 2.2.3, $c \notin \mathrm{fv}(t')$. By definition $\Psi^c(t,\mathcal{F})$ is not empty. Let $u \in \Psi^c(t,\mathcal{F})$ then there exist $u' \in \Psi^c(t',\mathcal{F}')$ and $p' \in \mathcal{R}_u^{\beta\eta}$ such that $u \xrightarrow{p'}_{\beta\eta} u'$ and $|\langle u, p'\rangle|^c = p$. By lemma 8.5, $\mathrm{hlr}(u') \subseteq \mathrm{hlr}(u)$. So, by lemma 8.4, $\mathrm{headlamred}(t',\mathcal{F}') \subseteq \mathrm{headlamred}(t,\mathcal{F})$.

However, this is not enough to match Klop's definition of λ-residuals. As a matter of fact, as we show below, we can find t and $\mathcal{F}$ such that, following Klop's definition, $p_0 \in \mathcal{R}_{t'}^{\beta\eta}$ and p_0 is a λ-residual of $\mathcal{F}$ via p but $p_0 \notin \mathcal{F}'$. Let $t = (\lambda_0 x.xy)(\lambda_1 z.yz) \xrightarrow{0}_{\beta\eta} (\lambda_1 z.yz)y = t'$ and let $\mathcal{F} = \{0, 2.0\}$. Then $\Psi^c(t,\mathcal{F}) = \{c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z))) \mid n_1,n_2,n_3,n_4 \geq 0\}$. Let $u \in \Psi^c(t,\mathcal{F})$, then $u = c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z)))$ such that $n_1,n_2,n_3,n_4 \geq 0$. We obtain $u = c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z))) \xrightarrow{p_0}_{\beta\eta} c^{n_1+n_2}(c^{n_3+3}(\lambda_1 z.c^{n_4+1}(y)z)y) = u'$ such that $p_0 = 2^{n_1}.0$. Then $\mathcal{F}' = \{1.0\}$ is the set of $\beta\eta$-residuals in t' of the set of redexes $\mathcal{F}$ in t relative to p. But 0 is a λ-residual of $\mathcal{F}$ via 0 and $0 \notin \mathcal{F}'$.

It turns out that, though our $\beta\eta$-residuals are λ-residuals, the opposite does not hold. For example: $t = \lambda_n\bar{x}.(\lambda_m\bar{y}.z\bar{y})\bar{x} \xrightarrow{1.0}_{\beta} \lambda_n\bar{x}.z\bar{x} = t'$ and $0 \in \mathcal{R}_{t'}^{\beta\eta}$, but $u = \lambda_n\bar{x}.(\lambda_m\bar{y}.cz(c(c\bar{y})))\bar{x} \in \Psi^c(t,\{0,1.0\})$ and $u = \lambda_n\bar{x}.(\lambda_m\bar{y}.cz(c(c\bar{y})))\bar{x} \xrightarrow{1.0}_{\beta\eta} \lambda_n\bar{x}.cz(c(c\bar{x})) = u'$ and $0 \notin \mathcal{R}_{u'}^{\beta\eta}$.

8.3. Confluence of $\beta\eta$-developments and hence of $\beta\eta$-reduction

The next lemma relates $\beta\eta$-reductions of frozen terms to $\beta\eta$-developments, and states that given a $\beta\eta$-development, one can always define a new development that allows at least the same reductions.

Lemma 8.6. 1. Let $M \in \Lambda$, where $c \notin \mathrm{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. Then:
$\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle \iff \exists N \in \Psi^c(M,\mathcal{F}).\ \exists N' \in \Psi^c(M',\mathcal{F}').\ N \rightarrow^*_{\beta\eta} N'$

2. Let $M \in \Lambda$, such that $c \notin \mathrm{fv}(M)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}_M^{\beta\eta}$. If $\langle M,\mathcal{F}_1\rangle \rightarrow_{\beta\eta d} \langle M',\mathcal{F}_1'\rangle$ then there exists $\mathcal{F}_2' \subseteq \mathcal{R}_{M'}^{\beta\eta}$ such that $\mathcal{F}_1' \subseteq \mathcal{F}_2'$ and $\langle M,\mathcal{F}_2\rangle \rightarrow_{\beta\eta d} \langle M',\mathcal{F}_2'\rangle$.

Proof: 1) Note that $\Psi^c(M,\mathcal{F}) \neq \varnothing$. Then, it is sufficient to prove:

- $\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle \Rightarrow \forall N \in \Psi^c(M,\mathcal{F}).\ \exists N' \in \Psi^c(M',\mathcal{F}').\ N \rightarrow^*_{\beta\eta} N'$ by induction on the reduction $\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle$.

- $\exists N \in \Psi^c(M, \mathcal{F}).\ \exists N' \in \Psi^c(M', \mathcal{F}').\ N \rightarrow^*_{\beta\eta} N' \Rightarrow \langle M, \mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$ by induction on the reduction $N \rightarrow^*_{\beta\eta} N'$ such that $N \in \Psi^c(M, \mathcal{F})$ and $N' \in \Psi^c(M', \mathcal{F}')$.

2) By lemma 8.2.1c, $\Psi^c(M, \mathcal{F}_1), \Psi^c(M, \mathcal{F}_2) \subseteq \Lambda\eta_c$. For all $N_1 \in \Psi^c(M, \mathcal{F}_1)$ and $N_2 \in \Psi^c(M, \mathcal{F}_2)$, by lemma 8.2.1g, $|N_1|^c = |N_2|^c$ and by lemma 8.2.1h, $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$.

If $\langle M, \mathcal{F}_1\rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}'_1\rangle$ then by 1), there exist $N_1 \in \Psi^c(M, \mathcal{F}_1)$ and $N'_1 \in \Psi^c(M', \mathcal{F}'_1)$ such that $N_1 \rightarrow_{\beta\eta} N'_1$. By definition, there exists p_1 such that $N_1 \xrightarrow{p_1}_{\beta\eta} N'_1$, and by lemma 2.2.8, $p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Let $p_0 = |\langle N_1, p_1\rangle|^c$, so by lemma 8.2.1h, $p_0 \in \mathcal{F}_1$. By lemma 5.8.7a and lemma 8.2.1g, $M \xrightarrow{p_0}_{\beta\eta} M'$.

By lemma 8.2.3 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ such that for all $P_1 \in \Psi^c(M, \mathcal{F}_1)$ there exist $P'_1 \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_{P_1}$ such that $P_1 \xrightarrow{p'}_{\beta\eta} P'_1$ and $|\langle P_1, p'\rangle|^c = p_0$.

Because, $N_1 \in \Psi^c(M, \mathcal{F}_1)$, there exist $P'_1 \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_{N_1}$ such that $N_1 \xrightarrow{p'}_{\beta\eta} P'_1$ and $|\langle N_1, p'\rangle|^c = p_0$. Since $p', p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$, by lemma 1, $p' = p_1$, so by lemma 2.2.9, $P'_1 = N'_1$. By lemma 8.2.1h, $\mathcal{F}' = |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c = \mathcal{F}'_1$.

By lemma 8.2.3 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $P_2 \in \Psi^c(M, \mathcal{F}_2)$ there exist $P'_2 \in \Psi^c(M', \mathcal{F}'_2)$ and $p_2 \in \mathcal{R}^{\beta\eta}_{P_2}$ such that $P_2 \xrightarrow{p_2}_{\beta\eta} P'_2$ and $|\langle P_2, p_2\rangle|^c = p_0$.

Since $\Psi^c(M, \mathcal{F}_2) \neq \varnothing$, let $N_2 \in \Psi^c(M, \mathcal{F}_2)$. So, there exist $N'_2 \in \Psi^c(M', \mathcal{F}'_2)$ and $p_2 \in \mathcal{R}^{\beta\eta}_{N_2}$ such that $N_2 \xrightarrow{p_2}_{\beta\eta} N'_2$ and $|\langle N_2, p_2\rangle|^c = p_0$. By lemma 8.2.1h, $\mathcal{F}'_2 = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c$.

Hence, by lemma 5.8.7c, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 8.6.1, $\langle M, \mathcal{F}_2\rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}'_2\rangle$. □

Lemma 8.7. (Confluence of the $\beta\eta$-developments)
Let $M \in \Lambda$ such that $c \notin \text{fv}(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exist $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$, $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ and $M_3 \in \Lambda$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta\eta d} M_3$.

Proof: If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exist $\mathcal{F}''_1, \mathcal{F}''_2$ such that $\langle M, \mathcal{F}_1\rangle \rightarrow^*_{\beta\eta d} \langle M_1, \mathcal{F}''_1\rangle$ and $\langle M, \mathcal{F}_2\rangle \rightarrow^*_{\beta\eta d} \langle M_2, \mathcal{F}''_2\rangle$. By definitions 8.3.1 and 8.3.2, $\mathcal{F}''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$. By lemma 8.6.2, there exist $\mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ such that $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta\eta d} \langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle$ and $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta\eta d} \langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle$. By lemma 7.6.1 there exist $T \in \Psi^c(M, \mathcal{F}_1 \cup \mathcal{F}_2)$, $T_1 \in \Psi^c(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $T_2 \in \Psi^c(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$ such that $T \rightarrow^*_{\beta\eta} T_1$ and $T \rightarrow^*_{\beta\eta} T_2$.

Because by lemma 8.2.1c, $T \in \Lambda\eta_c$ and by lemma 6.6.2, T is typable in the type system $\mathcal{D}$, so $T \in \mathsf{CR}^{\beta\eta}$ by corollary 6.5. So, by lemma 2.2a, there exists $T_3 \in \Lambda\eta_c$, such that $T_1 \rightarrow^*_{\beta\eta} T_3$ and $T_2 \rightarrow^*_{\beta\eta} T_3$. Let $\mathcal{F}_3 = |\langle T_3, \mathcal{R}^{\beta\eta}_{T_3}\rangle|^c$ and $M_3 = |T_3|^{\beta\eta}$, then by lemma 8.2.2a, $\mathcal{F}_3 \subseteq \mathcal{R}^{\beta\eta}_{M_3}$ and $T_3 \in \Psi^c(M_3, \mathcal{F}_3)$. Hence, by lemma 8.6.1, $\langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle \rightarrow^*_{\beta\eta d} \langle M_3, \mathcal{F}_3\rangle$ and $\langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle \rightarrow^*_{\beta\eta d} \langle M_3, \mathcal{F}_3\rangle$, i.e. $M_1 \xrightarrow{\mathcal{F}''_1 \cup \mathcal{F}'''_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}''_2 \cup \mathcal{F}'''_2}_{\beta\eta d} M_3$. □

Definition 8.8. Let $c \notin \text{fv}(M)$. We define the following one step reduction:

$$M \rightarrow_1 M' \iff \exists \mathcal{F}, \mathcal{F}', \langle M, \mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M', \mathcal{F}'\rangle$$

The next lemma is needed for the main proof of this section: the Church-Rosser property of the untyped λ-calculus w.r.t. $\beta\eta$-reduction and relates $\beta\eta$-developments to $\beta\eta$-reductions (lemma 8.9.5).

Lemma 8.9. 1. Let $c \notin \mathrm{fv}(M)$. $\forall P \in \Psi^c(M, \varnothing)$. $\mathcal{R}^{\beta\eta}_P = \varnothing$.

2. Let $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$ and $x \neq c$. $\forall P \in \Psi^c(M, \varnothing)$. $\forall Q \in \Psi^c(N, \varnothing)$. $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$.

3. Let $c \notin \mathrm{fv}(M)$. If $p \in \mathcal{R}^{\beta\eta}_M$, $P \in \Psi^c(M, \{p\})$ and $P \rightarrow_{\beta\eta} Q$ then $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

4. Let $c \notin \mathrm{fv}(M)$. If $M \xrightarrow{p}_{\beta\eta} M'$ then $\langle M, \{p\}\rangle \rightarrow_{\beta\eta d} \langle M', \varnothing\rangle$.

5. $\rightarrow^*_{\beta\eta} = \rightarrow^*_1$.

Proof: 1), 2) and 3) By induction on the structure of M.
4) By lemma 2.2.8, $p \in \mathcal{R}^{\beta\eta}_M$. By lemma 8.2.3, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $N \in \Psi^c(M, \{p\})$, there exists $N' \in \Psi^c(M', \mathcal{F}')$ such that $N \rightarrow_{\beta\eta} N'$. Note that $\Psi^c(M, \{p\}) \neq \varnothing$. Let $N \in \Psi^c(M, \{p\})$ then there exists $N' \in \Psi^c(M', \mathcal{F}')$ such that $N \rightarrow_{\beta\eta} N'$. By lemma 3, $\mathcal{R}^{\beta\eta}_{N'} = \varnothing$, so $|\langle N', \mathcal{R}^{\beta\eta}_{N'}\rangle|^c = \varnothing$ and by lemma 8.2.1h, $\mathcal{F}' = \varnothing$. Finally, by lemma 8.6.1, $\langle M, \{p\}\rangle \rightarrow_{\beta\eta d} \langle M', \varnothing\rangle$.
5) By definition $\rightarrow^*_1 \subseteq \rightarrow^*_{\beta\eta}$. We prove by induction on $M \rightarrow^*_{\beta\eta} M'$ that $\rightarrow^*_{\beta\eta} \subseteq \rightarrow^*_1$. □

Finally, the next lemma is the main result of this section.

Lemma 8.10. $\Lambda \subseteq \mathsf{CR}^{\beta\eta}$.

Proof: Let $M \in \Lambda$ and let $c \in \mathcal{V}$ such that $c \notin \mathrm{fv}(M)$. Let $M \rightarrow^*_{\beta\eta} M_1$ and $M \rightarrow^*_{\beta\eta} M_2$. Then by lemma 5, $M \rightarrow^*_1 M_1$ and $M \rightarrow^*_1 M_2$. We prove the statement by induction on $M \rightarrow^*_1 M_1$. □

9. Conclusion

Reducibility is a powerful concept which has been applied to prove a number of properties of the λ-calculus (Church-Rosser, strong normalisation, etc.) using a single method. This paper studied two reducibility methods which exploit the passage from typed (in an intersection type system) to untyped terms. We showed that the first method given by Ghilezan and Likavec [GL02] fails in its aim and we have only been able to provide a partial solution. We adapted the second method given by Koletsos and Stavrinos [KS08] from β to βI-reduction and we generalised it to $\beta\eta$-reduction. There are differences in the type systems chosen and the methods of reducibility used by Ghilezan and Likavec on one hand and by Koletsos and Stavrinos on the other. Koletsos and Stavrinos use system $\mathcal{D}$ [Kri90], which has elimination rules for intersection types whereas Ghilezan and Likavec use $\lambda\cap$ and $\lambda\cap^\Omega$ with subtyping. Moreover, Koletsos and Stavrinos's method depends on the inclusion of typable λ-terms in the set of λ-terms possessing the Church-Rosser property, whereas (the working part of) Ghilezan and Likavec's method aims to prove the inclusion of typable terms in an arbitrary subset of the untyped λ-calculus closed by some properties. Moreover, Ghilezan and Likavec consider the $\mathrm{VAR}(\mathcal{P})$, $\mathrm{SAT}(\mathcal{P})$, and $\mathrm{CLO}(\mathcal{P})$ predicates whereas Koletsos and Stavrinos use standard reducibility methods through saturated sets. Koletsos and Stavrinos prove the confluence of developments using the confluence of typable λ-terms in system $\mathcal{D}$ (the authors prove that even a simple type system is sufficient). The advantage of Koletsos and Stavrinos's proof of confluence of developments is that strong normalisation is not needed.

References

[Bar84] H. P. Barendregt. *The Lambda Calculus, its Syntax and Semantics*. North-Holland, revised edition, 1984.

[BBKV76] H. Barendregt, J. A. Bergstra, J. W. Klop, and H. Volken. Degrees, reductions and representability in the lambda calculus. Technical Report Preprint no. 22, University of Utrecht, Department of Mathematics, 1976.

[CDC80] M. Coppo and M. Dezani-Ciancaglini. An extension of the basic functionality theory for the λ-calculus. *Notre Dame Journal of Formal Logic.*, 21(4):685–693, 1980.

[CDCV80] M. Coppo, M. Dezani-Ciancaglini, and B. Venneri. Principal type schemes and λ-calculus semantic. In J. P. Seldin J. R. Hindley, editor, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Academic Press, 1980.

[CDCV81] M. Coppo, M. Dezani-Ciancaglini, and B. Venneri. Functional characters of solvable terms. *Mathematische Logik Und Grundlagen der Mathematik*, 27:45–58, 1981.

[CF58] H. B. Curry and R. Feys. *Combinatory Logic, vol. 1*. Elsevier, 1958.

[Gal90] J. Gallier. On Girard's "candidats de reductibilité". In P. Odifreddi, editor, *Logic and Computer Science*, pages 123–203. Academic Press, 1990.

[Gal97] J. Gallier. On the correspondance between proofs and λ-terms. *Cahiers du centre de logique*, 1997. Available at `http://www.cis.upenn.edu/~jean/gbooks/logic.html` (last visited 2008–02–6).

[Gal98] J. Gallier. Typing untyped λ-terms, or reducibility strikes again!. *Annals of Pure and Applied Logic*, 91(2-3):231–270, 1998.

[Gir72] J.-Y. Girard. *Interprétation Fonctionnelle et Elimination des Coupures de l'Arithmétique d'Ordre Supérieur*. PhD thesis, Université Paris 7, 1972.

[GL02] S. Ghilezan and S. Likavec. Reducibility: A ubiquitous method in lambda calculus with intersection types. *Electronic Notes in Theoretical Compututer Science*, 70(1), 2002.

[Kle45] S. C. Kleene. On the interpretation of intuitionistic number theory. *The Journal of Symbolic Logic*, 10(4):109–124, 1945.

[Klo80] J. W. Klop. *Combinatory Reductions Systems.* PhD thesis, Mathematisch Centrum, Amsterdam, 1980.

[Kol85] G. Koletsos. Church-Rosser theorem for typed functional systems. *Journal of Symbolic Logic*, 50(3):782–790, 1985.

[Kri90] J. L. Krivine. *Lambda-calcul, types et modèles.* Masson, 1990.

[KS08] G. Koletsos and G. Stavrinos. Church-Rosser property and intersection types. *Australian Journal of Logic*, 6:37–54, 2008.

[Mit90] John C. Mitchell. Type systems for programming languages. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Sematics (B)*, pages 365–458. MIT press, 1990.

[Mit96] John C. Mitchell. *Foundations for programming languages*. Foundation of computing series. MIT Press, 1996.

[Sta85] Richard Statman. Logical relations and the typed lambda-calculus. *Information and Control*, 65(2/3):85–97, 1985.

[Tai67] W. W. Tait. Intensional interpretations of functionals of finite type I. *Journal of Symbolic Logic*, 32(2):198–212, 1967.

A. Proofs of section 2

Proof(Lemma 2.2)**:**

1 We prove the lemma by induction on p.

- Let $p = 0$.

  Let $M \xrightarrow{0}_{\beta\eta} M'$ then either $M = (\lambda x.P)Q$ and $M' = P[x := Q]$ and so $M \xrightarrow{0}_{\beta} M'$. Or $M = \lambda x.M'x$ such that $x \notin \text{fv}(M')$ and so $M \xrightarrow{0}_{\eta} M'$.

  Let $M \rightarrow_{\eta} 0M'$ then $M = \lambda x.M'x$ such that $x \notin \text{fv}(M')$ and so $M \xrightarrow{0}_{\beta\eta} M'$.

  Let $M \rightarrow_{\beta} 0M'$ then $M = (\lambda x.P)Q$ and $M' = P[x := Q]$ and so $M \xrightarrow{0}_{\beta\eta} M'$.

- Let $p = 1.p'$.

  Let $M \xrightarrow{p}_{\beta\eta} M'$ then either $M = \lambda x.N$, $M' = \lambda x.N'$ and $N \xrightarrow{p'}_{\beta\eta} N'$. By IH, $N \xrightarrow{p}_{\beta} N'$ or $N \xrightarrow{p'}_{\eta} N'$. So $M \xrightarrow{p}_{\beta} M'$ or $M \xrightarrow{p}_{\eta} M'$. Or $M = PQ$, $M' = P'Q$ and $P \xrightarrow{p'}_{\beta\eta} P'$. By IH, $P \xrightarrow{p}_{\beta} P'$ or $P \xrightarrow{p'}_{\eta} P'$. So $M \xrightarrow{p}_{\beta} M'$ or $M \xrightarrow{p}_{\eta} M'$.

  Let $M \xrightarrow{p}_{\eta} M'$ then either $M = \lambda x.N$, $M' = \lambda x.N'$ and $N \xrightarrow{p'}_{\eta} N'$. By IH, $N \xrightarrow{p}_{\beta\eta} N'$, so $M \xrightarrow{p}_{\beta\eta} M'$. Or $M = PQ$, $M' = P'Q$ and $P \xrightarrow{p'}_{\eta} P'$. By IH, $P \xrightarrow{p}_{\beta\eta} P'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

  Let $M \xrightarrow{p}_{\beta} M'$ then either $M = \lambda x.N$, $M' = \lambda x.N'$ and $N \xrightarrow{p'}_{\beta} N'$. By IH, $N \xrightarrow{p}_{\beta\eta} N'$, so $M \xrightarrow{p}_{\beta\eta} M'$. Or $M = PQ$, $M' = P'Q$ and $P \xrightarrow{p'}_{\beta} P'$. By IH, $P \xrightarrow{p}_{\beta\eta} P'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

- Let $p = 2.p'$.

  Let $M \xrightarrow{p}_{\beta\eta} M'$ then $M = PQ$, $M' = PQ'$ and $Q \xrightarrow{p'}_{\beta\eta} Q'$. By IH, $Q \xrightarrow{p}_{\beta} Q'$ or $Q \xrightarrow{p'}_{\eta} Q'$. So $M \xrightarrow{p}_{\beta} M'$ or $M \xrightarrow{p}_{\eta} M'$.

  Let $M \xrightarrow{p}_{\eta} M'$ then $M = PQ$, $M' = PQ'$ and $Q \xrightarrow{p'}_{\eta} Q'$. By IH, $Q \xrightarrow{p}_{\beta\eta} Q'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

  Let $M \xrightarrow{p}_{\beta} M'$ then $M = PQ$, $M' = PQ'$ and $Q \xrightarrow{p'}_{\beta} Q'$. By IH, $Q \xrightarrow{p}_{\beta\eta} Q'$, so $M \xrightarrow{p}_{\beta\eta} M'$.

2 We prove this lemma by induction on the structure of M_1.

- Either $M_1 = x$, then $\text{fv}((\lambda x.M_1)M_2) = \text{fv}(M_2) = \text{fv}(M_1[x := M_2])$. If $(\lambda x.M_1)M_2 \in \Lambda\text{I}$ then $M_2 = M_1[x := M_2] \in \Lambda\text{I}$.
- Or $M_1 = \lambda y.M_0$ then $\text{fv}((\lambda x.\lambda y.M_0)M_2) = \text{fv}((\lambda x.M_0)M_2) \setminus \{y\} =^{IH} \text{fv}(M_0[x := M_2]) \setminus \{y\} = \text{fv}(M_1[x := M_2])$ such that $y \notin \text{fv}(M_2) \cup \{x\}$. If $(\lambda x.\lambda y.M_0)M_2 \in \Lambda\text{I}$ then $M_0, M_2 \in \Lambda\text{I}$ and $x, y \in \text{fv}(M_0)$. So $(\lambda x.M_0)M_2 \in \Lambda\text{I}$. By IH, $M_0[x := M_2] \in \Lambda\text{I}$. Hence, $M_1[x := M_2] \in \Lambda\text{I}$ such that $y \notin \text{fv}(M_2) \cup \{x\}$.
- Or $M_1 = PQ$ then $\text{fv}((\lambda x.PQ)M_2) = \text{fv}(\lambda x.P)M_2 \cup \text{fv}((\lambda x.Q)M_2) =^{IH} \text{fv}(P[x := M_2]) \cup \text{fv}(Q[x := M_2]) = \text{fv}((PQ)[x := M_2])$.

3. We prove the lemma by induction on the length of the reduction $M \rightarrow^*_{\beta\eta} M'$.

- If $M = M'$ then $\mathrm{fv}(M) = \mathrm{fv}(M')$
- Let $M \to^*_{\beta\eta} M'' \to_{\beta\eta} M'$. By IH, $\mathrm{fv}(M) \subseteq \mathrm{fv}(M'')$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta\eta} M'$. We prove that $\mathrm{fv}(M'') \subseteq \mathrm{fv}(M')$ by induction on p.
  - $*$ Let $p = 0$.
    - · either $M'' = (\lambda x.M_1)M_2$ and $M' = M_1[x := M_2]$. We prove that $\mathrm{fv}(M') \subseteq (\mathrm{fv}(M_1) \setminus \{x\}) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'')$ by induction on the structure of M_1.
      1. Let $M_1 = y$. If $y = x$ then $M' = M_2$ and $\mathrm{fv}(M') = \mathrm{fv}(M'')$. If $y \neq x$ then $M' = y$ and $\mathrm{fv}(M') = \{y\} \subseteq \{y\} \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'')$.
      2. Let $M_1 = \lambda y.M'_1$ then $M' = \lambda y.M'_1[x := M_2]$ such that $y \notin \mathrm{fv}(M_2) \cup \{x\}$. By IH, $\mathrm{fv}(M'_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M'_1)M_2)$. Hence, $\mathrm{fv}(M') = \mathrm{fv}(M'_1[x := M_2]) \setminus \{y\} \subseteq \mathrm{fv}((\lambda x.M'_1)M_2) \setminus \{y\} = (\mathrm{fv}(M'_1) \setminus \{x, y\}) \cup (\mathrm{fv}(M_2) \setminus \{y\}) = \mathrm{fv}(M'')$.
      3. Let $M_1 = M'_1 M''_1$ then $M' = M'_1[x := M_2]M''_1[x := M_2]$. By IH, $\mathrm{fv}(M'_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M'_1)M_2)$ and $\mathrm{fv}(M''_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M''_1)M_2)$.
         Hence, $\mathrm{fv}(M') = \mathrm{fv}(M'_1[x := M_2]) \cup \mathrm{fv}(M''_1[x := M_2]) \subseteq \mathrm{fv}((\lambda x.M'_1)M_2) \cup \mathrm{fv}((\lambda x.M''_1)M_2) = ((\mathrm{fv}(M'_1) \cup \mathrm{fv}(M''_1)) \setminus \{x\}) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'')$.
    - · Or $M'' = \lambda x.M'x$ such that $x \notin \mathrm{fv}(M')$, so $\mathrm{fv}(M'') = \mathrm{fv}(M')$.
  - $*$ Let $p = 1.p'$ then either $M'' = \lambda x.M_1$, $M' = \lambda x.M_2$ and $M_1 \xrightarrow{p'}_{\beta\eta} M_2$. By IH, $\mathrm{fv}(M_1) \subseteq \mathrm{fv}(M_2)$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \setminus \{x\} \subseteq \mathrm{fv}(M_2) \setminus \{x\} = \mathrm{fv}(M')$. Or $M'' = M_1 M_2$, $M' = M'_1 M_2$ and $M_1 \xrightarrow{p'}_{\beta\eta} M'_1$. By IH, $\mathrm{fv}(M_1) \subseteq \mathrm{fv}(M'_1)$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) \subseteq \mathrm{fv}(M'_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M')$.
  - $*$ Let $p = 2.p'$ then $M'' = M_1 M_2$, $M' = M_1 M'_2$ and $M_2 \xrightarrow{p'}_{\beta\eta} M'_2$. By IH, $\mathrm{fv}(M_2) \subseteq \mathrm{fv}(M'_2)$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) \subseteq \mathrm{fv}(M_1) \cup \mathrm{fv}(M'_2) = \mathrm{fv}(M')$.

4. We prove the lemma by induction on the length of the reduction $M \to^*_{\beta I} M'$.

- If $M = M'$ then $\mathrm{fv}(M) = \mathrm{fv}(M')$
- Let $M \to^*_{\beta I} M'' \to_{\beta I} M'$. By IH, $\mathrm{fv}(M) = \mathrm{fv}(M'')$ and if $M \in \Lambda\mathrm{I}$ then $M'' \in \Lambda\mathrm{I}$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta I} M'$. We prove that $\mathrm{fv}(M'') = \mathrm{fv}(M')$ and that if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$ by induction on p.
  - $*$ Let $p = 0$ then $M'' = (\lambda x.M_1)M_2$ and $M' = M_1[x := M_2]$ such that $x \in \mathrm{fv}(M_1)$. So, by lemmma 2.2.2, $\mathrm{fv}(M') = \mathrm{fv}(M'')$ and if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.
  - $*$ Let $p = 1.p'$ then either $M'' = \lambda x.M_1$, $M' = \lambda x.M_2$ and $M_1 \xrightarrow{p'}_{\beta I} M_2$. By IH, $\mathrm{fv}(M_1) = \mathrm{fv}(M_2)$ and if $M_1 \in \Lambda\mathrm{I}$ then $M_2 \in \Lambda\mathrm{I}$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \setminus \{x\} = \mathrm{fv}(M_2) \setminus \{x\} = \mathrm{fv}(M')$ and if $M'' \in \Lambda\mathrm{I}$ then $x \in \mathrm{fv}(M_1) = \mathrm{fv}(M_2)$ and so $M' \in \Lambda\mathrm{I}$.
    Or $M'' = M_1 M_2$, $M' = M'_1 M_2$ and $M_1 \xrightarrow{p'}_{\beta\eta} M'_1$. By IH, $\mathrm{fv}(M_1) = \mathrm{fv}(M'_1)$ and if $M_1 \in \Lambda\mathrm{I}$ then $M'_1 \in \Lambda\mathrm{I}$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M'_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M')$ and if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.
  - $*$ Let $p = 2.p'$ then $M'' = M_1 M_2$, $M' = M_1 M'_2$ and $M_2 \xrightarrow{p'}_{\beta\eta} M'_2$. By IH, $\mathrm{fv}(M_2) = \mathrm{fv}(M'_2)$ and if $M_2 \in \Lambda\mathrm{I}$ then $M'_2 \in \Lambda\mathrm{I}$, so $\mathrm{fv}(M'') = \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2) = \mathrm{fv}(M_1) \cup \mathrm{fv}(M'_2) = \mathrm{fv}(M')$ and if $M'' \in \Lambda\mathrm{I}$ then $M' \in \Lambda\mathrm{I}$.

5. $\Rightarrow$) Let $\lambda x.M \xrightarrow{p}_{\beta\eta} P$. We prove the result by case on p. Either $p = 0$ and $M = Px$ such that $x \notin \text{fv}(P)$. Or $p = 1.p'$, $P = \lambda x.M'$ and $M \xrightarrow{p'}_{\beta\eta} M'$.

   $\Leftarrow$) If $P = \lambda x.M'$ and $M \to_{\beta\eta} pM'$. So, $\lambda x.M \xrightarrow{1.p}_{\beta\eta} P$ and $\lambda x.M \to_{\beta\eta} P$. If $M = Px$ and $x \notin fvP$ then $\lambda x.M = \lambda x.Px \xrightarrow{0}_{\beta\eta} P$, so $\lambda x.M \to_{\beta\eta} P$.

6a. If $k = 0$ then $P = (\lambda x.M)N_1N_1 \ldots N_n$ is a direct r-reduct of $(\lambda x.M)N_0N_1 \ldots N_n$, absurd. So $k \geq 1$. Assume $k = 1$, we prove $P = M[x := N_0]N_1 \ldots N_n$ by induction on $n \geq 0$.

   – Let $n = 0$ and $r = \beta I$. By definition there exists p such that $(\lambda x.M)N_0 \xrightarrow{p}_{\beta I} P$. We prove the result by case on p.

     ∗ Let $p = 0$ then $P = M[x := N_0]$ and $x \in \text{fv}(M)$.

     ∗ Let $p = 1.p'$ then $\lambda x.M \xrightarrow{p'}_{\beta I} \lambda x.M'$ and $P = (\lambda x.M')N_0$ is a direct βI-reduct of $(\lambda x.M)N_0$, absurd.

     ∗ Let $p = 2.p'$ then $N_0 \xrightarrow{p'}_{\beta I} N'$ and $P = (\lambda x.M)N'$ is a direct βI-reduct of $(\lambda x.M)N_0$, absurd.

   – Let $n = 0$ and $r = \beta\eta$. By definition there exists p such that $(\lambda x.M)N_0 \xrightarrow{p}_{\beta I} P$. We prove the result by case on p.

     ∗ Let $p = 0$ then $P = M[x := N_0]$.

     ∗ Let $p = 1.p'$ then $\lambda x.M \xrightarrow{p'}_{\beta\eta} Q$ and $P = QN_0$. By lemma 2.2.5:

       · Either $p' = 1.p''$, $Q = \lambda x.M'$ and $M \xrightarrow{p''}_{\beta\eta} M'$. Hence $P = (\lambda x.M')N_0$ is a direct $\beta\eta$-reduct of $(\lambda x.M)N_0$, absurd.

       · Or $p = 0$, $M = Qx$ and $x \notin \text{fv}(Q)$. Hence, $P = QN_0 = M[x := N_0]$.

     ∗ Let $p = 2.p'$ then $N_0 \xrightarrow{p'}_{\beta\eta} N'$ and $P = (\lambda x.M)N'$ is a direct $\beta\eta$-reduct of $(\lambda x.M)N_0$, absurd.

   – Let $n = m+1$ where $m \geq 0$. By definition there exists p such that $(\lambda x.M)N_0 \ldots N_{m+1} \xrightarrow{p}_r P$. We prove the result by case on p.

     ∗ Either $p = 1.p'$ then $(\lambda x.M)N_0 \ldots N_m \xrightarrow{p'}_r Q$ and $P = QN_{m+1}$.

       · If Q is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_m$ then P is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_{m+1}$, absurd.

       · If Q is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_m$ then it is done by IH.

     ∗ Or $p = 2.p'$ then $N_{m+1} \xrightarrow{p'}_r N'_{m+1}$ and $P = (\lambda x.M)N_0 \ldots N_mN'_{m+1}$ which is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_{m+1}$, absurd.

6b. By 6a, $k \geq 1$. We prove the statement by induction on $k \geq 1$.

   – If $k = 1$ then we conclude by 6a.

   – Let $(\lambda x.M)N_0 \ldots N_n \to_r^* Q \to_r P$.

- $*$ If Q is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$, then $Q = (\lambda x.M')N'_0 \ldots N'_n$, such that $M \to_r^* M'$ and $\forall i \in \{0,\ldots,n\}$, $N_i \to_r^* N'_i$. Since P is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$, P is not a direct r-reduct of Q. Hence by 6a, $P = M'[x := N'_0]N'_1 \ldots N'_n$.
- $*$ If Q is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$, then by IH, there exists a direct r-reduct $(\lambda x.M')N'_0 \ldots N'_n$ of $(\lambda x.M)N_0 \ldots N_n$ such that $M'[x := N'_0]N'_1 \ldots N'_n \to_r^* Q \to_r P$.

7. If P is a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$ then $P = (\lambda x.M')N'_0 \ldots N'_n$ such that $M \to_r^* M'$ and $\forall i \in \{0,\ldots,n\}, N_i \to_r^* N'_i$. So $P \to_r M'[x := N'_0]N'_1 \ldots N'_n$ (if $r = \beta I$, note that $x \in \mathrm{fv}(M')$ by lemma 2.2.4) and $M[x := N_0]N_1 \ldots N_n \to_r^* M'[x := N'_0]N'_1 \ldots N'_n$. If P is not a direct r-reduct of $(\lambda x.M)N_0 \ldots N_n$ then by lemma 6.6b, there exists a direct r-reduct, $(\lambda x.M')N'_0 \ldots N'_n$, such that $M \to_r^* M'$ and $\forall i \in \{0,\ldots,n\}, N_i \to_r^* N'_i$, of $(\lambda x.M)N_0 \ldots N_n$. We have $M[x := N_0]N_1 \ldots N_n \to_r^* M'[x := N'_0]N'_1 \ldots N'_n \to_r^* P$.

8. We prove this lemma by induction on ths structure of p.
   - Let $p = 0$ it is done by definition.
   - Let $p = 1.p'$. Then:
     - $*$ Either $M = \lambda x.M_1 \xrightarrow{1.p'}_r \lambda x.M'_1 = M'$ such that $M_1 \xrightarrow{p'}_r M'_1$. By IH, $p' \in \mathcal{R}^r_{M_1}$. So $p \in \mathcal{R}^r_M$. If $p \in \mathcal{R}^r_M$ then $M|_p = M_1|_{p'} \in \mathcal{R}^r$. By IH, there exists M'_1 such that $M_1 \xrightarrow{p'}_r M'_1$, so $M \xrightarrow{p}_r \lambda x.M'_1$.
     - $*$ Or $M = M_1M_2 \xrightarrow{1.p}_r M'_1M_2 = M'$ such that $M_1 \xrightarrow{p'}_r M'_1$. By IH, $p' \in \mathcal{R}^r_{M_1}$. So $p \in \mathcal{R}^r_M$. If $p \in \mathcal{R}^r_M$ then $M|_p = M_1|_{p'} \in \mathcal{R}^r$. By IH, there exists M'_1 such that $M_1 \xrightarrow{p'}_r M'_1$, so $M \xrightarrow{p}_r M'_1M_2$.
   - Let $p = 2.p'$. Then, $M = M_1M_2 \xrightarrow{1.p}_r M_1M'_2 = M'$ such that $M_2 \xrightarrow{p'}_r M'_2$. By IH, $p' \in \mathcal{R}^r_{M_2}$. So $p \in \mathcal{R}^r_M$. If $p \in \mathcal{R}^r_M$ then $M|_p = M_2|_{p'} \in \mathcal{R}^r$. By IH, there exists M'_2 such that $M_2 \xrightarrow{p'}_r M'_2$, so $M \xrightarrow{p}_r M_1M'_2$.

9. We prove this lemma by induction on ths structure of p.
   - Let $p = 0$ it is done by definition.
   - Let $p = 1.p'$. Then either $M = \lambda x.M' \xrightarrow{1.p'}_r \lambda x.M'_1 = M_1$ such that $M' \xrightarrow{p'}_r M'_1$. By definition, $M_2 = \lambda x.M'_2$ and $M' \xrightarrow{p'}_r M'_2$. By IH, $M'_1 = M'_2$, so $M_1 = M_2$. Or $M = M'N \xrightarrow{1.p}_r M'_1N = M_1$ such that $M' \xrightarrow{p'}_r M'_1$. By definition, $M_2 = M'_2N$ and $M' \xrightarrow{p'}_r M'_2$. By IH, $M'_1 = M'_2$, so $M_1 = M_2$.
   - Let $p = 2.p'$. Then $M = NM' \xrightarrow{1.p}_r NM'_1 = M_1$ such that $M' \xrightarrow{p'}_r M'_1$. By definition, $M_2 = NM'_2$ and $M' \xrightarrow{p'}_r M'_2$. By IH, $M'_1 = M'_2$, so $M_1 = M_2$.

$\square$

Proof(Lemma 5.2)**:**

1. We prove the lemma by induction on the structure of M.
   - Let $M = y$.
     - Either $y = x$ then $M[x := c(cx)] = c(cx) \neq x$ and for any N, $M[x := c(cx)] = c(cx) \neq Nx$ because $cx \neq x$.
     - Or $y \neq x$ then $M[x := c(cx)] = y \neq x$ and for any N, $M[x := c(cx)] = y \neq Nx$.
   - Let $M = \lambda y.P$. Then, $M[x := c(cx)] = \lambda y.P[x := c(cx)] \neq x$ (such that $y \notin \{c, x\}$) and for any N, $M[x := c(cx)] \neq Nx$.
   - Let $M = PQ$. Then, $M[x := c(cx)] = P[x := c(cx)]Q[x := c(cx)] \neq x$. Assume $M[x := c(cx)] = Nx$, so $Q[x := c(cx)] = x$ and by IH, absurd.
2. We prove this lemma by induction on the structure of M.
   - Let $M = z$.
     - Either $z = y$ then $M[y := c(cx)] = c(cx) \neq x$ and for any N, $M[y := c(cx)] = c(cx) \neq Nx$ because $cx \neq x$.
     - Or $z \neq y$ then $M[y := c(cx)] = z \neq x$ by hypothesis and for any N, $M[y := c(cx)] = z \neq Nx$.
   - Let $M = \lambda z.P$. Then, $M[y := c(cx)] = \lambda z.P[y := c(cx)] \neq x$ (such that $y \notin \{c, x, y\}$) and for any N, $M[y := c(cx)] \neq Nx$.
   - Let $M = PQ$. Then, $M[y := c(cx)] = P[x := c(cx)]Q[x := c(cx)] \neq x$. Assume $M[y := c(cx)] = Nx$, so $Q[y := c(cx)] = x$ and by IH, absurd.
3. By cases on the derivation of $M \in \mathcal{M}_c$.
4. By cases on the structure of M using 3.
5. By cases on the derivation of $MN \in \mathcal{M}_c$.
6. We prove this result by induction on n.
   - If $n = 0$ then it is done.
   - Let $n = m + 1$ such that $m \geq 0$. By lemma 5.2.5, $c^m(M) \in \mathcal{M}_c$ then by IH, $M \in \mathcal{M}_c$.
7. Easy.
8. By cases on the derivation of $\lambda x.P \in \Lambda\eta_c$.
9. By cases on the derivation of $\lambda x.P \in \Lambda\mathrm{I}_c$.
10. We prove the lemma by induction on the structure of $M \in \mathcal{M}_c$.
    - Case (R1)1. Either $M = x$ then $M[x := N] = N \in \mathcal{M}_c$. Or $M = y \neq x$ then $M[x := N] = M \in \mathcal{M}_c$.

- Case (R1)2. Let $M = \lambda y.P \in \Lambda I_c$ such that $y \neq c$, $P \in \Lambda I_c$ and $y \in \text{fv}(P)$. We have $M[x := N] = \lambda y.M[x := N]$ such that $y \notin \text{fv}(N) \cup \{x\}$. By IH, $P[x := N] \in \Lambda I_c$, so $M[x := N] \in \Lambda I_c$.
- Case (R1)3. Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $y \neq c$ and $P \in \Lambda\eta_c$. By IH, $P[x := N] \in \Lambda\eta_c$. So by (R1).3 $M[x := N] = \lambda y.P[y := c(cy)][x := N] = \lambda y.P[x := N][y := c(cy)] \in \Lambda\eta_c$ such that $y \notin \text{fv}(N) \cup \{x\}$.
- Case (R1)4. Let $M = \lambda y.Py$ such that $Py \in \Lambda\eta_c$, $y \notin \text{fv}(P) \cup \{c\}$ and $P \neq c$. We have $M[x := N] = \lambda y.(Py)[x := N] = \lambda y.P[x := N]y$, such that $y \notin \text{fv}(N) \cup \{x\}$. By IH, $P[x := N]y \in \Lambda\eta_c$. By lemma 5.2.4, $P[x := N] \neq c$. Hence, because $y \notin \text{fv}(P[x := N])$, $M[x := N] \in \Lambda\eta_c$.
- Case (R2) Let $M = cM_1M_2$ such that $M_1, M_2 \in \mathcal{M}_c$. Then by IH, $M_1[x := N], M_2[x := N] \in \mathcal{M}_c$. Hence, $cM_1[x := N]M_2[x := N] \in \mathcal{M}_c$.
- Case (R3) Let $M = M_1M_2$ such that $M_1, M_2 \in \mathcal{M}_c$ and M_1 is a λ-abstraction. Then by IH, $M_1[x := N], M_2[x := N] \in \mathcal{M}_c$. Hence, $M_1[x := N]M_2[x := N] \in \mathcal{M}_c$, since $M_1[x := N]$ is a λ-abstraction.
- Case (R4) Let $M = cP$ such that $P \in \Lambda\eta_c$. Then by IH, $P[x := N] \in \Lambda\eta_c$ and by (R4), $M[x := N] \in \Lambda\eta_c$.

11. By case on the structure of M.
    - let $M \in \mathcal{V}$.
      - Either $M = x$ then, $M[x := c(cx)] = c(cx)$. Hence, $c(cx) \neq y$, $c(cx) \neq Py$ since $cx \neq y$, $c(cx) \neq \lambda y.P$ and $c(cx) \neq (\lambda y.P)Q$. If $M[x := c(cx)] = PQ$ then $P = c$ and $Q = cx$.
      - Or $M = z \neq x$ then $M[x := c(cx)] = z$. Hence, if $z = y$ then $M = y$, $z \neq Py$, $z \neq \lambda y.P$, $z \neq PQ$ and $z \neq (\lambda y.P)Q$.
    - Let $M = \lambda z.M'$ then $M[x := c(cx)] = \lambda z.M'[x := c(cx)]$, where $z \notin \{x, c\}$. Hence, $\lambda z.M'[x := c(cx)] \neq y$, $\lambda z.M'[x := c(cx)] \neq Py$, $\lambda z.M'[x := c(cx)] \neq PQ$ and $\lambda z.M'[x := c(cx)] \neq (\lambda y.P)Q$. Let $\lambda z.M'[x := c(cx)] = \lambda y.P$. By α-converions, assume $y = z$. So $M'[x := c(cx)] = P$.
    - Let $M = M_1M_2$ then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. Hence, $M_1[x := c(cx)]M_2[x := c(cx)] \neq y$ and $M_1[x := c(cx)]M_2[x := c(cx)] \neq \lambda y.P$. If $M_1[x := c(cx)]M_2[x := c(cx)] = Py$ then $P = M_1[x := c(cx)]$ and $M_2[x := c(cx)] = y$. So $M_2 = y$. If $M_1[x := c(cx)]M_2[x := c(cx)] = PQ$ then $P = M_1[x := c(cx)]$ and $Q = M_2[x := c(cx)]$. If $M_1[x := c(cx)]M_2[x := c(cx)] = (\lambda y.P)Q$ then $\lambda y.P = M_1[x := c(cx)]$ and $Q = M_2[x := c(cx)]$. So $M_1 = \lambda y.M_0$ and $P = M_0[x := c(cx)]$

12. 12a. By definition, $x \neq c$. By lemma 5.2.8, either $P = Nx$ where $Nx \in \Lambda\eta_c$ or $P = N[x := c(cx))]$ where $N \in \Lambda\eta_c$. In the second case since by (R4) $c(cx) \in \Lambda\eta_c$, we get by lemma 5.2.10 that $N[x := c(cx))] \in \Lambda\eta_c$.

    12b. By lemma 5.2.1 and lemma 5.2.8.

13. 13a. $\Rightarrow$) We prove the lemma by induction on the structure of p.

- Let $p = 0$ then:
  - either $M[x := c(cx)] = (\lambda y.P)Q$ and $M' = P[y := Q]$. By lemma 5.2.11, $M = (\lambda y.P')Q'$, $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$ such that $y \notin \{c, x\}$. So $M' = P'[y := Q'][x := c(cx)]$ and $M \xrightarrow{0}_{\beta\eta} P'[y := Q']$.
  - Or $M[x := c(cx)] = \lambda y.M'y$ such that $y \notin \text{fv}(M')$. By lemma 5.2.11, $M = \lambda y.N$ and $M'y = N[x := c(cx)]$ such that $y \notin \{x, c\}$. Again by lemma 5.2.11, $N = N'y$ and $M' = N'[x := c(cx)]$. Because $y \notin \text{fv}(M')$, we obatin $y \notin \text{fv}(N')$ and so $M = \lambda y.N'y \xrightarrow{0}_{\beta\eta} N'$.
- Let $p = 1.p'$. Then:
  - Either $M[x := c(cx)] = \lambda y.P \xrightarrow{1.p'}_{\beta\eta} \lambda y.P' = M'$ such that $P \xrightarrow{p'}_{\beta\eta} P'$. By lemma 5.2.11, $M = \lambda y.N$ and $P = N[x := c(cx)]$ such that $y \notin \{c, x\}$. By IH, $P' = N'[x := c(cx)]$ and $N \xrightarrow{p'}_{\beta\eta} N'$. So $M' = (\lambda y.N')[x := c(cx)]$ and $M \xrightarrow{1.p}_{\beta\eta} \lambda y.N'$.
  - Or $M[x := c(cx)] = PQ \xrightarrow{1.p'}_{\beta\eta} P'Q = M'$ such that $P \xrightarrow{p'}_{\beta\eta} P'$. Then by lemma 5.2.11, either $M = x$ and $P = c$ and $Q = cx$ but then $P \xrightarrow{p'}_{\beta\eta} P'$ is wrong. Or $M = P_0Q_0$, $P = P_0[x := c(cx)]$ and $Q = Q_0[x := c(cx)]$. By IH, $P' = P'_0[x := c(cx)]$ and $P_0 \xrightarrow{p'}_{\beta\eta} P'_0$. So $M' = (P'_0Q_0)[x := c(cx)]$ and $P_0Q_0 \xrightarrow{1.p'}_{\beta\eta} P'_0Q_0$.
- Let $p = 2.p'$ then $M[x := c(cx)] = PQ \xrightarrow{2.p'}_{\beta\eta} PQ' = M'$ such that $Q \xrightarrow{p'}_{\beta\eta} Q'$. Then by lemma 5.2.11, either $M = x$ and $P = c$ and $Q = cx$ but then $Q \xrightarrow{p'}_{\beta\eta} Q'$ is wrong. Or $M = P_0Q_0$, $P = P_0[x := c(cx)]$ and $Q = Q_0[x := c(cx)]$. By IH, $Q' = Q'_0[x := c(cx)]$ and $Q_0 \xrightarrow{p'}_{\beta\eta} Q'_0$. So $M' = (P_0Q'_0)[x := c(cx)]$ and $P_0Q_0 \xrightarrow{2.p'}_{\beta\eta} P_0Q'_0$.

$\Leftarrow$) We prove the lemma by induction on the structure of p.

- Let $p = 0$ then:
  - Either $M = \lambda y.Ny$ such that $y \notin \text{fv}(N)$. Then $M[x := c(cx)] = \lambda y.N[x := c(cx)]y \xrightarrow{0}_{\beta\eta} N[x := c(cx)]$ such that $y \notin \{c, x\}$.
  - Or $M = (\lambda y.P)Q$ and $M' = P[y := Q]$. Then $M[x := c(cx)] = (\lambda y.P[x := c(cx)])Q[x := c(cx)] \xrightarrow{0}_{\beta\eta} P[x := c(cx)][y := Q[x := c(cx)]] = P[y := Q][x := c(cx)]$ such that $y \notin \{c, x\}$.
- Let $p = 1.p'$.
  - Either $M = \lambda y.N \xrightarrow{p}_{\beta\eta} \lambda y.N' = M'$ such that $N \xrightarrow{p'}_{\beta\eta} N'$. By IH, $N[x := c(cx)] \xrightarrow{p'}_{\beta\eta} N'[x := c(cx)]$. So, $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'[x := c(cx)]$ such that $y \notin \{c, x\}$.
  - Or $M = PQ \xrightarrow{p}_{\beta\eta} P'Q = M'$ such that $P \xrightarrow{p'}_{\beta\eta} P'$. By IH, $P[x := c(cx)] \xrightarrow{p'}_{\beta\eta} P'[x := c(cx)]$. So, $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'[x := c(cx)]$.

- Let $p = 2.p'$ then $M = PQ \xrightarrow{p}_{\beta\eta} PQ' = M'$ such that $Q \xrightarrow{p'}_{\beta\eta} Q'$. By IH, $Q[x := c(cx)] \xrightarrow{p'}_{\beta\eta} Q'[x := c(cx)]$. So, $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'[x := c(cx)]$.

13b. We prove this lemma by induction on n.

- Let $n = 0$ then it is done.
- Let $n = m + 1$ such that $m \geq 0$. Then $c^n(M) = c(c^m(M)) \xrightarrow{p}_{\beta\eta} M'$. By case on p we obtain that $p = 2.p'$ and $M' = c(N')$ and $c^m(M) \xrightarrow{p'}_{\beta\eta} N'$. By IH, $p' = 2^m.p''$ and there exists $N'' \in \Lambda\eta_c$ such that $N' = c^m(N'')$ and $M \xrightarrow{p''}_{\beta\eta} N''$. So $p = 2^n.p''$ and $M' = c^n(N'')$.

□

Proof(Lemma 5.3)**:** We split the proof of this lemma in two.
We prove the first part of this lemma by case on the structure of M.

- Let $M \in \mathcal{V}$ and $p \in \mathcal{R}^r_M$. So $M|_p \in \mathcal{R}^r$. We prove by case on the structure of p that there is no such p.
  - Let $p = 0$ then $M|_p = M \notin \mathcal{R}^r$.
  - Let $p = 1.p'$ then $M|_p$ is undefined.
  - Let $p = 2.p'$ then $M|_p$ is undefined.
- Let $M = \lambda x.N$.
  - Let $M \in \mathcal{R}^r$. We prove by case on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{0\} \cup \{1.p' \mid p' \in \mathcal{R}^r_N\}$.
    * Let $p = 0$ then $M|_p = M \in \mathcal{R}^r$.
    * Let $p = 1.p'$ then $M|_p = N|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_N$.
    * Let $p = 2.p'$ then $M|_p$ is undefined.

    Let $p \in \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_N\}$, we prove that $p \in \mathcal{R}^r_M$.
    * Let $p = 0$. Since $M = M|_p \in \mathcal{R}^r$, by definition, $p \in \mathcal{R}^r_M$.
    * Let $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. By definition $M|_p = N|_{p'} \in \mathcal{R}^r$.
  - Let $M \notin \mathcal{R}^r$. We prove by case on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{1.p' \mid p' \in \mathcal{R}^r_N\}$.
    * Let $p = 0$ then $M|_p = M \notin \mathcal{R}^r$.
    * Let $p = 1.p'$ then $M|_p = N|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_N$.
    * Let $p = 2.p'$ then $M|_p$ is undefined.

    Let $p \in \{1.p' \mid p' \in \mathcal{R}^r_N\}$, we prove that $p \in \mathcal{R}^r_M$. Then, $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. By definition $M|_p = N|_{p'} \in \mathcal{R}^r$.
- Let $M = PQ$.
  - Let $M \in \mathcal{R}^r$. We prove by case on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{0\} \cup \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{2.p' \mid p' \in \mathcal{R}^r_Q\}$.

  * Let $p = 0$ then $M|_p = M \in \mathcal{R}^r$.
  * Let $p = 1.p'$ then $M|_p = P|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_P$.
  * Let $p = 2.p'$ then $M|_p = Q|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_Q$.

  Let $p \in \{0\} \cup \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{2.p' \mid p' \in \mathcal{R}^r_Q\}$, we prove that $p \in \mathcal{R}^r_M$.

  * Let $p = 0$. Since $M|_p = M \in \mathcal{R}^r$, so $p \in \mathcal{R}^r_M$.
  * Let $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$. Since $M|_p = P|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$
  * Let $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$. Since $M|_p = Q|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$

- Let $M \notin \mathcal{R}^r$. We prove by induction on the structure of p that if $p \in \mathcal{R}^r_M$ then $p \in \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{1.p' \mid p' \in \mathcal{R}^r_Q\}$.

  * Let $p = 0$ then $M|_p = M \notin \mathcal{R}^r$.
  * Let $p = 1.p'$ then $M|_p = P|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_P$.
  * Let $p = 2.p'$ then $M|_p = Q|_{p'} \in \mathcal{R}^r$, so $p' \in \mathcal{R}^r_Q$.

  Let $p \in \{1.p' \mid p' \in \mathcal{R}^r_P\} \cup \{2.p' \mid p' \in \mathcal{R}^r_Q\}$, we prove that $p \in \mathcal{R}^r_M$.

  * Let $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$. Since $M|_p = P|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$
  * Let $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$. Since $M|_p = Q|_{p'} \in \mathcal{R}^r$, $p \in \mathcal{R}^r_M$

We prove the second part of this lemma by case on the structure of M.

- Let $M \in \mathcal{V}$, by lemma 5.3, $\mathcal{R}^r_M = \varnothing$, so $\mathcal{F} = \varnothing$.

- Let $M = \lambda y.N$ then by lemma 5.3:

  - If $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_N\}$. Let $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\}$. Let $p \in \mathcal{F}'$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_N$.
    * Let $p \in \mathcal{F} \setminus \{0\}$ then $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. So $p' \in \mathcal{F}'$ and it is done.
    * Let $p \in \{1.p' \mid p' \in \mathcal{F}'\}$ then $p = 1.p'$ such that $p' \in \mathcal{F}'$. So $1.p' = p \in \mathcal{F} \setminus \{0\}$.
  - If $M \notin \mathcal{R}^r$ then $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_N\}$. Let $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\}$. Let $p \in \mathcal{F}'$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_N$.
    * Let $p \in \mathcal{F}$ then $p = 1.p'$ such that $p' \in \mathcal{R}^r_N$. So $p' \in \mathcal{F}'$ and it is done.
    * Let $p \in \{1.p' \mid p' \in \mathcal{F}'\}$ then $p = 1.p'$ such that $p' \in \mathcal{F}'$. So $1.p' = p \in \mathcal{F}$.

- Let $M = PQ$ then by lemma 5.3:

  - If $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. Let $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{2.p \mid p \in \mathcal{F}\}$. Let $p \in \mathcal{F}_1$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_P$. Let $p \in \mathcal{F}_2$ then $2.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_Q$.
    * Let $p \in \mathcal{F} \setminus \{0\}$. Either $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$, so $p' \in \mathcal{F}_1$ and it is done. Or $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$, so $p' \in \mathcal{F}_2$ and it is done.
    * Let $p \in \{1.p' \mid p' \in \mathcal{F}_1\} \cup \{2.p' \mid p' \in \mathcal{F}_2\}$. Either $p = 1.p'$ such that $p' \in \mathcal{F}_1$, so $1.p' \in \mathcal{F} \setminus \{0\}$. Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$, so $2.p' \in \mathcal{F} \setminus \{0\}$.

- If $M \notin \mathcal{R}^r$ then $\mathcal{R}^r_M = \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \mid p \in \mathcal{R}^r_Q\}$. Let $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$. Let $p \in \mathcal{F}_1$ then $1.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_P$. Let $p \in \mathcal{F}_2$ then $2.p \in \mathcal{F}$, so $p \in \mathcal{R}^r_Q$.
  - Let $p \in \mathcal{F}$. Either $p = 1.p'$ such that $p' \in \mathcal{R}^r_P$, so $p' \in \mathcal{F}_1$ and it is done. Or $p = 2.p'$ such that $p' \in \mathcal{R}^r_Q$, so $p' \in \mathcal{F}_2$ and it is done.
  - Let $p \in \{1.p' \mid p' \in \mathcal{F}_1\} \cup \{2.p' \mid p' \in \mathcal{F}_2\}$. Either $p = 1.p'$ such that $p' \in \mathcal{F}_1$, so $1.p' \in \mathcal{F}$. Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$, so $2.p' \in \mathcal{F}$.

□

Proof(Lemma 5.4)**:**

1. By case on the structure of M.
   - Let $M \in \mathcal{V}$ then $M, M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$.
   - Let $M = \lambda y.N$ then $M[x := c(cx)] = \lambda y.N[x := c(cx)]$, where $y \notin \{x, c\}$.
     - If $M \in \mathcal{R}^{\beta\eta}$ then $N = Py$ such that $y \notin \mathrm{fv}(P)$. $N[x := c(cx)] = P[x := c(cx)]y$ and $y \notin \mathrm{fv}(P[x := c(cx)])$, so $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
     - If $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ then $N[x := c(cx)] = Py$ such that $y \notin \mathrm{fv}(P)$. By 5.2.11, $N = Qy$ and $P = Q[x := c(cx)]$. So $M = \lambda y.Qy$. Because $y \notin \mathrm{fv}(P)$, we obtain $y \notin \mathrm{fv}(Q)$ and so $M \in \mathcal{R}^{\beta\eta}$.
   - Let $M = M_1M_2$ then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$.
     - If $M \in \mathcal{R}^{\beta\eta}$ then $M_1 = \lambda y.M_0$. So $M[x := c(cx)] = (\lambda y.M_0[x := c(cx)])M_2[x := c(cx)] \in \mathcal{R}^{\beta\eta}$, where $y \notin \{x, c\}$.
     - If $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ then $M_1[x := c(cx)] = \lambda y.P$. By 5.2.11, $M_1 = \lambda y.M_0$ and $P = M_0[x := c(cx)]$ such that $y \notin \{c, x\}$. So, $M \in \mathcal{R}^{\beta\eta}$
2. We prove this result by inducion on the structure of M.
   - If $M \in \mathcal{V}$ then by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \varnothing$.
   - Let $M = \lambda y.M'$. Then $M[x := c(cx)] = \lambda y.M'[x := c(cx)]$ where $y \notin \{x, c\}$. By lemma 5.3:
     - If $M \in \mathcal{R}^{\beta\eta}$ then let $p = 0$. Then, $M[x := c(cx)]|_p = M[x := c(cx)] = M|_p[x := c(cx)]$
     - Let $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M'}$. Then, $M[x := c(cx)]|_p = M'[x := c(cx)]|_{p'} =^{IH} M'|_{p'}[x := c(cx)] = M|_p[x := c(cx)]$.
   - Let $M = M_1M_2$. Then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. By lemma 5.3:
     - If $M \in \mathcal{R}^{\beta\eta}$ then let $p = 0$. Then, $M[x := c(cx)]|_p = M[x := c(cx)] = M|_p[x := c(cx)]$
     - Let $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. Then, $M[x := c(cx)]|_p = M_1[x := c(cx)]|_{p'} =^{IH} M_1|_{p'}[x := c(cx)] = M|_p[x := c(cx)]$.
     - Let $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. Then, $M[x := c(cx)]|_p = M_2[x := c(cx)]|_{p'} =^{IH} M_2|_{p'}[x := c(cx)] = M|_p[x := c(cx)]$.

3. $\Rightarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$. By lemma 5.2.1, $\lambda x.M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$ so by lemma 5.3, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.

   $\Leftarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. By lemma 5.3, $1.p \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$.

4. $\Rightarrow$) Let $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. We prove the statement by induction on the structure of M

   - $M \notin \mathcal{V}$ since by lemma 5.3, $\mathcal{R}^{\beta\eta}_{M[x:=c(cx)]} = \varnothing$.
   - Let $M = \lambda y.N$ so $M[x := c(cx)] = \lambda y.N[x := c(cx)]$, where $y \notin \{x, c\}$. By lemma 5.3:
     * Either if $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$, $p = 0$. By 1, $M \in \mathcal{R}^{\beta\eta}$, so $p \in \mathcal{R}^{\beta\eta}_M$.
     * Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}$. By IH, $p' \in \mathcal{R}^{\beta\eta}_N$. Hence by lemma 5.3, $p = 1.p' \in \mathcal{R}^{\beta\eta}_M$.
   - Let $M = M_1M_2$ so $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. By lemma 5.3:
     * Either if $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$, $p = 0$. By 1, $M \in \mathcal{R}^{\beta\eta}$, so $0 \in \mathcal{R}^{\beta\eta}_M$.
     * Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1[x:=c(cx)]}$. By IH, $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. Hence by lemma 5.3, $p = 1.p' \in \mathcal{R}^{\beta\eta}_M$.
     * Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2[x:=c(cx)]}$. By IH, $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. Hence by lemma 5.3, $p = 2.p' \in \mathcal{R}^{\beta\eta}_M$.

   $\Leftarrow$) Let $p \in \mathcal{R}^r_M$. Then by definition $M|_p \in \mathcal{R}^{\beta\eta}$. By 1, $M|_p[x := c(cx)] \in \mathcal{R}^{\beta\eta}$. By 2, $M[x := c(cx)]|_p \in \mathcal{R}^{\beta\eta}$. So $p \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.

5. We prove this statement by induction on $n \geq 0$.

   - Let $n = 0$ then trivial.
   - Let $n = m + 1$ such that $m \geq 0$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_{c^m(M)} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_c\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} =^{IH} \{2^n.p \mid p \in \mathcal{R}^{\beta\eta}_M\}$.

□

Proof(Lemma 5.5.1a)**:** We prove the statement by case on r.

- Either $r = \beta I$. Since $M \in \Lambda\mathrm{I}_c$, $M \in \Lambda\mathrm{I}$, so $\lambda x.P, Q \in \Lambda\mathrm{I}$. Hence, $x \in \mathrm{fv}(P)$ and $M \in \mathcal{R}^{\beta I}$.
- Or $r = \beta\eta$. Trivial.

□

Proof(Lemma 5.5.1b)**:** We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$. By lemma 5.3, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda x.N \in \Lambda\mathrm{I}_c$ such that $N \in \Lambda\mathrm{I}_c$ and let $p \in \mathcal{R}^{\beta I}_M$. Since $M \notin \mathcal{R}^{\beta I}$, by lemma 5.3, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_N$. So by IH, $M|_p = N|_{p'} \in \Lambda\mathrm{I}_c$.

- Let $M = \lambda x.N[x := c(cx)] \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$ and let $p \in \mathcal{R}_M^{\beta\eta}$. By lemma 5.4.3, $p = 1.p'$ and $p' \in \mathcal{R}_{N[x:=c(cx)]}^{\beta\eta}$. By lemma 5.4.4, $p' \in \mathcal{R}_N^{\beta\eta}$. By IH, $N|_{p'} \in \Lambda\eta_c$. So, $M|_p = N[x := c(cx)]|_{p'} =^{5.4.2} N|_{p'}[x := c(cx)]$. By lemma 5.2.10, $N|_{p'}[x := c(cx)] \in \Lambda\eta_c$.

- Let $M = \lambda x.Nx \in \Lambda\eta_c$ such that $Nx \in \Lambda\eta_c$, $x \notin \text{fv}(N)$ and $c \neq N$. Let $p \in \mathcal{R}_M^{\beta\eta}$. Since $M \in \mathcal{R}^{\beta\eta}$, by lemma 5.3:

  – Either $p = 0$ so $M|_p = M \in \Lambda\eta_c$.
  – Or $p = 1.p'$ such that $p' \in \mathcal{R}_{Nx}^{\beta\eta}$. By IH, $M|_p = (Nx)|_{p'} \in \Lambda\eta_c$.

- Let $M = cNP \in \mathcal{M}_c$ such that $N, P \in \mathcal{M}_c$. Let $p \in \mathcal{R}_M^r$. Since $M, cN \notin \mathcal{R}^r$, by lemma 5.3:

  – Either $p = 1.2.p'$ such that $p' \in \mathcal{R}_N^r$. By IH, $M|_p = N|_{p'} \in \mathcal{M}_c$.
  – Or $p = 2.p'$ such that $p' \in \mathcal{R}_r^P$. By IH, $M|_p = P|_{p'} \in \mathcal{M}_c$.

- Let $M = (\lambda x.N)P \in \mathcal{M}_c$ such that $\lambda x.N, P \in \mathcal{M}_c$. Let $p \in \mathcal{R}_M^r$. Since by lemma 1a, $M \in \mathcal{R}^r$, by lemma 5.3:

  – Either $p = 0$ so $M|_p = M \in \mathcal{M}_c$.
  – Or $p = 1.p'$ such that $p' \in \mathcal{R}_{\lambda x.N}^r$. By IH, $M|_p = (\lambda x.N)|_{p'} \in \mathcal{M}_c$.
  – Or $p = 2.p'$ such that $p' \in \mathcal{R}_P^r$. By IH, $M|_p = P|_{p'} \in \mathcal{M}_c$.

- Let $M = cN \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Let $p \in \mathcal{R}_M^{\beta\eta}$. Since $M \notin \mathcal{R}^{\beta\eta}$, by lemma 5.3, $p = 2.p'$ such that $p' \in \mathcal{R}_N^{\beta\eta}$. By IH, $M|_p = N|_{p'} \in \Lambda\eta_c$.

$\square$

Proof(Lemma 5.5.2)**:**

2a. Let $M \in \Lambda\eta_c$ and $M \to_{\beta\eta} M'$. Then there exists p such that $M \xrightarrow{p}_{\beta\eta} M'$. We prove that $M' \in \Lambda\eta_c$ by induction on the structure of p.

- Let $p = 0$. Then:
  – either $M = \lambda x.M'x$ such that $x \notin \text{fv}(M')$. Because $M \in \Lambda\eta_c$, then $M'x \in \Lambda\eta_c$ and $x \neq c$. By lemma 5.2.8, $M' \in \Lambda\eta_c$.
  – or $M = (\lambda x.N)P$ and $M' = N[x := P]$. Since $M \in \Lambda\eta_c$ then $\lambda x.N, P \in \Lambda\eta_c$. By definition and lemmas 5.2.10, $N \in \Lambda\eta_c$ and $x \neq c$. By lemma 5.2.10, $M' \in \Lambda\eta_c$.
- Let $p = 1.p'$. Then:
  – either $M = \lambda x.N \xrightarrow{p}_{\beta\eta} \lambda x.N' = M'$ such that $N \xrightarrow{p'}_{\beta\eta} N'$. Since $M \in \Lambda\eta_c$:
    * Either $N = P[x := c(cx)]$ where $P \in \Lambda\eta_c$ and $x \neq c$. So by lemma 5.2.13a, $N' = N''[x := c(cx)]$ and $P \to_{\beta\eta} N''$. By IH, $N'' \in \Lambda\eta_c$ so by (R1).3, $M' = \lambda x.N''[x := c(cx)] \in \Lambda\eta_c$.
    * Or $N = Px$ where $Px \in \Lambda\eta_c$, $x \notin \text{fv}(P) \cup \{c\}$, $P \neq c$. By IH, $N' \in \Lambda\eta_c$. By lemma 5.2.8, $P \in \Lambda\eta_c$. By case on p':

  - · Either $p' = 0$, $P = (\lambda y.Q)$ and $N' = Q[y := x]$. Hence $M' = \lambda x.Q[y := x] = P \in \Lambda\eta_c$.
    - · Or $p' = 1.p''$, $N' = P'x$ and $P \xrightarrow{p''}_{\beta\eta} P'$. By lemma 2.2.3, $x \notin \text{fv}(P')$. By IH, $P' \in \Lambda\eta_c$, so by lemma 5.2.3, $P' \neq c$. Hence, $M' = \lambda x.P'x \in \Lambda\eta_c$.
  - or $M = M_1M_2 \xrightarrow{p}_{\beta\eta} M_1'M_2 = M'$ such that $M_1 \xrightarrow{p'}_{\beta\eta} M_1'$. By lemma 5.2.5, $M_2 \in \Lambda\eta_c$ and because $M_1 \neq c$ we obtain:
    - ∗ Either $M_1 = cM_0$ and $M_0 \in \Lambda\eta_c$. By case on p' we obtain $p' = 2.p''$, $M_1' = cM_0'$ and $M_0 \xrightarrow{p''}_{\beta\eta} M_0'$. By IH, $M_0' \in \Lambda\eta_c$, so by (R2), $M' = cM_0'M_2 \in \Lambda\eta_c$.
    - ∗ Or $M_1 = \lambda x.M_0$ and $M_1 \in \Lambda\eta_c$. By IH, $M_1' \in \Lambda\eta_c$. By lemma 5.2.12a, $M_0 \in \Lambda\eta_c$. lemma 5.2.8, $x \neq c$. By case on p':
      - · Either $p' = 0$ and $M_0 = M_1'x$ such that $x \notin \text{fv}(M_1')$. Because $M_0 = M_1'x \in \Lambda\eta_c$, by definition and lemma 5.2.5 we obtain $M' = M_1'M_2 \in \Lambda\eta_c$.
      - · Or $p' = 1.p''$ and $M_1' = \lambda x.M_0'$ such that $M_0 \xrightarrow{p''}_{\beta\eta} M_0'$. So $M' = (\lambda x.M_0')M_2 \in \Lambda\eta_c$.
- Let $p = 2.p'$. Then $M = M_1M_2 \xrightarrow{p}_{\beta\eta} M_1M_2' = M'$ such that $M_2 \xrightarrow{p'}_{\beta\eta} M_2'$. By lemma 5.2.5, $M_2 \in \Lambda\eta_c$ so by IH, $M_2' \in \Lambda\eta_c$. Because $M = M_1M_2 \in \Lambda\eta_c$, again by lemma 5.2.5 $M' = M_1M_2' \in \Lambda\eta_c$.

2b. By induction on $M \rightarrow_{\beta I} M'$ in a similar fashion to the above.

□

Proof(Lemma 5.7.1)**:** We prove the statement by induction on $n \geq 0$.

- Let $n = 0$ then by definition $|c^n(M)|^c = |M|^c$.
- Let $n = m + 1$ such that $m \geq 0$ then $|c^n(M)|^c = |c(c^m(M))|^c = |c^m(M)|^c =^{IH} |M|^c$.

□

Proof(Lemma 5.7.2)**:** We prove the lemma by induction on n.

- If $n = 0$ then it is done.
- Let $n = m+1$ such that $m \geq 0$. Then, $|\langle c^n(M), \mathcal{R}^{\beta\eta}_{c^n(M)}\rangle|^c = \{|\langle c^n(M), p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^n(M)}\} =^{5.3} \{|\langle c^n(M), 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} = \{|\langle c^m(M), p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} =^{IH} |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$.

□

Proof(Lemma 5.7.3)**:** We prove the lemma by induction on n.

- If $n = 0$ then it is done.
- Let $n = m + 1$ such that $m \geq 0$. Then, $|\langle c^n(M), 2^n.p\rangle|^c = |\langle c^m(M), 2^m.p\rangle|^c =^{IH} |\langle M, p\rangle|^c$

□

Proof(Lemma 5.7.4)**:**

- let $P \in \mathcal{V}$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M = P$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c \neq P$.
  - Let $M = M_1 M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(P)$. If $M_1 \neq c$ then $|M|^c = |M_1|^c |M_2|^c \neq P$.
- Let $P = \lambda x.Q$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M \neq \lambda x.Q$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c$ so $|N|^c = Q$.
  - Let $M = M_1 M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(\lambda x.N)$ and $|N|^c = Q$. If $M_1 \neq c$ then $|M|^c = |M_1|^c |M_2|^c \neq \lambda x.Q$.
- Let $P = P_1 P_2$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M \neq P_1 P_2$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c \neq P_1 P_2$.
  - Let $M = M_1 M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(M_2' M_2'')$, $M_2' \neq c$, $|M_2'|^c = P_1$ and $|M_2''|^c = P_2$. If $M_1 \neq c$ then $|M|^c = |M_1|^c |M_2|^c = P_1 P_2$ so $|M_1|^c = P_1$ and $|M_2|^c = P_2$.

$\square$

Proof(Lemma 5.8.1)**:** We prove the statement by induction on M.

- Let $M \in \mathcal{V}$ then by lemma 5.3, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda x.N$ then by lemma 5.3:
  - Either $M \in \mathcal{R}^r$ then:
    * Either $p = p' = 0$ so it is done.
    * Or $p = 0$ and $p' = 1.p_1'$ such that $p_1' \in \mathcal{R}^r_N$. Then, $|\langle M, 0\rangle|^c = 0 \neq |\langle M, p'\rangle|^c = 1.|\langle N, p_1'\rangle|^c$.
    * Or $p = 1.p_1$ and $p' = 1.p_1'$ such that $p_1, p_1 \in \mathcal{R}^r_N$. By hypothesis, $|\langle M, p\rangle|^c = 1.|\langle N, p_1\rangle|^c = 1.|\langle N, p_1'\rangle|^c = |\langle M, p'\rangle|^c$. So $|\langle N, p_1\rangle|^c = |\langle N, p_1'\rangle|^c$ and by IH, $p_1 = p_1'$ so $p = p'$.
  - Or $M \notin \mathcal{R}^r$ then $p = 1.p_1$ and $p' = 1.p_1'$ such that $p_1, p_1 \in \mathcal{R}^r_N$. By hypothesis, $|\langle M, p\rangle|^c = 1.|\langle N, p_1\rangle|^c = 1.|\langle N, p_1'\rangle|^c = |\langle M, p'\rangle|^c$. So $|\langle N, p_1\rangle|^c = |\langle N, p_1'\rangle|^c$ and by IH, $p_1 = p_1'$ so $p = p'$.
- Let $M = PQ$ then by lemma 5.3:
  - Either $M \in \mathcal{R}^r$, so P is a λ-abstraction and:
    * Either $p = p' = 0$ so it is done.

    * Or $p = 0$ and $p' = 1.p'_1$ such that $p'_1 \in \mathcal{R}^r_P$. Then $|\langle M, 0\rangle|^c = 0 \neq |\langle M, p'\rangle|^c = 1.|\langle P, p'_1\rangle|^c$.
    * Or $p = 0$ and $p' = 2.p'_1$ such that $p'_1 \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, $|\langle M, 0\rangle|^c = 0 \neq |\langle M, p'\rangle|^c = 2.|\langle Q, p'_1\rangle|^c$.
    * Or $p = 1.p_1$ and $p' = 1.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_P$. Since by hypothesis, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c = 1.|\langle P, p'_1\rangle|^c = |\langle M, p'\rangle|^c$, then $|\langle P, p_1\rangle|^c = |\langle P, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.
    * Or $p = 1.p_1$ and $p' = 2.p'_1$ such that $p_1 \in \mathcal{R}^r_P$ and $p'_1 \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c \neq 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$.
    * Or $p = 2.p_1$ and $p' = 2.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, by hypothesis, $|\langle M, p\rangle|^c = 2.|\langle Q, p_1\rangle|^c = 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$ so $|\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.
  - Or $M \notin \mathcal{R}^r$, then:
    * Or $p = 1.p_1$ and $p' = 1.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_P$. Since by hypothesis, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c = 1.|\langle P, p'_1\rangle|^c = |\langle M, p'\rangle|^c$, then $|\langle P, p_1\rangle|^c = |\langle P, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.
    * Or $p = 1.p_1$ and $p' = 2.p'_1$ such that $p_1 \in \mathcal{R}^r_P$ and $p'_1 \in \mathcal{R}^r_Q$. $P =\neq c$, otherwise, by lemma 5.3, $\mathcal{R}^r_P = \varnothing$. Moreover, $|\langle M, p\rangle|^c = 1.|\langle P, p_1\rangle|^c \neq 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$.
    * Or $p = 2.p_1$ and $p' = 2.p'_1$ such that $p_1, p'_1 \in \mathcal{R}^r_Q$. If $P \neq c$ then, by hypothesis, $|\langle M, p\rangle|^c = 2.|\langle Q, p_1\rangle|^c = 2.|\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$ so $|\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$. If $P = c$ then, by hypothesis, $|\langle M, p\rangle|^c = |\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c = |\langle M, p'\rangle|^c$ so $|\langle Q, p_1\rangle|^c = |\langle Q, p'_1\rangle|^c$. By IH, $p_1 = p'_1$ so $p = p'$.

□

Proof(Lemma 5.8.2)**:** We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$
  - Let $M = x$ then $|M[x := c(cx)]|^c = |c(cx)|^c = |x|^c$.
  - Let $M = y \neq x$ then $|M[x := c(cx)]|^c = |M|^c$.
- Let $M = \lambda y.N$ then $|M[x := c(cx)]|^c = \lambda y.|N[x := c(cx)]|^c =^{IH} \lambda y.|N|^c = |M|^c$, where $y \notin \{x, c\}$.
- Let $M = NP$.
  - Either $N = c$, so $N[x := c(cx)] = c$. Then, $|M[x := c(cx)]|^c = |P[x := c(cx)]|^c =^{IH} |P|^c = |M|^c$.
  - Or $N \neq c$, so $N[x := c(cx)] \neq c$. Then, $|M[x := c(cx)]|^c = |N[x := c(cx)]|^c |P[x := c(cx)]|^c =^{IH} |N|^c|P|^c = |M|^c$.

□

Proof(Lemma 5.8.3)**:** We prove the statement by induction on the structure of M

- Let $M = y$ then by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \varnothing$.

- Let $M = \lambda y.N$. Then by lemma 5.3:
  - Either $p = 0$ if $M \in \mathcal{R}^{\beta\eta}$. Then, $|\langle M[x := c(cx)], 0\rangle|^c = 0 = |\langle M, 0\rangle|^c$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. Then $|\langle M[x := c(cx)], p\rangle|^c = 1.|\langle N[x := c(cx)], p'\rangle|^c =^{IH} 1.|\langle N, p'\rangle|^c = |\langle M, p\rangle|^c$ such that $y \notin \{x, c\}$.
- Let $M = M_1M_2$. Then by lemma 5.3:
  - Either $p = 0$ if $M \in \mathcal{R}^{\beta\eta}$. Then, $|\langle M[x := c(cx)], 0\rangle|^c = 0 = |\langle M, 0\rangle|^c$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. Then $|\langle M[x := c(cx)], p\rangle|^c = 1.|\langle M_1[x := c(cx)], p'\rangle|^c =^{IH} 1.|\langle M_1, p'\rangle|^c = |\langle M, p\rangle|^c$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$.
    * If $M_1 = c$ then $M_1[x := c(cx)] = c$ and $|\langle M[x := c(cx)], p\rangle|^c = |\langle M_2[x := c(cx)], p'\rangle|^c =^{IH} |\langle M_2, p'\rangle|^c = |\langle M, p\rangle|^c$.
    * If $M_1 \neq c$ then $M_1[x := c(cx)] \neq c$ and $|\langle M[x := c(cx)], p\rangle|^c = 2.|\langle M_2[x := c(cx)], p'\rangle|^c =^{IH} 2.|\langle M_2, p'\rangle|^c = |\langle M, p\rangle|^c$.

□

Proof(Lemma 5.8.4)**:** We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $|M|^c = M$ and $\mathrm{fv}(M) \setminus \{c\} = \{M\} = \mathrm{fv}(|M|^c)$.
- Let $M = \lambda y.P \in \Lambda\mathrm{I}_c$ such that $P \in \Lambda\mathrm{I}_c$ and $y \neq c$. Then $|M|^c = \lambda y.|P|^c$ and $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(P) \setminus \{y, c\} =^{IH} \mathrm{fv}(|P|^c) \setminus \{y\} = \mathrm{fv}(|M|^c)$.
- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$ and $y \neq c$. Then $|M|^c = \lambda y.|P[y := c(cy)]|^c =^2 \lambda y.|P|^c$ and $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(P[y := c(cy)]) \setminus \{c, y\} = \mathrm{fv}(P) \setminus \{c, y\} =^{IH} \mathrm{fv}(|P|^c) \setminus \{y\} = \mathrm{fv}(|M|^c)$.
- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin \mathrm{fv}(P) \cup \{c\}$ and $c \neq N$. Then $|M|^c = \lambda y.|Py|^c$ and $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(Py) \setminus \{c, y\} =^{IH} \mathrm{fv}(|Py|^c) \setminus \{y\} = \mathrm{fv}(|M|^c)$.
- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. Then $|M|^c = |P|^c|Q|^c$ and $\mathrm{fv}(M) \setminus \{c\} = (\mathrm{fv}(P) \cup \mathrm{fv}(Q)) \setminus \{c\} = (\mathrm{fv}(P) \setminus \{c\}) \cup (\mathrm{fv}(Q) \setminus \{c\}) =^{IH} \mathrm{fv}(|P|^c) \cup \mathrm{fv}(|Q|^c) = \mathrm{fv}(|M|^c)$.
- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. Then $|M|^c = |\lambda y.P|^c|Q|^c$ and $\mathrm{fv}(M)\setminus\{c\} = (\mathrm{fv}(\lambda y.P) \cup \mathrm{fv}(Q)) \setminus \{c\} = (\mathrm{fv}(\lambda y.P) \setminus \{c\}) \cup (\mathrm{fv}(Q) \setminus \{c\}) =^{IH} \mathrm{fv}(|\lambda y.P|^c) \cup \mathrm{fv}(|Q|^c) = \mathrm{fv}(|M|^c)$.
- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Then $|M|^c = |P|^c$ and $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(P) \setminus \{c\} =^{IH} \mathrm{fv}(|P|^c) = \mathrm{fv}(|M|^c)$.

□

Proof(Lemma 5.8.5)**:** We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$.

– Either $M = x$ then $|M[x := N]|^c = |N|^c = M[x := |N|^c] = |M|^c[x := |N|^c]$.

– Or $M = y \neq x$ then $|M[x := N]|^c = |M|^c = M = M[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = \lambda y.P \in \Lambda I_c$ such that $P \in \Lambda I_c$ and $y \neq c$. Then $|M[x := N]|^c = \lambda y.|P[x := N]|^c =^{IH} \lambda y.|P|^c[x := |N|^c] = |M|^c[x := |N|^c]$, where $y \notin \text{fv}(N) \cup \{x\}$ and so by lemma 4, $y \notin \text{fv}(|N|^c)$.

- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$ and $y \neq c$. Then $|M[x := N]|^c = \lambda y.|P[y := c(cy)][x := N]|^c = \lambda y.|P[x := N][y := c(cy)]|^c =^2 \lambda y.|P[x := N]|^c =^{IH} \lambda y.|P|^c[x := |N|^c] =^2 \lambda y.|P[y := c(cy)]|^c[x := |N|^c] = |M|^c[x := |N|^c]$, where $y \notin \text{fv}(N) \cup \{x\}$ and so by lemma 4, $y \notin \text{fv}(|N|^c)$.

- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin \text{fv}(P) \cup \{c\}$ and $c \neq P$. $|M[x := N]|^c = \lambda y.|(Py)[x := N]|^c =^{IH} \lambda y.|Py|^c[x := |N|^c] = |M|^c[x := |N|^c]$, where $y \notin \text{fv}(N) \cup \{x\}$ and so by lemma 4, $y \notin \text{fv}(|N|^c)$.

- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. $|M[x := N]|^c = |P[x := N]|^c|Q[x := N]|^c =^{IH} |P|^c[x := |N|^c]|Q|^c[x := |N|^c] = (|P|^c|Q|^c)[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. $|M[x := N]|^c = |(\lambda y.P)[x := N]|^c|Q[x := N]|^c =^{IH} |\lambda y.P|^c[x := |N|^c]|Q|^c[x := |N|^c] = (|\lambda y.P|^c|Q|^c)[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. $|M[x := N]|^c = |P[x := N]|^c =^{IH} |P|^c[x := |N|^c] = |M|^c[x := |N|^c]$.

□

Proof(Lemma 5.8.6)**:** We prove the lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $|M|^c = M \in \mathcal{V} \setminus \{c\} \subseteq \Lambda I$.

- let $M = \lambda x.N$ such that $N \in \Lambda I_c$ and $x \in \text{fv}(N)$ and $x \neq c$. Then $|M|^c = \lambda x.|N|^c$ and by IH $|N|^c \in \Lambda I$. Since $x \in \text{fv}(N)$, by lemma 4, $x \in \text{fv}(|N|^c)$, so $|M|^c \in \Lambda I$.

- Let $M = cPQ$ such that $P, Q \in \Lambda I_c$ then $|M|^c = |P|^c|Q|^c$ and by IH, $|P|^c, |Q|^c \in \Lambda I$, hence $|M|^c \in \Lambda I$.

- Let $M = (\lambda x.P)Q$ such that $\lambda x.P, Q \in \Lambda I_c$ then $|M|^c = |\lambda x.P|^c|Q|^c$ and by IH, $|\lambda x.P|^c, |Q|^c \in \Lambda I$, hence $|M|^c \in \Lambda I$.

□

Proof(Lemma 5.8.7a)**:** Let $p \in \mathcal{R}^r_M$, then by definition, $M|_p \in \mathcal{R}^r$. We prove the result by induction on the structure of p.

- Let $p = 0$.

  – Let $r = \beta I$ then $M = (\lambda x.M_1)M_2$ such that $x \in \text{fv}(M_1)$ and $\lambda x.M_1, M_2 \in \Lambda I_c$ and $M' = M_1[x := M_2]$. By definition $M_1 \in \Lambda I_c$, $x \in \text{fv}(M_1)$ and $x \neq c$. Then $|M|^c = (\lambda x.|M_1|^c)|M_2|^c$ and $|M'|^c = |M_1[x := M_2]|^c =^5 |M_1|^c[x := |M_2|^c]$. By lemma 4, $x \in \text{fv}(|M_1|^c)$. So, $|M|^c \xrightarrow{0}_{\beta I} |M'|^c$ and $|\langle M, 0\rangle|^c = 0$.

– Let $r = \beta\eta$.

  * Either $M = (\lambda x.M_1)M_2$ such that $\lambda x.M_1, M_2 \in \Lambda\eta_c$ and $M' = M_1[x := M_2]$. By lemma 5.2, $M_1 \in \Lambda I_c$ and $x \neq c$. Then $|M|^c = (\lambda x.|M_1|^c)|M_2|^c$ and $|M'|^c = |M_1[x := M_2]|^c =^5 |M_1|^c[x := |M_2|^c]$. So, $|M|^c \xrightarrow{0}_\beta |M'|^c$ and $|\langle M, 0\rangle|^c = 0$.
  * Or $M = \lambda x.M'x$ such that $M'x \in \Lambda\eta_c$, $x \notin \text{fv}(M')$, $x \neq c$ and $M' \neq c$. Then $|M|^c = \lambda x.|M'|^c x$. By lemma 4, $x \in \text{fv}(|M'|^c)$. So, $|M|^c \xrightarrow{0}_\beta |M'|^c$ and $|\langle M, 0\rangle|^c = 0$.

- Let $p = 1.p'$.

  – Either $M = \lambda x.M_1$ and $M' = \lambda x.M_1'$ such that $M_1 \xrightarrow{p'}_r M_1'$. By lemma 5.3, $p' \in \mathcal{R}^r_{M_1}$. By lemma 5.2, $M_1 \in \mathcal{M}_c$ and $x \neq c$. By IH, $|M_1|^c \xrightarrow{p''}_r |M_1'|^c$ such that $p'' = |\langle M_1, p'\rangle|^c$. So $|M|^c \xrightarrow{1.p''}_r |M'|^c$ and $1.p'' = |\langle M, p\rangle|^c$.

  – Or $M = M_1M_2$ and $M' = M_1'M_2$ such that $M_1 \xrightarrow{p'}_r M_1'$. By lemma 5.3, $p' \in \mathcal{R}^r_{M_1}$. By lemma 5.3, $M_1 \neq c$. By lemma 5.2.5:

    * Either $M_1 = cM_0$ where $M_0 \in \mathcal{M}_c$. By lemma 5.3, $p' = 2.p_0'$ such that $p_0' \in \mathcal{R}^r_{M_0}$. So by definition $M_1' = cM_0'$ such that $M_0 \xrightarrow{p_0'}_r M_0'$. By IH, $|M_0|^c \xrightarrow{p_0''}_r |M_0'|^c$ such that $p_0'' = |\langle M_0, p_0'\rangle|^c$. Hence $|M|^c \xrightarrow{1.p_0''}_r |M'|^c$ and $|\langle M, p\rangle|^c = |\langle cM_0M_2, 1.2.p_0'\rangle|^c = 1.|\langle cM_0, 2.p_0'\rangle|^c = 1.|\langle M_0, p_0'\rangle|^c = 1.p_0''$
    * Or $M_1 = \lambda x.M_0 \in \mathcal{M}_c$. By IH, $|M_1|^c \xrightarrow{p''}_r |M_1'|^c$ such that $p'' = |\langle M_1, p'\rangle|^c$. By lemma 2, $M_1' \in \mathcal{M}_c$ and by lemma 5.2.3, $M_1' \neq c$. So, $|M|^c \xrightarrow{1.p''}_r |M'|^c$ and $|\langle M, p\rangle|^c = 1.|\langle M_1, p'\rangle|^c = 1.p''$.

- Let $p = 2.p'$ then $M = M_1M_2$ and $M' = M_1M_2'$ such that $M_2 \xrightarrow{p'}_r M_2'$. By lemma 5.3, $p' \in \mathcal{R}^r_{M_2}$. By lemma 5.2.5, $M_2 \in \mathcal{M}_c$. By IH, $|M_2|^c \xrightarrow{p''}_r |M_2'|^c$ such that $p'' = |\langle M_2, p'\rangle|^c$.

  – If $M_1 = c$ then $|M|^c \xrightarrow{p''}_r |M'|^c$ and $|\langle M, p\rangle|^c = |\langle M_2, p'\rangle|^c = p''$.

  – Otherwise $|M|^c \xrightarrow{2.p''}_r |M'|^c$ and $|\langle M, p\rangle|^c = 2.|\langle M_2, p'\rangle|^c = 2.p''$.

□

Proof(Lemma 5.8.7b)**:** The proof is by induction on the structure of M_1.

- Let $M_1 \in \mathcal{V} \setminus \{c\}$. Then $M_1 = |M_1|^c = |M_2|^c$. By lemma 4, $M_2 = c^n(M_1)$.

  – Either $M_1 = x$, then $M_1[x := N_1] = N_1$ and $M_2[x := N_2] = c^n(N_2)$. By hypothesis $|\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c =^2 |\langle c^n(N_2), \mathcal{R}^r_{c^n(N_2)}\rangle|^c$

  – Or $M_1 = y \neq x$ then $M_1[x := N_1] = y$ and $M_2[x := N_2] = c^n(y)$. We conclude using lemma 2.

- Let $M_1 = \lambda y.M_1' \in \Lambda I_c$ such that $y \in \text{fv}(M_1')$, $y \neq c$ and $M_1' \in \Lambda I_c$ then $|M_1|^c = \lambda y.M_1' = |M_2|^c$. By lemma 4 and because $M_2 \in \Lambda I_c$, $M_2 = \lambda y.M_2'$, $y \in \text{fv}(M_2')$, $M_2' \in \Lambda I_c$ and

$|M_2'|^c = |M_1'|^c$. By lemma 5.3, $\mathcal{R}^{\beta I}_{M_1} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_1'}\}$ and $\mathcal{R}^{\beta I}_{M_2} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_2'}\}$. So, $|\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c = \{1.p \mid p \in |\langle M_1', \mathcal{R}^{\beta I}_{M_1'}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c = \{1.p \mid p \in |\langle M_2', \mathcal{R}^{\beta I}_{M_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta I}_{M_1'}\rangle|^c$, then $1.p \in |\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^{\beta I}_{M_2'}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta I}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta I}_{M_2'}\rangle|^c$.
By IH, $|\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c$.

Since $M_1[x := N_1] = \lambda y.M_1'[x := N_1]$ and $M_2[x := N_2] = \lambda y.M_2'[x := N_2]$ where $y \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, by lemma 5.3, $\mathcal{R}^{\beta I}_{M_1[x:=N_1]} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta I}_{M_2[x:=N_2]} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\}$.
So $|\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c = \{1.p \mid p \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c = \{1.p \mid p \in |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = \lambda y.M_1'[y := c(cy)] \in \Lambda\eta_c$ such that $M_1' \in \Lambda\eta_c$ and $y \neq c$, then $|M_1|^c =^2 \lambda y.|M_1'|^c$. Because $|M_2|^c = \lambda y.|M_1'|^c$, then by lemma 4, $M_2 = c^n(\lambda y.P)$ such that $|P|^c = |M_1'|^c$. By lemma 5.2.6, $\lambda y.P \in \Lambda\eta_c$. By lemma 5.2.12a, $P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda y.P$.

  – Either $\lambda y.P = \lambda y.M_2'[y := c(cy)]$ such that $M_2' \in \Lambda\eta_c$. Hence $|M_2'|^c =^2 |M_2'[y := c(cy)]|^c = |M_1'|^c$. We also have $\mathcal{R}^{\beta\eta}_{M_1} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda y.P} =^{5.4.3} \{1.p \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'}\}$. So $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^3 \{1.p \mid p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c =^3 \{1.p \mid p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.
    By IH, $|\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\rangle|^c$.
    Because $M_1[x := N_1] = \lambda y.M_1'[y := c(cy)][x := N_1] = \lambda y.M_1'[x := N_1][y := c(cy)]$ and $(\lambda y.P)[x := N_2] = \lambda y.M_2'[y := c(cy)][x := N_2] = \lambda y.M_2'[x := N_2][y := c(cy)]$ such that $y \notin \text{fv}(N_1) \cup \text{fv}(N_2) \cup \{x\}$, we obtain $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1][y:=c(cy)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2][y:=c(cy)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\}$.
    So $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c =^3 \{1.p \mid p \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c =^2 |\langle(\lambda y.P)[x := N_2], \mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]}\rangle|^c =^3 \{1.p \mid p \in |\langle M_2'[x := N_2], \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle M_2'[x := N_2], \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\rangle|^c$. Hence, $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

  – Let $\lambda y.P = \lambda y.M_2'y$ such that $P = M_2'y \in \Lambda\eta_c$, $y \notin \text{fv}(M_2')$ and $M_2' \neq c$. So we have $|M_2'y|^c = |M_1'|^c$. We already showed that $\mathcal{R}^{\beta\eta}_{M_1} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'}\}$. Since $\lambda y.P \in \mathcal{R}^{\beta\eta}$, by lemma 5.3, $\mathcal{R}^{\beta\eta}_{\lambda y.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'y}\}$. So $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^3 \{1.p \mid p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$.

By IH, $|\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c = |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c$.

Because $M_1[x := N_1] = \lambda y.M_1'[y := c(cy)][x := N_1] = \lambda y.M_1'[x := N_1][y := c(cy)]$, $(\lambda y.P)[x := N_2] = \lambda y.(M_2'y)[x := N_2] = \lambda y.M_2'[x := N_2]y$ such that $y \notin \text{fv}(N_1) \cup \text{fv}(N_2) \cup \{x\}$, we obtain $(\lambda y.P)[x := N_2] \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1][y:=c(cy)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\}$.

So $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c =^3 \{1.p \mid p \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta\eta}_{M_1'[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c =^2 |\langle (\lambda y.P)[x := N_2], \mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle M_1'[x := N_1], \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\rangle|^c \subseteq |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta I}_{(M_2'y)[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = \lambda y.M_1'y \in \Lambda\eta_c$ such that $M_1'y \in \Lambda\eta_c$, $M_1' \neq c$ and $y \notin \text{fv}(M_1') \cup \{c\}$, then $|M_1|^c = \lambda y.|M_1'y|^c$. Because $|M_2|^c = \lambda y.|M_1'y|^c$, then by lemma 4, $M_2 = c^n(\lambda y.P)$ such that $|P|^c = |M_1'y|^c$. By lemma 5.2.6, $\lambda y.P \in \Lambda\eta_c$. By lemma 5.2.12a, $P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda y.P$.

  - Either $\lambda y.P = \lambda y.M_2'[y := c(cy)]$ such that $M_2' \in \Lambda\eta_c$. Since $M_1 \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1} =^{5.3} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'y}\}$. Moreover, $\mathcal{R}^{\beta\eta}_{\lambda y.P} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\}$, so $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c = \{1.p \mid p \in |\langle M_2'[y := c(cy)], \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\rangle|^c\}$. We have $0 \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c$ but $0 \notin |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$.
  - Or $\lambda y.P = \lambda y.M_2'y$ such that $M_2'y \in \Lambda\eta_c$, $y \notin \text{fv}(M_2') \cup \{x\}$ and $M_2' \neq c$. So we have $|M_2'y|^c = |M_1'y|^c$. Because $M_1, \lambda y.P \in \mathcal{R}^{\beta\eta}$, by lemma 5.3, $\mathcal{R}^{\beta\eta}_{M_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1'y}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda y.P} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2'y}\}$. So $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda y.P, \mathcal{R}^{\beta\eta}_{\lambda y.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c\}$. Let $p \in |\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$, i.e. $|\langle M_1'y, \mathcal{R}^{\beta\eta}_{M_1'y}\rangle|^c \subseteq |\langle M_2'y, \mathcal{R}^{\beta\eta}_{M_2'y}\rangle|^c$. By IH, $|\langle (M_1'y)[x := N_1], \mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}\rangle|^c = |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c$.

    Because $M_1[x := N_1] = \lambda y.(M_1'y)[x := N_1] = \lambda y.M_1'[x := N_1]y$, $(\lambda y.P)[x := N_2] = \lambda y.(M_2'y)[x := N_2] = \lambda y.M_2'[x := N_2]y$ and $y \notin \text{fv}(N_1) \cup \text{fv}(N_2)$ such that $y \notin \text{fv}(N_1) \cup \text{fv}(N_2) \cup \{x\}$, we have $M_1[x := N_1], (\lambda y.P)[x := N_2] \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} = \{0\} \cup$

$\{1.p \mid p \in \mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle (M_1'y)[x := N_1], \mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c =^2 |\langle (\lambda y.P)[x := N_2], \mathcal{R}^{\beta\eta}_{(\lambda y.P)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^{\beta I}_{M_1[x:=N_1]}\rangle|^c$ then either $p = 0 \in |\langle M_2[x := N_2], \mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}\rangle|^c$ or $p = 1.p'$ such that $p' \in |\langle (M_1'y)[x := N_1], \mathcal{R}^{\beta I}_{(M_1'y)[x:=N_1]}\rangle|^c \subseteq |\langle (M_2'y)[x := N_2], \mathcal{R}^{\beta I}_{(M_2'y)[x:=N_2]}\rangle|^c$.
So $p \in |\langle M_2[x := N_2], \mathcal{R}^{\beta I}_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = cP_1Q_1 \in \mathcal{M}_c$ such that $P_1, Q_2 \in \mathcal{M}_c$ then $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. Note that $M_1 \notin \mathcal{R}^r$. Because $|M_2|^c = |P_1|^c|Q_1|^c$, then by lemma 4, $M_2 = c^n(PQ)$ such that $P \neq c$, $|P|^c = |P_1|^c$ and $|Q|^c = |Q_1|^c$. By lemma 5.2.6, $PQ \in \mathcal{M}_c$. We prove the lemma by case on PQ.

  - Either $P, Q \in \mathcal{M}_c$ and P is a λ-abstraction $\lambda y.P'$. Because $PQ \in \mathcal{M}_c$, by lemma 1a, $PQ = (\lambda y.P')Q \in \mathcal{R}^r$. By lemma 5.3, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{PQ} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. So $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^2 |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P, \mathcal{R}^r_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P, \mathcal{R}^r_P\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P, \mathcal{R}^r_P\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q, \mathcal{R}^r_Q\rangle|^c$. By IH, $|\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$ and $|\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$.

    Because $M_1[x := N_1] = cP_1[x := N_1]Q_1[x := N_1]$ and $(PQ)[x := N_2] = (\lambda y.P'[x := N_2])Q[x := N_2] \in^{5.2.10} \mathcal{M}_c$ such that $y \notin \text{fv}(N_2)$, we obtain $M_1[x := N_1] \notin \mathcal{R}^r$ and $(PQ)[x := N_2] \in^{1a} \mathcal{R}^r$. So by lemma 5.3 we have $\mathcal{R}^r_{M_1[x:=N_1]} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{(PQ)[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P[x:=N_2]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q[x:=N_2]}\}$.
    So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c = \{1.p \mid p \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c =^2 |\langle (PQ)[x := N_2], \mathcal{R}^r_{(PQ)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c$ then either $p = 1.p'$ such that $p' \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

  - Or $P = cP'$ such that $P', Q \in \mathcal{M}_c$, then $|P|^c = |P'|^c = |P_1|^c$. Since $M_1, PQ \notin \mathcal{R}^r$, by lemma 5.3, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{PQ} = \{1.2.p \mid p \in \mathcal{R}^r_{P'}\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. So $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^2 |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{1.p \mid p \in |\langle P', \mathcal{R}^r_{P'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P', \mathcal{R}^r_{P'}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P', \mathcal{R}^r_{P'}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q, \mathcal{R}^r_Q\rangle|^c$. By IH, $|\langle P_1[x :=$

$N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P'[x := N_2], \mathcal{R}^r_{P'[x:=N_2]}\rangle|^c$ and $|\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$.

Because $M_1[x := N_1] = cP_1[x := N_1]Q_1[x := N_1]$ and $(PQ)[x := N_2] = cP'[x := N_2]Q[x := N_2]$, we obtain $M_1[x := N_1], (PQ)[x := N_2] \notin \mathcal{R}^r$. So by lemma 5.3 we have $\mathcal{R}^r_{M_1[x:=N_1]} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{(PQ)[x:=N_2]} = \{1.2.p \mid p \in \mathcal{R}^r_{P'[x:=N_2]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c = \{1.p \mid p \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c =^2 |\langle (PQ)[x := N_2], \mathcal{R}^r_{(PQ)[x:=N_2]}\rangle|^c = \{1.p \mid p \in |\langle P'[x := N_2], \mathcal{R}^r_{P'[x:=N_2]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c\}$.

Let $p \in |\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c$ then either $p = 1.p'$ such that $p' \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P'[x := N_2], \mathcal{R}^r_{P'[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = P_1Q_1 \in \mathcal{M}_c$ such that $P_1, Q_1 \in \mathcal{M}_c$ and P_1 is a λ-abstraction $\lambda y.P_0$. Then $|M_1|^c = |P_1|^c|Q_1|^c$. Note that because $M_1 \in \mathcal{M}_c$ then by lemma 1a, $M_1 \in \mathcal{R}^r$. So by lemma 5.3, $0 \in \mathcal{R}^r_{M_1}$, so $0 \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c$. Because $|M_2|^c = |P_1|^c|Q_1|^c$, then by lemma 4, $M_2 = c^n(PQ)$ such that $P \neq c$, $|P|^c = |P_1|^c$ and $|Q|^c = |Q_1|^c$. By lemma 5.2.6, $PQ \in \mathcal{M}_c$. We prove the lemma by case on PQ.

  – Either $P = cP'$ such that $P', Q \in \mathcal{M}_c$, so $PQ \notin \mathcal{R}^r$. Hence, by lemma 5.3, $\mathcal{R}^r_{PQ} = \{1.2.p \mid p \in \mathcal{R}^r_{P'}\} \cup \{2.p \mid p \in \mathcal{R}^r_Q\}$. So $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^2 |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{1.p \mid p \in |\langle P', \mathcal{R}^r_{P'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Hence $0 \notin |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$.

  – Or $P, Q \in \mathcal{M}_c$ and P is a λ-abstraction $\lambda y.P'$. Because $PQ = (\lambda y.P')Q \in \mathcal{M}_c$ then by lemma 1a, $PQ \in \mathcal{R}^r$. By lemma 5.3, $\mathcal{R}^r_{M_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{PQ} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_P\} \cup \{2.p \in \mathcal{R}^r_Q\}$. So, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c =^2 |\langle PQ, \mathcal{R}^r_{PQ}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P, \mathcal{R}^r_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P, \mathcal{R}^r_P\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P, \mathcal{R}^r_P\rangle|^c$. let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So, $p \in |\langle Q, \mathcal{R}^r_Q\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q, \mathcal{R}^r_Q\rangle|^c$.

    By IH, $|\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$ and $|\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$.

    By lemma 5.2.10, $M_1[x := N_1] \in \mathcal{M}_c$ and by lemma 1a, $M_1[x := N_1] = (\lambda y.P_0[x := N_1])Q_1[x := N_1] \in \mathcal{R}^r$. By lemma 5.2.10, $(PQ)[x := N_2] \in \mathcal{M}_c$ and by lemma 1a, $(PQ)[x := N_2] = (\lambda y.P'[x := N_2])Q[x := N_2] \in \mathcal{R}^r$. So by lemma 5.3 we have $\mathcal{R}^r_{M_1[x:=N_1]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{(PQ)[x:=N_2]} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P[x:=N_2]}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q[x:=N_2]}\}$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c = \{0\}\{1.p \mid p \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c\}$ and $|\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c =^2 |\langle (PQ)[x := N_2], \mathcal{R}^r_{(PQ)[x:=N_2]}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c\}$. Let $p \in |\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c$ then either $p = 0 \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or

$p = 1.p'$ such that $p' \in |\langle P_1[x := N_1], \mathcal{R}^r_{P_1[x:=N_1]}\rangle|^c \subseteq |\langle P[x := N_2], \mathcal{R}^r_{P[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1[x := N_1], \mathcal{R}^r_{Q_1[x:=N_1]}\rangle|^c \subseteq |\langle Q[x := N_2], \mathcal{R}^r_{Q[x:=N_2]}\rangle|^c$. So $p \in |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

- Let $M_1 = cM'_1 \in \Lambda\eta_c$ such that $M'_1 \in \Lambda\eta_c$. So $|M'_1|^c = |M_1|^c$. By lemm 2, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = |\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c$. By IH, $|\langle M'_1[x := N_1], \mathcal{R}^r_{M'_1[x:=N_1]}\rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$. Since $M_1[x := N_1] = cM'_1[x := N_1]$ then by lemm 2, $|\langle M_1[x := N_1], \mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}\rangle|^c = |\langle M'_1[x := N_1], \mathcal{R}^{\beta\eta}_{M'_1[x:=N_1]}\rangle|^c$. So $|\langle M_1[x := N_1], \mathcal{R}^r_{M_1[x:=N_1]}\rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}^r_{M_2[x:=N_2]}\rangle|^c$.

□

Proof(Lemma 5.8.7c)**:** By lemma 8, $p_1 \in \mathcal{R}^r_{M_1}$ and $p_2 \in \mathcal{R}^r_{M_2}$. We prove this lemma by induction on the structure of M_1.

1. Let $M_1 \in \mathcal{V} \setminus \{c\}$ then nothing to prove since M_1 does not reduce.
2. Let $M_1 = \lambda x.N_1 \in \Lambda\mathrm{I}_c$ such that $x \neq c$. So $|M_1|^c = \lambda x.|N_1|^c = |M_2|^c$. By lemma 4, because $M_2 \in \Lambda\mathrm{I}_c$ and by lemma 5.2, $M_2 = \lambda x.N_2$ and $|N_2|^c = |N_1|^c$. So $N_2 \in \Lambda\mathrm{I}_c$. Since $M_1, M_2 \notin \mathcal{R}^{\beta I}$, by lemma 5.3, $\mathcal{R}^{\beta I}_{M_1} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{N_1}\}$ and $\mathcal{R}^{\beta I}_{M_2} = \{1.p \mid p\mathcal{R}^{\beta I}_{N_2}\}$ so $|\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c = \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta I}_{N_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c = \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta I}_{N_2}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta I}_{N_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta I}_{M_1}\rangle|^c$, so by hypothesis, $1.p \in |\langle M_2, \mathcal{R}^{\beta I}_{M_2}\rangle|^c$. Hence, $p \in |\langle N_2, \mathcal{R}^{\beta I}_{N_2}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta I}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta I}_{N_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^{\beta I}_{M_1}$, $p_1 = 1.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta I}_{N_1}$. Since $p_2 \in \mathcal{R}^{\beta I}_{M_2}$, $p_2 = 1.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta I}_{N_2}$. Since $|\langle M_1, p\rangle|^c = |\langle M_2, p\rangle|^c$ then $|\langle N_1, p'_1\rangle|^c = |\langle N_2, p'_2\rangle|^c$. Hence, $M_1 = \lambda x.N_1 \xrightarrow{p_1}_{\beta I} \lambda x.N'_1 = M'_1$ such that $N_1 \xrightarrow{p'_1}_{\beta I} N'_1$ and $M_2 = \lambda x.N_2 \xrightarrow{p_2}_{\beta I} \lambda x.N'_2 = M'_2$ such that $N_2 \xrightarrow{p'_2}_{\beta I} N'_2$. By IH, $|\langle N'_1, \mathcal{R}^{\beta I}_{N'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta I}_{N'_2}\rangle|^c$. By lemma 5.3, $\mathcal{R}^{\beta I}_{M'_1} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{N'_1}\}$ and $\mathcal{R}^{\beta I}_{M'_2} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{N'_2}\}$, so $|\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle N'_1, \mathcal{R}^{\beta I}_{N'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c = \{1.p \mid p \in |\langle N'_2, \mathcal{R}^{\beta I}_{N'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^{\beta I}_{M'_1}\rangle|^c$, then $p = 1.p'$ such that $p' \in |\langle N'_1, \mathcal{R}^{\beta I}_{N'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta I}_{N'_2}\rangle|^c$, so $p \in |\langle M'_2, \mathcal{R}^{\beta I}_{M'_2}\rangle|^c$.
3. Let $M_1 = \lambda x.N_1[x := c(cx)] \in \Lambda\eta_c$ such that $N_1 \in \Lambda\eta_c$ and $x \neq c$ then $|M_1|^c = \lambda x.|N_1[x := c(cx)]|^c =^2 \lambda x.|N_1|^c$. Because $|M_2|^c = \lambda x.|N_1|^c$, then by lemma 4, $M_2 = c^n(\lambda x.P)$ such that $|P|^c = |N_1|^c$. By lemma 5.2.6, $\lambda x.P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda x.P$.
   - Either $\lambda x.P = \lambda x.N_2[x := c(cx)]$ such that $N_2 \in \Lambda\eta_c$. Then, $|N_1|^c = |P|^c = |N_2[x := c(cx)]|^c =^2 |N_2|^c$ and $\mathcal{R}^{\beta\eta}_{M_1} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^3 \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c =^3 \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Because $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, we obtain $p_1 = 1.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and by lemma 5.4.5 we obtain $p_2 = 2^n.1.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. Because $1.|\langle N_1, p'_1\rangle|^c =^3 |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c =^{3,3} 1.|\langle N_2, p'_2\rangle|^c$,

we obtain $|\langle N_1, p_1'\rangle|^c = |\langle N_2, p_2'\rangle|^c$. So $M_1 = \lambda x.N_1[x := c(cx)] \stackrel{p_1}{\to}_{\beta\eta} \lambda x.P_1 = M_1'$ and $M_2 = c^n(\lambda x.N_2[x := c(cx)]) \stackrel{p_2}{\to}_{\beta\eta} c^n(\lambda x.P_2) = M_2'$ such that $N_1[x := c(cx)] \stackrel{p_1'}{\to}_{\beta\eta} P_1$ and $N_2[x := c(cx)] \stackrel{p_2'}{\to}_{\beta\eta} P_2$. By lemma 5.2.13a, $P_1 = N_1'[x := c(cx)]$, $P_2 = N_2'[x := c(cx)]$, $N_1 \stackrel{p_1'}{\to}_{\beta\eta} N_1'$ and $N_2 \stackrel{p_2'}{\to}_{\beta\eta} N_2'$. By IH, $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$. Hence, $\mathcal{R}^{\beta\eta}_{M_1'} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P_2} =^{5.4.3} \{1.p \in \mathcal{R}^{\beta\eta}_{N_2'[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. So, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c =^3 \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c =^2 |\langle \lambda x.P_2, \mathcal{R}^{\beta\eta}_{\lambda x.P_2}\rangle|^c =^3 \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, so $p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

- Let $\lambda x.P = \lambda x.N_2x$ such that $N_2x \in \Lambda\eta_c$, $x \notin \text{fv}(N_2)$ and $N_2 \neq c$, then $\lambda x.P \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{5.3} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2x}\}$. By lemma 5.4.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{5.3} \{2^n.0\} \cup \{2^n.1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2x}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c =^3 \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2x, \mathcal{R}^{\beta\eta}_{N_2x}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle N_2x, \mathcal{R}^{\beta\eta}_{N_2x}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2x, \mathcal{R}^{\beta\eta}_{N_2x}\rangle|^c$. Since $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, $p_1 = 1.p_1'$ such that $p_1' \in \mathcal{R}^{\beta\eta}_{N_1}$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $1.|\langle N_1, p_1'\rangle|^c =^3 |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, then $p_2 = 2^n.1.p_2'$ such that $p_2' \in \mathcal{R}^{\beta\eta}_{N_2x}$. Because $1.|\langle N_1, p_1'\rangle|^c =^3 |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c =^3 |\langle \lambda x.N_2x, 1.p_2'\rangle|^c = 1.|\langle N_2x, p_2'\rangle|^c$ then $|\langle N_1, p_1'\rangle|^c = |\langle N_2x, p_2'\rangle|^c$. So $M_1 = \lambda x.N_1[x := c(cx)] \stackrel{p_1}{\to}_{\beta\eta} \lambda x.P_1 = M_1'$ and $M_2 = c^n(\lambda x.N_2x) \stackrel{p_2}{\to}_{\beta\eta} c^n(\lambda x.N_2') = M_2'$ such that $N_1[x := c(cx)] \stackrel{p_1'}{\to}_{\beta\eta} P_1$ and $N_2x \stackrel{p_2'}{\to}_{\beta\eta} N_2'$. By lemma 5.2.13a, $P_1 = N_1'[x := c(cx)]$, and $N_1 \stackrel{p_1'}{\to}_{\beta\eta} N_1'$. By IH, $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$. Moreover, $\mathcal{R}^{\beta\eta}_{M_1'} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.N_2'} \setminus \{0\} =^{5.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. So, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c =^3 \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c \setminus \{0\} =^2 |\langle \lambda x.N_2', \mathcal{R}^{\beta\eta}_{\lambda x.N_2'}\rangle|^c \setminus \{0\} = \{1.p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c$ then $p = 1.p'$ such that $p' \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, so $p \in |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c \setminus \{0\}$, i.e. $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

4. Let $M_1 = \lambda x.N_1x \in \Lambda\eta_c$ such that $N_1x \in \Lambda\eta_c$, $x \notin \text{fv}(N_1) \cup \{c\}$ and $N_1 \neq c$, then $M_1 \in \mathcal{R}^{\beta\eta}$ and $|M_1|^c = \lambda x.|N_1x|^c = \lambda x.|N_1|^c x$. Because $|M_2|^c = \lambda x.|N_1|^c x$, then by lemma 4, $M_2 = c^n(\lambda x.P)$ such that $|P|^c = |N_1|^c x$. By lemma 5.2.6, $\lambda x.P \in \Lambda\eta_c$. We prove the lemma by case on $\lambda x.P$.

   (a) Let $\lambda x.P = \lambda x.N_2[x := c(cx)]$ such that $N_2 \in \Lambda\eta_c$ then $\mathcal{R}^{\beta\eta}_{M_1} =^{5.3} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1x}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{5.4.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2[x:=c(cx)]}\} =^{5.4.4} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1x, \mathcal{R}^{\beta\eta}_{N_1x}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c =^3 \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c\}$. Hence, $0 \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c$ but $0 \notin |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$.

(b) Let $\lambda x.P = \lambda x.N_2 x$ such that $N_2 x \in \Lambda\eta_c$, $x \notin \text{fv}(N_2)$ and $N_2 \neq c$, then $M_2 \in \mathcal{R}^{\beta\eta}$. Since $|M_2|^c = \lambda x.|N_2 x|^c = \lambda x.|N_2|^c x$, $|N_1 x|^c = |N_2 x|^c$ and $|N_1|^c = |N_2|^c$. Moreover, $\mathcal{R}^{\beta\eta}_{M_1} =^{5.3} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1 x}\}$, $\mathcal{R}^{\beta\eta}_{\lambda x.P} =^{5.3} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2 x}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} =^{5.4.5} \{2^n.p \mid p \in \mathcal{R}^{\beta\eta}_{\lambda x.P}\} =^{5.3} \{2^n.0\} \cup \{2^n.1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2 x}\}$. So, $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle \lambda x.P, \mathcal{R}^{\beta\eta}_{\lambda x.P}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c\}$. Let $p \in |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$, so $p \in |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$, i.e. $|\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \subseteq |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$. Moreover, $\mathcal{R}^{\beta\eta}_{N_1 x} \setminus \{0\} =^{5.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{N_2 x} \setminus \{0\} =^{5.3} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c\}$ and $|\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c\}$. Let $p \in |\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c$ then $1.p \in |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \setminus \{0\} \subseteq |\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \subseteq |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$, so $p \in |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$, i.e. $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$:

- Either $p_1 = 0$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, we obtain $p_2 = 2^n.0$. So $M_1 \xrightarrow{0}_{\beta\eta} N_1$ and $M_2 = c^n(\lambda x.N_2 x) \xrightarrow{p_2}_{\beta\eta} c^n(N_2)$. It is done since $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c =^2 |\langle c^n(N_2), \mathcal{R}^{\beta\eta}_{c^n(N_2)}\rangle|^c$.
- Or $p_1 = 1.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta\eta}_{N_1 x}$. Becasue $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, we obtain $p_2 = 2^n.1.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2 x}$. Becasue $1.|\langle N_1 x, p'_1\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c =^3 |\langle \lambda x.N_2 x, 1.p'_2\rangle|^c = 1.|\langle N_2 x, p'_2\rangle|^c$, we obtain $|\langle N_1 x, p'_1\rangle|^c = |\langle N_2 x, p'_2\rangle|^c$. So $M_1 = \lambda x.N_1 x \xrightarrow{p_1}_{\beta\eta} \lambda x.N'_1 = M'_1$ and $M_2 = c^n(\lambda x.N_2 x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N'_2) = M'_2$ such that $N_1 x \xrightarrow{p'_1}_{\beta\eta} N'_1$ and $N_2 x \xrightarrow{p'_2}_{\beta\eta} N'_2$. By IH, $|\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c$.
  - Either $N_1 x \in \mathcal{R}^{\beta\eta}$, so $N_1 = \lambda y.P_1$ and by lemma 5.3, $\mathcal{R}^{\beta\eta}_{N_1 x} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$. Because $|\langle N_1 x, \mathcal{R}^{\beta\eta}_{N_1 x}\rangle|^c \subseteq |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$, we obtain $0 \in |\langle N_2 x, \mathcal{R}^{\beta\eta}_{N_2 x}\rangle|^c$. Hence, $0 \in \mathcal{R}^{\beta\eta}_{N_2 x}$ and by lemma 5.3, $\mathcal{R}^{\beta\eta}_{N_2 x} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$. Hence, $N_2 x \in \mathcal{R}^{\beta\eta}$ and by lemma 4, $N_2 = \lambda y.P_2$ such that $|P_1|^c = |P_2|^c$.
    - ∗ Either $p'_1 = 0$. Because $|\langle N_1 x, p'_1\rangle|^c = |\langle N_2 x, p'_2\rangle|^c$, we obtain $p'_2 = 0$. So $M_1 = \lambda x.(\lambda y.P_1)x \xrightarrow{p_1}_{\beta\eta} \lambda x.P_1[y := x] = M'_1$ and $M_2 = c^n(\lambda x.(\lambda y.P_2)x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.P_2[y := x]) = M'_2$. Because $x \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, we obtain $M'_1 = N_1$ and $M'_2 = c^n(N_2)$. It is done since $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c =^2 |\langle c^n(N_2), \mathcal{R}^{\beta\eta}_{c^n(N_2)}\rangle|^c$.
    - ∗ Let $p'_1 = 1.p''_1$ such that $p''_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Because $|\langle N_1 x, p'_1\rangle|^c = |\langle N_2 x, p'_2\rangle|^c$, we obtain $p'_2 = 1.p''_2$ such that $p''_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So $M_1 = \lambda x.N_1 x \xrightarrow{p_1}_{\beta\eta} \lambda x.N''_1 x = M'_1$ and $M_2 = c^n(\lambda x.N_2 x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N''_2 x) = M'_2$ such that $N_1 \xrightarrow{p''_1}_{\beta\eta} N''_1$ and $N_2 \xrightarrow{p''_2}_{\beta\eta} N''_2$. because $x \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, by lemma 2.2.3, we obtain $x \notin \text{fv}(N''_1) \cup \text{fv}(N''_2)$. So, $M'_1, \lambda x.N''_2 x \in \mathcal{R}^{\beta\eta}$ and by lemma 5.3, $\mathcal{R}^{\beta\eta}_{M'_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N'_1}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.N''_2 x} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N'_2}\}$. Hence, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c = \{0\} \cup \{\lambda x.C \mid C \in |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c =^2$

$|\langle\lambda x.N_2''x, \mathcal{R}^{\beta\eta}_{\lambda x.N_2''x}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$.
Because $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, we obtain $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\} \subseteq \{0\} \cup \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\} = |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

– Else by lemma 5.3, $\mathcal{R}^{\beta\eta}_{N_1 x} = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$. Let $p_1' = 1.p_1''$ such that $p_1'' \in \mathcal{R}^{\beta\eta}_{N_1}$. Then, $p_2' = 1.p_2''$ such that $p_2'' \in \mathcal{R}^{\beta\eta}_{N_2}$. So $M_1 = \lambda x.N_1 x \xrightarrow{p_1}_{\beta\eta} \lambda x.N_1''x = M_1'$ and $M_2 = c^n(\lambda x.N_2 x) \xrightarrow{p_2}_{\beta\eta} c^n(\lambda x.N_2''x) = M_2'$ such that $N_1 \xrightarrow{p_1''}_{\beta\eta} N_1''$ and $N_2 \xrightarrow{p_2''}_{\beta\eta} N_2''$. Because $x \notin \mathrm{fv}(N_1) \cup \mathrm{fv}(N_2)$, by lemma 2.2.3 we obtain, $x \notin \mathrm{fv}(N_1'') \cup \mathrm{fv}(N_2'')$. So, $M_1', \lambda x.N_2'x \in \mathcal{R}^{\beta\eta}$ and by lemma 5.3, $\mathcal{R}^{\beta\eta}_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{\lambda x.N_2'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. Hence, $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c =\!=^2 |\langle\lambda x.N_2', \mathcal{R}^{\beta\eta}_{\lambda x.N_2'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\}$. Because $|\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c \subseteq |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c$, we obtain $|\langle M_1', \mathcal{R}^{\beta\eta}_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1', \mathcal{R}^{\beta\eta}_{N_1'}\rangle|^c\} \subseteq \{0\} \cup \{1.p \mid p \in |\langle N_2', \mathcal{R}^{\beta\eta}_{N_2'}\rangle|^c\} = |\langle M_2', \mathcal{R}^{\beta\eta}_{M_2'}\rangle|^c$.

5. Let $M_1 = cP_1Q_1 \in \mathcal{M}_c$ such that $P_1, P_2 \in \mathcal{M}_c$. So $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. We prove the statement by induction on the structure of M_2:

- Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = \lambda x.N_2 \in \Lambda\mathrm{I}_c$ such that $N_2 \in \Lambda\mathrm{I}_c$ and $x \neq c$ then $|M_2|^c = \lambda x.|N_2|^c \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = \lambda x.N_2[x := c(cx)] \in \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$ and $x \neq c$then $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = \lambda x.N_2 x \in \Lambda\eta_c$ such that $N_2 x \in \Lambda\mathrm{I}_c$ and $x \notin \mathrm{fv}(N_2) \cup \{c\}$ and $N_2 \neq c$ then $|M_2|^c = \lambda x.|N_2 x|^c \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$, then $|cP_2|^c = |P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. Since $M_1, cP_2 \notin \mathcal{R}^r$, by lemma 5.3, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$. So, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$. Again by lemma 5.3, since $M_2 \notin \mathcal{R}^r$, $\mathcal{R}^r_{M_2} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$. So, $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^r_{M_1}$:
  – Either $p_1 = 1.2.p_1'$ such that $p_1' \in \mathcal{R}^r_{P_1}$ and so $1.|\langle P_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Hence, because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 1.2.p_2'$ such that $|\langle P_1, p_1'\rangle|^c = |\langle P_2, p_2'\rangle|^c$ and $p_2' \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = cP_1Q_1 \xrightarrow{p_1}_r cP_1'Q_1 = M_1'$ and $M_2 = cP_2Q_2 \xrightarrow{p_2}_r cP_2'Q_2 = M_2'$ such that $P_1 \xrightarrow{p_1'}_r P_1'$ and $P_2 \xrightarrow{p_2'}_r P_2'$. By IH, $|\langle P_1', \mathcal{R}^r_{P_1'}\rangle|^c \subseteq |\langle P_2', \mathcal{R}^r_{P_2'}\rangle|^c$. By lemma 5.3, $\mathcal{R}^r_{M_1'} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{1.p \mid p \in |\langle P_1', \mathcal{R}^r_{P_1'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c = \{1.p \mid p \in |\langle P_2', \mathcal{R}^r_{P_2'}\rangle|^c\} \cup \{2.p \mid p \in$

$|\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P_1', \mathcal{R}^r_{P_1'}\rangle|^c \subseteq |\langle P_2', \mathcal{R}^r_{P_2'}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

– Or $p_1 = 2.p_1'$ such that $p_1' \in \mathcal{R}^r_{Q_1}$ and so $2.|\langle Q_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 2.p_2'$ such that $|\langle Q_1, p_1'\rangle|^c = |\langle Q_2, p_2'\rangle|^c$. Hence, $M_1 = cP_1Q_1 \overset{p_1}{=\!\!\rightarrow}_r cP_1Q_1' = M_1'$ and $M_2 = cP_2Q_2 \overset{p_2}{\rightarrow}_r cP_2Q_2' = M_2'$ such that $Q_1 \overset{p_1'}{\rightarrow}_r Q_1'$ and $Q_2 \overset{p_2'}{\rightarrow}_r Q_2'$. By IH, $|\langle Q_1', \mathcal{R}^r_{Q_1'}\rangle|^c \subseteq |\langle Q_2', \mathcal{R}^r_{Q_2'}\rangle|^c$. By lemma 5.3, $\mathcal{R}^r_{M_1'} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1'}\}$ and $\mathcal{R}^r_{M_2'} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2'}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1', \mathcal{R}^r_{Q_1'}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2', \mathcal{R}^r_{Q_2'}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1', \mathcal{R}^r_{Q_1'}\rangle|^c \subseteq |\langle Q_2', \mathcal{R}^r_{Q_2'}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

- Let $M_2 = P_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$ and P_2 is a λ-abstraction. Then $|P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. Since $M_1 \notin \mathcal{R}^r$, by lemma 5.3, $\mathcal{R}^r_{M_1} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$. So, $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$. Again by lemma 5.3, since $M_2 \in \mathcal{R}^r$ by lemma 1a, $\mathcal{R}^r_{M_2} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$. So, $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{0\} \cup = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^r_{M_1}$:
  – Either $p_1 = 1.2.p_1'$ such that $p_1' \in \mathcal{R}^r_{P_1}$ and so $1.|\langle P_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 1.p_2'$ such that $|\langle P_1, p_1'\rangle|^c = |\langle P_2, p_2'\rangle|^c$ and $p_2' \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = cP_1Q_1 \overset{p_1}{\rightarrow}_r cP_1'Q_1 = M_1'$ and $M_2 = P_2Q_2 \overset{p_2}{\rightarrow}_r P_2'Q_2 = M_2'$ such that $P_1 \overset{p_1'}{\rightarrow}_r P_1'$ and $P_2 \overset{p_2'}{\rightarrow}_r P_2'$. By IH, $|\langle P_1', \mathcal{R}^r_{P_1'}\rangle|^c \subseteq |\langle P_2', \mathcal{R}^r_{P_2'}\rangle|^c$. Because $P_2 \in \mathcal{M}_c$, then by lemma 2, $P_2' \in \mathcal{M}_c$. By lemma 5.2.3, $P_2' \neq c$. By lemma 5.3, $\mathcal{R}^r_{M_1'} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} \setminus \{0\} = \{1.p \mid p \in \mathcal{R}^r_{P_2'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{1.p \mid p \in |\langle P_1', \mathcal{R}^r_{P_1'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle P_2', \mathcal{R}^r_{P_2'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P_1', \mathcal{R}^r_{P_1'}\rangle|^c \subseteq |\langle P_2', \mathcal{R}^r_{P_2'}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.
  – Or $p_1 = 2.p_1'$ such that $p_1' \in \mathcal{R}^r_{Q_1}$ and so $2.|\langle Q_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 2.p_2'$ such that $|\langle Q_1, p_1'\rangle|^c = |\langle Q_2, p_2'\rangle|^c$. Hence, $M_1 = cP_1Q_1 \overset{p_1}{\rightarrow}_r cP_1Q_1' = M_1'$ and $M_2 = P_2Q_2 \overset{p_2}{\rightarrow}_r P_2Q_2' = M_2'$ such that $Q_1 \overset{p_1'}{\rightarrow}_r Q_1'$ and $Q_2 \overset{p_2'}{\rightarrow}_r Q_2'$. By IH, $|\langle Q_1', \mathcal{R}^r_{Q_1'}\rangle|^c \subseteq |\langle Q_2', \mathcal{R}^r_{Q_2'}\rangle|^c$. By lemma 5.3, $\mathcal{R}^r_{M_1'} = \{1.2.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1'}\}$ and $\mathcal{R}^r_{M_2'} \setminus \{0\} = \{1.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid$

$p \in \mathcal{R}^r_{Q'_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.

- Let $M_2 = cN_2 \in \mathcal{M}_c = \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 5.4.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$ and $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 \xrightarrow{p_2}_{\beta\eta} cN'_2 = M'_2$ such that $N_2 \xrightarrow{p'_2}_{\beta\eta} N'_2$. Because $|\langle N_2, p'_2\rangle|^c =^3 |\langle M_2, p_2\rangle|^c = |\langle M_1, p_1\rangle|^c$, by IH, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c =^2 |\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c$.

6. Let $M_1 = (\lambda x.P_1)Q_1 \in \mathcal{M}_c$ such that $\lambda x.P_1, Q_1 \in \mathcal{M}_c$. By lemma 5.2.8, lemma 5.2.12a and lemma 5.2.9, $P_1 \in \mathcal{M}_c$ and $x \neq c$. So $|M_1|^c = |\lambda x.P_1|^c|Q_1|^c = |M_2|^c = (\lambda x.|P_1|^c)|Q_1|^c$. By lemma 1a, $M_1 \in \mathcal{R}^r$, so by lemma 5.3, $\mathcal{R}^r_{M_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_1} \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in \mathcal{R}^r_{P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$. So $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$. We prove this statement by induction on the structure of M_2:

   - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2 \in \Lambda\mathrm{I}_c$ such that $N_2 \in \Lambda\mathrm{I}_c$ and $x \neq c$ then $|M_2|^c = \lambda x.|N_2|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2[x := c(cx)] \in \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$ and $x \neq c$ then $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2x \in \Lambda\eta_c$ such that $N_2x \in \Lambda\eta_c$, $N_2 \neq c$ and $x \notin \mathrm{fv}(N_2) \cup \{c\}$ then $|M_2|^c = \lambda x.|N_2x|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$. By lemma 5.3, $\mathcal{R}^r_{M_2} = \{1.2.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Because $0 \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c$ and $0 \notin |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$, we obtain $|\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \not\subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$.
   - Let $M_2 = (\lambda x.P_2)Q_2 \in \mathcal{M}_c$ such that $\lambda x.P_2, Q_2 \in \mathcal{M}_c$, then $|P_1|^c = |P_2|^c$ and $|Q_1|^c = |Q_2|^c$. By lemma 5.2.8, lemma 5.2.12a and lemma 5.2.9, $P_2 \in \mathcal{M}_c$. By lemma 5.3, $\mathcal{R}^r_{M_2} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$ and $\mathcal{R}^r_{M_2} \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in \mathcal{R}^r_{P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$. So $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$ and $|\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c \setminus \{1.0\} = \{0\} \cup \{1.1.p \mid p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c$ then $1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c$, i.e. $|\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c \subseteq |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c$. Let $p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c$ then $1.1.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$, i.e. $|\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c \subseteq |\langle P_2, \mathcal{R}^r_{P_2}\rangle|^c$. Let $p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c$ then $2.p \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. So $p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$, i.e. $|\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. Since $p_1 \in \mathcal{R}^r_{M_1}$:
     - Either $p_1 = 0$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 0$. Hence, $M_1 = (\lambda x.P_1)Q_1 \xrightarrow{0}_r$

$P_1[x := Q_1] = M_1'$ and $M_2 = (\lambda x.P_2)Q_2 \xrightarrow{0}_r P_2[x := Q_2] = M_2'$. By lemma 7b, $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c \subseteq |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

– Or $p_1 = 1.p_1'$ such that $p_1' \in \mathcal{R}^r_{\lambda x.P_1}$ and so $1.|\langle \lambda x.P_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 1.p_2'$ such that $|\langle \lambda x.P_1, p_1'\rangle|^c = |\langle \lambda x.P_2, p_2'\rangle|^c$ and $p_2' \in \mathcal{R}^r_{\lambda x.P_2}$. By lemma 5.3:
  * Either $\lambda x.P_1 = \lambda x.N_1 x \in \mathcal{R}^r$ such that $x \notin \mathrm{fv}(N_1)$, $\mathcal{M}_c = \Lambda\eta_c$ and $p_1' = 0$. So, $|\langle \lambda x.P_2, p_2'\rangle|^c = 0$. Hence, $p_2' = 0$ and $\lambda x.P_2 = \lambda x.N_2 x$ such that $x \notin \mathrm{fv}(N_2)$. Hence, $M_1 = (\lambda x.N_1 x)Q_1 \xrightarrow{p_1}_r N_1 Q_1 = M_1'$ and $M_2 = (\lambda x.N_2 x)Q_2 \xrightarrow{p_2}_r N_2 Q_2 = M_2'$ such that $\lambda x.N_1 x \xrightarrow{p_1'}_r N_1$ and $\lambda x.N_2 x \xrightarrow{p_2'}_r N_2$. By IH, $|\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$.
    · If N_1 is a λ-abstraction then by lemma 1a, $N_1 x \in \mathcal{R}^r$. So $1.1.0 \in \mathcal{R}^r_{M_1}$ and $|\langle M_2, 1.1.0\rangle|^c = 1.1.0 = |\langle M_1, 1.1.0\rangle|^c \in |\langle M_1, \mathcal{R}^r_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^r_{M_2}\rangle|^c$. Hence, $1.1.0 \in \mathcal{R}^r_{M_2}$. So N_2 is a λ-abstraction. So $\mathcal{R}^r_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{N_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{N_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 0 \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.
    · Otherwise $\mathcal{R}^r_{M_1'} = \{1.p \mid p \in \mathcal{R}^r_{N_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} \setminus \{0\} = \{1.p \mid p \in \mathcal{R}^r_{N_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{1.p \mid p \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c \setminus \{0\} = \{1.p \mid p \in |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 1.p'$ such that $p' \in |\langle N_1, \mathcal{R}^r_{N_1}\rangle|^c \subseteq |\langle N_2, \mathcal{R}^r_{N_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.
  * Or $p_1' = 1.p_1''$ such that $p_1'' \in \mathcal{R}^r_{P_1}$. So $p_2' = 1.p_2''$ such that $p_2'' \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = (\lambda x.P_1)Q_1 \xrightarrow{p_1}_r (\lambda x.P_1')Q_1 = M_1'$ and $M_2 = (\lambda x.P_2)Q_2 \xrightarrow{p_2}_r (\lambda x.P_2')Q_2 = M_2'$ such that $\lambda x.P_1 \xrightarrow{p_1'}_r \lambda x.P_1'$ and $\lambda x.P_2 \xrightarrow{p_2'}_r \lambda x.P_2'$. By IH, $|\langle \lambda x.P_1', \mathcal{R}^r_{\lambda x.P_1'}\rangle|^c \subseteq |\langle \lambda x.P_2', \mathcal{R}^r_{\lambda x.P_2'}\rangle|^c$. Since $M_1, M_2 \in \mathcal{M}_c$, by lemma 2, $M_1', M_2' \in \mathcal{M}_c$. By lemma 5.3 and lemma 1a, $\mathcal{R}^r_{M_1'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_1'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_2'}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q_2}\}$, so $|\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_1', \mathcal{R}^r_{\lambda x.P_1'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c\}$ and $|\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_2', \mathcal{R}^r_{\lambda x.P_2'}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c\}$. Let $p \in |\langle M_1', \mathcal{R}^r_{M_1'}\rangle|^c$. Either $p = 0$ then $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle \lambda x.P_1', \mathcal{R}^r_{\lambda x.P_1'}\rangle|^c \subseteq |\langle \lambda x.P_2', \mathcal{R}^r_{\lambda x.P_2'}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q_1, \mathcal{R}^r_{Q_1}\rangle|^c \subseteq |\langle Q_2, \mathcal{R}^r_{Q_2}\rangle|^c$. So $p \in |\langle M_2', \mathcal{R}^r_{M_2'}\rangle|^c$.

– Or $p_1 = 2.p_1'$ such that $p_1' \in \mathcal{R}^r_{Q_1}$ and so $2.|\langle Q_1, p_1'\rangle|^c = |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$. Because $p_2 \in \mathcal{R}^r_{M_2}$, we obtain $p_2 = 2.p_2'$ such that $|\langle Q_1, p_1'\rangle|^c = |\langle Q_2, p_2'\rangle|^c$. Hence,

$M_1 = (\lambda x.P_1)Q_1 \xrightarrow{p_1}_r (\lambda x.P_1)Q'_1 = M'_1$ and $M_2 = (\lambda x.P_2)Q_2 \xrightarrow{p_2}_r (\lambda x.P_2)Q'_2 = M'_2$ such that $Q_1 \xrightarrow{p'_1}_r Q'_1$ and $Q_2 \xrightarrow{p'_2}_r Q'_2$. By IH, $|\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. Since $M_1, M_2 \in \mathcal{M}_c$, by lemma 2, $M'_1, M'_2 \in \mathcal{M}_c$. By lemma 5.3 and lemma 1a, $\mathcal{R}^r_{M'_1} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_1}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^r_{\lambda x.P_2}\} \cup \{2.p \mid p \in \mathcal{R}^r_{Q'_2}\}$, so $|\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle P_1, \mathcal{R}^r_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c\}$ and $|\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c\} \cup \{2.p \mid p \in |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c\}$. Let $p \in |\langle M'_1, \mathcal{R}^r_{M'_1}\rangle|^c$. Either $p = 0 \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 1.p'$ such that $p' \in |\langle \lambda x.P_1, \mathcal{R}^r_{\lambda x.P_1}\rangle|^c \subseteq |\langle \lambda x.P_2, \mathcal{R}^r_{\lambda x.P_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$. Or $p = 2.p'$ such that $p' \in |\langle Q'_1, \mathcal{R}^r_{Q'_1}\rangle|^c \subseteq |\langle Q'_2, \mathcal{R}^r_{Q'_2}\rangle|^c$. So $p \in |\langle M'_2, \mathcal{R}^r_{M'_2}\rangle|^c$.

- Let $M_2 = cN_2 \in \mathcal{M}_c = \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 5.4.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{N_2}\}$ and $|\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c =^2 |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$. Because $p_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, we obtain $p_2 = 2.p'_2$ such that $p'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 \xrightarrow{p_2}_{\beta\eta} cN'_2 = M'_2$ such that $N_2 \xrightarrow{p'_2}_{\beta\eta} N'_2$. Since $|\langle N_2, p'_2\rangle|^c =^3 |\langle M_2, p_2\rangle|^c = |\langle M_1, p_1\rangle|^c$, by IH, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c \subseteq |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c =^2 |\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c$.

7. Let $M_1 = cN_1 \in \mathcal{M}_c = \Lambda\eta_c$ such that $N_1 \in \Lambda\eta_c$. So $|N_1|^c = |M_1|^c = |M_2|^c$. By lemma 5.4.5, $\mathcal{R}^{\beta\eta}_{M_1} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c =^2 |\langle M_1, \mathcal{R}^{\beta\eta}_{M_1}\rangle|^c \subseteq |\langle M_2, \mathcal{R}^{\beta\eta}_{M_2}\rangle|^c$. Because $p_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, we obtain $p_1 = 2.p'_1$ such that $p'_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. So, $M_1 = cN_1 \xrightarrow{p_1}_{\beta\eta} cN'_1 = M'_1$ such that $N_1 \xrightarrow{p'_1}_{\beta\eta} N'_1$. Because $|\langle N_1, p'_1\rangle|^c =^3 |\langle M_1, p_1\rangle|^c = |\langle M_2, p_2\rangle|^c$, by IH, $|\langle M'_1, \mathcal{R}^{\beta\eta}_{M'_1}\rangle|^c =^2 |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c \subseteq |\langle M'_2, \mathcal{R}^{\beta\eta}_{M'_2}\rangle|^c$.

□

B. Proofs of section 3

Proof(Remark 3.3)**:**

- Commutativity: by (in_R), $\tau_1 \cap \tau_2 \leq^2 \tau_2$ and by (in_L), $\tau_1 \cap \tau_2 \leq^2 \tau_1$ so by (mon'), $\tau_1 \cap \tau_2 \leq^2 \tau_2 \cap \tau_1$. By (in_L), $\tau_2 \cap \tau_1 \leq^2 \tau_2$ and by (in_R), $\tau_2 \cap \tau_1 \leq^2 \tau_1$ so by (mon'), $\tau_2 \cap \tau_1 \leq^2 \tau_1 \cap \tau_2$. Hence, $\tau_1 \cap \tau_2 \sim^2 \tau_2 \cap \tau_1$.

- Associativity: by (in_R), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_3$, by (in_L), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_1 \cap \tau_2$, by (in_R), $\tau_1 \cap \tau_2 \leq^2 \tau_2$, by (in_L), $\tau_1 \cap \tau_2 \leq^2 \tau_1$, so by (tr), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_1$ and $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_2$. By (mon'), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_2 \cap \tau_3$ and again by (mon'), $(\tau_1 \cap \tau_2) \cap \tau_3 \leq^2 \tau_1 \cap (\tau_2 \cap \tau_3)$. By (in_L), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_1$, by (in_R), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_2 \cap \tau_3$, by (in_L), $\tau_2 \cap \tau_3 \leq^2 \tau_2$, by (in_R), $\tau_2 \cap \tau_3 \leq^2 \tau_3$, so by (tr), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_2$ and $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_3$. By (mon'), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 \tau_1 \cap \tau_2$ and again by (mon'), $\tau_1 \cap (\tau_2 \cap \tau_3) \leq^2 (\tau_1 \cap \tau_2) \cap \tau_3$. Hence, $(\tau_1 \cap \tau_2) \cap \tau_3 \sim^2 \tau_1 \cap (\tau_2 \cap \tau_3)$.

- Idempotence: by (in_L), $\tau \cap \tau \leq^2 \tau$ and by (ref) and (mon'), $\tau \leq^2 \tau \cap \tau$, hence, $\tau \sim^2 \tau \cap \tau$.

□

Proof(Lemma 3.5)**:**

1. By induction on the size derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last rule of the derivation.

   - (ref): $\tau \leq \tau$. By $\tau \in$ TypeOmega.
   - (tr): $(\tau_1 \leq^2 \tau_2 \wedge \tau_2 \leq^2 \tau_3) \Rightarrow \tau_1 \leq^2 \tau_3$. By IH twice, $\tau_3 \in$ TypeOmega.
   - (in_L): $\tau_1 \cap \tau_2 \leq^2 \tau_1$. By definition $\tau_1 \in$ TypeOmega.
   - (in_R): $\tau_1 \cap \tau_2 \leq^2 \tau_2$. By definition $\tau_2 \in$ TypeOmega.
   - $(\rightarrow$-$\cap)$: $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \leq^2 \tau_1 \rightarrow (\tau_2 \cap \tau_3)$. If $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \in$ TypeOmega then by definition $\tau_1 \rightarrow \tau_2, \tau_1 \rightarrow \tau_3 \in$ TypeOmega which is false.
   - (mon'): $(\tau_1 \leq^2 \tau_2 \wedge \tau_1 \leq^2 \tau_3) \Rightarrow \tau_1 \leq^2 \tau_2 \cap \tau_3$. By IH $\tau_2, \tau_3 \in$ TypeOmega. Hence, $\tau_2 \cap \tau_3 \in$ TypeOmega.
   - (mon): $(\tau_1 \leq^2 \tau_1' \wedge \tau_2 \leq^2 \tau_2') \Rightarrow \tau_1 \cap \tau_2 \leq^2 \tau_1' \cap \tau_2'$. By definition $\tau_1, \tau_2 \in$ TypeOmega. By IH, $\tau_1', \tau_2' \in$ TypeOmega. So $\tau_1' \cap \tau_2' \in$ TypeOmega.
   - $(\rightarrow$-$\eta)$: $(\tau_1 \leq^2 \tau_1' \wedge \tau_2' \leq^2 \tau_2) \Rightarrow \tau_1' \rightarrow \tau_2' \leq^2 \tau_1 \rightarrow \tau_2$. By $\tau_1' \rightarrow \tau_2' \notin$ TypeOmega.
   - (Ω): $\tau \leq^2 \Omega$. By definition $\Omega \in$ TypeOmega.
   - $(\Omega'$-$lazy)$: $\tau \rightarrow \Omega \leq^2 \Omega \rightarrow \Omega$. It is done since $\tau \rightarrow \Omega \notin$ TypeOmega.

2. Let $\tau \leq^2 \tau'$. Assume $\tau \sim^2 \Omega$. Then $\Omega \leq^2 \tau$ and by transitivity $\Omega \leq^2 \tau'$. Moreover, by (Ω), $\tau' \leq^2 \Omega$. So $\tau' \sim^2 \Omega$.

3. By (Ω), $\tau \cap \tau' \leq^2 \Omega$. let $\tau \sim^2 \Omega$ and $\tau' \sim^2 \Omega$, so $\Omega \leq^2 \tau$ and $\Omega \leq^2 \tau'$ and by (mon'), $\Omega \leq^2 \tau \cap \tau'$.

4. By (Ω), $\tau \leq^2 \Omega$ and by transitivity, $\tau \leq^2 \tau'$ because $\Omega \leq^2 \tau'$. By (ref), $\tau \leq^2 \tau$ and by (mon'), $\tau \leq^2 \tau \cap \tau'$.

5. We prove the lemma by induction on the size derivation of $\tau \leq^2 \tau'$ and then by case on the last rule of the derivation.

   - (ref): $\tau \leq \tau$. Then it is done with $n = 1$, $\tau_1'' = \tau_2$ and $\tau_1' = \tau_1$.
   - (tr): $(\tau_1 \leq^2 \tau_2 \wedge \tau_2 \leq^2 \tau_3) \Rightarrow \tau_1 \leq^2 \tau_3$. Let τ, τ' such that $\mathrm{inInter}(\tau \rightarrow \tau', \tau_3)$ and $\tau' \not\sim^2 \Omega$. By IH there exist $n \geq 1$ and $\tau_1', \tau_1'', \ldots, \tau_n', \tau_n''$ such that for all $i \in \{1, \ldots, n\}$, $\mathrm{inInter}(\tau_i' \rightarrow \tau_i'', \tau_2)$ and $\tau_i'' \not\sim^2 \Omega$ and $\tau_1'' \cap \cdots \cap \tau_n'' \leq^2 \tau'$. Again by IH, for all $i \in \{1, \ldots, n\}$, there exist $m_i \geq 1$ and $\tau_{1,i}''', \tau_{1,i}'''', \ldots, \tau_{m_i,i}''', \tau_{m_i,i}'''' \in$ Type^2 such that for all $j \in \{1, \ldots, m_i\}$, $\mathrm{inInter}(\tau_{j,i}''' \rightarrow \tau_{j,i}'''', \tau_1)$ and $\tau_{j,i}'''' \not\sim^2 \Omega$ and $\tau_1'''' \cap \cdots \cap \tau_m'''' \leq^2 \tau_i''$. Using rule (mon), associativity and commutativity, $\tau_{1,1}'''' \cap \cdots \cap \tau_{m_1,1}'''' \cap \cdots \cap \tau_{1,n}'''' \cap \cdots \cap \tau_{m_n,n}'''' \leq^2 \tau'$.

     Let $\tau \sim^2 \Omega$. Then by IH, for all $i \in \{1, \ldots, n\}$, $\tau_i' \sim^2 \Omega$. Again by IH, for all $i \in \{1, \ldots, n\}$, for all $j \in \{1, \ldots, m_i\}$, $\tau_{j,i}''' \sim^2 \Omega$.
   - (in_L): $\tau_1 \cap \tau_2 \leq^2 \tau_1$. Let τ, τ' such that $\mathrm{inInter}(\tau \rightarrow \tau', \tau_1)$ and $\tau' \not\sim^2 \Omega$ then it is done with $n = 1$, $\tau_1'' = \tau'$ and $\tau_1' = \tau$.
   - (in_R): $\tau_1 \cap \tau_2 \leq^2 \tau_2$. Let τ, τ' such that $\mathrm{inInter}(\tau \rightarrow \tau', \tau_2)$ and $\tau' \not\sim^2 \Omega$ then it is done with $n = 1$, $\tau_1'' = \tau'$ and $\tau_1' = \tau$.

- ($\rightarrow$ -$\cap$): $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \leq^2 \tau_1 \rightarrow (\tau_2 \cap \tau_3)$. Let τ, τ' such that $\text{inInter}(\tau \rightarrow \tau', \tau_1 \rightarrow (\tau_2 \cap \tau_3))$ and $\tau' \not\sim^2 \Omega$ then $\tau = \tau_1$ and $\tau' = \tau_2 \cap \tau_3$. $\tau_2 \not\sim^2 \Omega$ or $\tau_3 \not\sim^2 \Omega$ because $\tau' \not\sim^2 \Omega$ and using lemma 3.5.3. If $\tau_2 \not\sim^2 \Omega$ and $\tau_3 \not\sim^2 \Omega$ then it is done with $n = 2$, $\tau_1' = \tau_2' = \tau_1$ and $\tau_1'' = \tau_2$ and $\tau_2'' = \tau_3$. If $\tau_2 \not\sim^2 \Omega$ and $\tau_3 \sim^2 \Omega$ then it is done with $n = 1$, $\tau_1' = \tau_1$ and $\tau_1'' = \tau_2$ because $\tau_2 \leq^2 \tau_2 \cap \tau_3$ by lemma 3.5.4. If $\tau_2 \sim^2 \Omega$ and $\tau_3 \not\sim^2 \Omega$ then it is done with $n = 1$, $\tau_1' = \tau_1$ and $\tau_1'' = \tau_3$ because $\tau_3 \leq^2 \tau_2 \cap \tau_3$ by lemma 3.5.4 and commutativity.
- (mon'): $(\tau_1 \leq^2 \tau_2 \wedge \tau_1 \leq^2 \tau_3) \Rightarrow \tau_1 \leq^2 \tau_2 \cap \tau_3$. Let τ, τ' such that $\text{inInter}(\tau \rightarrow \tau', \tau_2 \cap \tau_3)$ and $\tau' \not\sim^2 \Omega$. Either $\text{inInter}(\tau \rightarrow \tau', \tau_2)$ and we conclude by IH. Or $\text{inInter}(\tau \rightarrow \tau', \tau_3)$ and we conclude by IH.
- (mon): $(\tau_1 \leq^2 \tau_1' \wedge \tau_2 \leq^2 \tau_2') \Rightarrow \tau_1 \cap \tau_2 \leq^2 \tau_1' \cap \tau_2'$. Let τ, τ' such that $\text{inInter}(\tau \rightarrow \tau', \tau_1' \cap \tau_2')$. Either $\text{inInter}(\tau \rightarrow \tau', \tau_1')$ and it is done by IH. Or $\text{inInter}(\tau \rightarrow \tau', \tau_2')$ and it is done by IH.
- ($\rightarrow$ -η): $(\tau_1 \leq^2 \tau_1' \wedge \tau_2' \leq^2 \tau_2) \Rightarrow \tau_1' \rightarrow \tau_2' \leq^2 \tau_1 \rightarrow \tau_2$. Let τ, τ' such that $\text{inInter}(\tau \rightarrow \tau', \tau_1 \rightarrow \tau_2)$ and $\tau' \not\sim^2 \Omega$ then $\tau = \tau_1$ and $\tau' = \tau_2$ and it is done with $n = 1$ and $\tau_1'' = \tau_2'$ because $\tau_2' \not\sim^2 \Omega$ by lemma 3.5.2 and because if $\tau_1 \sim^2 \Omega$ then $\tau_1' \sim^2 \Omega$.
- (Ω): $\tau_0 \leq^2 \Omega$. There is no τ, τ' such that $\text{inInter}(\tau \rightarrow \tau', \Omega)$.
- (Ω'-$lazy$): $\tau_0 \rightarrow \Omega \leq^2 \Omega \rightarrow \Omega$. there is no $\tau' \not\sim^2 \Omega$ such that $\text{inInter}(\tau \rightarrow \tau', \Omega \rightarrow \Omega)$.

6. let $\tau' \in \mathsf{Type}^2$. First we prove that $\Omega \rightarrow \tau' \not\sim^2 \Omega$. Assume $\Omega \rightarrow \tau' \not\sim^2 \Omega$ then $\Omega \leq^2 \Omega \rightarrow \tau'$. By lemma 3.5.1, $\Omega \rightarrow \tau' \in \mathsf{TypeOmega}$ which is false.

   Let $\tau \sim^2 \Omega$. Assume $\alpha \rightarrow \Omega \rightarrow \tau' \sim^2 \Omega \rightarrow \tau$ then $\Omega \rightarrow \tau \leq^2 \alpha \rightarrow \Omega \rightarrow \tau'$. By lemma 3.5.5, $\tau \leq^2 \Omega \rightarrow \tau'$ which is false.

   Let $\tau \not\sim^2 \Omega$. Assume $\alpha \rightarrow \Omega \rightarrow \tau' \sim^2 \Omega \rightarrow \tau$ then $\alpha \rightarrow \Omega \rightarrow \tau' \leq^2 \Omega \rightarrow \tau$. By lemma 3.5.5, $\alpha \sim^2 \Omega$ because $\Omega \sim^2 \Omega$, which is false.

□

C. Proofs of section 4

Proof(Lemma 4.4)**:**

1. If $\tau_1 \cap \tau_2 \in \mathsf{NTType}^3$ then it is done by definition. Otherwise $\tau_1, \tau_2 \notin \mathsf{NTType}^3$, so $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = \Lambda = \Lambda \cap \Lambda = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}}$.
2. We prove this result by induction on the structure of ρ.
   - Let $\rho = \alpha$ then $[\![\rho]\!]^3_{\mathcal{P}} = \mathcal{P}$.
   - Let $\rho = \tau \rightarrow \rho'$, then by definition, $[\![\rho]\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
   - Let $\rho = \tau \cap \rho'$, then by IH, $[\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$. So $[\![\rho]\!]^3_{\mathcal{P}} = [\![\tau]\!]^3_{\mathcal{P}} \cap [\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
   - Let $\rho = \rho' \cap \tau$, then by IH, $[\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$. So $[\![\rho]\!]^3_{\mathcal{P}} = [\![\tau]\!]^3_{\mathcal{P}} \cap [\![\rho']\!]^3_{\mathcal{P}} \subseteq \mathcal{P}$.
3. By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.
   - (ref): $\tau \leq \tau$. This case is trivial.

- (Ω): $\tau \leq \Omega$. This case is trivial since $\Omega \notin \mathsf{NTType}^3$.
- (tr): $\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_3$. We conclude using IH twice.
- $(\Omega'\text{-}lazy)$: $\tau \to \Omega \leq \Omega \to \Omega$. This case is trivial since $\Omega \to \Omega \notin \mathsf{NTType}^3$.
- (in_L): $\tau_1 \cap \tau_2 \leq \tau_1$. This case is trivial.
- (in_R): $\tau_1 \cap \tau_2 \leq \tau_2$. This case is trivial.
- $(\to\text{-}\cap)$: $(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \leq \tau_1 \to (\tau_2 \cap \tau_3)$. if $\tau_1 \to (\tau_2 \cap \tau_3) \in \mathsf{NTType}^3$ then $\tau_2 \in \mathsf{NTType}^3$ or $\tau_3 \in \mathsf{NTType}^3$. Hence $\tau_1 \to \tau_2 \in \mathsf{NTType}^3$ or $\tau_1 \to \tau_3 \in \mathsf{NTType}^3$, so $(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \in \mathsf{NTType}^3$.
- (mon'): $\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$. If $\tau_2 \cap \tau_3 \in \mathsf{NTType}^3$ then $\tau_2 \in \mathsf{NTType}^3$ or $\tau_3 \in \mathsf{NTType}^3$, so by IH, $\tau_1 \in \mathsf{NTType}^3$.
- (mon): $\tau_1 \leq \tau_1' \wedge \tau_2 \leq \tau_2' \Rightarrow \tau_1 \cap \tau_2 \leq \tau_1' \cap \tau_2'$. If $\tau_1' \cap \tau_2' \in \mathsf{NTType}^3$ then $\tau_1' \in \mathsf{NTType}^3$ or $\tau_2' \in \mathsf{NTType}^3$. So by IH, $\tau_1 \in \mathsf{NTType}^3$ or $\tau_2 \in \mathsf{NTType}^3$, hence $\tau_1 \cap \tau_2 \in \mathsf{NTType}^3$.
- $(\to\text{-}\eta)$: $\tau_1 \leq \tau_1' \wedge \tau_2' \leq \tau_2 \Rightarrow \tau_1' \to \tau_2' \leq \tau_1 \to \tau_2$. If $\tau_1 \to \tau_2 \in \mathsf{NTType}^3$ then $\tau_2 \in \mathsf{NTType}^3$, so by IH, $\tau_2' \in \mathsf{NTType}^3$, hence $\tau_1' \to \tau_2' \in \mathsf{NTType}^3$.

4. By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.

- (ref): $\tau \leq \tau$. This case is trivial.
- (Ω): $\tau \leq \Omega$. This case is trivial since $[\![\Omega]\!]^3_{\mathcal{P}} = \Lambda$.
- (tr): $\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_3$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$ and $[\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_3]\!]^3_{\mathcal{P}}$, so $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_3]\!]^3_{\mathcal{P}}$.
- $(\Omega'\text{-}lazy)$: $\tau \to \Omega \leq \Omega \to \Omega$. This case is trivial since $[\![\tau \to \Omega]\!]^3_{\mathcal{P}} = [\![\Omega \to \Omega]\!]^3_{\mathcal{P}} = \Lambda$.
- (in_L): $\tau_1 \cap \tau_2 \leq \tau_1$. By 1, $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1]\!]^3_{\mathcal{P}}$.
- (in_R): $\tau_1 \cap \tau_2 \leq \tau_2$. By 1, $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3 \cap [\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$.
- $(\to\text{-}\cap)$: $(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3) \leq \tau_1 \to (\tau_2 \cap \tau_3)$.
  - If $\tau_1 \to \tau_2, \tau_1 \to \tau_3 \in \mathsf{NTType}^3$ then $\tau_2, \tau_3, \tau_2 \cap \tau_3 \in \mathsf{NTType}^3$, so $[\![(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_1 \to \tau_3]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\} \cap \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_3]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_3]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \to (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}}$.
  - If $\tau_1 \to \tau_2 \in \mathsf{NTType}^3$ and $\tau_1 \to \tau_3 \notin \mathsf{NTType}^3$, then $\tau_2, \tau_2 \cap \tau_3 \in \mathsf{NTType}^3$ and $\tau_3 \notin \mathsf{NTType}^3$, so $[\![(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_1 \to \tau_3]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \to (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}}$.
  - If $\tau_1 \to \tau_2 \notin \mathsf{NTType}^3$ and $\tau_1 \to \tau_3 \in \mathsf{NTType}^3$, then $\tau_3, \tau_2 \cap \tau_3 \in \mathsf{NTType}^3$ and $\tau_2 \notin \mathsf{NTType}^3$, so $[\![(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \to \tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_1 \to \tau_3]\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_3]\!]^3_{\mathcal{P}}\} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \to (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}}$.
  - If $\tau_1 \to \tau_2, \tau_1 \to \tau_3 \notin \mathsf{NTType}^3$, then $\tau_2, \tau_3, \tau_2 \cap \tau_3 \notin \mathsf{NTType}^3$, so $[\![(\tau_1 \to \tau_2) \cap (\tau_1 \to \tau_3)]\!]^3_{\mathcal{P}} = [\![\tau_1 \to (\tau_2 \cap \tau_3)]\!]^3_{\mathcal{P}} = \Lambda$.

- (mon'): $\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3 \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$ and $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_3]\!]^3_{\mathcal{P}}$. So by 1, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}} \cap [\![\tau_3]\!]^3_{\mathcal{P}} = [\![\tau_2 \cap \tau_3]\!]^3_{\mathcal{P}}$.
- (mon): $\tau_1 \leq \tau_1' \wedge \tau_2 \leq \tau_2' \Rightarrow \tau_1 \cap \tau_2 \leq \tau_1' \cap \tau_2'$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1']\!]^3_{\mathcal{P}}$ and $[\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_2']\!]^3_{\mathcal{P}}$. So by 1, $[\![\tau_1 \cap \tau_2]\!]^3_{\mathcal{P}} = [\![\tau_1]\!]^3_{\mathcal{P}} \cap [\![\tau_2]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1']\!]^3_{\mathcal{P}} \cap [\![\tau_2']\!]^3_{\mathcal{P}} = [\![\tau_1' \cap \tau_2']\!]^3_{\mathcal{P}}$.
- $(\rightarrow$ -$\eta)$: $\tau_1 \leq \tau_1' \wedge \tau_2' \leq \tau_2 \Rightarrow \tau_1' \rightarrow \tau_2' \leq \tau_1 \rightarrow \tau_2$. By IH, $[\![\tau_1]\!]^3_{\mathcal{P}} \subseteq [\![\tau_1']\!]^3_{\mathcal{P}}$ and $[\![\tau_2']\!]^3_{\mathcal{P}} \subseteq [\![\tau_2]\!]^3_{\mathcal{P}}$. If $\tau_1 \rightarrow \tau_2 \in \mathsf{NTType}^3$ then $\tau_2 \in \mathsf{NTType}^3$ and by 3, $\tau_2' \in \mathsf{NTType}^3$, so $\tau_1' \rightarrow \tau_2' \in \mathsf{NTType}^3$ and $[\![\tau_1' \rightarrow \tau_2']\!]^3_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1']\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2']\!]^3_{\mathcal{P}}\} \subseteq \{M \in \mathcal{P} \mid \forall N \in [\![\tau_1]\!]^3_{\mathcal{P}}.\ MN \in [\![\tau_2]\!]^3_{\mathcal{P}}\} = [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}}$. Otherwise, $[\![\tau_1' \rightarrow \tau_2']\!]^3_{\mathcal{P}} \subseteq [\![\tau_1 \rightarrow \tau_2]\!]^3_{\mathcal{P}} = \Lambda$.

5. Assume $\mathrm{VAR}(\mathcal{P}, \mathcal{P})$. Let $n \geq 0$, $x \in \mathcal{V}$ and for all $i \in \{1, \ldots, n\}, M_i \in \mathcal{P}$. By the hypothesis, $xM_1 \cdots M_n \in \mathcal{P}$. We prove that $xM_1 \cdots M_n \in [\![\varphi]\!]^3_{\mathcal{P}}$ by induction on the structure of φ.

   - If $\varphi = \alpha$ then $xM_1 \cdots M_n \in \mathcal{P} = [\![\alpha]\!]^3_{\mathcal{P}}$.
   - If $\varphi = \Omega$ then $xM_1 \cdots M_n \in \Lambda = [\![\Omega]\!]^3_{\mathcal{P}}$.
   - If $\varphi = \tau \cap \varphi'$. By IH, $xM_1 \cdots M_n \in [\![\varphi']\!]^3_{\mathcal{P}}$, so by 1, $xM_1 \cdots M_n \in [\![\tau]\!]^3_{\mathcal{P}} \cap [\![\varphi']\!]^3_{\mathcal{P}} = [\![\tau \cap \varphi']\!]^3_{\mathcal{P}}$.
   - If $\varphi = \varphi' \cap \tau$. By IH, $xM_1 \cdots M_n \in [\![\varphi']\!]^3_{\mathcal{P}}$, so by 1, $xM_1 \cdots M_n \in [\![\varphi']\!]^3_{\mathcal{P}} \cap [\![\tau]\!]^3_{\mathcal{P}} = [\![\varphi' \cap \tau]\!]^3_{\mathcal{P}}$.
   - If $\varphi = \rho \rightarrow \varphi'$.
     - If $\varphi \in \mathsf{NTType}^3$ then $\varphi' \in \mathsf{NTType}^3$. Let $N \in [\![\rho]\!]^3_{\mathcal{P}}$, so by 2, $N \in \mathcal{P}$. By IH, $xM_1 \cdots M_n N \in [\![\varphi']\!]^3_{\mathcal{P}}$. So $xM_1 \cdots M_n \in [\![\rho \rightarrow \varphi']\!]^3_{\mathcal{P}}$.
     - If $\varphi \notin \mathsf{NTType}^3$ then $xM_1 \cdots M_n \in [\![\rho \rightarrow \varphi']\!]^3_{\mathcal{P}} = \Lambda$.

6. Assume $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$. Let $n \geq 0$, $x \in \mathcal{V}$, $M, N \in \Lambda$ and for all $i \in \{1, \ldots, n\}, N_i \in \Lambda$. We prove that if $M[x := N]N_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$ then $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$ by induction on the structure of τ.

   - If $\tau = \alpha$ then $[\![\alpha]\!]^3_{\mathcal{P}} = \mathcal{P}$ and we conclude using the hypothesis $\mathrm{SAT}(\mathcal{P}, \mathcal{P})$.
   - If $\tau = \Omega$ then $(\lambda x.M)NN_1 \cdots N_n \in \Lambda = [\![\Omega]\!]^3_{\mathcal{P}}$.
   - If $\tau = \tau_1 \cap \tau_2$. Assume $M[x := N]N_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}} =^1 [\![\tau_1]\!]^3 \cap [\![\tau_2]\!]^3$, then by IH, $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau_1]\!]^3 \cap [\![\tau_2]\!]^3 =^1 [\![\tau]\!]^3$.
   - If $\tau = \tau_1 \rightarrow \tau_2$.
     - If $\tau \in \mathsf{NTType}^3$ then $\tau_2 \in \mathsf{NTType}^3$. Let $P \in [\![\tau_1]\!]^3_{\mathcal{P}}$ and $M[x := N]N_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$ then by 2, $M[x := N]N_1 \cdots N_n \in \mathcal{P}$. By hypothesis, $(\lambda x.M)NN_1 \cdots N_n \in \mathcal{P}$. Moreover, $M[x := N]N_1 \cdots N_n P \in [\![\tau_2]\!]^3_{\mathcal{P}}$. By IH, $(\lambda x.M)NN_1 \cdots N_n P \in [\![\tau_2]\!]^3_{\mathcal{P}}$, so $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}}$.
     - Let $\tau \notin \mathsf{NTType}^3$ then $(\lambda x.M)NN_1 \cdots N_n \in [\![\tau]\!]^3_{\mathcal{P}} = \Lambda$.

□

D. Proofs of section 6

Proof(Lemma 6.2)**:** 1. By induction on $\Gamma \vdash^{\beta I} M : \sigma$. 2. By induction on $\Gamma \vdash^{\beta\eta} M : \sigma$.
3. First prove (*): if $\Gamma \vdash^{r} M : \sigma$, and $\sigma \sqsubseteq \sigma'$ then $\Gamma \vdash^{r} M : \sigma'$ by induction on $\sigma \sqsubseteq \sigma'$. Then, do the proof of 3. by induction on $\Gamma \vdash^{r} M : \sigma$. For the latter we do:

- Case (ax): If $\Gamma, x : \sigma \vdash^{\beta\eta} x : \sigma$, $\Gamma', x : \sigma' \sqsubseteq \Gamma, x : \sigma$ and $\sigma \sqsubseteq \sigma''$ then $\sigma' \sqsubseteq \sigma$ and so $\sigma' \sqsubseteq \sigma''$. By (ax) $\Gamma', x : \sigma' \vdash^{\beta\eta} x : \sigma'$. By (*), $\Gamma', x : \sigma' \vdash^{\beta\eta} x : \sigma''$.

- Case $(\to_{E^I})$: If $\frac{\Gamma \vdash^{\beta I} M : \sigma \to \tau \quad \Delta \vdash^{\beta I} N : \sigma}{\Gamma \sqcap \Delta \vdash^{\beta I} MN : \tau}$, $\Gamma = \Gamma_1, \Gamma_2$, $\Delta = \Delta_1, \Delta_2$, $\Gamma \sqcap \Delta = \Gamma_3, \Gamma_2, \Delta_2$, $\Gamma' = \Gamma'_3, \Gamma'_2, \Delta'_2 \sqsubseteq \Gamma$ where, $\Gamma_1 = (x_i : \sigma_i)_n$, $\Gamma_2 = (y_j, \tau_j)_m$, $\Gamma_3 = (x_i : \sigma_i \cap \sigma'_i)_n$, $\Delta_1 = (x_i : \sigma'_i)_n$, $\Delta_2 = (z_l, \rho_l)_k$, $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Delta_2) = \varnothing$, $\Gamma'_3 = (x_i : \overline{\sigma}_i)_n$, $\Gamma'_2 = (y_j, \overline{\tau}_j)_m$, $\Delta'_2 = (z_l, \overline{\rho}_l)_k$, $\overline{\sigma_i} \sqsubseteq \sigma_i \cap \sigma'_i$, $\overline{\tau_j} \sqsubseteq \tau_j$ and $\overline{\rho_l} \sqsubseteq \rho_l$ then $\Gamma'_3, \Gamma'_2 \sqsubseteq \Gamma$ and $\Gamma'_3, \Delta'_2 \sqsubseteq \Delta$. By IH, $\Gamma'_3, \Gamma'_2 \vdash^{\beta I} M : \sigma \to \tau$ and $\Gamma'_3, \Delta'_2 \vdash^{\beta I} N : \sigma$, so by $(\to_{E^I})$, $\Gamma'_3 \sqcap \Gamma'_3, \Gamma'_2, \Delta'_2 \vdash^{\beta I} MN : \tau$. By (*), and since $\Gamma'_3 \sqcap \Gamma'_3 = \Gamma'_3$, we have: $\Gamma'_3, \Gamma'_2, \Delta'_2 \vdash^{\beta I} MN : \tau$.

□

Proof(Lemma 6.3)**:** When $M \to_r^* N$ and $M \to_r^* P$, we write $M \to_r^* \{N, P\}$.

1. By induction on $\sigma \in \mathsf{Type}^1$.
   - If $\sigma \in \mathcal{A}$ then $\mathsf{CR}_0^r \subseteq \mathsf{CR}^r = [\![\sigma]\!]^r$.
   - If $\sigma = \tau \cap \rho$ then by IH, $\mathsf{CR}_0^r \subseteq [\![\tau]\!]^r, [\![\rho]\!]^r \subseteq \mathsf{CR}^r$, so $\mathsf{CR}_0^r \subseteq [\![\tau \cap \rho]\!]^r \subseteq \mathsf{CR}^r$.
   - If $\sigma = \tau \to \rho$ then by IH, $\mathsf{CR}_0^r \subseteq [\![\tau]\!]^r, [\![\rho]\!]^r \subseteq \mathsf{CR}^r$ and $[\![\sigma]\!]^r \subseteq \mathsf{CR}^r$ by definition. Let $M \in \mathsf{CR}_0^r$, so $M = xN_1 \dots N_n$ such that $n \geq 0$ and $N_1, \dots, N_n \in \mathsf{CR}^r$. Let $P \in [\![\tau]\!]^r$ so $P \in \mathsf{CR}^r$, hence, $MP \in \mathsf{CR}_0^r \subseteq [\![\rho]\!]^r$ and $M \in [\![\sigma]\!]^r$.

2. Let $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta I}$ where $n \geq 0$, $x \in \mathrm{fv}(M)$ and $(\lambda x.M)NN_1 \dots N_n \to_{\beta I}^* \{M_1, M_2\}$.
   By lemma 2.2.7, there exist M'_1 and M'_2 such that $M_1 \to_{\beta I}^* M'_1$, $M[x := N]N_1 \dots N_n \to_{\beta I}^* M'_1$, $M_2 \to_{\beta I}^* M'_2$ and $M[x := N]N_1 \dots N_n \to_{\beta I}^* M'_2$. Then we conclude using $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta I}$.

3. Let $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta\eta}$ where $n \geq 0$ and $(\lambda x.M)NN_1 \dots N_n \to_{\beta\eta}^* \{M_1, M_2\}$. By lemma 2.2.7, there exist M'_1 and M'_2 such that $M_1 \to_{\beta\eta}^* M'_1$, $M[x := N]N_1 \dots N_n \to_{\beta\eta}^* M'_1$, $M_2 \to_{\beta\eta}^* M'_2$ and $M[x := N]N_1 \dots N_n \to_{\beta\eta}^* M'_2$. Then we conclude using $M[x := N]N_1 \dots N_n \in \mathsf{CR}^{\beta\eta}$.

4. By induction on σ.
   - If $\sigma \in \mathcal{A}$, then the statement is true by 2.
   - If $\sigma = \tau \cap \rho$, then by IH, $[\![\tau]\!]^{\beta I}$ and $[\![\rho]\!]^{\beta I}$ are I-saturated. Let M, N, N_1,. . . , $N_n \in \Lambda$, $x \in \mathrm{fv}(M)$, $n \geq 0$, and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta I} = [\![\tau]\!]^{\beta I} \cap [\![\rho]\!]^{\beta I}$. Then by I-saturation, $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta I}$ and $(\lambda x.M)NN_1 \dots N_n \in [\![\rho]\!]^{\beta I}$. Done.

- If $\sigma = \tau \to \rho$, then by IH, $[\![\tau]\!]^{\beta I}$ and $[\![\rho]\!]^{\beta I}$ are I-saturated. Let $n \geq 0$, $M, N, N_1, \dots, N_n \in \Lambda$, $x \in \text{fv}(M)$, and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta I}$. Let $P \in [\![\tau]\!]^{\beta I} \neq \varnothing$, then $M[x := N]N_1 \dots N_n P \in [\![\rho]\!]^{\beta I}$.
  By I-saturation, $(\lambda x.M)NN_1 \dots N_n P \in [\![\rho]\!]^{\beta I}$ so $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta I} \Rightarrow [\![\rho]\!]^{\beta I}$.
  Since, $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta I} \subseteq CR^{\beta I}$ and $CR^{\beta I}$ is saturated by 2,
  then $(\lambda x.M)NN_1 \dots N_n \in CR^{\beta I}$.

5. By induction on σ.
   - If $\sigma \in \mathcal{A}$, then the statement is true by 3.
   - If $\sigma = \tau \cap \rho$, then by IH, $[\![\tau]\!]^{\beta\eta}$ and $[\![\rho]\!]^{\beta\eta}$ are saturated.
     Let $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta\eta} = [\![\tau]\!]^{\beta\eta} \cap [\![\rho]\!]^{\beta\eta}$.
     Then by saturation, $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta\eta}$ and $(\lambda x.M)NN_1 \dots N_n \in [\![\rho]\!]^{\beta\eta}$. Done.
   - If $\sigma = \tau \to \rho$, then by IH, $[\![\tau]\!]^{\beta\eta}$ and $[\![\rho]\!]^{\beta\eta}$ are saturated. Let $n \geq 0$, $M, N, N_1, \dots, N_n \in \Lambda$, $x \in \mathcal{V}$, and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta\eta}$. Let $P \in [\![\tau]\!]^{\beta\eta} \neq \varnothing$, then $M[x := N]N_1 \dots N_n P \in [\![\rho]\!]^{\beta\eta}$. By saturation, $(\lambda x.M)NN_1 \dots N_n P \in [\![\rho]\!]^{\beta\eta}$ so
     $(\lambda x.M)NN_1 \dots N_n \in [\![\tau]\!]^{\beta\eta} \Rightarrow [\![\rho]\!]^{r}$. Since, $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^{\beta\eta} \subseteq CR^{\beta\eta}$ and $CR^{\beta\eta}$ is saturated by 3, then $(\lambda x.M)NN_1 \dots N_n \in CR^{\beta\eta}$.

□

Proof(Lemma 6.4)**:** By induction on $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash^r M : \sigma$.

- If the last rule is (ax) or (ax^I), use the hypothesis.
- If the last rule is $(\to_{E^I})$. Let $\Gamma_1 \sqcap \Gamma_2 = (x_i : \sigma_i \cap \sigma'_i)_n, (y_i : \tau_i)_p, (z_i : \rho_i)_q$ such that $\Gamma_1 = (x_i : \sigma_i)_n, (y_i : \tau_i)_p$ and $\Gamma_2 = (x_i : \sigma'_i)_n, (z_i : \rho_i)_q$. Let $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i \cap \sigma'_i]\!]^{\beta I}$ so $N_i \in [\![\sigma_i]\!]^{\beta I}$ and $N_i \in [\![\sigma'_i]\!]^{\beta I}$, $\forall i \in \{1, \dots, p\}, P_i \in [\![\tau_i]\!]^{\beta I}$ and $\forall i \in \{1, \dots, q\}, P'_i \in [\![\rho_i]\!]^{\beta I}$. So by IH, $M[(x_i := N_i)_n, (y_i := P_i)_p] \in [\![\sigma \to \tau]\!]^{\beta I}$ and $N[(x_i := N_i)_n, (z_i := P'_i)_q] \in [\![\sigma]\!]^{\beta I}$. Hence, $(MN)[(x_i := N_i)_n, (y_i := P_i)_p, (z_i := P'_i)_q] \in [\![\tau]\!]^{\beta I}$.
- If the last rule is $(\to_E)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^{\beta\eta}$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \to \tau]\!]^{\beta\eta}$ and $N[(x_i := N_i)_n] \in [\![\sigma]\!]^{\beta\eta}$. Hence, $(MN)[(x_i := N_i)_n] \in [\![\tau]\!]^{\beta\eta}$.
- If the last rule is $(\to_I)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^{r}$. Let $P \in [\![\sigma]\!]^{r} \neq \varnothing$. So by IH, $M[(x_i := N_i)_n, x := P] \in [\![\tau]\!]^{r}$. Moreover $((\lambda x.M)[(x_i := N_i)_n])P = (\lambda x.M[(x_i := N_i)_n])P$.
  - For $\vdash^{\beta I}$, since $x \in \text{fv}(M)$ by lemma 2.2.4, $(\lambda x.M[(x_i := N_i)_n]) \to_{\beta I} M[(x_i := N_i)_n, x := P]$ and since by lemma 6.3, $[\![\tau]\!]^{\beta I}$ is I-saturated, $((\lambda x.M)[(x_i := N_i)_n])P \in [\![\tau]\!]^{\beta I}$.
  - For $\vdash^{\beta\eta}$, $(\lambda x.M[(x_i := N_i)_n]) \to_{\beta} M[(x_i := N_i)_n, x := P]$ and since by lemma 6.3, $[\![\tau]\!]^{\beta\eta}$ is saturated, $((\lambda x.M)[(x_i := N_i)_n])P \in [\![\tau]\!]^{\beta\eta}$.

  So $(\lambda x.M)[(x_i := N_i)_n] \in [\![\sigma]\!]^{r} \Rightarrow [\![\tau]\!]^{r}$. Since $x \in [\![\sigma]\!]^{r}$, $M[(x_i := N_i)_n] \in [\![\tau]\!]^{r} \subseteq CR^{r}$, so $\lambda x.M[(x_i := N_i)_n] = (\lambda x.M)[(x_i := N_i)_n] \in CR^{r}$.
- If the last rule is $(\cap_I)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^{r}$. So by IH, $M[(x_i := N_i)_n] \in [\![\tau]\!]^{r}$ and $M[(x_i := N_i)_n] \in [\![\rho]\!]^{r}$. So $M[(x_i := N_i)_n] \in [\![\sigma]\!]^{r}$.

- If the last rule is $(\cap_{E1})$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \cap \tau]\!]^r$, so $M[(x_i := N_i)_n] \in [\![\sigma]\!]^r$.

- If the last rule is $(\cap_{E2})$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \dots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \cap \tau]\!]^r$, so $M[(x_i := N_i)_n] \in [\![\tau]\!]^r$.

□

Proof(Lemma 6.6)**:** By induction on M. Note that by Lemma 5.2, $M \neq c$.

- Let $M = x \neq c$. Then $\Gamma = \Gamma_1, x : \tau$, $\Gamma' = x : \tau$, $\Gamma' \vdash^{\beta I} x : \tau$ and $\forall \sigma, \Gamma_1, x : \tau, c : \sigma \vdash^{\beta\eta} x : \tau$.

- Let $M = \lambda x.N \in \Lambda I_c$ then by lemma 5.2, $N \in \Lambda I_c$ and $x \in \mathrm{fv}(N)$. $\forall \rho$:
  - If $c \in \mathrm{fv}(M)$ then $c \in \mathrm{fv}(N)$ and by IH, $\exists \sigma, \tau$ where $\Gamma', x : \rho, c : \sigma \vdash^{\beta I} N : \tau$, hence $\Gamma', c : \sigma \vdash^{\beta I} \lambda x.N : \rho \to \tau$.
  - If $c \notin \mathrm{fv}(M)$ then by IH, $\exists \tau$ where $\Gamma', x : \rho \vdash^{\beta I} N : \tau$, hence $\Gamma' \vdash^{\beta I} \lambda x.N : \tau$.

- Let $M = \lambda x.N \in \Lambda\eta_c$ then by lemma 5.2.12.12a, $N \in \Lambda\eta_c$. By IH, $\forall \rho$, $\exists \sigma, \tau$ such that $\Gamma, x : \rho, c : \sigma \vdash^{\beta\eta} N : \tau$. Hence, $\Gamma, c : \sigma \vdash^{\beta\eta} \lambda x.N : \tau$.

- Let $M = cNP$ where $N, P \in \Lambda I_c$. Let $\Gamma'_1 = \Gamma \upharpoonright \mathrm{fv}(N)$ and $\Gamma'_2 = \Gamma \upharpoonright \mathrm{fv}(P)$. Note that $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(cNP) = \Gamma'_1 \sqcap \Gamma'_2$.
  - If $c \notin \mathrm{fv}(N) \cup \mathrm{fv}(P)$ then by IH, $\exists \tau_1, \tau_2$ such that $\Gamma'_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and $\sigma = \tau_1 \to \tau_2 \to \rho$. By $(\to_{E_I})$ twice, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} cNP : \rho$.
  - If $c \in \mathrm{fv}(N)$ and $c \notin \mathrm{fv}(P)$ then by IH, $\exists \sigma_1, \tau_1, \tau_2$ such that $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\tau_1 \to \tau_2 \to \rho)$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta I} c : \tau_1 \to \tau_2 \to \rho$. By lemma 6.2.3, $\Gamma'_1, c : \sigma \vdash^{\beta I} N : \tau_1$. By $(\to_{E_I})$ twice, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} cNP : \rho$.
  - If $c \in \mathrm{fv}(N) \cap \mathrm{fv}(P)$ then by IH, $\exists \sigma_1, \sigma_2, \tau_1, \tau_2$ such that $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma'_2, c : \sigma_2 \vdash^{\beta I} N : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\sigma_2 \cap (\tau_1 \to \tau_2 \to \rho))$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta I} c : \tau_1 \to \tau_2 \to \rho$. By lemma 6.2.3, $\Gamma'_1, c : \sigma \vdash^{\beta I} N : \tau_1$, and $\Gamma'_2, c : \sigma \vdash^{\beta I} P : \tau_2$. By $(\to_{E_I})$ twice, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} cNP : \rho$.

- Let $M = cNP$ where $N, P \in \Lambda\eta_c$. by IH, $\exists \sigma_1, \sigma_2, \tau_1, \tau_2$ such that $\Gamma, c : \sigma_1 \vdash^{\beta\eta} N : \tau_1$ and $\Gamma, c : \sigma_2 \vdash^{\beta\eta} N : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\sigma_2 \cap (\tau_1 \to \tau_2 \to \rho))$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta\eta} c : \tau_1 \to \tau_2 \to \rho$. By lemma 6.2.3, $\Gamma, c : \sigma \vdash^{\beta\eta} N : \tau_1$, and $\Gamma, c : \sigma \vdash^{\beta\eta} P : \tau_2$. By $(\to_{E_I})$ twice, $\Gamma, c : \sigma \vdash^{\beta\eta} cNP : \rho$.

- Let $M = NP$ where $N, P \in \Lambda I_c$ and $N = \lambda x.N_0$. So $N_0 \in \Lambda I_c$ and $x \in \mathrm{fv}(N_0)$. Let $\Gamma'_1 = \Gamma \upharpoonright \mathrm{fv}(N)$ and $\Gamma'_2 = \Gamma \upharpoonright \mathrm{fv}(P)$. Note that $\Gamma' = \Gamma \upharpoonright \mathrm{fv}(NP) = \Gamma'_1 \sqcap \Gamma'_2$. By BC, $x \neq c$ and $x \notin \mathrm{fv}(P)$.
  - If $c \notin \mathrm{fv}(\lambda x.N_0) \cup \mathrm{fv}(P)$ then by IH, $\exists \tau_2$ such that $\Gamma'_2 \vdash^{\beta I} P : \tau_2$ and again by IH, $\exists \tau_1$ such that $\Gamma'_1, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$ and $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2 \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.

– If $c \in \text{fv}(\lambda x.N_0)$ and $c \notin \text{fv}(P)$ then by IH, $\exists \tau_2$ such that $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Again by IH, $\exists \sigma, \tau_1$ such that $\Gamma'_1, c : \sigma, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$ and $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.

– If $c \in \text{fv}(\lambda x.N_0) \cap \text{fv}(P)$, then by IH, $\exists \sigma_2, \tau_2$ such that $\Gamma'_2, c : \sigma_2 \vdash^{\beta I} P : \tau_2$ and again by IH, $\exists \sigma_1, \tau_1$ such that $\Gamma'_1, c : \sigma_1, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$, $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} \lambda x N_0 : \tau_2 \to \tau_1$. By $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma_1 \cap \sigma_2 \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.

- Let $M = NP$ where $N, P \in \Lambda\eta_c$ and $N = \lambda x.N_0$ then by lemma 5.2.12.12a, $N_0 \in \Lambda\eta_c$. By IH, $\exists \sigma_2, \tau_2$ such that $\Gamma, c : \sigma_2 \vdash^{\beta\eta} P : \tau_2$ and again by IH, $\exists \sigma_1, \tau_1$ such that $\Gamma, c : \sigma_1, x : \tau_2 \vdash^{\beta\eta} N_0 : \tau_1$. By $(\to_I)$, $\Gamma, c : \sigma_1 \vdash^{\beta\eta} \lambda x.N_0 : \tau_2 \to \tau_1$. Let $\sigma = \sigma_1 \cap \sigma_2$. By Lemma 6.2.3, $\Gamma, c : \sigma \vdash^{\beta\eta} \lambda x.N_0 : \tau_2 \to \tau_1$ and $\Gamma, c : \sigma \vdash^{\beta\eta} P : \tau_2$. Hence, by $(\to_E)$, $\Gamma, c : \sigma \vdash^{\beta\eta} (\lambda x.N_0)P : \tau_1$.
- Let $M = cN$ where $N \in \Lambda\eta_c$. By IH, $\exists \sigma, \tau$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} N : \tau$. Let $\rho \in \mathsf{Type}^1$ and $\sigma' = \sigma \cap (\tau \to \rho)$. By Lemma 6.2.3, $\Gamma, c : \sigma' \vdash^{\beta\eta} N : \tau$ and $\Gamma, c : \sigma' \vdash^{\beta\eta} c : \tau \to \rho$. Hence, by $(\to_E)$, $\Gamma, c : \sigma' \vdash^{\beta\eta} cN : \rho$.

□

E. Proofs of section 7

Proof(Lemma 7.2)**:**

1. 1a. By induction on the structure of $M \in \Lambda\text{I}$.
   - Let $M = x \neq c$. Then $\Phi^c(x, \mathcal{F}) = x$, $\mathcal{F} = \varnothing$ and $\text{fv}(x) = \text{fv}(x) \setminus \{c\}$.
   - Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. Then, $\text{fv}(M) = \text{fv}(N) \setminus \{x\} =^{IH} \text{fv}(\Phi^c(N, \mathcal{F}')) \setminus \{c, x\} = \text{fv}(\lambda x.\Phi^c(N, \mathcal{F}')) \setminus \{c\} = \text{fv}(\Phi^c(M, \mathcal{F})) \setminus \{c\}$.
   - Let $M = M_1 M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
     – If $0 \in \mathcal{F}$ then, $\Phi^c(M, \mathcal{F}) = \Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$.
     – Else, $\Phi^c(M, \mathcal{F}) = c\Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$.
     In both cases, $\text{fv}(M) = \text{fv}(M_1) \cup \text{fv}(M_2) =^{IH} (\text{fv}(\Phi^c(M_1, \mathcal{F}_1)) \setminus \{c\}) \cup (\text{fv}(\Phi^c(M_2, \mathcal{F}_2)) \setminus \{c\}) = \text{fv}(\Phi^c(M, \mathcal{F})) \setminus \{c\}$.

   1b. By induction on the structure of $M \in \Lambda\text{I}$.
   - Let $M \in \mathcal{V}$, then $M \neq c$. So $\mathcal{F} = \varnothing$ and $\Phi^c(M, \mathcal{F}) = M \in \Lambda\text{I}_c$.
   - Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. By IH, $\Phi^c(N, \mathcal{F}') \in \Lambda\text{I}_c$. By lemma 7.2.1a, $x \in \text{fv}(\Phi^c(N, \mathcal{F}'))$. Hence, $\Phi^c(M, \mathcal{F}) = \lambda x.\Phi^c(N, \mathcal{F}') \in \Lambda\text{I}_c$.
   - Let $M = M_1 M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
     – If $0 \in \mathcal{F}$ then $\Phi^c(M, \mathcal{F}) = \Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$.
     By IH, $\Phi^c(M_1, \mathcal{F}_1), \Phi^c(M_2, \mathcal{F}_2) \in \Lambda\text{I}_c$ and as M_1 is a λ-abstraction, $\Phi^c(M_1, \mathcal{F}_1)$ is a λ-abstraction. Hence $\Phi^c(M, \mathcal{F}) \in \Lambda\text{I}_c$.
     – Else, $\Phi^c(M, \mathcal{F}) = c\Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$. By IH, $\Phi^c(M_1, \mathcal{F}_1), \Phi^c(M_2, \mathcal{F}_2) \in \Lambda\text{I}_c$, hence, $\Phi^c(M, \mathcal{F}) \in \Lambda\text{I}_c$.

   1c. By induction on the structure of $M \in \Lambda\text{I}$.

- Let $M = x \neq c$. Then, $\mathcal{F} = \varnothing$ and $\Phi^c(x, \mathcal{F}) = x = |x|^c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. Then, $|\Phi^c(M, \mathcal{F})|^c = |\lambda x.\Phi^c(N, \mathcal{F}')|^c = \lambda x.|\Phi^c(N, \mathcal{F}')|^c =^{IH} \lambda x.N$.
- Let $M = M_1 M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $0 \in \mathcal{F}$ then M_1 is a λ-abstraction, hence, $\Phi^c(M_1, \mathcal{F}_1)$ is a λ-abstraction. So, $|\Phi^c(M, \mathcal{F})|^c = |\Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)|^c = |\Phi^c(M_1, \mathcal{F}_1)|^c|\Phi^c(M_2, \mathcal{F}_2)|^c =^{IH} M_1 M_2 = M$.
  - Else, $|\Phi^c(M, \mathcal{F})|^c = |c\Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)|^c = |\Phi^c(M_1, \mathcal{F}_1)|^c|\Phi^c(M_2, \mathcal{F}_2)|^c =^{IH} M_1 M_2 = M$.

1d. By induction on the structure of $M \in \Lambda\mathrm{I}$.

- If $M = x \neq c$ then $\Phi^c(M, \mathcal{F}) = M$ and $\mathcal{F} = \varnothing =^{5.3} |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. Then $\mathcal{F} =^{5.3} \{1.p \mid p \in \mathcal{F}'\} =^{IH} \{1.p \mid p \in |\langle \Phi^c(N, \mathcal{F}'), \mathcal{R}^{\beta I}_{\Phi^c(N,\mathcal{F}')}\rangle|^c\} = \{1.|\langle \Phi^c(N, \mathcal{F}'), p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(N,\mathcal{F}')}\} = \{|\langle \Phi^c(M, \mathcal{F}), 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(N,\mathcal{F}')}\} =^{5.3} |\langle \Phi^c(M, \mathcal{F}), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}\rangle|^c$.
- Let $M = M_1 M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $0 \in \mathcal{F}$ then $\Phi^c(M, \mathcal{F}) = \Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction then $\Phi^c(M_1, \mathcal{F}_1)$ too. By lemma 7.2.1b, $\Phi^c(M, \mathcal{F}) \in \Lambda\mathrm{I}_c$ then $\Phi^c(M, \mathcal{F}) \in \mathcal{R}^{\beta I}$. Hence, $\mathcal{F} =^{5.3} \{0\} \cup \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{IH} \{0\} \cup \{1.p \mid p \in |\langle \Phi^c(M_1, \mathcal{F}_1), \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\rangle|^c\} \cup \{2.p \mid p \in |\langle \Phi^c(M_2, \mathcal{F}_2), \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\rangle|^c\} = \{0\} \cup \{1.|\langle \Phi^c(M_1, \mathcal{F}_1), p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{2.|\langle \Phi^c(M_2, \mathcal{F}_2), p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} = \{0\} \cup \{|\langle \Phi^c(M, \mathcal{F}), 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{|\langle \Phi^c(M, \mathcal{F}), 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} =^{5.3} |\langle \Phi^c(M, \mathcal{F}), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}\rangle|^c$.
  - Else, $\Phi^c(M, \mathcal{F}) = c\Phi^c(M_1, \mathcal{F}_1)\Phi^c(M_2, \mathcal{F}_2)$. Then, $\mathcal{F} =^{5.3} \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{IH} \{1.p \mid p \in |\langle \Phi^c(M_1, \mathcal{F}_1), \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\rangle|^c\} \cup \{2.p \mid p \in |\langle \Phi^c(M_2, \mathcal{F}_2), \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\rangle|^c\} = \{1.|\langle \Phi^c(M_1, \mathcal{F}_1), p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{2.|\langle \Phi^c(M_2, \mathcal{F}_2), p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} =$
    $\{|\langle \Phi^c(M, \mathcal{F}), 1.2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\mathcal{F}_1)}\} \cup \{|\langle \Phi^c(M, \mathcal{F}), 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\mathcal{F}_2)}\} =^{5.3} |\langle \Phi^c(M, \mathcal{F}), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}\rangle|^c$.

2. 2a. By induction on the construction of $M \in \Lambda\mathrm{I}_c$. By lemma 6, $|M|^c \in \Lambda\mathrm{I}$

- Let $M \in \mathcal{V} \setminus \{c\}$. Hence $|M|^c = M$, by lemma 5.3, $|\langle M, \mathcal{R}^{\beta I}_M\rangle|^c = \varnothing = \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c)$.
- Let $M = \lambda x.P$ such that $x \neq c$, $P \in \Lambda\mathrm{I}_c$ and $x \in \mathrm{fv}(P)$. Then, $|M|^c = \lambda x.|P|^c$. By IH, $|\langle P, \mathcal{R}^{\beta I}_P\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|P|^c}$ and $P = \Phi^c(|P|^c, |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c)$. Hence, $|\langle M, \mathcal{R}^{\beta I}_M\rangle|^c =^{5.3} \{|\langle M, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_P\} = \{1.p \mid p \in |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c\} \subseteq \{1.p \mid p \in \mathcal{R}^{\beta I}_{|P|^c}\} =^{5.3} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover, $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c)$.
- Let $M = cPQ$ where $P, Q \in \Lambda\mathrm{I}_c$ then $|M|^c = |P|^c|Q|^c$. By IH, $|\langle P, \mathcal{R}^{\beta I}_P\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|P|^c}$, $|\langle Q, \mathcal{R}^{\beta I}_Q\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|Q|^c}$, $P = \Phi^c(|P|^c, |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c)$ and $Q = \Phi^c(|Q|^c, |\langle Q, \mathcal{R}^{\beta I}_Q\rangle|^c)$.

Hence, $|\langle M, \mathcal{R}^{\beta I}_M\rangle|^c =^{5.3} \{|\langle M, 1.2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_P\} \cup \{|\langle M, 2.p\rangle|^c \mid p\mathcal{R}^{\beta I}_Q\} = \{1.p \mid p \in |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^{\beta I}_Q\rangle|^c\} \subseteq \{1.p \mid p \in \mathcal{R}^{\beta I}_{|P|^c}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{|Q|^c}\} \subseteq^{5.3} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover $M = \Phi^c(|M|^{\beta I}, |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c)$.

- Let $M = PQ$ where $P, Q \in \Lambda\mathrm{I}_c$ and P is a λ-abstraction. Then, $|M|^c = |P|^c|Q|^c$, where $|P|^c$ is a λ-abstraction. By IH, $|\langle P, \mathcal{R}^{\beta I}_P\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|P|^c}$, $|\langle Q, \mathcal{R}^{\beta I}_Q\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|Q|^c}$, $P = \Phi^c(|P|^c, |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c)$ and $Q = \Phi^c(|Q|^c, |\langle Q, \mathcal{R}^{\beta I}_Q\rangle|^c)$. Hence, $|\langle M, \mathcal{R}^{\beta I}_M\rangle|^c =^{5.3} \{0\} \cup \{|\langle M, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_P\} \cup \{|\langle M, 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta I}_Q\} = \{0\} \cup \{1.p \mid p \in |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c\} \cup \{2.p \mid p \in |\langle Q, \mathcal{R}^{\beta I}_Q\rangle|^c\} \subseteq \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta I}_{|P|^c}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{|Q|^c}\} =^{5.3} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover $M = \Phi^c(|M|^{\beta I}, |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c)$.

2b. By lemma 6, $|M|^c \in \Lambda\mathrm{I}$. By lemma 4 $c \notin \mathrm{fv}(|M|^c)$. By lemma 7.2.2a, $|\langle M, \mathcal{R}^{\beta I}_M\rangle|^c \subseteq \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c)$. To prove unicity, assume that $\langle N', \mathcal{F}'\rangle$ is another such pair. So $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{N'}$ and $M = \Phi^c(N', \mathcal{F}')$. Then, $|M|^c = |\Phi^c(N', \mathcal{F}')|^c =^{7.2.1c} N'$ and $\mathcal{F}' =^{7.2.1d} |\langle \Phi^c(N', \mathcal{F}'), \mathcal{R}^{\beta I}_{\Phi^c(N', \mathcal{F}')}\rangle|^c = |\langle M, \mathcal{R}^{\beta I}_M\rangle|^c$.

□

Proof(Lemma 7.3)**:** By lemma 7.2.1c and lemma 1, there exists a unique $p' \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}$, such that $|\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}, p'\rangle|^c = p$. By lemma 2.2.8, there exists P such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} P$. By lemma 5.8.7a, $M =^{7.2.1c} |\Phi^c(M, \mathcal{F})|^c \xrightarrow{p_0}_{\beta I} |P|^c$, such that $|\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}, p'\rangle|^c = p_0$. So $p = p_0$ and by lemma 2.2.9, $M' = |P|^c$. Let $\mathcal{F}' = |\langle P, \mathcal{R}^{\beta I}_P\rangle|^c$. Because, $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} P$, by lemma 2 and lemma 7.2.1b, $P \in \Lambda\mathrm{I}_c$. By lemma 7.2.2a, $P = \Phi^c(M', \mathcal{F}')$ and $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$. By lemma 7.2.2b, $\mathcal{F}'$ is unique. □

Proof(Lemma 7.6.1)**:** It sufficient to prove:

$$\langle M, \mathcal{F}\rangle \to_{\beta Id} \langle M', \mathcal{F}'\rangle \iff \Phi^c(M, \mathcal{F}) \to_{\beta I} \Phi^c(M', \mathcal{F}')$$

- $\Rightarrow$) let $\langle M, \mathcal{F}\rangle \to_{\beta Id} \langle M', \mathcal{F}'\rangle$. Then by definition 7.5, there exists $p \in \mathcal{F}$ such that $M \xrightarrow{p}_{\beta I} M'$ and $\mathcal{F}'$ is the set of βI-residuals in M' of the set of redexes $\mathcal{F}$ in M relative to p. By definition 7.4 we obtain $\Phi^c(M, \mathcal{F}) \to_{\beta I} \Phi^c(M', \mathcal{F}')$.

- $\Leftarrow$) Let $\Phi^c(M, \mathcal{F}) \to_{\beta I} \Phi^c(M', \mathcal{F}')$ then by lemma 2.2.8, there exists $p\mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F})}$ such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p}_{\beta I} \Phi^c(M', \mathcal{F}')$. Because, by lemma 7.2.1b, $\Phi^c(M, \mathcal{F}) \in \Lambda\mathrm{I}_c$, by lemma 5.8.7a and lemma 7.2.1c, $M = |\Phi^c(M, \mathcal{F})|^c \xrightarrow{p_0}_{\beta I} |\Phi^c(M', \mathcal{F}')|^c = M'$ such that $|\langle \Phi^c(M, \mathcal{F}), p_0\rangle|^c = p$. By definition 7.4, $\mathcal{F}'$ is the set of βI-residuals in M' of the set of redexes $\mathcal{F}$ in M relative to p_0. By definition 7.5 we obtain $\langle M, \mathcal{F}\rangle \to_{\beta d} \langle M', \mathcal{F}'\rangle$.

□

Proof(Lemma 7.6.2)**:** By lemma 7.2.1b, $\Phi^c(M, \mathcal{F}_1), \Phi^c(M, \mathcal{F}_2) \in \Lambda\mathrm{I}_c$. By lemma 7.2.1c, $|\Phi^c(M, \mathcal{F}_1)|^c = |\Phi^c(M, \mathcal{F}_2)|^c$. By lemma 7.2.1d, $|\langle \Phi^c(M, \mathcal{F}_1), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}\rangle|^c = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\langle \Phi^c(M, \mathcal{F}_2), \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_2)}\rangle|^c$.

If $\langle M, \mathcal{F}_1\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_1\rangle$ then by lemma 7.6.1, $\Phi^c(M, \mathcal{F}_1) \rightarrow_{\beta I} \Phi^c(M', \mathcal{F}'_1)$. By lemma 2.2.8, there exists $p_1 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$ such that $\Phi^c(M, \mathcal{F}_1) \xrightarrow{p_1}_{\beta I} \Phi^c(M', \mathcal{F}'_1)$. Let $p_0 = |\langle \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}, p_1\rangle|^c$, so by lemma 7.2.1d, $p_0 \in \mathcal{F}_1$. By lemma 5.8.7a and lemma 7.2.1c, $M \xrightarrow{p_0}_{\beta I} M'$.

By lemma 7.3 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M, \mathcal{F}_1) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}_1), p'\rangle|^c = p_0$. By lemma 2.2.8, $p' \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$. Since $p', p_1 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\mathcal{F}_1)}$, by lemma 1, $p' = p_1$. So, by lemma 2.2.9, $\Phi^c(M', \mathcal{F}') = \Phi^c(M', \mathcal{F}'_1)$. By lemma 7.2.1d, $\mathcal{F}' = \mathcal{F}'_1$ and $\mathcal{F}'_1 = |\langle \Phi^c(M', \mathcal{F}'_1), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}'_1)}\rangle|^c$.

By lemma 7.3 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M, \mathcal{F}_2) \xrightarrow{p_2}_{\beta I} \Phi^c(M', \mathcal{F}'_2)$ and $|\langle \Phi^c(M, \mathcal{F}_2), p_2\rangle|^c = p_0$.
By lemma 2.2.8, $p_2 \in \Phi^c(M, \mathcal{F}_2)$. By lemma 7.2.1d, $\mathcal{F}'_2 = |\langle \Phi^c(M', \mathcal{F}'_2), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}'_2)}\rangle|^c$.

Hence, by lemma 5.8.7c, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 7.6.1, $\langle M, \mathcal{F}_2\rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_2\rangle$. □

Proof(Lemma 7.7)**:** If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exists $\mathcal{F}''_1, \mathcal{F}''_2$ such that $\langle M, \mathcal{F}_1\rangle \rightarrow^*_{\beta Id} \langle M_1, \mathcal{F}''_1\rangle$ and $\langle M, \mathcal{F}_2\rangle \rightarrow^*_{\beta Id} \langle M_2, \mathcal{F}''_2\rangle$. By definitions 7.4 and 7.5, $\mathcal{F}''_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}''_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$. Note that by definition 7.5 and lemma 2.2.4, $M_1, M_2 \in \Lambda\mathrm{I}$. By lemma 2, there exist $\mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ such that $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta Id} \langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle$ and $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta Id} \langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle$. By lemma 7.6.1, $T \rightarrow^*_{\beta I} T_1$ and $T \rightarrow^*_{\beta I} T_2$ where $T = \Phi^c(M, \mathcal{F}_1 \cup \mathcal{F}_2)$, $T_1 = \Phi^c(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $T_2 = \Phi^c(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$. Since by lemma 7.2.1b, $T \in \Lambda\mathrm{I}_c$ and by lemma 6.6.1, T is typable in the type system $\mathcal{D}_I$, so $T \in \mathsf{CR}^{\beta I}$ by corollary 6.5. So, by lemma 2.2b, there exists $T_3 \in \Lambda\mathrm{I}_c$, such that $T_1 \rightarrow^*_{\beta I} T_3$ and $T_2 \rightarrow^*_{\beta I} T_3$. Let $\mathcal{F}_3 = |\langle T_3, \mathcal{R}^{\beta I}_{T_3}\rangle|^c$ and $M_3 = |T_3|^{\beta I}$, then by lemma 7.2.2b, $T_3 = \Phi^c(M_3, \mathcal{F}_3)$. Hence, by lemma 7.6.1, $\langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle \rightarrow^*_{\beta Id} \langle M_3, \mathcal{F}_3\rangle$ and $\langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle \rightarrow^*_{\beta Id} \langle M_3, \mathcal{F}_3\rangle$, i.e. $M_1 \xrightarrow{\mathcal{F}''_1 \cup \mathcal{F}'''_1}_{\beta Id} M_3$ and $M_2 \xrightarrow{\mathcal{F}''_2 \cup \mathcal{F}'''_2}_{\beta Id} M_3$. □

Proof(Lemma 7.9.1)**:** Note that $\varnothing \subseteq \mathcal{R}^{\beta I}_M$. We prove this statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then $\Phi^c(M, \varnothing) = M$ and $\mathcal{R}^{\beta I}_M = \varnothing$ by lemma 5.3.
- Let $M = \lambda x.N$ such that $x \neq c$ then $\Phi^c(M, \varnothing) = \lambda x.\Phi^c(N, \varnothing)$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(N,\varnothing)} = \varnothing$ and by lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)} = \varnothing$.
- Let $M = M_1 M_2$ then $\Phi^c(M, \varnothing) = c\Phi^c(M_1, \varnothing)\Phi^c(M_2, \varnothing)$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \varnothing$ and $\mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$ and by lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)} = \varnothing$.

□

Proof(Lemma 7.9.2)**:** We prove the statement by induction on the structure of M.

- let $M \in \mathcal{V}$, then $\Phi^c(M, \varnothing) = M$.
  - Either $M = x$, then $\Phi^c(M, \varnothing)[x := \Phi^c(N, \varnothing)] = \Phi^c(N, \varnothing)$ and by lemma 1, $\mathcal{R}^{\beta I}_{\Phi^c(N,\varnothing)} = \varnothing$.

– Or $M \neq x$, then $\Phi^c(M,\varnothing)[x := \Phi^c(N,\varnothing)] = M$ and by lemma 5.3, $\mathcal{R}^{\beta I}_M = \varnothing$.

- Let $M = \lambda y.M'$ such that $y \neq c$ then $\Phi^c(M,\varnothing) = \lambda y.\Phi^c(M',\varnothing)$. So, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \mathcal{R}^{\beta I}_{\lambda y.\Phi^c(M',\varnothing)[x:=\Phi^c(N,\varnothing)]}$ such that $y \notin \mathrm{fv}(\Phi^c(N,\varnothing)) \cup \{x\}$. By IH, $\mathcal{R}^{\beta I}_{\Phi^c(M',\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$. By lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$.

- Let $M = M_1M_2$ then $\Phi^c(M,\varnothing) = c\Phi^c(M_1,\varnothing)\Phi^c(M_2,\varnothing)$.
So, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \mathcal{R}^{\beta I}_{c\Phi^c(M_1,\varnothing)[x:=\Phi^c(N,\varnothing)]\Phi^c(M_2,\varnothing)[x:=\Phi^c(N,\varnothing)]}$.
By IH, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$
and by lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\varnothing)[x:=\Phi^c(N,\varnothing)]} = \varnothing$.

□

Proof(Lemma 7.9.3)**:** We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then by lemma 5.3, $\mathcal{R}^{\beta I}_M = \varnothing$.

- Let $M = \lambda x.N$ such that $x \neq c$ then by lemma 5.3, $\mathcal{R}^{\beta I}_M = \{1.p \mid p \in \mathcal{R}^{\beta I}_N\}$. Let $p \in \mathcal{R}^{\beta I}_M$, then $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_N$. Then, $\Phi^c(M,\{p\}) = \lambda x.\Phi^c(N,\{p'\})$ By lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{1.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(N,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} P$ then $p_0 = 1.p_1$, $P = \lambda x.P'$ and $\Phi^c(N,\{p'\}) \xrightarrow{p_1}_{\beta I} P'$. By IH, $\mathcal{R}^{\beta I}_{P'} = \varnothing$, so by lemma 5.3, $\mathcal{R}^{\beta I}_P = \varnothing$.

- Let $M = M_1M_2$.

  – Let $M \in \mathcal{R}^{\beta I}$, then $M_1 = \lambda x.M_0$ and by lemma 5.3, $\mathcal{R}^{\beta I}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{M_2}\}$.

    ∗ Either $p = 0$ then $\Phi^c(M,\{0\}) = \Phi^c(M_1,\varnothing)\Phi^c(M_2,\varnothing)$. By lemma 1, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$. Because $\Phi^c(M,\{0\}) \to_{\beta I} M'$ then by definition there exists p_0 such that $\Phi^c(M,\{0\}) \xrightarrow{p_0}_{\beta I} M'$. By lemma 2.2.8, $p_0 \in \mathcal{R}^{\beta I}_{\Phi^c(M,\{0\})}$. Because $\Phi^c(M_1,\varnothing) = \lambda x.\Phi^c(M_0,\varnothing)$ such that $x \neq c$, by lemma 5.3, we obtain:
    $\mathcal{R}^{\beta I}_{\Phi^c(M,\{0\})} = \{0\}$ if $\Phi^c(M,\{0\}) \in \mathcal{R}^{\beta I}$, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{0\})} = \varnothing$ otherwise. So p_0 and $\Phi^c(M,\{0\}) \in \mathcal{R}^{\beta I}$. Hence, $M' = \Phi^c(M_0,\varnothing)[x := \Phi^c(M_2,\varnothing)]$ and by lemma 2, $\mathcal{R}^{\beta I}_{\Phi^c(M_0,\varnothing)[x:=\Phi^c(M_2,\varnothing)]} = \varnothing$.

    ∗ Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_1}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\{p'\})\Phi^c(M_2,\varnothing)$. By lemma 1, $\mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$. By lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{1.2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 1.2.p_0'$, $p_0' \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}$, $M' = cM_1'\Phi^c(M_2,\varnothing)$ and $\Phi^c(M_1,\{p'\}) \xrightarrow{p_0'}_{\beta I} M_1'$. By IH, $\mathcal{R}^{\beta I}_{M_1'} = \varnothing$ and by lemma 5.3, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

    ∗ Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_2}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\varnothing)\Phi^c(M_2,\{p'\})$. By lemma 1, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \varnothing$. By lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}\}$.

So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 2.p_0'$, $p_0' \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}$, $M' = c\Phi^c(M_1,\varnothing)M_2'$ and $\Phi^c(M_2,\{p'\}) \xrightarrow{p_0'}_{\beta I} M_2'$. By IH, $\mathcal{R}^{\beta I}_{M_2'} = \varnothing$ and by lemma 5.3, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

– Let $M \notin \mathcal{R}^{\beta I}$, then by lemma 5.3, $\mathcal{R}^{\beta I}_M = \{1.p \mid p \in \mathcal{R}^{\beta I}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta I}_{M_2}\}$.

  * Either $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_1}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\{p'\})\Phi^c(M_2,\varnothing)$. By lemma 1, $\mathcal{R}^{\beta I}_{\Phi^c(M_2,\varnothing)} = \varnothing$. By lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{1.2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 1.2.p_0'$, $p_0' \in \mathcal{R}^{\beta I}_{\Phi^c(M_1,\{p'\})}$, $M' = cM_1'\Phi^c(M_2,\varnothing)$ and $\Phi^c(M_1,\{p'\}) \xrightarrow{p_0'}_{\beta I} M_1'$. By IH, $\mathcal{R}^{\beta I}_{M_1'} = \varnothing$ and by lemma 5.3, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.
  * Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta I}_{M_2}$. So, $\Phi^c(M,\{p\}) = c\Phi^c(M_1,\varnothing)\Phi^c(M_2,\{p'\})$. By lemma 1, $\mathcal{R}^{\beta I}_{\Phi^c(M_1,\varnothing)} = \varnothing$. By lemma 5.3, $\mathcal{R}^{\beta I}_{\Phi^c(M,\{p\})} = \{2.p \mid p \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}\}$. So, By lemma 2.2.8, if $\Phi^c(M,\{p\}) \xrightarrow{p_0}_{\beta I} M'$ then $p_0 = 2.p_0'$, $p_0' \in \mathcal{R}^{\beta I}_{\Phi^c(M_2,\{p'\})}$, $M' = c\Phi^c(M_1,\varnothing)M_2'$ and $\Phi^c(M_2,\{p'\}) \xrightarrow{p_0'}_{\beta I} M_2'$. By IH, $\mathcal{R}^{\beta I}_{M_2'} = \varnothing$ and by lemma 5.3, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

□

Proof(Lemma 7.9.4)**:** By lemma 2.2.8, $p \in \mathcal{R}^{\beta I}_M$. By lemma 7.3, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^c(M,\{p\}) \to_{\beta I} \Phi^c(M',\mathcal{F}')$. By lemma 3, $\mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}')} = \varnothing$, so $|\langle \Phi^c(M',\mathcal{F}'), \mathcal{R}^{\beta I}_{\Phi^c(M',\mathcal{F}')}\rangle|^c = \varnothing$ and by lemma 7.2.1d, $\mathcal{F}' = \varnothing$. Finally, by lemma 7.6.1, $\langle M,\{p\}\rangle \to_{\beta Id} \langle M',\varnothing\rangle$. □

Proof(Lemma 7.9.5)**:** It is obvious that $\to^*_{1I} \subseteq \to^*_{\beta I}$. We only prove that $\to^*_{\beta I} \subseteq \to^*_{1I}$. Let $M, M' \in \Lambda\mathrm{I}$ such that $M \to^*_{\beta I} M'$. We prove this claim by induction on the length of $M \to^*_{\beta I} M'$.

- Let $M = M'$ then it is done since $\langle M,\mathcal{F}\rangle \to^*_{\beta Id} \langle M,\mathcal{F}\rangle$ for some $\mathcal{F}$.
- Let $M \to^*_{\beta I} M'' \to_{\beta I} M'$. By IH, $M \to^*_{1I} M''$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta I} M'$ then by lemma 4 $\langle M'',\{p\}\rangle \to_{\beta Id} \langle M',\varnothing\rangle$, so $M'' \to_{1I} M'$. Hence $M \to^*_{1I} M'' \to_{1I} M'$.

□

Proof(Lemma 7.10)**:** Let $M \in \Lambda\mathrm{I}$ and c be a variable such that $c \notin \mathrm{fv}(M)$. Assume $M \to^*_{\beta I} M_1$ and $M \to^*_{\beta I} M_2$. Then by lemma 5, $M \to^*_{1I} M_1$ and $M \to^*_{1I} M_2$. We prove the statement by induction on the length of $M \to^*_{1I} M_1$.

- Let $M = M_1$. Hence $M_1 \to^*_{1I} M_2$ and $M_2 \to^*_{1I} M_2$.
- Let $M \to^*_{1I} M_1' \to_{1I} M_1$. By IH, $\exists M_3', M_1' \to^*_{1I} M_3'$ and $M_2 \to^*_{1I} M_3'$. We prove that $\exists M_3, M_1 \to^*_{1I} M_3$ and $M_3' \to_{1I} M_3$, by induction on $M_1' \to^*_{1I} M_3'$.
  – let $M_1' = M_3'$, hence $M_3' \to_{1I} M_1$ and $M_1 \to^*_{1I} M_1$.
  – Let $M_1' \to^*_{1I} M_3'' \to_{1I} M_3'$. By IH, $\exists M_3''', M_1 \to^*_{1I} M_3'''$ and $M_3'' \to_{1I} M_3'''$. By lemma 2.2.4, $c \notin \mathrm{fv}(M_3'')$. Since $M_3'' \to_{1I} M_3'$ and $M_3'' \to_{1I} M_3'''$, by lemma 7.7, $\exists M_3, M_3' \to_{1I} M_3$ and $M_3''' \to_{1I} M_3$.

□

F. Proofs of section 8

Proof(Lemma 8.2)**:**

1. 1a. By induction on the structure of M.
   - Let $M \in \mathcal{V} \setminus \{c\}$, then $\mathcal{F} =^{5.3} \varnothing$ and $\Psi_0^c(M, \varnothing) = \{M\} = \{c^0(M)\} \subseteq \Psi^c(M, \varnothing)$.
   - Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{5.3} \mathcal{R}_N^{\beta\eta}$.
     - If $0 \in \mathcal{F}$ then $\Psi_0^c(M, \mathcal{F}) = \{\lambda x.N' \mid N' \in \Psi_0^c(N, \mathcal{F}')\} = \{c^0(\lambda x.N') \mid N' \in \Psi_0^c(N, \mathcal{F}')\} \subseteq \Psi^c(M, \mathcal{F})$.
     - Else $\Psi_0^c(M, \mathcal{F}) = \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} = \{c^0(\lambda x.N'[x := c(cx)]) \mid N' \in \Psi^c(N, \mathcal{F}')\} \subseteq \Psi^c(M, \mathcal{F})$.
   - Let $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq^{5.3} \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq^{5.3} \mathcal{R}_P^{\beta\eta}$.
     - If $0 \in \mathcal{F}$ then $\Psi_0^c(M, \mathcal{F}) = \{N'P' \mid N' \in \Psi_0^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\} = \{c^0(N'P') \mid N' \in \Psi_0^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\}$. By IH, $\Psi_0^c(P, \mathcal{F}_2) \subseteq \Psi^c(P, \mathcal{F}_2)$, so by definition, $\Psi_0^c(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.
     - Else $\Psi_0^c(M, \mathcal{F}) = \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\}$ $= \{c^0(cN'P') \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\}$. By IH, $\Psi_0^c(P, \mathcal{F}_2) \in \Psi^c(P, \mathcal{F}_2)$, so by definition, $\Psi_0^c(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.

   1b. By induction on the structure of M.
   - Let $M \in \mathcal{V} \setminus \{c\}$, then $\mathcal{F} = \varnothing$, $\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n \geq 0\}$ and $\forall N \in \Psi^c(M, \mathcal{F})$. $\text{fv}(M) = \{M\} = \text{fv}(N) \setminus \{c\}$.
   - Let $M = \lambda x.N$ such that $x \neq x$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$.
     - If $0 \in \mathcal{F}$ then $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi_0^c(N, \mathcal{F}')\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$ and $N' \in \Psi_0^c(N, \mathcal{F}')$ such that $P = c^n(\lambda x.N')$. Hence, $\text{fv}(M) = \text{fv}(N) \setminus \{x\} =^{IH,1a} \text{fv}(N') \setminus \{c, x\} = \text{fv}(P) \setminus \{c\}$.
     - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}')\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$ and $\exists N' \in \Psi^c(N, \mathcal{F}')$ such that, $P = c^n(\lambda x.N'[x := c(cx)])$. Hence, $\text{fv}(M) = \text{fv}(N) \setminus \{x\} =^{IH} \text{fv}(N') \setminus \{c, x\} = \text{fv}(P) \setminus \{c\}$.
   - Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_{M_1}^{\beta\eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_{M_2}^{\beta\eta}$.
     - If $0 \in \mathcal{F}$ then, $\Psi^c(M, \mathcal{F}) =$ $\{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi_0^c(M_1, \mathcal{F}_1) \wedge P' \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$, $N' \in \Psi_0^c(M_1, \mathcal{F}_1)$ and $P' \in \Psi^c(M_2, \mathcal{F}_2)$ such that $P = c^n(N'P')$. Hence, $\text{fv}(M) = \text{fv}(M_1) \cup \text{fv}(M_2) =^{IH,1a} (\text{fv}(N') \setminus \{c\}) \cup (\text{fv}(P') \setminus \{c\}) = (\text{fv}(N') \cup \text{fv}(P')) \setminus \{c\} = \text{fv}(P) \setminus \{c\}$.
     - Else $\Psi^c(M, \mathcal{F}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(M_1, \mathcal{F}_1) \wedge P' \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $P \in \Psi^c(M, \mathcal{F})$, so $\exists n \geq 0$, $N' \in \Psi^c(M_1, \mathcal{F}_1)$ and $P' \in \Psi^c(M_2, \mathcal{F}_2)$ such that $P = c^n(cN'P')$. Hence, $\text{fv}(M) = \text{fv}(M_1) \cup \text{fv}(M_2) =^{IH} (\text{fv}(N') \cup \text{fv}(P')) \setminus \{c\} = \text{fv}(P) \setminus \{c\}$.

   1c. By induction on the structure of M.
   - If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \varnothing$ and $\Psi^c(M, \mathcal{F}) = \{c^n(M) | n \geq 0\}$. Use lemma 5.2.7.
   - Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$.

  - If $0 \in \mathcal{F}$, then $N = Px$ such that $x \notin \text{fv}(P)$ and $\Psi^c(M,\mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi_0^c(N,\mathcal{F}')\}$. Let $\mathcal{F}'' = \{p \mid 1.p \in \mathcal{F}'\} \subseteq^{5.3} \mathcal{R}_P^{\beta\eta}$.
    * If $0 \in \mathcal{F}'$ then, $\Psi_0^c(N,\mathcal{F}') = \{P'x \mid P' \in \Psi_0^c(P,\mathcal{F}'')\}$. Let $M' \in \Psi^c(M,\mathcal{F})$, so $M' = c^n(\lambda x.P'x)$ where $n \geq 0$ and $P' \in \Psi_0^c(P,\mathcal{F}'')$. Since $x \notin \text{fv}(P)$, by lemmas 8.2.1b and 8.2.1a, $x \notin \text{fv}(P')$. By IH and lemma 8.2.1a, $P', P'x \in \Lambda\eta_c$. By lemma 5.2, $P' \neq c$. Hence, by $(R1).4$, $\lambda x.P'x \in \Lambda\eta_c$. We conclude using lemma 5.2.7.
    * Else $\Psi_0^c(N,\mathcal{F}') = \{cP'x \mid P' \in \Psi^c(P,\mathcal{F}'')\}$. Let $M' \in \Psi^c(M,\mathcal{F})$, so $M' = c^n(\lambda x.cP'x)$ where $n \geq 0$ and $P' \in \Psi^c(P,\mathcal{F}'')$. Since $x \notin \text{fv}(P)$, by lemmas 8.2.1b, $x \notin \text{fv}(P')$, so $x \notin \text{fv}(cP')$. By IH and lemma 8.2.1a, $cP'x \in \Lambda\eta_c$. Since $cP' \neq c$, by $(R1).4$, $\lambda x.cP'x \in \Lambda\eta_c$. We conclude using lemma 5.2.7.
  - Else $\Psi^c(M,\mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N,\mathcal{F}')\}$. Let $N' \in \Psi^c(N,\mathcal{F}')$ and $n \geq 0$. Since by IH $N' \in \Lambda\eta_c$, by lemma 5.2.7 and $(R1).3$, $c^n(\lambda x.N'[x := c(cx)]) \in \Lambda\eta_c$.
- Let $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta\eta}$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M,\mathcal{F}) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi_0^c(N,\mathcal{F}_1) \wedge P' \in \Psi^c(P,\mathcal{F}_2)\}$. Let $P = c^n(N'P') \in \Psi^c(M,\mathcal{F})$ such that $n \geq 0$, $N' \in \Psi_0^c(N,\mathcal{F}_1)$ and $P' \in \Psi^c(P,\mathcal{F}_2)$. By IH and lemma 8.2.1a, $N', P' \in \Lambda\eta_c$. Since N is a λ-abstraction then by definition N' too. Hence, by $(R3)$, $N'P' \in \Lambda\eta_c$. By lemma 5.2.7, $c^n(N'P') \in \Lambda\eta_c$.
  - Else $\Psi^c(M,\mathcal{F}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N,\mathcal{F}_1) \wedge P' \in \Psi^c(P,\mathcal{F}_2)\}$. Let $c^n(cN'P') \in \Psi^c(M,\mathcal{F})$ such that $n \geq 0$, $N' \in \Psi^c(N,\mathcal{F}_1)$ and $P' \in \Psi^c(P,\mathcal{F}_2)$. By IH, $N', P' \in \Lambda\eta_c$. Hence by $(R2)$, $cN'P' \in \Lambda\eta_c$ and by lemma 5.2.7, $c^n(cN'P') \in \Lambda\eta_c$.

1d. We prove this lemma by case on the belonging of 0 in $\mathcal{F}$. Let $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$.

- If $0 \in \mathcal{F}$ then $\Psi_0^c(Nx,\mathcal{F}) = \{N'x \mid N' \in \Psi_0^c(N,\mathcal{F}')\}$. Hence, $P = N'x$ such that $N' \in \Psi_0^c(N,\mathcal{F}')$. Since $x \notin \text{fv}(N)$, by lemmas 8.2.1b and 8.2.1a, $x \notin \text{fv}(N')$. So $\lambda x.P = \lambda x.N'x \in \mathcal{R}^{\beta\eta}$ and by lemma 5.3, $\mathcal{R}_{\lambda x.P}^{\beta\eta} = \{0\} \cup \{1.p \mid p \in \mathcal{R}_P^{\beta\eta}\}$.
- Else $\Psi_0^c(Nx,\mathcal{F}) = \{cN'x \mid N' \in \Psi^c(N,\mathcal{F}')\}$ and $P = cN'x$ such that $N' \in \Psi^c(N,\mathcal{F}')$. Since $x \notin \text{fv}(N)$, by lemmas 8.2.1b, $x \notin \text{fv}(N')$ and so $x \notin \text{fv}(cN')$. Since $\lambda x.cN'x \in \mathcal{R}^{\beta\eta}$, by lemma 5.3, $\mathcal{R}_{\lambda x.P}^{\beta\eta} = \{0\} \cup \{1.p \mid p \in \mathcal{R}_P^{\beta\eta}\}$.

1e. Let $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_x^{\beta\eta} =^{5.3} \varnothing$. We prove this lemma by case on the belonging of 0 in $\mathcal{F}$.

- If $0 \in \mathcal{F}$ then $\Psi^c(Nx,\mathcal{F}) = \{c^n(N'Q) \mid n \geq 0 \wedge N' \in \Psi_0^c(N,\mathcal{F}_1) \wedge Q \in \Psi^c(x,\mathcal{F}_2)\}$. So $Px = c^n(N'Q)$ such that $n \geq 0$, $N' \in \Psi_0^c(N,\mathcal{F}_1)$ and $Q \in \Psi^c(x,\mathcal{F}_2)$. So $n = 0$, $N' = P$ and $Q = x$. Since $x \in \Psi_0^c(x,\varnothing)$, $Px \in \Psi_0^c(Nx,\mathcal{F})$.
- Else $\Psi^c(Nx,\mathcal{F}) = \{c^n(cN'Q) \mid n \geq 0 \wedge N' \in \Psi_0^c(N,\mathcal{F}_1) \wedge Q \in \Psi^c(x,\mathcal{F}_2)\}$. So $Px = c^n(cN'Q)$ such that $n \geq 0$, $N' \in \Psi_0^c(N,\mathcal{F}_1)$ and $Q \in \Psi^c(x,\mathcal{F}_2)$. So $n = 0$, $cN' = P$ and $Q = x$. Since $x \in \Psi_0^c(x,\varnothing)$, $Px \in \Psi_0^c(Nx,\mathcal{F})$.

1f. Easy by case on the structure of M and induction on n.

1g. By induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$. Then $\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n \geq 0\}$ and $\mathcal{F} = \varnothing$. Now, use lemma 1.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\}$. Let $c^n(\lambda x.N') \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Psi^c_0(N, \mathcal{F}')$. Then, $|c^n(\lambda x.N')|^c =^1 |\lambda x.N'|^c = \lambda x.|N'|^c =^{IH,1a} \lambda x.N$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}')\}$. Let $c^n(\lambda x.N'[x := c(cx)]) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Psi^c(N, \mathcal{F}')$. Then, $|c^n(\lambda x.N'[x := c(cx)])|^c =^1 |\lambda x.N'[x := c(cx)]|^c = \lambda x.|N'[x := c(cx)]|^c =^2 \lambda x.|N'|^c =^{IH} \lambda x.N$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If 0 then $\Psi^c(M, \mathcal{F}) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(M_1, \mathcal{F}_1) \wedge P' \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $c^n(N'P') \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$, $N' \in \Psi^c_0(M_1, \mathcal{F}_1)$ and $P' \in \Psi^c(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction, by definition N' too. Then, $|c^n(N'P')|^c =^1 |N'P'|^c = |N'|^c|P'|^c =^{IH,1a} M_1M_2$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \mathcal{F}_1) \wedge P_2 \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $c^n(cP_1P_2) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$, $P_1 \in \Psi^c(M_1, \mathcal{F}_1)$ and $P_2 \in \Psi^c(M_2, \mathcal{F}_2)$. Then $|c^n(cP_1P_2)|^c =^1 |cP_1P_2|^c = |cP_1|^c|P_2|^c = |P_1|^c|P_2|^c =^{IH} M_1M_2$.

1h. We prove the statement by induction on M.

- Let $M \in \mathcal{V} \setminus \{c\}$. Then $\Psi^c(M, \mathcal{F}) = \{c^n(x) \mid n \geq 0\}$ and $\mathcal{F} = \varnothing$. If $P \in \Psi^c(M, \mathcal{F})$ then $\mathcal{R}^{\beta\eta}_P =^{5.4.5} \varnothing$. Hence, $\mathcal{F} = |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c$.
- Let $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $0 \in \mathcal{F}$ then $N = Px$ where $x \notin \text{fv}(P)$ and $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(N, \mathcal{F}')\}$. Let $N_0 = c^n(\lambda x.N') \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Psi^c_0(N, \mathcal{F}')$. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N_0}\} =^{5.4.5} \{|\langle\lambda x.N', p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{\lambda x.N'}\} =^{1d} \{0\} \cup \{|\langle\lambda x.N', 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N'}\} = \{0\} \cup \{1.|\langle N', p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N'}\} = \{0\} \cup \{1.p \mid p \in |\langle N', \mathcal{R}^{\beta\eta}_{N'}\rangle|^c\} =^{IH,1a} \{0\} \cup \{1.p \mid p \in \mathcal{F}'\} =^{5.3} \mathcal{F}$.
  - Else $\Psi^c(M, \mathcal{F}) = \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Psi^c(N, \mathcal{F}')\}$. Let $N_0 = c^n(\lambda x.P[x := c(cx)]) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$ and $P \in \Psi^c(N, \mathcal{F}')$. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N_0}\} =^{5.4.5} \{|\langle\lambda x.P[x := c(cx)], p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{\lambda x.P[x:=c(cx)]}\} =^{5.4.3} \{|\langle\lambda x.P[x := c(cx)], 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P[x:=c(cx)]}\} =^{5.4.4} \{|\langle\lambda x.P[x := c(cx)], 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} = \{1.|\langle P[x := c(cx)], p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} =^3 \{1.|\langle P, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} = \{1.p \mid p \in |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c\} =^{IH} \{1.p \mid p \in \mathcal{F}'\} =^{5.3} \mathcal{F}$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If $0 \in \mathcal{F}$ then $\Psi^c(M, \mathcal{F}) = \{c^n(NP) \mid n \geq 0 \wedge N \in \Psi^c_0(M_1, \mathcal{F}_1) \wedge P \in \Psi^c(M_2, \mathcal{F}_2)\}$. Let $N_0 = c^n(NP) \in \Psi^c(M, \mathcal{F})$ where $n \geq 0$, $N \in \Psi^c_0(M_1, \mathcal{F}_1)$ and $P \in \Psi^c(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction, by definition N too. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{c^n(NP)}\} =^{5.4.5} \{|\langle NP, p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{NP}\} =^{5.3} \{0\} \cup \{|\langle NP, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_N\} \cup \{|\langle NP, 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} = \{0\} \cup \{1.|\langle N, p\rangle|^c \mid$

$p \in \mathcal{R}^{\beta\eta}_N\} \cup \{2.|\langle P,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_P\} = \{0\} \cup \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\} \cup \{2.p \mid p \in |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c\} =^{IH} \{0\} \cup \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{5.3} \mathcal{F}$.

– Else $\Psi^c(M,\mathcal{F}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1,\mathcal{F}_1) \wedge P_2 \in \Psi^c(M_2,\mathcal{F}_2)\}$. Let $N_0 = c^n(cP_1P_2) \in \Psi^c(M,\mathcal{F})$ where $n \geq 0$, $P_1 \in \Psi^c(M_1,\mathcal{F}_1)$ and $P_2 \in \Psi^c(M_2,\mathcal{F}_2)$. Then, $|\langle N_0, \mathcal{R}^{\beta\eta}_{N_0}\rangle|^c = \{|\langle N_0,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N_0}\} =^{5.4.5} \{|\langle cP_1P_2,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{cP_1P_2}\} =^{5.3} \{|\langle cP_1P_2, 1.2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\} \cup \{|\langle cP_1P_2, 2.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\} = \{1.|\langle P_1,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\} \cup \{2.|\langle P_2,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\} = \{1.p \mid p \in |\langle P_1, \mathcal{R}^{\beta\eta}_{P_1}\rangle|^c\} \cup \{2.p \mid p \in |\langle P_2, \mathcal{R}^{\beta\eta}_{P_2}\rangle|^c\} =^{IH} \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\} =^{5.3} \mathcal{F}$.

2. 2a. By induction on the construction of M.

- Let $M \in \mathcal{V} \setminus \{c\}$. So $|M|^c = M$, by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \varnothing = \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \Psi^c(M,\varnothing) = \{c^n(M) \mid n \geq 0\}$.
- Let $M = \lambda x.N[x := c(cx)]$ such that $x \neq c$ and $N \in \Lambda\eta_c$. Then, $|M|^c = \lambda x.|N|^c$ and $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{|\langle M,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_M\} =^{5.4.3} \{|\langle M, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}\} =^{5.4.4} \{|\langle M, 1.p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_N\} =^3 \{1.|\langle N,p\rangle|^c \mid p \in \mathcal{R}^{\beta\eta}_N\} = \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\} \subseteq^{IH} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|N|^c}\} =^2 \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|N[x:=c(cx)]|^c}\} \subseteq^{5.3} \mathcal{R}^{\beta\eta}_{\lambda x.|N[x:=c(cx)]|^c} = \mathcal{R}^{\beta\eta}_{|\lambda x.N[x:=c(cx)]|^c}$.

  We just proved that $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\}$, so $0 \notin |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$ and $|\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)\}$. By IH, $N \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)$, so $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.
- Let $M = \lambda x.Nx$ such that $Nx \in \Lambda\eta_c$, $N \neq c$ and $x \notin \mathrm{fv}(N) \cup \{c\}$. By lemma 5.2.8, $N \in \Lambda\eta_c$ and by lemma 4, $x \notin \mathrm{fv}(|N|^c)$. $|M|^c = \lambda x.|Nx|^c = \lambda x.|N|^c x$. Since $M, |M|^c \in \mathcal{R}^{\beta\eta}$, by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{Nx}\}$, so $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c\} \subseteq^{IH} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|Nx|^c}\} = \mathcal{R}^{\beta\eta}_{|M|^c}$.

  We proved $|\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$ and $0 \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi^c_0(|Nx|^c, |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c)\}$. By IH, $Nx \in \Psi^c(|Nx|^{\beta\eta}, |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c)$, so by lemma 8.2.1e,
  $Nx \in \Psi^c_0(|Nx|^{\beta\eta}, |\langle Nx, \mathcal{R}^{\beta\eta}_{Nx}\rangle|^c)$. Hence $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.
- Let $M = cNP$ where $N, P \in \Lambda\eta_c$, so $cN \in \Lambda\eta_c$. $|M|^c = |cN|^c|P|^c = |N|^c|P|^c$. Because $M, cN \notin \mathcal{R}^{\beta\eta}$, By lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{1.2.p \mid p \in \mathcal{R}^{\beta\eta}_N\} \cup \{2.p \mid\in \mathcal{R}^{\beta\eta}_P\}$. So $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\} \cup \{2.p \mid p \in |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c\} \subseteq^{IH} \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|N|^c}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{|P|^c}\} \subseteq^{5.3} \mathcal{R}^{\beta\eta}_{|M|^c}$.

  We just proved that $0 \notin |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$ and $|\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$ and $|\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c = \{p \mid 2.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c) \wedge P' \in \Psi^c(|P|^c, |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c)\}$. By IH, $N \in \Psi^c(|N|^{\beta\eta}, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)$ and $P \in \Psi^c(|P|^{\beta\eta}, |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c)$,
  so $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.

- Let $M = NP$ where $N, P \in \Lambda\eta_c$ and N is a λ-abstraction. So by definition $|N|^c$ is a λ-abstraction too and $|M|^c = |N|^c|P|^c$. Since $M \in \mathcal{R}^{\beta\eta}$, By lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_N\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_P\}$. So $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = \{0\} \cup \{1.p \mid p \in |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c\} \cup \{2.p \mid p \in |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c\} \subseteq^{IH} \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{|N|^c}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{|P|^c}\} =^{5.3} \mathcal{R}^{\beta\eta}_{|M|^c}$.
  We just proved that $0 \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$, $|\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c = \{p \mid 1.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$ and $|\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c = \{p \mid 2.p \in |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c\}$. By definition, $\Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi^c_0(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c) \wedge P' \in \Psi^c(|P|^c, |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c)\}$. By IH, $N \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)$ and $P \in \Psi^c(|P|^c, |\langle P, \mathcal{R}^{\beta\eta}_P\rangle|^c)$,
  so $N \in \Psi^c_0(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c)$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.
- Let $M = cN$ where $N \in \Lambda\eta_c$ then $|M|^c = |N|^c$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_N\}$ so $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c = |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c \subseteq^{IH} \mathcal{R}^{\beta\eta}_{|N|^c} = \mathcal{R}^{\beta\eta}_{|M|^c}$.
  By IH, $N \in \Psi^c(|N|^c, |\langle N, \mathcal{R}^{\beta\eta}_N\rangle|^c) = \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$, so by lemma 8.2.1f, $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$.

2b. By lemma 4, $c \notin \mathrm{fv}(|M|^c)$. By lemma 8.2.2a, $|\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c \subseteq \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c)$. To prove unicity, assume that $\langle N', \mathcal{F}'\rangle$ is another such pair. So $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{N'}$ and $M \in \Psi^c(N', \mathcal{F}')$. By lemma 8.2.1g, $|M|^c = N'$ and by lemma 8.2.1h, $\mathcal{F}' = |\langle M, \mathcal{R}^{\beta\eta}_M\rangle|^c$.

□

Proof(Lemma 8.2.3)**:** Let $N_1 \in \Psi^c(M, \mathcal{F})$. By lemma 8.2.1c, $N_1 \in \Lambda\eta_c$. By lemma 8.2.1h and lemma 1, there exists a unique $p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$, such that $|\langle N_1, p_1\rangle|^c = p$. By lemma 2.2.8, there exists N'_1 such that $N_1 \xrightarrow{p_1}_{\beta\eta} N'_1$. By lemma 2, $N'_1 \in \Lambda\eta_c$. By lemma 5.8.7a, $|N_1|^c \xrightarrow{p'_1}_{\beta\eta} |N'_1|^c$ such that $p'_1 = |\langle N_1, p_1\rangle|^c = p$. By lemma 8.2.1g, $M = |N_1|^c$. So by lemma 2.2.9, $M' = |N'_1|^c$. Let $\mathcal{F}' = |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c$. By lemma 8.2.2b, $(M', \mathcal{F}')$ is the one and only pair such that $c \notin \mathrm{fv}(M')$, $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ and $N'_1 \in \Psi^c(M', \mathcal{F}')$.

Let $N_2 \in \Psi^c(M, \mathcal{F})$. By lemma 8.2.1c, $N_2 \in \Lambda\eta_c$. By lemma 8.2.1h and lemma 1, there exists a unique $p_2 \in \mathcal{R}^{\beta\eta}_{N_2}$, such that $|\langle N_2, p_2\rangle|^c = p$. By lemma 2.2.8, there exists N'_2 such that $N_2 \xrightarrow{p_2}_{\beta\eta} N'_2$. By lemma 2, $N'_2 \in \Lambda\eta_c$. By lemma 5.8.7a, $|N_2|^c \xrightarrow{p'_2}_{\beta\eta} |N'_2|^c$ such that $p'_2 = |\langle N_2, p_2\rangle|^c = p$. By lemma 8.2.1g, $M = |N_2|^c$. So by lemma 2.2.9, $M' = |N'_2|^c$. Let $\mathcal{F}'' = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c$. By lemma 8.2.2b, $(M', \mathcal{F}'')$ is the one and only pair such that $c \notin \mathrm{fv}(M')$, $\mathcal{F}'' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ and $N'_2 \in \Psi^c(M', \mathcal{F}'')$.

Because $N_1, N_2 \in \Psi^c(M, \mathcal{F})$, by lemma 8.2.1h, $|\langle N_1, \mathcal{R}^{\beta\eta}_{N_1}\rangle|^c = |\langle N_2, \mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$ and by lemma 8.2.1g, $|N_1|^c = |N_2|^c$. Finally, by lemma 5.8.7c, $\mathcal{F}' = |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c = \mathcal{F}''$. □

Lemma F.1. If $p \in \mathcal{R}^{\beta\eta}_t$ then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t|_p)$.

Proof: We prove this lemma by induction on the structure of t.

- Let $t \in \mathcal{V}$ then by lemma 5.3, $\mathcal{R}^{\beta\eta}_t = \varnothing$.

- Let $t = \lambda_n y.t'$ then by lemma 5.3:
  - Either $p = 0$ if $t' = t''y$ and $y \notin \mathrm{fv}(t'')$. Then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(\lambda_n y.t''[\bar{x} := c(c\bar{x})]y) = \langle 2, n\rangle = \mathrm{headlam}(t)$ such that $y \notin \{c, \bar{x}\}$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t'}$. Then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t'|_{p'}[\bar{x} := c(c\bar{x})]) =^{IH} \mathrm{headlam}(t'|_{p'}) = \mathrm{headlam}(t|_p)$.
- Let $t = t_1 t_2$ then by lemma 5.3:
  - Either $p = 0$ if $t_1 = \lambda_n y.t_0$. Then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}((\lambda_n y.t_0[\bar{x} := c(c\bar{x})])t_2[\bar{x} := c(c\bar{x})]) = \langle 1, n\rangle = \mathrm{headlam}(t)$ such that $y \notin \{c, \bar{x}\}$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t_1|_{p'}[\bar{x} := c(c\bar{x})]) =^{IH} \mathrm{headlam}(t_1|_{p'}) = \mathrm{headlam}(t|_p)$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_2}$. Then $\mathrm{headlam}(t|_p[\bar{x} := c(c\bar{x})]) = \mathrm{headlam}(t_2|_{p'}[\bar{x} := c(c\bar{x})]) =^{IH} \mathrm{headlam}(t_2|_{p'}) = \mathrm{headlam}(t|_p)$.

□

Lemma F.2. Let $t \in \bar{\Lambda}$ and $\mathcal{F} \subseteq \mathcal{R}^{\beta\eta}_t$.

- If $t = x$ then $\mathrm{headlamred}(t, \mathcal{F}) = \mathrm{hlr}(t) = \varnothing$.
- If $t = \lambda_n x.t_1$ then if $t \in \mathcal{R}^{\beta\eta}$ then $\mathrm{hlr}(t) = \mathrm{hlr}(t_1) \cup \{\langle 2, n\rangle\}$ else $\mathrm{hlr}(t) = \mathrm{hlr}(t_1)$.
- If $t = \lambda_n x.t_1$ and $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ then if $0 \in \mathcal{F}$ then $\mathrm{headlamred}(t, \mathcal{F}) = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \{\langle 2, n\rangle\}$ else $\mathrm{headlamred}(t, \mathcal{F}) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$.
- If $t = t_1 t_2$ then if $t \in \mathcal{R}^{\beta\eta}$ then $\mathrm{hlr}(t) = \mathrm{hlr}(t_1) \cup \mathrm{hlr}(t_2) \cup \{\mathrm{headlam}(t)\}$ else $\mathrm{hlr}(t) = \mathrm{hlr}(t_1) \cup \mathrm{hlr}(t_2)$.
- If $t = t_1 t_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$ then if $0 \in \mathcal{F}$ then $\mathrm{headlamred}(t, \mathcal{F}) = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \mathrm{headlamred}(t_2, \mathcal{F}_2) \cup \{\mathrm{headlam}(t)\}$ else $\mathrm{headlamred}(t, \mathcal{F}) = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \mathrm{headlamred}(t_2, \mathcal{F}_2)$.
- If $t = \lambda_n \bar{x}.t_1[\bar{x} := c(c\bar{x})]$ then $\mathrm{hlr}(t) = \mathrm{hlr}(t_1)$.
- If $t = c^n(t_1)$, then $\mathrm{hlr}(t) = \mathrm{hlr}(t_1)$.

Proof: By definition $\mathrm{hlr}(t) = \{\langle i, n\rangle \mid \exists p \in \mathcal{R}^{\beta\eta}_t.\ \mathrm{headlam}(t|_p) = \langle i, n\rangle\}$ and $\mathrm{headlamred}(t, \mathcal{F}) = \{\langle i, n\rangle \mid \exists p \in \mathcal{F}.\ \mathrm{headlam}(t|_p) = \langle i, n\rangle\}$. We prove the frist three items of this lemma by induction on the size of t and then by case on the structure of t.

- Let $t = x$. By lemma 5.3, $\mathcal{F} = \mathcal{R}^{\beta\eta}_x = \varnothing$, then $\mathrm{headlamred}(x, \mathcal{F}) = \mathrm{hlr}(x) = \varnothing$.
- Let $t = \lambda_n x.t_1$.
  - Let $t \in \mathcal{R}^{\beta\eta}$ then $t_1 = t_0 x$ such that $x \notin \mathrm{fv}(t_0)$.

- Let $\langle j, m\rangle \in \mathrm{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\mathrm{headlam}(t|_p) = \langle j, m\rangle$. By lemma 5.3:
  - Either $p = 0$, so $\langle j, m\rangle = \mathrm{headlam}(t|_0) = \mathrm{headlam}(t) = \langle 2, n\rangle$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \mathrm{hlr}(t_1)$.
- Let $\langle j, m\rangle \in \mathrm{hlr}(t_1) \cup \{\langle 2, n\rangle\}$.
  - Either $\langle j, m\rangle \in \mathrm{hlr}(t_1)$. Then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\mathrm{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 5.3, $1.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.
  - Or $\langle j, m\rangle = \langle 2, n\rangle$. By lemma 5.3, $0 \in \mathcal{R}^{\beta\eta}_t$ and $\mathrm{headlam}(t|_0) = \mathrm{headlam}(t) = \langle 2, n\rangle$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.

– Let $t \notin \mathcal{R}^{\beta\eta}$.

- Let $\langle j, m\rangle \in \mathrm{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\mathrm{headlam}(t|_p) = \langle j, m\rangle$. By lemma 5.3, $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \mathrm{hlr}(t_1)$.
- Let $\langle j, m\rangle \in \mathrm{hlr}(t_1)$ then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\mathrm{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 5.3, $1.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.

• Let $t = \lambda_n x.t_1$ and $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$.

– Let $0 \in \mathcal{F}$ then $t \in \mathcal{R}^{\beta\eta}$.

- Let $\langle j, m\rangle \in \mathrm{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\mathrm{headlam}(t|_p) = \langle j, m\rangle$. By lemma 5.3:
  - Either $p = 0$, so $\langle j, m\rangle = \mathrm{headlam}(t|_0) = \mathrm{headlam}(t) = \langle 2, n\rangle$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Then, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \mathrm{headlamred}(t_1, \mathcal{F}_1)$.
- Let $\langle j, m\rangle \in \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \{\langle 2, n\rangle\}$.
  - Either $\langle j, m\rangle \in \mathrm{headlamred}(t_1, \mathcal{F}_1)$. Then there exists $p \in \mathcal{F}_1$ such that $\mathrm{headlam}(t_1|_p) = \langle j, m\rangle$. So, $1.p \in \mathcal{F}$ and $\langle j, m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. Hence, $\langle j, m\rangle \in \mathrm{headlamred}(t, \mathcal{F})$.
  - Or $\langle j, m\rangle = \langle 2, n\rangle$. Because $0 \in \mathcal{F}$ and $\mathrm{headlam}(t|_0) = \mathrm{headlam}(t) = \langle 2, n\rangle$ then $\langle j, m\rangle \in \mathrm{headlamred}(t, \mathcal{F})$.

– Let $0 \notin \mathcal{F}$.

- Let $\langle j, m\rangle \in \mathrm{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\mathrm{headlam}(t|_p) = \langle j, m\rangle$. By lemma 5.3, $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Then, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \mathrm{headlamred}(t_1, \mathcal{F}_1)$.
- Let $\langle j, m\rangle \in \mathrm{headlamred}(t_1, \mathcal{F}_1)$ then there exists $p \in \mathcal{F}_1$ such that $\mathrm{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 5.3, $1.p \in \mathcal{F}$ and $\langle j, m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \mathrm{headlamred}(t, \mathcal{F})$.

- Let $t = t_1 t_2$.
  - Let $t \in \mathcal{R}^{\beta\eta}$ then $t_1 = \lambda_n x.t_0$. So $\langle 1, n\rangle = \mathrm{headlam}(t)$.
    - Let $\langle j, m\rangle \in \mathrm{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\mathrm{headlam}(t|_p) = m$. By lemma 5.3:
      - Either $p = 0$, so $\langle j, m\rangle = \mathrm{headlam}(t|_0) = \mathrm{headlam}(t) = \langle 1, n\rangle$.
      - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Then, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \mathrm{hlr}(t_1)$.
      - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_2}$.
        Moreover, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_2|_{p'})$. So $\langle j, m\rangle \in \mathrm{hlr}(t_2)$.
    - Let $\langle j, m\rangle \in \mathrm{hlr}(t_1) \cup \mathrm{hlr}(t_2) \cup \{\langle 1, n\rangle\}$.
      - Either $\langle j, m\rangle \in \mathrm{hlr}(t_1)$. Then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\mathrm{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 5.3, $1.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.
      - Or $\langle j, m\rangle \in \mathrm{hlr}(t_2)$. Then there exists $p \in \mathcal{R}^{\beta\eta}_{t_2}$ such that $\mathrm{headlam}(t_2|_p) = \langle j, m\rangle$. By lemma 5.3, $2.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \mathrm{headlam}(t_2|_p) = \mathrm{headlam}(t|_{2.p})$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.
      - Or $\langle j, m\rangle = \langle 1, n\rangle$. By lemma 5.3, $0 \in \mathcal{R}^{\beta\eta}_t$ and $\mathrm{headlam}(t|_0) = \mathrm{headlam}(t) = \langle 1, n\rangle$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.
  - Let $t \notin \mathcal{R}^{\beta\eta}$.
    - Let $\langle j, m\rangle \in \mathrm{hlr}(t)$ then there exists $p \in \mathcal{R}^{\beta\eta}_t$ such that $\mathrm{headlam}(t|_p) = \langle j, m\rangle$. By lemma 5.3:
      - Either $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_1}$. Moreover, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j, m\rangle \in \mathrm{hlr}(t_1)$.
      - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{t_2}$.
        Moreover, $\langle j, m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_2|_{p'})$. So $\langle j, m\rangle \in \mathrm{hlr}(t_2)$.
    - Let $\langle j, m\rangle \in \mathrm{hlr}(t_1) \cup \mathrm{hlr}(t_2)$.
      - Either $\langle j, m\rangle \in \mathrm{hlr}(t_1)$. Then there exists $p \in \mathcal{R}^{\beta\eta}_{t_1}$ such that $\mathrm{headlam}(t_1|_p) = \langle j, m\rangle$. By lemma 5.3, $1.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.
      - Or $\langle j, m\rangle \in \mathrm{hlr}(t_2)$. Then there exists $p \in \mathcal{R}^{\beta\eta}_{t_2}$ such that $\mathrm{headlam}(t_2|_p) = \langle j, m\rangle$. By lemma 5.3, $2.p \in \mathcal{R}^{\beta\eta}_t$ and $\langle j, m\rangle = \mathrm{headlam}(t_2|_p) = \mathrm{headlam}(t|_{2.p})$. So $\langle j, m\rangle \in \mathrm{hlr}(t)$.
- Let $t = t_1 t_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$.
  - Let $0 \in \mathcal{F}$ then $t \in \mathcal{R}^{\beta\eta}$.
    - Let $\langle j, m\rangle \in \mathrm{headlamred}(t, \mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\mathrm{headlam}(t|_p) = m$. By lemma 5.3:
      - Either $p = 0$, so $\langle j, m\rangle = \mathrm{headlam}(t|_0) = \mathrm{headlam}(t)$.

- Or $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Then, $\langle j,m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j,m\rangle \in \mathrm{headlamred}(t_1,\mathcal{F}_1)$.
- Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$. Then, $\langle j,m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_2|_{p'})$. So $\langle j,m\rangle \in \mathrm{headlamred}(t_2,\mathcal{F}_2)$.

* Let $\langle j,m\rangle \in \mathrm{headlamred}(t_1,\mathcal{F}_1) \cup \mathrm{headlamred}(t_2,\mathcal{F}_2) \cup \{\mathrm{headlam}(t)\}$.
  - Either $\langle j,m\rangle \in \mathrm{headlamred}(t_1,\mathcal{F}_1)$. Then there exists $p \in \mathcal{F}_1$ such that $\mathrm{headlam}(t_1|_p) = \langle j,m\rangle$. So, $1.p \in \mathcal{F}$ and $\langle j,m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. Hence, $\langle j,m\rangle \in \mathrm{headlamred}(t,\mathcal{F})$.
  - Or $\langle j,m\rangle \in \mathrm{headlamred}(t_2,\mathcal{F}_2)$. Then there exists $p \in \mathcal{F}_2$ such that $\mathrm{headlam}(t_2|_p) = \langle j,m\rangle$. So, $2.p \in \mathcal{F}$ and $\langle j,m\rangle = \mathrm{headlam}(t_2|_p) = \mathrm{headlam}(t|_{2.p})$. Hence, $\langle j,m\rangle \in \mathrm{headlamred}(t,\mathcal{F})$.
  - Or $\langle j,m\rangle = \mathrm{headlam}(t)$. Because $0 \in \mathcal{F}$ and $\mathrm{headlam}(t|_0) = \mathrm{headlam}(t)$, then $\langle j,m\rangle \in \mathrm{headlamred}(t,\mathcal{F})$.

– Let $0 \notin \mathcal{F}$.

* Let $\langle j,m\rangle \in \mathrm{headlamred}(t,\mathcal{F})$ then there exists $p \in \mathcal{F}$ such that $\mathrm{headlam}(t|_p) = \langle j,m\rangle$. By lemma 5.3:
  - Either $p = 1.p'$ such that $p' \in \mathcal{F}_1$. Moreover, $\langle j,m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1|_{p'})$. So $\langle j,m\rangle \in \mathrm{headlamred}(t_1,\mathcal{F}_1)$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{F}_2$. Moreover, $\langle j,m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_2|_{p'})$. So $\langle j,m\rangle \in \mathrm{headlamred}(t_2,\mathcal{F}_2)$.
* Let $\langle j,m\rangle \in \mathrm{headlamred}(t_1,\mathcal{F}_1) \cup \mathrm{headlamred}(t_2,\mathcal{F}_2)$.
  - Either $\langle j,m\rangle \in \mathrm{headlamred}(t_1,\mathcal{F}_1)$. Then there exists $p \in \mathcal{F}_1$ such that $\mathrm{headlam}(t_1|_p) = \langle j,m\rangle$. So, $1.p \in \mathcal{F}$ and $\langle j,m\rangle = \mathrm{headlam}(t_1|_p) = \mathrm{headlam}(t|_{1.p})$. Hence, $\langle j,m\rangle \in \mathrm{headlamred}(t,\mathcal{F})$.
  - Or $\langle j,m\rangle \in \mathrm{headlamred}(t_2,\mathcal{F}_2)$. Then there exists $p \in \mathcal{F}_2$ such that $\mathrm{headlam}(t_2|_p) = \langle j,m\rangle$. So, $2.p \in \mathcal{F}$ and $\langle j,m\rangle = \mathrm{headlam}(t_2|_p) = \mathrm{headlam}(t|_{2.p})$. Hence, $\langle j,m\rangle \in \mathrm{headlamred}(t,\mathcal{F})$.

Let $t = \lambda_n \bar{x}.t_1[\bar{x} := c(c\bar{x})]$.

- Let $\langle j,m\rangle \in \mathrm{hlr}(t)$ then there exists $p \in \mathcal{R}_t^{\beta\eta}$ such that $\mathrm{headlam}(t|_p) = \langle j,m\rangle$. By lemma 5.4.3 and lemma 5.4.4, $p = 1.p'$ such that $p' \in \mathcal{R}_{t_1}^{\beta\eta}$. Moreover, $\langle j,m\rangle = \mathrm{headlam}(t|_p) = \mathrm{headlam}(t_1[\bar{x} := c(c\bar{x})]|_{p'}) =^{5.4.2} \mathrm{headlam}(t_1|_{p'}[\bar{x} := c(c\bar{x})]) =^{F.1} \mathrm{headlam}(t_1|_{p'})$. So $\langle j,m\rangle \in \mathrm{hlr}(t_1)$.

- Let $\langle j,m\rangle \in \mathrm{hlr}(t_1)$ then there exists $p \in \mathcal{R}_{t_1}^{\beta\eta}$ such that $\mathrm{headlam}(t_1|_p) = \langle j,m\rangle$. By lemma 5.4.3 and lemma 5.4.4, $1.p \in \mathcal{R}_t^{\beta\eta}$. Moreover, $\langle j,m\rangle = \mathrm{headlam}(t_1|_p) =^{F.1} \mathrm{headlam}(t_1|_p[\bar{x} := c(c\bar{x})]) =^{5.4.2} \mathrm{headlam}(t_1[\bar{x} := c(c\bar{x})]|_p) = \mathrm{headlam}(t|_{1.p})$. So $\langle j,m\rangle \in \mathrm{hlr}(t)$.

Let $t = c^n(t_1)$. We prove that $\mathrm{hlr}(t) = \mathrm{hlr}(t_1)$ by induction on n.

- Let $n = 0$ then it is done.

- Let $n = m + 1$ such that $m \geq 0$ then $\mathrm{hlr}(t) =^{F.2} \mathrm{hlr}(c^m(t_1)) =^{IH} \mathrm{hlr}(t_1)$.

□

Proof(Lemma 8.4)**:** We prove this lemma by induction on the structure of t.

- Let $t = x \neq c$ then by lemma 5.3, $\mathcal{F} = \varnothing$ and $u = c^n(x)$ such that $n \geq 0$. Then, $\mathrm{hlr}(u) =^{F.2} \varnothing = \mathrm{headlamred}(t, \mathcal{F})$.
- Let $t = \lambda_n x.t_1$ such that $x \neq c$ and $\mathcal{F}_1 = p \mid 1.p \in \mathcal{F}$.
  - If $0 \in \mathcal{F}$ then $t_1 = t'_1 x$ such that $x \notin \mathrm{fv}(t'_1)$, and $u = c^n(\lambda_n x.u_1)$ such that $n \geq 0$ and $u_1 \in \Psi^c_0(t_1, \mathcal{F}_1)$. By IH and lemma 8.2.1a, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$. Then, $\mathrm{hlr}(u) =^{8.2.1d, F.2} \mathrm{hlr}(u_1) \cup \{\langle 2, n\rangle\} = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \{\langle 2, n\rangle\} =^{F.2} \mathrm{headlamred}(t, \mathcal{F})$.
  - Else, $u = c^n(\lambda_n x.u_1[x := c(cx)])$ such that $n \geq 0$ and $u_1 \in \Psi^c(t_1, \mathcal{F}_1)$. By IH, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$.
    Then, $\mathrm{hlr}(u) =^{F.2} \mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1) =^{F.2} \mathrm{headlamred}(t, \mathcal{F})$.
- Let $t = t_1 t_2$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\}$.
  - If $0 \in \mathcal{F}$ then $t_1 = \lambda_n y.t'_1$, and $u = c^n(u_1 u_2)$ such that $n \geq 0$, $u_1 \in \Psi^c_0(t_1, \mathcal{F}_1)$ and $u_2 \in \Psi^c(t_2, \mathcal{F}_2)$. By definition, $u_1 = \lambda_n y.u'_1$. By IH and lemma 8.2.1a, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$ and $\mathrm{hlr}(u_2) = \mathrm{headlamred}(t_2, \mathcal{F}_2)$.
    Then, $\mathrm{hlr}(u) =^{F.2} \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \mathrm{headlamred}(t_2, \mathcal{F}_2) \cup \{\langle 1, n\rangle\} =^{F.2} \mathrm{headlamred}(t, \mathcal{F})$.
  - Else, $u = c^n(c u_1 u_2)$ such that $n \geq 0$, $u_1 \in \Psi^c(t_1, \mathcal{F}_1)$ and $u_2 \in \Psi^c(t_2, \mathcal{F}_2)$. By IH, $\mathrm{hlr}(u_1) = \mathrm{headlamred}(t_1, \mathcal{F}_1)$ and $\mathrm{hlr}(u_2) = \mathrm{headlamred}(t_2, \mathcal{F}_2)$. Then, $\mathrm{hlr}(u) =^{F.2} \mathrm{hlr}(u_1) \cup \mathrm{hlr}(u_2) = \mathrm{headlamred}(t_1, \mathcal{F}_1) \cup \mathrm{headlamred}(t_2, \mathcal{F}_2) =^{F.2} \mathrm{headlamred}(t, \mathcal{F})$.

□

Lemma F.3. $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) \subseteq \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$.

Proof: We prove the lemma by induction on the size of u_1 and then by case on the structure of u_1.

- Let $u_1 \in \mathcal{V}$. Either $u_1 = \bar{x}$ then $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(c(cu_2)) =^{F.2} \mathrm{hlr}(u_2) \subseteq^{F.4} \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$. Or $u_1 = y \neq \bar{x}$ then $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(u_1) \subseteq^{F.4, F.2} \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$.
- Let $u_1 = \lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})]$. Then $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}((\lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})])[\bar{x} := c(cu_2)]) = \mathrm{hlr}(\lambda_m \bar{y}.u'_1[\bar{x} := c(cu_2)][\bar{y} := c(c\bar{y})]) =^{F.2} \mathrm{hlr}(u'_1[\bar{x} := c(cu_2)]) \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.u'_1[\bar{x} := c(c\bar{x})])u_2) =^{F.2} \mathrm{hlr}(u'_1) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{F.2} \mathrm{hlr}(\lambda_m \bar{y}.u'_1[\bar{y} := c(c\bar{y})]) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{F.2} \mathrm{hlr}((\lambda_n \bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2)$ such that $\bar{y} \notin \mathrm{fv}(u_2) \cup \{\bar{x}\}$.
- Let $u_1 = \lambda_m \bar{y}.w\bar{y}$ such that $\bar{y} \notin \mathrm{fv}(w)$. Then, $\mathrm{hlr}(u_1[\bar{x} := c(cu_2)]) = \mathrm{hlr}(\lambda_m \bar{y}.(w\bar{y})[\bar{x} := c(cu_2)]) =^{F.2} \mathrm{hlr}((w\bar{y})[\bar{x} := c(cu_2)]) \cup \{\langle 2, m\rangle\} \subseteq^{IH} \mathrm{hlr}((\lambda_n \bar{x}.(w\bar{y})[\bar{x} := c(c\bar{x})])u_2) \cup \{\langle 2, m\rangle\} =^{F.2} \mathrm{hlr}(w\bar{y}) \cup \mathrm{hlr}(u_2) \cup \{\langle 1, n\rangle, \langle 2, m\rangle\} =^{F.2} \mathrm{hlr}((\lambda_n \bar{x}.(\lambda_m \bar{y}.w\bar{y})[\bar{x} := c(c\bar{x})])u_2)$ such that $\bar{y} \notin \mathrm{fv}(u_2) \cup \{\bar{x}\}$.

- Let $u_1 = cu'_1u''_1$. Then, $\text{hlr}(u_1[\bar{x} := c(cu_2)]) = \text{hlr}(cu'_1[\bar{x} := c(cu_2)]u''_1[\bar{x} := c(cu_2)]) =^{F.2} \text{hlr}(u'_1[\bar{x} := c(cu_2)]) \cup \text{hlr}(u''_1[\bar{x} := c(cu_2)]) \subseteq^{IH} \text{hlr}((\lambda_n\bar{x}.u'_1[\bar{x} := c(c\bar{x})])u_2) \cup \text{hlr}((\lambda_n\bar{x}.u''_1[\bar{x} := c(c\bar{x})])u_2) =^{F.2} \text{hlr}(u'_1) \cup \text{hlr}(u''_1) \cup \text{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{F.2} \text{hlr}((\lambda_n\bar{x}.(cu'_1u''_1)[\bar{x} := c(c\bar{x})])u_2)$.

- Let $u_1 = vu''_1$ (such that $v = \lambda_m\bar{y}.w\bar{y}$ and $\bar{y} \notin \text{fv}(w)$ or $v = \lambda_m\bar{y}.u'_1[\bar{y} := c(c\bar{y})]$). Then, $\text{hlr}(u_1[\bar{x} := c(cu_2)]) = \text{hlr}(v[\bar{x} := c(cu_2)]u''_1[\bar{x} := c(cu_2)]) =^{F.2} \text{hlr}(v[\bar{x} := c(cu_2)]) \cup \text{hlr}(u''_1[\bar{x} := c(cu_2)]) \cup \{\langle 1, m\rangle\} \subseteq^{IH} \text{hlr}((\lambda_n\bar{x}.v[\bar{x} := c(c\bar{x})])u_2) \cup \text{hlr}((\lambda_n\bar{x}.u''_1[\bar{x} := c(c\bar{x})])u_2) \cup \{\langle 1, m\rangle\} =^{F.2} \text{hlr}(v) \cup \text{hlr}(u''_1) \cup \text{hlr}(u_2) \cup \{\langle 1, n\rangle, \langle 1, m\rangle\} =^{F.2} \text{hlr}((\lambda_n\bar{x}.(vu''_1)[\bar{x} := c(c\bar{x})])u_2)$.

- Let $u_1 = cu'_1$. Then, $\text{hlr}(u_1[\bar{x} := u_2]) = \text{hlr}(cu'_1[\bar{x} := c(cu_2)]) =^{F.2} \text{hlr}(u'_1[\bar{x} := c(cu_2)]) \subseteq^{IH} \text{hlr}((\lambda_n\bar{x}.u'_1[\bar{x} := c(c\bar{x})])u_2) =^{F.2} \text{hlr}(u'_1) \cup \text{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{F.2} \text{hlr}((\lambda_n\bar{x}.(cu'_1)[\bar{x} := c(c\bar{x})])u_2)$.

□

Lemma F.4. If $t_1 \subseteq t_2$ then $\text{hlr}(t_1) \subseteq \text{hlr}(t_2)$.

Proof: We prove the lemma by induction on the structure of t_2.

- Let $t_2 = x$, then it is done because by definition $t_1 = x$.
- Let $t_2 = \lambda_n x.t_0$ then by definition:
  - Either $t_1 = t_2$ so it is done.
  - Or $t_1 \subseteq t_0$. Then $\text{hlr}(t_1) \subseteq^{IH} \text{hlr}(t_0) \subseteq^{F.2} \text{hlr}(t_2)$.
- Let $t_2 = t_3t_4$ then by definition:
  - Either $t_1 = t_2$ so it is done.
  - Or $t_1 \subseteq t_3$. Then $\text{hlr}(t_1) \subseteq^{IH} \text{hlr}(t_3) \subseteq^{F.2} \text{hlr}(t_2)$.
  - Or $t_1 \subseteq t_4$. Then $\text{hlr}(t_1) \subseteq^{IH} \text{hlr}(t_4) \subseteq^{F.2} \text{hlr}(t_2)$.

□

Proof(Lemma 8.5)**:** We prove this lemma by induction on the size of u and then by case on the structure of u.

- Let $u = \bar{x}$ then it is done because $\bar{x}$ does not reduce by $\rightarrow_{\beta\eta}$.

- Let $u = \lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]$. Because $u \xrightarrow{p}_{\beta\eta} u'$, then by lemma 2.2.8, lemma 5.4.3 and lemma 5.2.13a, $p = 1.p'$, $u' = \lambda_n\bar{x}.u'_1[\bar{x} := c(c\bar{x})]$ and $u_1 \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(u_1)$. So, by lemma F.2, $\text{hlr}(u') = \text{hlr}(u'_1) \subseteq \text{hlr}(u_1) = \text{hlr}(u)$.

- Let $u = \lambda_n\bar{x}.w\bar{x}$ and $\bar{x} \notin \text{fv}(w)$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 5.3:
  - Either $p = 0$ and $u' = w$. So $\text{hlr}(u') \subseteq^{F.4} \text{hlr}(u)$.
  - Or $p = 1.p'$, $w\bar{x} \xrightarrow{p'}_{\beta\eta} u'_1$ and $u' = \lambda_n\bar{x}.u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(w\bar{x})$. So, $\text{hlr}(u') \subseteq^{F.2} \text{hlr}(u'_1) \cup \{\langle 2, n\rangle\} \subseteq \text{hlr}(w\bar{x}) \cup \{\langle 2, n\rangle\} =^{F.2} \text{hlr}(t)$.

- Let $u = (\lambda_n\bar{x}.w\bar{x})u_1$ such that $\bar{x} \notin \text{fv}(w)$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 5.3:
  - Either $p = 0$. So $u' = wu_1$. By case on w:
    - Either w is a v and so $u' \in \mathcal{R}^{\beta\eta}$. Let $\langle 1, m\rangle = \text{headlam}(u')$ then $\text{hlr}(u') =^{F.2} \text{hlr}(w) \cup \text{hlr}(u_1) \cup \{\langle 1, m\rangle\} \subseteq^{F.2} \text{hlr}(u)$.
    - Or $w = cu_2$ and so $u' \notin \mathcal{R}^{\beta\eta}$. Then $\text{hlr}(u') =^{F.2} \text{hlr}(w) \cup \text{hlr}(u_1) \subseteq^{F.2} \text{hlr}(u)$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{\lambda_n\bar{x}.w\bar{x}}$. So $u' = u'_1u_1$ such that $\lambda_n\bar{x}.w\bar{x} \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(\lambda_n\bar{x}.w\bar{x})$. By lemma 5.3:
    - Either $p' = 0$ and $u'_1 = w$, so $u' = wu_1$. By case on w:
      - Either w is a v and so $u' \in \mathcal{R}^{\beta\eta}$. Let $\langle 1, m\rangle = \text{headlam}(u')$ then $\text{hlr}(u') =^{F.2} \text{hlr}(w) \cup \text{hlr}(u_1) \cup \{\langle 1, m\rangle\} \subseteq^{F.2} \text{hlr}(u)$.
      - Or $w = cu_2$ and so $u' \notin \mathcal{R}^{\beta\eta}$. Then $\text{hlr}(u') =^{F.2} \text{hlr}(w) \cup \text{hlr}(u_1) \subseteq^{F.2} \text{hlr}(u)$.
    - Or $p' = 1.p''$, $u'_1 = \lambda_n\bar{x}.u_2$ and $w\bar{x} \xrightarrow{p''}_{\beta\eta} u_2$. Then, $\text{hlr}(u') =^{F.2} \text{hlr}(u'_1) \cup \text{hlr}(u_1) \cup \{\langle 1, n\rangle\} \subseteq \text{hlr}(\lambda_n\bar{x}.w\bar{x}) \cup \text{hlr}(u_1) \cup \{\langle 1, n\rangle\} =^{F.2} \text{hlr}(t)$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_1}$. So $u' = (\lambda_n\bar{x}.wx)u'_1$ such that $u_1 \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(u_1)$. So, $\text{hlr}(u') =^{F.2} \text{hlr}(\lambda_n\bar{x}.w\bar{x}) \cup \text{hlr}(u'_1) \cup \{\langle 1, n\rangle\} \subseteq \text{hlr}(\lambda_n\bar{x}.w\bar{x}) \cup \text{hlr}(u_1) \cup \{\langle 1, n\rangle\} =^{F.2} \text{hlr}(u)$.
- Let $u = (\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})])u_2$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 5.3:
  - Either $p = 0$. So $u' = u_1[\bar{x} := c(cu_2)]$. By lemma F.3, $\text{hlr}(u') \subseteq \text{hlr}(u)$.
  - Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{\lambda_n\bar{x}.u_1[\bar{x}:=c(c\bar{x})]}$. So $u' = u'_1u_2$ such that $\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})] \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})])$. By lemma 2.2.8, lemma 5.4.3, lemma 5.4.4 and lemma 5.2.13a, $p' = 1.p''$, $u'_1 = \lambda_n\bar{x}.u''_1[\bar{x} := c(c\bar{x})]$ and $u_1 \xrightarrow{p''}_{\beta\eta} u''_1$. Then, $\text{hlr}(u') =^{F.2} \text{hlr}(u'_1) \cup \text{hlr}(u_2) \cup \{\langle 1, n\rangle\} \subseteq \text{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]) \cup \text{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{F.2} \text{hlr}(u)$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_2}$. So $u' = (\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})])u'_2$ such that $u_2 \xrightarrow{p'}_{\beta\eta} u'_2$. By IH, $\text{hlr}(u'_2) \subseteq \text{hlr}(u_2)$. So, $\text{hlr}(u') =^{F.2} \text{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]) \cup \text{hlr}(u'_2) \cup \{\langle 1, n\rangle\} \subseteq \text{hlr}(\lambda_n\bar{x}.u_1[\bar{x} := c(c\bar{x})]) \cup \text{hlr}(u_2) \cup \{\langle 1, n\rangle\} =^{F.2} \text{hlr}(u)$.
- Let $u = cu_1u_2$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 5.3:
  - Either $p = 1.2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_1}$. So $u' = cu'_1u_2$ such that $u_1 \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(u_1)$. So, $\text{hlr}(u') =^{F.2} \text{hlr}(u'_1) \cup \text{hlr}(u_2) \subseteq \text{hlr}(u_1) \cup \text{hlr}(u_2) =^{F.2} \text{hlr}(u)$.
  - Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_2}$. So $u' = cu_1u'_2$ such that $u_2 \xrightarrow{p'}_{\beta\eta} u'_2$. By IH, $\text{hlr}(u'_2) \subseteq \text{hlr}(u_2)$. So, $\text{hlr}(u') =^{F.2} \text{hlr}(u_1) \cup \text{hlr}(u'_2) \subseteq \text{hlr}(u_1) \cup \text{hlr}(u_2) =^{F.2} \text{hlr}(u)$.
- Let $u = cu_1$. Because $u \xrightarrow{p}_{\beta\eta} u'$, by lemma 2.2.8 and lemma 5.3 $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{u_1}$. So $u' = cu'_1$ such that $u_1 \xrightarrow{p'}_{\beta\eta} u'_1$. By IH, $\text{hlr}(u'_1) \subseteq \text{hlr}(u_1)$. So, $\text{hlr}(u') =^{F.2} \text{hlr}(u'_1) \subseteq \text{hlr}(u_1) =^{F.2} \text{hlr}(u)$.

□

Proof(Lemma 8.6.1)**:** Note that $\Psi^c(M,\mathcal{F}) \neq \varnothing$. Then, it is sufficient to prove:

- $\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle \Rightarrow \forall N \in \Psi^c(M,\mathcal{F}).\ \exists N' \in \Psi^c(M',\mathcal{F}').\ N \rightarrow^*_{\beta\eta} N'$ by induction on the reduction $\langle M,\mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle$.
  - If $\langle M,\mathcal{F}\rangle = \langle M',\mathcal{F}'\rangle$ then it is done.
  - Let $\langle M,\mathcal{F}\rangle \rightarrow_{\beta\eta d} \langle M'',\mathcal{F}''\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle$. By IH: $\forall N'' \in \Psi^c(M'',\mathcal{F}'').\ \exists N' \in \Psi^c(M',\mathcal{F}').\ N \rightarrow^*_{\beta\eta} N''$. By definition 8.3.2, there exist $p \in \mathcal{F}$ such that $M \xrightarrow{p}_{\beta\eta} M''$ and $\mathcal{F}''$ is the set of $\beta\eta$-residuals in M'' of the set of redexes $\mathcal{F}$ in M relative to p. By definition 1 we obtain: $\forall N \in \Psi^c(M,\mathcal{F}).\ \exists N'' \in \Psi^c(M'',\mathcal{F}'').\ N \rightarrow_{\beta\eta} N''$.
- $\exists N \in \Psi^c(M,\mathcal{F}).\ \exists N' \in \Psi^c(M',\mathcal{F}').\ N \rightarrow^*_{\beta\eta} N' \Rightarrow \langle M,\mathcal{F}\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle$ by induction on the reduction $N \rightarrow^*_{\beta\eta} N'$ such that $N \in \Psi^c(M,\mathcal{F})$ and $N' \in \Psi^c(M',\mathcal{F}')$.
  - If $N = N'$ then by lemma 8.2.2b, $M = M'$ and $\mathcal{F} = \mathcal{F}'$.
  - Let $N \rightarrow_{\beta\eta} N'' \rightarrow^*_{\beta\eta} N'$. By lemma 8.2.1c, $N \in \Lambda\eta_c$, so by lemma 2, $N'' \in \Lambda\eta_c$. By lemma 8.2.2b, $\langle |N''|^c, |\langle N'',\mathcal{R}^{\beta\eta}_{N''}\rangle|^c\rangle$ is the one and only pair such that $c \notin FV(|N''|^c)$, $|\langle N'',\mathcal{R}^{\beta\eta}_{N''}\rangle|^c \subseteq \mathcal{R}^{\beta\eta}_{|N''|^c}$ and $N'' \in \Psi^c(|N''|^c, |\langle N'',\mathcal{R}^{\beta\eta}_{N''}\rangle|^c)$.
    So by IH, $\langle |N''|^c, |\langle N'',\mathcal{R}^{\beta\eta}_{N''}\rangle|^c\rangle \rightarrow^*_{\beta\eta d} \langle M',\mathcal{F}'\rangle$. By definition, there exists p such that $N \xrightarrow{p}_{\beta\eta} N''$ and by lemma 2.2.8, $p \in \mathcal{R}^{\beta\eta}_N$. By lemmas 5.8.7a and lemma 8.2.1g, $M = |N|^c \xrightarrow{p_0}_{\beta\eta} |N''|^c$ such that $|\langle N,p\rangle|^c = p_0$. So by lemma 2.2.8, $p_0 \in \mathcal{R}^{\beta\eta}_M$. By definition 1, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{|N''|^c}$, such that for all $P \in \Psi^c(M,\mathcal{F})$, there exist $P' \in \Psi^c(|N''|^c,\mathcal{F}')$ and $p'_0 \in \mathcal{R}^{\beta\eta}_P$ such that $P \xrightarrow{p'_0}_{\beta\eta} P'$ and $|\langle P,p'_0\rangle|^c = p_0 = |\langle N,p\rangle|^c$. Moreover, $\mathcal{F}'$ is called the set of $\beta\eta$-residuals in $|N''|^c$ of the set of redexes $\mathcal{F}$ in M relative to $|\langle N,p\rangle|^c$. Since $N \in \Psi^c(M,\mathcal{F})$, there exist $P' \in \Psi^c(|N''|^c,\mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_N$ such that $N \xrightarrow{p'}_{\beta\eta} P'$ and $|\langle N,p'\rangle|^c = |\langle N,p\rangle|^c$. By lemma 1, $p = p'$, so by lemma 2.2.9, $P' = N''$. Since $N'' \in \Psi^c(|N''|^c,\mathcal{F}')$, by lemma 8.2.2b, $\mathcal{F}' = |\langle N'',\mathcal{R}^{\beta\eta}_{N''}\rangle|^c$. Finally, by definition 8.3.2, $\langle M,\mathcal{F}\rangle \rightarrow_{\beta\eta d} \langle |N''|^c, |\langle N'',\mathcal{R}^{\beta\eta}_{N''}\rangle|^c\rangle$.

□

Proof(Lemma 8.6.2)**:** By lemma 8.2.1c, $\Psi^c(M,\mathcal{F}_1), \Psi^c(M,\mathcal{F}_2) \subseteq \Lambda\eta_c$. For all $N_1 \in \Psi^c(M,\mathcal{F}_1)$ and $N_2 \in \Psi^c(M,\mathcal{F}_2)$, by lemma 8.2.1g, $|N_1|^c = |N_2|^c$ and by lemma 8.2.1h, $|\langle N_1,\mathcal{R}^{\beta\eta}_{N_1}\rangle|^c = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\langle N_2,\mathcal{R}^{\beta\eta}_{N_2}\rangle|^c$.

If $\langle M,\mathcal{F}_1\rangle \rightarrow_{\beta\eta d} \langle M',\mathcal{F}'_1\rangle$ then by lemma 8.6.1, there exist $N_1 \in \Psi^c(M,\mathcal{F}_1)$ and $N'_1 \in \Psi^c(M',\mathcal{F}'_1)$ such that $N_1 \rightarrow_{\beta\eta} N'_1$. By definition, there exists p_1 such that $N_1 \xrightarrow{p_1}_{\beta\eta} N'_1$, and by lemma 2.2.8, $p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Let $p_0 = |\langle N_1,p_1\rangle|^c$, so by lemma 8.2.1h, $p_0 \in \mathcal{F}_1$. By lemma 5.8.7a and lemma 8.2.1g, $M \xrightarrow{p_0}_{\beta\eta} M'$.

By lemma 8.2.3 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$ such that for all $P_1 \in \Psi^c(M,\mathcal{F}_1)$ there exist $P'_1 \in \Psi^c(M',\mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_{P_1}$ such that $P_1 \xrightarrow{p'}_{\beta\eta} P'_1$ and $|\langle P_1,p'\rangle|^c = p_0$.

Because, $N_1 \in \Psi^c(M, \mathcal{F}_1)$, there exist $P'_1 \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}^{\beta\eta}_{N_1}$ such that $N_1 \xrightarrow{p'}_{\beta\eta} P'_1$ and $|\langle N_1, p'\rangle|^c = p_0$. Since $p', p_1 \in \mathcal{R}^{\beta\eta}_{N_1}$, by lemma 1, $p' = p_1$, so by lemma 2.2.9, $P'_1 = N'_1$. By lemma 8.2.1h, $\mathcal{F}' = |\langle N'_1, \mathcal{R}^{\beta\eta}_{N'_1}\rangle|^c = \mathcal{F}'_1$.

By lemma 8.2.3 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $P_2 \in \Psi^c(M, \mathcal{F}_2)$ there exist $P'_2 \in \Psi^c(M', \mathcal{F}'_2)$ and $p_2 \in \mathcal{R}^{\beta\eta}_{P_2}$ such that $P_2 \xrightarrow{p_2}_{\beta\eta} P'_2$ and $|\langle P_2, p_2\rangle|^c = p_0$.

Since $\Psi^c(M, \mathcal{F}_2) \neq \varnothing$, let $N_2 \in \Psi^c(M, \mathcal{F}_2)$. So, there exist $N'_2 \in \Psi^c(M', \mathcal{F}'_2)$ and $p_2 \in \mathcal{R}^{\beta\eta}_{N_2}$ such that $N_2 \xrightarrow{p_2}_{\beta\eta} N'_2$ and $|\langle N_2, p_2\rangle|^c = p_0$. By lemma 8.2.1h, $\mathcal{F}'_2 = |\langle N'_2, \mathcal{R}^{\beta\eta}_{N'_2}\rangle|^c$.

Hence, by lemma 5.8.7c, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 8.6.1, $\langle M, \mathcal{F}_2\rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}'_2\rangle$. □

Proof(Lemma 8.7)**:** If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exist $\mathcal{F}''_1, \mathcal{F}''_2$ such that $\langle M, \mathcal{F}_1\rangle \rightarrow^*_{\beta\eta d} \langle M_1, \mathcal{F}''_1\rangle$ and $\langle M, \mathcal{F}_2\rangle \rightarrow^*_{\beta\eta d} \langle M_2, \mathcal{F}''_2\rangle$. By definitions 8.3.1 and 8.3.2, $\mathcal{F}''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$. By lemma 8.6.2, there exist $\mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ such that $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta\eta d} \langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle$ and $\langle M, \mathcal{F}_1 \cup \mathcal{F}_2\rangle \rightarrow^*_{\beta\eta d} \langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle$. By lemma 8.6.1 there exist $T \in \Psi^c(M, \mathcal{F}_1 \cup \mathcal{F}_2)$, $T_1 \in \Psi^c(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $T_2 \in \Psi^c(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$ such that $T \rightarrow^*_{\beta\eta} T_1$ and $T \rightarrow^*_{\beta\eta} T_2$.

Because by lemma 8.2.1c, $T \in \Lambda\eta_c$ and by lemma 6.6.2, T is typable in the type system $\mathcal{D}$, so $T \in \mathsf{CR}^{\beta\eta}$ by corollary 6.5. So, by lemma 2.2a, there exists $T_3 \in \Lambda\eta_c$, such that $T_1 \rightarrow^*_{\beta\eta} T_3$ and $T_2 \rightarrow^*_{\beta\eta} T_3$. Let $\mathcal{F}_3 = |\langle T_3, \mathcal{R}^{\beta\eta}_{T_3}\rangle|^c$ and $M_3 = |T_3|^{\beta\eta}$, then by lemma 8.2.2a, $\mathcal{F}_3 \subseteq \mathcal{R}^{\beta\eta}_{M_3}$ and $T_3 \in \Psi^c(M_3, \mathcal{F}_3)$. Hence, by lemma 8.6.1, $\langle M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1\rangle \rightarrow^*_{\beta\eta d} \langle M_3, \mathcal{F}_3\rangle$ and $\langle M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2\rangle \rightarrow^*_{\beta\eta d} \langle M_3, \mathcal{F}_3\rangle$, i.e. $M_1 \xrightarrow{\mathcal{F}''_1 \cup \mathcal{F}'''_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}''_2 \cup \mathcal{F}'''_2}_{\beta\eta d} M_3$. □

Proof(Lemma 8.9.1)**:** Note that $\varnothing \subseteq \mathcal{R}^{\beta\eta}_M$. We prove this statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $\Psi^c(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$ and $\mathcal{R}^{\beta\eta}_{c^n(M)} = \varnothing$, where $n \geq 0$, by lemma 5.3 and lemma 5.4.5.
- Let $M = \lambda x.N$ such that $x \neq c$ then $\Psi^c(M, \varnothing) = \{c^n(\lambda x.Q[x := c(cx)]) \mid n \geq 0 \wedge Q \in \Psi^c(N, \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$, then $P = c^n(\lambda x.Q[x := c(cx)])$ such that $n \geq 0$ and $Q \in \Psi^c(N, \varnothing)$ By IH, $\mathcal{R}^{\beta\eta}_Q = \varnothing$ and by lemma 5.4.4, lemma 5.4.3 and lemma 5.4.5, $\mathcal{R}^{\beta\eta}_P = \varnothing$.
- Let $M = M_1M_2$ then $\Psi^c(M, \varnothing) = \{c^n(cQ_1Q_2) \mid n \geq 0 \wedge Q_1 \in \Psi^c(M_1, \varnothing) \wedge Q_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$, then $P = c^n(cQ_1Q_2)$ such that $n \geq 0$, $Q_1 \in \Psi^c(M_1, \varnothing)$ and $Q_2 \in \Psi^c(M_2, \varnothing)$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$ and by lemma 5.3 and lemma 5.4.5, $\mathcal{R}^{\beta\eta}_P = \varnothing$.

□

Proof(Lemma 8.9.2)**:** We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$, then $\Psi^c(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$. Let $P \in \Psi^c(M, \varnothing)$ and $Q \in \Psi^c(N, \varnothing)$, then $P = c^n(M)$ where $n \geq 0$.
  - Either $M = x$, then $P[x := Q] = c^n(Q)$ and by lemma 8.2.1f and lemma 1, $\mathcal{R}^{\beta\eta}_{c^n(Q)} = \varnothing$.
  - Or $M \neq x$, then $P[x := Q] = P$ and by lemma 1, $\mathcal{R}^{\beta\eta}_P = \varnothing$.

- Let $M = \lambda y.M'$ such that $y \neq c$ then $\Psi^c(M, \varnothing) = \{c^n(\lambda y.P'[y := c(cy)]) \mid n \geq 0 \wedge P' \in \Psi^c(M', \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$ and $Q \in \Psi^c(N, \varnothing)$, then $P = c^n(\lambda y.P'[y := c(cy)])$ where $n \geq 0$ and $P' \in \Psi^c(M', \varnothing)$. So, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \mathcal{R}^{\beta\eta}_{c^n(\lambda y.P'[x:=Q][y:=c(cy)])}$, such that $y \notin \text{fv}(Q) \cup \{x\}$. By IH, $\mathcal{R}^{\beta\eta}_{P'[x:=Q]} = \varnothing$ and by lemmas 5.4.4, 5.4.3 and 5.4.5, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$.

- Let $M = M_1M_2$ then $\Psi^c(M, \varnothing) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \varnothing) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \varnothing)$ and $Q \in \Psi^c(N, \varnothing)$ then $P = c^n(cP_1P_2)$ where $n \geq 0$, $P_1 \in \Psi^c(M_1, \varnothing)$ and $P_2 \in \Psi^c(M_2, \varnothing)$. So, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \mathcal{R}^{\beta\eta}_{c^n(cP_1[x:=Q]P_2[x:=Q])}$. By IH, $\mathcal{R}^{\beta\eta}_{P_1[x:=Q]} = \mathcal{R}^{\beta\eta}_{P_2[x:=Q]} = \varnothing$ and by lemmas 5.3 and 5.4.5, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$.

$\square$

Proof(Lemma 8.9.3)**:** We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then nothing to prove since by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \varnothing$.

- Let $M = \lambda x.N$ such that $x \neq c$.

  – If $M \in \mathcal{R}^{\beta\eta}$ then $N = N_0x$ such that $x \notin FV(N_0)$ and by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_N\}$. Let $p \in \mathcal{R}^{\beta\eta}_M$ then:

    ∗ Either $p = 0$, then $\Psi^c(M, \{p\}) = \{c^n(\lambda x.P') \mid n \geq 0 \wedge P' \in \Psi^c_0(N, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(\lambda x.P')$ such that $n \geq 0$ and $P' \in \Psi^c_0(N, \varnothing)$. So $P' = cP'_0x$ such that $P'_0 \in \Psi^c(N_0, \varnothing)$. By lemmas 1 and 8.2.1a, $\mathcal{R}^{\beta\eta}_{P'} = \varnothing$. If $P \rightarrow_{\beta\eta} Q$ then by definition, there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b and lemma 2.2.8, $Q = c^n(Q')$, $p_0 = 2^n.p'_0$ and $\lambda x.P' \xrightarrow{p'_0}_{\beta\eta} Q'$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{\lambda x.P'}$. By lemma 8.2.1b, $x \notin \text{fv}(cP'_0)$. By lemmas 5.3, $\mathcal{R}^{\beta\eta}_{\lambda x.P'} = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{P'}\} = \{0\}$. So $p'_0 = 0$ and $Q' = cP'_0$. By lemma 1, $\mathcal{R}^{\beta\eta}_{P'_0} = \varnothing$ and by lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

    ∗ Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. So $\Psi^c(M, \{p\}) = \{c^n(\lambda x.P'[x := c(cx)]) \mid n \geq 0 \wedge P' \in \Psi^c(N, \{p'\})\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(\lambda x.P'[x := c(cx)])$ such that $n \geq 0$ and $P' \in \Psi^c(N, \{p'\})$. If $P \rightarrow_{\beta\eta} Q$ then there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b, lemma 2.2.8, lemma 5.4.3 and lemma 5.2.13a, $p_0 = 2^n.1.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{P'}$ and $Q = c^n(\lambda x.Q'[x := c(cx)])$ such that $P' \xrightarrow{p'_0}_{\beta\eta} Q'$. By IH, $\mathcal{R}^{\beta\eta}_{Q'} = \varnothing$, so by lemma 5.4.4, lemma 5.4.3 and lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

  – Else, by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_N\}$. Let $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_N$. So $\Psi^c(M, \{p\}) = \{c^n(\lambda x.P'[x := c(cx)]) \mid n \geq 0 \wedge P' \in \Psi^c(N, \{p'\})\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(\lambda x.P'[x := c(cx)])$ such that $n \geq 0$ and $P' \in \Psi^c(N, \{p'\})$. If $P \rightarrow_{\beta\eta} Q$ then there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b, lemma 2.2.8, lemma 5.4.3 and lemma 5.2.13a, $p_0 = 2^n.1.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{P'}$ and $Q = c^n(\lambda x.Q'[x := c(cx)])$ such that $P' \xrightarrow{p'_0}_{\beta\eta} Q'$. By IH, $\mathcal{R}^{\beta\eta}_{Q'} = \varnothing$, so by lemma 5.4.4, lemma 5.4.3 and lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

- Let $M = M_1M_2$.

– Let $M \in \mathcal{R}^{\beta\eta}$, then $M_1 = \lambda x.M_0$ such that $x \neq c$ and by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{0\} \cup \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2}\}$. Let $p \in \mathcal{R}^{\beta\eta}_M$ then:

  ∗ Either $p = 0$ then $\Psi^c(M, \{p\}) = \{c^n(P_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c_0(M_1, \varnothing) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(P_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c_0(M_1, \varnothing)$ and $P_2 \in \Psi^c(M_2, \varnothing)$. By lemma 1 and lemma 8.2.1a, $\mathcal{R}^{\beta\eta}_{P_1} = \mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. Since $P_1 \in \Psi^c_0(M_1, \varnothing)$, $P_1 = \lambda x.P_0[x := c(cx)]$ such that $P_0 \in \Psi^c(M_0, \varnothing)$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b and lemma 2.2.8, $Q = c^n(Q')$, $p_0 = 2^n.p'_0$ and $P_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{P_1P_2}$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_{P_1P_2} = \{0\}$. So $p'_0 = 0$ and $Q = c^n(P_0[x := c(cP_2)])$. Because $c(cP_2) \in \Psi^c(M_2, \varnothing)$, by lemma 2 and lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

  ∗ Or $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{p'\}) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{p'\})$ and $P_2 \in \Psi^c(M_2, \varnothing)$. By lemma 1, $\mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{1.2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\}$. So $p'_0 = 1.2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_1}$. So $Q' = cQ_1P_2$ and $P_1 \xrightarrow{p''_0}_{\beta\eta} Q_1$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \varnothing$, so by lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

  ∗ Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{\varnothing\}) \wedge P_2 \in \Psi^c(M_2, p')\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{\varnothing\})$ and $P_2 \in \Psi^c(M_2, p')$. By lemma 1, $\mathcal{R}^{\beta\eta}_{P_1} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\}$. So $p'_0 = 2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_2}$. So $Q' = cP_1Q_2$ and $P_2 \xrightarrow{p''_0}_{\beta\eta} Q_2$. By IH, $\mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$, so by lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

– Let $M \notin \mathcal{R}^{\beta\eta}$, then by lemma 5.3, $\mathcal{R}^{\beta\eta}_M = \{1.p \mid p \in \mathcal{R}^{\beta\eta}_{M_1}\} \cup \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{M_2}\}$.

  ∗ Either $p = 1.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_1}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{p'\}) \wedge P_2 \in \Psi^c(M_2, \varnothing)\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{p'\})$ and $P_2 \in \Psi^c(M_2, \varnothing)$. By lemma 1, $\mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{1.2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_1}\}$. So $p'_0 = 1.2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_1}$. So $Q' = cQ_1P_2$ and $P_1 \xrightarrow{p''_0}_{\beta\eta} Q_1$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \varnothing$, so by lemma 5.4.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

  ∗ Or $p = 2.p'$ such that $p' \in \mathcal{R}^{\beta\eta}_{M_2}$. So, $\Psi^c(M, \{p\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Psi^c(M_1, \{\varnothing\}) \wedge P_2 \in \Psi^c(M_2, p')\}$. Let $P \in \Psi^c(M, \{p\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Psi^c(M_1, \{\varnothing\})$ and $P_2 \in \Psi^c(M_2, p')$. By lemma 1, $\mathcal{R}^{\beta\eta}_{P_1} = \varnothing$. If $P \to_{\beta\eta} Q$ then by definition there exists p_0 such that $P \xrightarrow{p_0}_{\beta\eta} Q$. By lemma 5.2.13b and lemma 2.2.8, $p_0 = 2^n.p'_0$ such that $p'_0 \in \mathcal{R}^{\beta\eta}_{cP_1P_2}$ and $Q = c^n(Q')$ such that $cP_1P_2 \xrightarrow{p'_0}_{\beta\eta} Q'$. By lemma 5.3, $\mathcal{R}^{\beta\eta}_{cP_1P_2} = \{2.p \mid p \in \mathcal{R}^{\beta\eta}_{P_2}\}$. So $p'_0 = 2.p''_0$ such that $p''_0 \in \mathcal{R}^{\beta\eta}_{P_2}$. So $Q' = cP_1Q_2$ and $P_2 \xrightarrow{p''_0}_{\beta\eta} Q_2$. By IH, $\mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$, so by lemma 5.4.5, $\mathcal{R}^{\beta\eta}_{Q} = \varnothing$.

□

Proof(Lemma 8.9.4)**:** By lemma 2.2.8, $p \in \mathcal{R}^{\beta\eta}_{M}$. By lemma 8.2.3, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that for all $N \in \Psi^c(M, \{p\})$, there exists $N' \in \Psi^c(M', \mathcal{F}')$ such that $N \to_{\beta\eta} N'$. Note that $\Psi^c(M, \{p\}) \neq \varnothing$. Let $N \in \Psi^c(M, \{p\})$ then there exists $N' \in \Psi^c(M', \mathcal{F}')$ such that $N \to_{\beta\eta} N'$. By lemma 3, $\mathcal{R}^{\beta\eta}_{N'} = \varnothing$, so $|\langle N', \mathcal{R}^{\beta\eta}_{N'}\rangle|^c = \varnothing$ and by lemma 8.2.1h, $\mathcal{F}' = \varnothing$. Finally, by lemma 8.6.1, $\langle M, \{p\}\rangle \to_{\beta\eta d} \langle M', \varnothing\rangle$. □

Proof(Lemma 8.9.5)**:** By definition $\to^*_1 \subseteq \to^*_{\beta\eta}$. We prove that $\to^*_{\beta\eta} \subseteq \to^*_1$. Let $M, M' \in \Lambda$ such that $c \notin \mathrm{fv}(M)$ and $M \to^*_{\beta\eta} M'$. We prove this claim by induction on $M \to^*_{\beta\eta} M'$.

- Let $M = M'$ then it is done since $\langle M, \mathcal{F}\rangle \to^*_{\beta\eta d} \langle M, \mathcal{F}\rangle$.
- Let $M \to^*_{\beta\eta} M'' \to_{\beta\eta} M'$. By IH, $M \to^*_1 M''$. By definition there exists p such that $M'' \xrightarrow{p}_{\beta\eta} M'$. By lemma 2.2.3, $c \notin \mathrm{fv}(M'')$. By lemma 4, $\langle M'', \{p\}\rangle \to_{\beta\eta d} \langle M', \varnothing\rangle$, so $M'' \to_1 M'$. Hence $M \to^*_1 M'' \to_1 M'$.

□

Proof(Lemma 8.10)**:** Let $M \in \Lambda$ and let $c \in \mathcal{V}$ such that $c \notin \mathrm{fv}(M)$. Let $M \to^*_{\beta\eta} M_1$ and $M \to^*_{\beta\eta} M_2$. Then by lemma 5, $M \to^*_1 M_1$ and $M \to^*_1 M_2$. We prove the statement by induction on $M \to^*_1 M_1$.

- Let $M = M_1$. Hence $M_1 \to^*_1 M_2$ and $M_2 \to^*_1 M_2$.
- Let $M \to^*_1 M'_1 \to_1 M_1$. By IH, $\exists M'_3, M'_1 \to^*_1 M'_3$ and $M_2 \to^*_1 M'_3$. We prove that $\exists M_3, M_1 \to^*_1 M_3$ and $M'_3 \to_1 M_3$, by induction on $M'_1 \to^*_1 M'_3$.
  - let $M'_1 = M'_3$, hence $M'_3 \to_1 M_1$ and $M_1 \to^*_1 M_1$.
  - Let $M'_1 \to^*_1 M''_3 \to_1 M'_3$. By IH, $\exists M'''_3, M_1 \to^*_1 M'''_3$ and $M''_3 \to_1 M'''_3$. By lemma 2.2.3, $c \notin \mathrm{fv}(M''_3)$. Since $M''_3 \to_1 M'_3$ and $M''_3 \to_1 M'''_3$, By lemma 8.7, $\exists M_3, M'_3 \to_1 M_3$ and $M'''_3 \to_1 M_3$.

□