Explicit substitutions calculi with de Bruijn indices and intersection type systems

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Abstract

Explicit substitution calculi propose solutions to the main drawback of the λ-calculus: substitution defined as a meta-operation in the system. By making explicit the process of substitution, the theoretical system gets closer to an eventual implementation. Furthermore, for implementation purposes, many explicit substitution systems are written with de Bruijn indices. The λ-calculus with de Bruijn indices, called λ_{dB}, assembles each α-class of λ-terms in a unique term, which is more *"machine-friendly"* than the classical version with variables. Intersection types (IT) provide finitary type polymorphism satisfying important properties like principal typing (PT), which allows the type system to include features such as data abstraction (modularity) and separate compilation. Although some explicit substitution calculi with simple type systems are well investigated, providing nice applications such as specialised implementations of higher order unification, more elaborated type systems such as IT have not been proposed/studied for these calculi. In an earlier work, we introduced IT systems for two explicit substitution calculi, $\lambda\sigma$ and λs_e, conjecturing them to satisfy the basic property of subject reduction, which guarantees the preservation of types during computations. In this paper, we take a deeper look at these systems, providing an insight into their development which helps us construct for the first time the proofs of subject reduction omitted before. This new result also 1) enables us to prove another new result: subject reduction for an IT system for λ_{dB}, and 2) allows us to introduce for the first time an IT system for the $\lambda\upsilon$-calculus.

Keywords: intersection types, lambda calculus, explicit substitution, de Bruijn indices

1 Introduction

In the λ-calculus [8], β-contraction is defined with an implicit notion of substitution. Explicit substitution calculi are extensions of the λ-calculus which include the specification of how the substitution process is to be performed, breaking down the whole process in minor steps. The λ-calculus à la de Bruijn [12], λ_{dB} for short, was invented by the Dutch mathematician N.G. de Bruijn in the context of the project Automath [45] and de Bruijn presented in [13] the first explicit substitution calculus, called

$C\lambda\xi\phi$, based on λ_{dB}. Term variables are represented by indices instead of names in λ_{dB}, assembling each α-class of terms in the λ-calculus into a unique term with de Bruijn indices, thus turning it into more *"machine-friendly"* than its counterparts. De Bruijn indices have been adopted for several calculi of explicit substitutions ever since, e.g. [1, 10, 31]. The $\lambda\sigma$- [1] and the λs_e-calculus [31] have different approaches for dealing with substitutions and both have applications of the respective simply typed versions in higher order unification, HOU for short [24, 3]. Although simply typed versions of $\lambda\sigma$ and λs_e have been studied in detail, neither calculus has been extended with an intersection type system before our work. This paper studies the type systems introduced in [58], which to the best of our knowledge are the first systems with intersection types proposed for both $\lambda\sigma$ and λs_e. Proofs of subject reduction for both intersection type systems are given here for the first time. We also establish subject reduction for an intersection type system for λ_{dB} and introduce for the first time such a system for the $\lambda\upsilon$-calculus [10].

Intersection types, IT for short, were introduced as an extension to simple types, in order to provide a characterisation of strongly normalising λ-terms [16, 17, 46]. In programming, the IT discipline is of interest because λ-terms corresponding to correct programs not typeable in the standard Curry type assignment system [20], or in some polymorphic extensions as the Hindley/Milner type system[1] [43], HM for short, are typeable with IT. For instance, a strongly normalising λ-term not typeable in system F_ω [55] and typeable with IT is presented in [14]. Moreover, some IT systems satisfy the principal typing property, PT for short, meaning that for any typeable term M there is a type judgement $\Gamma \vdash M : \tau$ representing all possible typings $\langle \Gamma' \vdash \tau' \rangle$ of M in the corresponding type system. T. Jim discussed in [28] the importance of this property in computational type systems, in providing support to features such as *separate compilation*, including *smartest recompilation* [53, 2], and *recursive definitions* [27]. In [59] J. Wells proved that HM does not have the PT property. Principal typings have been studied for some IT systems [18, 48, 49, 5, 35] and it was shown in [18, 48] that for a term M, the principal typing of M's β-normal form, β-nf for short, is principal for M itself.

IT in the typing systems presented here are non-idempotent, i.e. $\sigma \wedge \sigma \neq \sigma$. D. de Carvalho established in [15] a relation between the size of a typing derivation in a non-idempotent IT system for the λ-calculus and the head/weak-normalisation execution time of head/weak-normalising λ-terms, respectively, through abstract machines. Resource-aware semantics rising from such systems have been explored [11, 22], in order to prove normalisation properties about typeable terms by such a combinatorial argument, i.e. *counting things*, instead of the usual *reducibility argument* [38, 29, 6][2]. Non-idempotent IT are represented by multiset of types in [15] as the idempotent ones are represented by set of types in [17] and, in [41, 42], is pointed out that multisets are the proper abstraction regarding the relevant implication. In fact, in [23] a tight relation is showed between the minimal positive relevant logic B_+ and the intersection types discipline, even though an idempotent IT is presented. For this reason the *rel-*

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[1] Hindley/Milner is the typing system present in the SML [44].

[2] The *reducibility argument* in order to prove termination of typed terms in [54], then called convertibility, was extended by Girard to the notion of reducible candidates to prove strong normalisation for the System F [25]. Krivine presented in [37] (the original French version is from 1990) a more general argument, sometimes referenced as realisability, to prove a variety of termination properties, such as head-/weak-normalisation, beside strong normalisation. The technique in [37] is also called stable sets (cf. [9]).

evance in the sense of [21], a property allowing to obtain a non-trivial typing system "as restricted as possible", should be considered while developing a non-idempotent IT system. Contexts in relevant type systems have only types that are needed while inferring a typing for some term thus no weakening rule is admissible[3]. For instance, in a non-relevant IT system both $x : \alpha \vdash \lambda y.x : \beta \to \alpha$ and $\emptyset \vdash \lambda x.x : (\alpha \wedge \beta) \to \alpha$ are derivable while in relevant systems only $x : \alpha \vdash \lambda y.x : \omega \to \alpha$ and $\emptyset \vdash \lambda x.x : \alpha \to \alpha$ can be derived, where ω is the universal type of [19] (*cf.* [6]). Relevance is also explored in the study of PT for IT systems [18] and in a functional characterisation of terms in the λ-calculus [19].

Previous work

In [56] we introduced an IT system for λ_{dB}, based on the type system given in [29] which characterises termination in the λ-calculus, and proved it to satisfy the subject reduction property, SR for short. SR states preservation of typings under β-reduction: whenever $\Gamma \vdash M : \sigma$ and M β-reduces into N, then $\Gamma \vdash N : \sigma$. However, the system in [56] is not *relevant* in the sense of [21], a property of the system in [29], due to the interaction between sequential type contexts and the subtyping relation[4]. Hence, to avoid this drawback, in [57] we introduced a relevant IT system λ_{dB}^{SM} for λ_{dB}. The system in [57] is a de Bruijn version of the system originally introduced in [50] to characterise the syntactic structure of PT for β-nfs in the λ-calculus. We also established a characterisation of PT for β-nfs in λ_{dB}. In [58] we introduced the IT systems for $\lambda\sigma$ and λs_e, based on the IT system $\lambda_{dB}^{\wedge}$ thus called $\lambda\sigma^{\wedge}$ and $\lambda {s_e}^{\wedge}$, respectively. The system $\lambda_{dB}^{\wedge}$ is a de Bruijn version of the system in [52] and was presented as a variation of λ_{dB}^{SM}, in a discussion about the SR property. We focused on SR and relevance properties, in order to obtain a proper non-idempotent IT system. Furthermore, an IT system λs^{SM} was proposed for λs, based on λ_{dB}^{SM}, as an intermediate step towards the IT system for λs_e.

Present results

In this paper, IT systems for λs and λs_e are presented with proof-sketches of SR, omitted in [58]. Besides, different from [58], the IT systems for λ_{dB} are presented here as restrictions of those systems to pure terms, i.e. terms without pending substitutions, thus deriving their properties from the systems for λs and λs_e. Although $\lambda_{dB}^{\wedge}$ was already introduced in [58], the SR property is proved here for the first time, derived from the system $\lambda {s_e}^{\wedge}$. The build up to how the IT system for $\lambda\sigma$ was obtained is also given and the proof for its SR, omitted in [58], is included here for the first time. Furthermore, $\lambda\upsilon^{\wedge}$ is introduced, an IT system for $\lambda\upsilon$ [10] that is proved to satisfy SR. Below, we describe the work done in each section:

- In Section 2 the untyped versions of the λ_{dB}, λs, λs_e, $\lambda\sigma$ and $\lambda\upsilon$ calculi are presented. The notion of *available indices* for λs is presented and proved to be an extension of the *free indices* concept introduced in [32][5] for λ_{dB}. Those are the counterparts of *available* [38] and *free variables* notions for the λ-calculus with names. They play an essential role in the discussion of relevance for the systems

[3] An admissible rule in a formal system is a rule that can be derived from the inference rules in the system.

[4] The subtyping relation in [29] is crucial in obtaining a complete IT system with respect to realisability semantics but also guarantees SR on both systems (see Remark 4.14 for further discussion about the issue).

[5] Under the name of free variables.

presented here and the definition of an appropriate notion of SR for relevant type systems.

- In Section 3, the set of intersection types used by all IT systems in the present work, and the sequential contexts required in those systems, are presented.
- The following two sections have basically the same structure, consisting of two subsections, in each of which an IT system and its properties are presented. Hence:
  - In Section 4, the IT system λs^{SM} is presented in Subsection 4.1 while in Subsection 4.2 the IT system $\lambda {s_e}^{\wedge}$ is presented.
  - In Section 5, the system λ_{dB}^{SM} is presented in Subsection 5.1 and in Subsection 5.2 the system $\lambda_{dB}^{\wedge}$.
- In Section 6, the process of obtaining the IT system $\lambda\sigma^{\wedge}$ is presented in Subsection 6.1 while in Subsection 6.2 its properties, and respective proofs, are given.
- Finally, Section 7 gives the system $\lambda\upsilon^{\wedge}$ and its properties.
- Conclusion and future work are presented in Section 8.

Related work
In [38] an IT system is presented for λx, an ES calculus without composition, and in [33] an IT system is presented for λex, the ES calculus with safe composition which preserves strong normalisation, PSN for short. Each IT system proposed characterises strong normalisation in the corresponding calculus. However, both calculi are defined with named variables while in the present work calculi with de Bruijn indices and explicit substitutions with compositions not satisfying PSN are investigated.

About non-idempotent IT, in [15] a relevant IT system characterising head/weak-normalisation was used to prove the relation between the size of typing derivations and execution time in the respective normalisation process as mentioned before, while in [11, 22] IT systems were used to prove the relation between typing derivations and the number of steps for the normalisation of strong-normalising terms in the λ-calculus. Since the system in [11] is non-relevant, the notions of *optimality* and *principality* for typing derivations are introduced, which are in fact relevant derivations, to present the quantitative aspects of typeable terms. There are also some results about explicit substitution/resource calculi [11, 34]. However, only calculi with names were investigated with such an IT system and, in the explicit versions, only composition free substitutions were considered.

2 Type free calculi

Calculi in the present section are defined as term rewriting systems, TRS for short, and standard rewriting notions and notations are used [4]. For instance, given a TRS $\mathfrak{R}$, $\rightarrow_{\mathfrak{R}}^{+}$ denotes its transitive closure while $\rightarrow_{\mathfrak{R}}^{*}$ denotes its reflexive transitive closure.

2.1 λ-calculus with de Bruijn indices

Definition 2.1 (Set Λ_{dB})
The set of λ_{dB}-term, denoted by Λ_{dB}, is inductively defined for $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ by:

$$M, N \in \Lambda_{dB} ::= \underline{n} \,|\, (M\ N) \,|\, \lambda.M.$$

Terms like $((\ldots((M_1\ M_2)\ M_3)\ldots)\ M_n)$ are written $(M_1\, M_2\, \cdots\, M_n)$, as usual. An index $\underline{i}$ is bound if it occurs inside the scope of at least i λ's and it is free otherwise. The following subsets are introduced in order to formally define the set of free indices of a term.

Definition 2.2
Let $N \subset \mathbb{N}^*$ and $k \geq 0$. Define:

$$1. N\backslash k = \{n-k \,|\, n \in N\} \quad 3. N+k = \{n+k \,|\, n \in N\}$$
$$2. N_{>k} = \{n \in N \,|\, n > k\} \quad 4. N_{\leq k} = \{n \in N \,|\, n \leq k\},\ N_{<k} = \{n \in N \,|\, n < k\}$$

Definition 2.3
$FI(M)$, the **set of free indices** of $M \in \Lambda_{dB}$, is defined by:

$$FI(\,\underline{n}\,) = \{\,\underline{n}\,\} \quad FI(M_1\ M_2) = FI(M_1) \cup FI(M_2) \quad FI(\lambda.M) = FI(M)\backslash 1$$

Free indices correspond to the notion of free variables in the λ-calculus with names and M is thus called closed whenever $FI(M) = \emptyset$. The greatest index value of $FI(M)$ is denoted by $sup(M)$.

In this notation, a β-contraction definition needs a mechanism which detects and updates free indices of terms. Intuitively, the **i-lift** of M, denoted by M^{+i}, corresponds to an increment by 1 of all free indices greater than i occurring in M. A more general mechanism is introduced in [30, 31], presented below.

Definition 2.4
Updating functions $U^i_k : \Lambda_{dB} \to \Lambda_{dB}$ for $i \in \mathbb{N}^*$ and $k \in \mathbb{N}$ are inductively defined as follows:

$$1.\ U^i_k(M\ N) = (U^i_k(M)\ U^i_k(N)) \qquad 3.\ U^i_k(\,\underline{n}\,) = \begin{cases} \underline{n+i-1}\,, & \text{if } n > k \\ \underline{n}\,, & \text{if } n \leq k. \end{cases}$$
$$2.\ U^i_k(\lambda.M) = \lambda.U^i_{k+1}(M)$$

Therefore, $U^i_k(M)$ represents $i-1$ applications of the k-lift on term M. Now, it is possible to present the substitution definition used by β-contractions as introduced in [30, 31].

Definition 2.5
Let $m, n \in \mathbb{N}^*$. The ***β*-substitution** for free occurrences of $\underline{n}$ in $M \in \Lambda_{dB}$ by term N, denoted as $\{\underline{n}\,/N\}M$, is defined inductively by

$$1.\ \{\,\underline{n}\,/N\}(M_1\ M_2) = (\{\,\underline{n}\,/N\}M_1\ \{\underline{n}\,/N\}M_2) \quad 3.\ \{\,\underline{n}\,/N\}\,\underline{m} = \begin{cases} \underline{m-1}\,, & \text{if } m > n \\ U^n_0(N), & \text{if } m = n \\ \underline{m}\,, & \text{if } m < n \end{cases}$$
$$2.\ \{\,\underline{n}\,/N\}(\lambda.M_1) = \lambda.\{\,\underline{n+1}\,/N\}M_1$$

β-contraction can then be defined.

Definition 2.6
***β*-contraction** in λ_{dB} is defined by:

$$(\lambda.M\ N) \to_\beta \{\underline{1}\,/N\}M$$

Item 3 in Definition 2.5 is the mechanism which does the substitution and updates free indices in M as a consequence of the elimination of the lead abstractor. The updating function is used to avoid the capture of free indices in N. The formal definition of $\boldsymbol{\beta}$-reduction is given below.

DEFINITION 2.7
β**-reduction** in λdB is defined by:

$$\frac{}{(\lambda.M\ N) \rightarrow_\beta \{\,\underline{1}\,/N\}M} \qquad \frac{M \rightarrow_\beta N}{\lambda.M \rightarrow_\beta \lambda.N}$$

$$\frac{M_1 \rightarrow_\beta N_1}{(M_1\ M_2) \rightarrow_\beta (N_1\ M_2)} \qquad \frac{M_2 \rightarrow_\beta N_2}{(M_1\ M_2) \rightarrow_\beta (M_1\ N_2)}$$

In other words, the $\boldsymbol{\beta}$-reduction is defined to be the λ-compatible closure of β-contraction. A term is in $\boldsymbol{\beta}$**-normal form**, β-nf for short, if there is no β-reduction to be done.

If $\underline{i} \notin FI(M)$ then one has $\{\,\underline{i}\,/N\}M = M^{-i}$, where M^{-i} is the term M in which indices greater than i are decreased by one. We call this an **empty substitution** because no index is replaced by an instance of term N. A β-contraction $(\lambda.M\ N)$ when $\underline{1} \notin FI(M)$ is thus called an **empty application**.

2.2 *The λs_e-calculus*

The λs-calculus is a proper extension of λ_{dB}. Two operators σ and φ are introduced for substitution and updating, respectively, to control the atomisation of β-substitutions by arithmetic constraints.

DEFINITION 2.8 (Set Λs)
The set of λs-terms, denoted by Λs, is inductively defined for $n, i, j \in \mathbb{N}^*$ and $k \in \mathbb{N}$ by:

$$M, N \in \Lambda s ::= \underline{n} \mid (M\ N) \mid \lambda.M \mid M\sigma^i N \mid \varphi^j_k M$$

A term of the form $M\sigma^i N$ represents the procedure to obtain the term $\{\,\underline{i}\,/N\}M$; i.e., the substitution of the free occurrences of $\underline{i}$ in M by N, updating the free indices on both terms. Similarly, the term $\varphi^j_k M$ represents the procedure for $U^j_k(M)$. Table 1 contains the rewriting rules of λs_e as given in [31]. The bottom six rules of Table 1 are those which extend λs [30] to λs_e [31]. They ensure the confluence of the λs_e-calculus on open terms thus its application to the HOU problem [3]. In this paper we work with the same set Λs of terms for both calculi.

An associated substitution calculus, denoted by s_e, is induced by all the rules except (σ-generation). The rewriting system obtained by removing from s_e the bottom six rules presented in Table 1 is called the s-calculus, which is the substitution calculus associated with λs. For any $M \in \Lambda s$, by confluence and strong normalisation of s [30] there exists a unique s-normal form, denoted by $s(M)$. The set of s-nfs is exactly the set Λ_{dB} [30] then called **pure terms**. The following lemma states significant relations between s and both the term structure and the β-substitution.

LEMMA 2.9 ([30])
Let $M, N \in \Lambda s$:

TABLE 1. The rewriting system of the λs_e-calculus

$(\lambda.M\ N)$	$\longrightarrow$	$M\,\sigma^1 N$	(σ-generation)
$(\lambda.M)\sigma^i N$	$\longrightarrow$	$\lambda.(M\sigma^{i+1}N)$	(σ-λ-transition)
$(M_1\ M_2)\sigma^i N$	$\longrightarrow$	$((M_1\sigma^i N)\ (M_2\sigma^i N))$	(σ-app-trans.)
$\underline{n}\,\sigma^i N$	$\longrightarrow$	$\begin{cases} \underline{n-1} & \text{if } n > i \\ \varphi_0^i N & \text{if } n = i \\ \underline{n} & \text{if } n < i \end{cases}$	(σ-destruction)
$\varphi_k^i(\lambda.M)$	$\longrightarrow$	$\lambda.(\varphi_{k+1}^i M)$	(φ-λ-trans.)
$\varphi_k^i(M_1\ M_2)$	$\longrightarrow$	$((\varphi_k^i M_1)\ (\varphi_k^i M_2))$	(φ-app-trans.)
$\varphi_k^i\,\underline{n}$	$\longrightarrow$	$\begin{cases} \underline{n+i-1} & \text{if } n > k \\ \underline{n} & \text{if } n \leq k \end{cases}$	(φ-destruction)
$(M_1\sigma^i M_2)\sigma^j N$	$\longrightarrow$	$(M_1\sigma^{j+1}N)\sigma^i(M_2\sigma^{j-i+1}N)$ if $i \leq j$	(σ-σ-trans.)
$(\varphi_k^i M)\sigma^j N$	$\longrightarrow$	$\varphi_k^{i-1}M$ if $k < j < k+i$	(σ-φ-trans. 1)
$(\varphi_k^i M)\sigma^j N$	$\longrightarrow$	$\varphi_k^i(M\sigma^{j-i+1}N)$ if $k+i \leq j$	(σ-φ-trans. 2)
$\varphi_k^i(M\sigma^j N)$	$\longrightarrow$	$(\varphi_{k+1}^i M)\sigma^j(\varphi_{k+1-j}^i N)$ if $j \leq k+1$	(φ-σ-trans.)
$\varphi_k^i(\varphi_l^j M)$	$\longrightarrow$	$\varphi_l^j(\varphi_{k+1-j}^i M)$ if $l+j \leq k$	(φ-φ-trans. 1)
$\varphi_k^i(\varphi_l^j M)$	$\longrightarrow$	$\varphi_l^{j+i-1}M$ if $l \leq k < l+j$	(φ-φ-trans. 2)

1. $s(M\ N) = (s(M)\ s(N))$.
2. $s(\lambda.M) = \lambda.s(M)$.
3. $s(\varphi_k^i M) = U_k^i(s(M))$.
4. $s(M\sigma^i N) = \{\,\underline{i}/s(N)\}s(M)$.

In order to have a syntactic characterisation related to empty applications and substitutions, as with free indices for λ_{dB}, we present the definition of *available indices*, a notion analogous to that of available variables introduced in [38].

DEFINITION 2.10
$AI(M)$, the **set of available indices** of $M \in \Lambda s$ is defined by:

$$AI(\,\underline{n}\,) = \{\,\underline{n}\,\} \quad AI(\lambda.M) = AI(M)\backslash 1 \quad AI(M_1\ M_2) = AI(M_1) \cup AI(M_2)$$

and

$$AI(\varphi_k^i M) = AI(M)_{\leq k} \cup (AI(M)_{>k} + (i-1))$$
$$AI(M\sigma^i N) = \begin{cases} AI(M^{-i}) \cup AI(\varphi_0^i N), & \text{if } i \in AI(M) \\ AI(M^{-i}), & \text{if } i \notin AI(M) \end{cases}$$

where $AI(M^{-i})$ denotes $AI(M)_{<i} \cup (AI(M)_{>i})\backslash 1$.

The greatest value of $AI(M)$ is denoted by $sav(M)$. Below, the relation between AI and FI is presented.

LEMMA 2.11
If $M \in \Lambda s$ then $AI(M) = FI(s(M))$.

PROOF. By induction on the structure of $M \in \Lambda s$. ∎

COROLLARY 2.12
If $M \in \Lambda_{dB}$, then $AI(M) = FI(M)$.

Below, lemmas stating the relation between set AI, its greatest value sav and the structure of terms are presented.

Lemma 2.13

1. $\underline{n} \in AI(\lambda.M)$ iff $\underline{n{+}1} \in AI(M)$.
2. $sav(M_1\ M_2) = max(sav(M_1), sav(M_2))$.
3. If $sav(M)=0$, then $sav(\lambda.M)=0$. Otherwise, $sav(\lambda.M)=sav(M)-1$.
4. If $sav(N)>k$ then $sav(\varphi^i_k N) = sav(N)+(i{-}1)$. If $sav(N)\leq k$ then $sav(\varphi^i_k N) = sav(N)$.
5. Let $sav(M^{-i})=max(AI(M^{-i}))=max(AI(M)_{<i} \cup (AI(M)_{>i})\backslash 1)$. If $sav(M)<i$, then $sav(M^{-i})=sav(M)$. If $sav(M)>i$, then $sav(M^{-i})=sav(M)-1$.
6. If $\underline{i} \in sav(M)$ then $sav(M\sigma^i N)=max(sav(M^{-i}), sav(\varphi^i_0 N))$ and $sav(M\sigma^i N) = sav(M^{-i})$ otherwise.

Proof. 1. By Definition 2.10. 2,3,4,5,6. Analysing the AI definition and its relation with the max function. ∎

Remark 2.14
By Corollary 2.12 the statements 1, 2, 3 and 5 above are valid for $FI(M)$ and $sup(M)$ when $M \in \Lambda_{dB}$.

Properties relating free indices and β-reduction in λ_{dB} can then be derived from Lemmas 2.9 and 2.11.

Lemma 2.15
Let $M, N \in \Lambda_{dB}$:

1. If $\underline{i} \notin FI(M)$ then $FI(\{\underline{i}/N\}M) = FI(M^{-i})$.
2. If $\underline{i} \in FI(M)$ then $FI(\{\underline{i}/N\}M) = FI(M^{-i}) \cup FI(U^i_0(N))$.

Proof. By Lemma 2.9.4 and Lemma 2.11 one has $AI(M\sigma^i N)=FI(\{\underline{i}/N\}M)$ and by Corollary 2.12 one has $FI(M)=AI(M)$ and $FI(N)=AI(N)$.

1. Suppose $\underline{i} \notin FI(M)$. Therefore, $FI(\{\underline{i}/N\}M) = AI(M\sigma^i N) = AI(M^{-i}) = FI(M^{-i})$.
2. Suppose $\underline{i} \in FI(M)$. Therefore, $FI(\{\underline{i}/N\}M)=AI(M\sigma^i N)=AI(M^{-i}) \cup AI(\varphi^i_0 N) = FI(M^{-i}) \cup FI(s(\varphi^i_0 N)) = FI(M^{-i}) \cup FI(U^i_0(N))$.

∎

Corollary 2.16
If $\underline{1} \in FI(M)$, then $FI(\{\underline{1}/N\}M) = FI(\lambda.M\ \ N)$. Otherwise, $FI(\{\underline{1}/N\}M) = FI(\lambda.M)$.

Therefore, we can state that the β-reduction in λ_{dB} does not create new indices.

Lemma 2.17
If $M \to_\beta N$ then $FI(N) \subseteq FI(M)$.

Proof. By induction on the derivation of $M \to_\beta N$ and Corollary 2.16. ∎

2.3 The $\lambda\sigma$-calculus

The $\lambda\sigma$-calculus is given by a first-order rewriting system, which makes substitutions explicit by extending the language with two sorts of objects: **terms** and **substitutions** which are called $\lambda\sigma$-expressions.

DEFINITION 2.18 (Set $\Lambda\sigma$)
The set of $\lambda\sigma$-expressions, denoted by $\Lambda\sigma$, is formed by the set $\Lambda\sigma^t$ of terms and the set $\Lambda\sigma^s$ of substitutions, inductively defined by:

$$M, N \in \Lambda\sigma^t ::= \underline{1} \,|\, (M\ N) \,|\, \lambda.M \,|\, M[S] \qquad S \in \Lambda\sigma^s ::= id \,|\, \uparrow \,|\, M.S \,|\, S \circ S$$

Substitutions can intuitively be thought of as lists of the form $N/\underline{i}$ indicating that index $\underline{i}$ ought to be replaced by term N. The expression id represents a substitution of the form $\{\underline{1}/\underline{1}, \underline{2}/\underline{2}, \dots\}$ whereas the **shift**, denoted by $\uparrow$, is the substitution $\{\underline{i+1}/\underline{i} \,|\, i \in \mathbb{N}^*\}$. The expression $S \circ S$ represents the composition of substitutions. Moreover, $\underline{1}[\uparrow^n]$, where $n \in \mathbb{N}^*$, codifies the de Bruijn index $\underline{n+1}$ and $\underline{i}[S]$ represents the value of $\underline{i}$ through the substitution S, which can be seen as a function $S(i)$. The substitution $M.S$ has the form $\{M/\underline{1}, S(i)/\underline{i+1}\}$ and is called the **cons of** M **in** S. $M[N.id]$ starts the β-reduction simulation of $(\lambda.M\ N)$ in $\lambda\sigma$. Thus, in addition to the substitution of free occurrences of index $\underline{1}$ by the corresponding term, free occurrences of any other index should be decremented because of the elimination of the abstractor. Table 2 lists the rewriting system of the $\lambda\sigma$-calculus, as presented in [24], without the (*Eta*) rule.

TABLE 2. The rewriting system for the $\lambda\sigma$-calculus

$(\lambda.M\ N)$	$\longrightarrow$	$M[N.id]$	(*Beta*)
$(M\ N)[S]$	$\longrightarrow$	$(M[S]\ N[S])$	(*App*)
$\underline{1}[M.S]$	$\longrightarrow$	M	(*VarCons*)
$(\lambda.M)[S]$	$\longrightarrow$	$\lambda.(M[\underline{1}.(S\circ\uparrow)])$	(*Abs*)
$M[id]$	$\longrightarrow$	M	(*Id*)
$(M[S])[T]$	$\longrightarrow$	$M[S \circ T]$	(*Clos*)
$id \circ S$	$\longrightarrow$	S	(*IdL*)
$\uparrow \circ (M.S)$	$\longrightarrow$	S	(*ShiftCons*)
$(S_1 \circ S_2) \circ S_3$	$\longrightarrow$	$S_1 \circ (S_2 \circ S_3)$	(*AssEnv*)
$(M.S) \circ T$	$\longrightarrow$	$M[T].(S \circ T)$	(*MapEnv*)
$S \circ id$	$\longrightarrow$	S	(*IdR*)
$\underline{1}.\uparrow$	$\longrightarrow$	id	(*VarShift*)
$\underline{1}[S].(\uparrow\circ S)$	$\longrightarrow$	S	(*Scons*)

This system is equivalent to that of [1]. An associated substitution calculus, denoted by σ, is induced by all the rules except (Beta).

2.4 The $\lambda\upsilon$-calculus

P. Lescanne introduced in [39] the $\lambda\upsilon$-calculus. The calculus was originally presented as a rewriting system with three sort of objects where, besides terms and substitutions, an inductively defined set $\mathcal{N}$ represents the natural numbers. However, $\lambda\upsilon$ is presented here as a two sorted calculus similar to $\lambda\sigma$.

Definition 2.19 (Set $\Lambda\upsilon$)
The set of $\lambda\upsilon$-expressions, denoted by $\Lambda\upsilon$, is formed by the set $\Lambda\upsilon^t$ of terms and the set $\Lambda\upsilon^s$ of substitutions inductively defined for $n \in \mathbb{N}^*$ by:

$$M, N \in \Lambda\upsilon^t ::= \underline{n} \,|\, (M\ N) \,|\, \lambda.M \,|\, M[S] \qquad S \in \Lambda\upsilon^s ::= M/ \,|\, \Uparrow(S) \,|\, \uparrow$$

The $\lambda\upsilon$-calculus is intended to describe the minimal substitution mechanism, where features such as composition of substitutions and the representation of compound substitution as lists are considered to be implementation choices. Hence, the cons (.) and the composition ($\circ$), essential to satisfy the confluence property for open terms, i.e. terms with meta-variables, are removed and new constructors for substitutions are introduced, in order to have as few forms of substitutions as possible. Therefore, $M/$ can be seen as the substitution $M.id$ and the **lift**, denoted by $\Uparrow(S)$, as the substitution $\underline{1}.(S \circ \uparrow)$. The rewriting rules of $\lambda\upsilon$ are in Table 3.

Table 3. The rewriting system for the $\lambda\upsilon$-calculus

$(\lambda.M\ N)$	$\longrightarrow$	$M[N/]$	(B)
$(M\ N)[S]$	$\longrightarrow$	$(M[S]\ N[S])$	(App)
$(\lambda.M)[S]$	$\longrightarrow$	$\lambda.(M[\Uparrow(S)])$	$(Lambda)$
$\underline{1}\,[M/]$	$\longrightarrow$	M	$(FVar)$
$\underline{n{+}1}\,[M/]$	$\longrightarrow$	$\underline{n}$	$(RVar)$
$\underline{1}\,[\Uparrow(S)]$	$\longrightarrow$	$\underline{1}$	$(FVarLift)$
$\underline{n{+}1}[\Uparrow(S)]$	$\longrightarrow$	$\underline{n}\,[S][\uparrow]$	$(RVarLift)$
$\underline{n}[\uparrow]$	$\longrightarrow$	$\underline{n{+}1}$	$(VarShift)$

An associated substitution calculus, denoted by υ, is induced by all the rules of Table 3 but (B). Lescanne et al. proved in [10] the properties of $\lambda\upsilon$ such as the simulation of the β-reduction, confluence for terms without meta-variable and the preservation of strong normalisation, PSN for short. The PSN property means that any strongly normalising term, SN for short, in the λ-calculus is SN in the $\lambda\upsilon$-calculus. Although it seems to be a property any calculus intended to simulate β-reduction should satisfy, after some years of the introduction of the $\lambda\sigma$-calculus P.-A. Melliès presented in [40] a counter-example where some term, corresponding to a simply typed term in the λ-calculus, has an infinity reduction strategy. B. Guillaume presented in [26] an analogous counter-example for the λs_e-calculus.

The proof of PSN in [10] relies on the fact that only the rule (B) creates new closures thus any closure occurring in a term can have the corresponding (B) rule traced back. Interesting enough, Melliès pointed out the rule $(MapEnv)$ as the responsible for the failure of PSN by the $\lambda\sigma$-calculus. In [47], E. Ritter proved that the kind of composition of substitutions allowed in $\lambda\sigma$, and in λs_e, was the characteristic determining the failure w.r.t. PSN.

Analogous to the s-calculus, for any $M \in \Lambda\upsilon^t$ and by confluence and termination of υ [10] there is a unique υ-nf, denoted by $\upsilon(M)$. Note that $\upsilon(M)$ is a pure term, i.e. a term without closures [10].

3 The non-idempotent intersection types

All intersection type systems presented in this paper have the same set of types $\mathcal{T}$, of the so called restricted intersection types, in which intersections do not occur immediately on the right of an $\rightarrow$. Moreover, the intersection is non-idempotent, i.e. $\sigma \wedge \sigma \neq \sigma$. In addition, type contexts in type systems with de Bruijn indices are sequences of types instead of sets of type assignments. Below, all these concepts are defined.

DEFINITION 3.1

1. Let $\mathcal{A}$ be a denumerably infinite **set of type variables** and let α, β range over $\mathcal{A}$.
2. The set $\mathcal{T}$ of **non-idempotent intersection types** is defined by:

$$\tau, \sigma \in \mathcal{T} ::= \mathcal{A} \,|\, \mathcal{U} \rightarrow \mathcal{T} \qquad u \in \mathcal{U} ::= \omega \,|\, \mathcal{U} \wedge \mathcal{U} \,|\, \mathcal{T}$$

   Types are quotiented by taking $\wedge$ to be commutative, associative and to have ω as the neutral element.
3. **Sequential contexts** are ordered lists of $u \in \mathcal{U}$, defined by:

$$\Gamma ::= nil \,|\, u.\Gamma$$

   Γ_i denotes the i-th element of Γ and $|\Gamma|$ denotes the length of Γ, where $|nil| = 0$. $\Gamma_{<i}$ denotes the first $i-1$ types in the sequence. $\Gamma_{>i}$, $\Gamma_{\leq i}$ and $\Gamma_{\geq i}$ are similarly defined. If $i = 0$, then $\Gamma_{\leq 0}.\Gamma = \Gamma_{<0}.\Gamma = \Gamma$.

   An **omega context** $\omega^{\underline{n}}$ denotes the context $\omega.\omega.\cdots.\omega.nil$ of length n. If $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$, then $\Gamma \neq \Delta.\omega^{\underline{m}}$ denotes that Γ does not end with an omega context different than nil and $\Gamma \neq \Delta.\omega^{\underline{n}}$ stands for $\Gamma \neq nil$ and $\Gamma \neq \Delta.\omega^{\underline{m}}$.

   An extension of $\wedge$ to contexts is obtained by taking nil as the neutral element and $(u_1.\Gamma) \wedge (u_2.\Delta) = (u_1 \wedge u_2).(\Gamma \wedge \Delta)$. Hence, $\wedge$ is commutative and associative on contexts.
4. Let $u' \sqsubseteq u$ if there exists v such that $u = u' \wedge v$ and $u' \sqsubset u$ if $v \neq \omega$. Let $\Gamma' \sqsubseteq \Gamma$ if there exists Δ such that $\Gamma = \Gamma' \wedge \Delta$, where neither Γ' nor Δ are omega contexts different than nil and $\Gamma' \sqsubset \Gamma$ if $\Delta \neq nil$.

The set $\mathcal{T}$ defined here is equivalent to the one originally defined in [50] and also used in [15]. Type judgements will be of the form $M : \langle \Gamma \vdash_{\mathfrak{S}} \tau \rangle$, instead of the usual $\Gamma \vdash_{\mathfrak{S}} M : \tau$ notation, meaning M has type τ with context Γ in system $\mathfrak{S}$. Briefly, M has type τ with Γ in $\mathfrak{S}$ or $\langle \Gamma \vdash \tau \rangle$ is a typing of M in $\mathfrak{S}$. The subscript $\mathfrak{S}$ is omitted whenever it is clear to which system the typing belongs.

Below, some properties about the extension of $\wedge$ to contexts, straightforward from its definition, are presented.

LEMMA 3.2
Let $\Gamma^1, \ldots, \Gamma^m$ be contexts different than nil:

1. $\Gamma^1 \wedge \cdots \wedge \Gamma^m = (\Gamma^1_1 \wedge \cdots \wedge \Gamma^m_1).(\Gamma^1_{>1} \wedge \cdots \wedge \Gamma^m_{>1})$.
2. If $i \leq min(|\Gamma^1|, \ldots, |\Gamma^m|)$, then $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_i = \Gamma^1_i \wedge \cdots \wedge \Gamma^m_i$. Else, $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_i = \Gamma^{j_1}_i \wedge \cdots \wedge \Gamma^{j_k}_i$, where $k \leq m$ and $\forall 1 \leq l \leq k$, $\Gamma^{j_l}_i \in \mathcal{U}$.

3. $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_{<i} = \Gamma^1_{<i} \wedge \cdots \wedge \Gamma^m_{<i}$. If $i \geq |\Gamma^j|$ then $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_{<i} = \Gamma^1_{<i} \wedge \cdots \wedge \Gamma^j \wedge \cdots \wedge \Gamma^m_{<i}$. $(\Gamma \wedge \Delta)_{\leq i}$ has similar properties.
4. $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_{>i} = \Gamma^1_{>i} \wedge \cdots \wedge \Gamma^m_{>i}$. If $i \geq |\Gamma^j|$ then $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_{>i} = \Gamma^1_{>i} \wedge \cdots \wedge \Gamma^{j-1}_{>i} \wedge \Gamma^{j+1}_{>i} \wedge \cdots \wedge \Gamma^m_{>i}$. $(\Gamma \wedge \Delta)_{\geq i}$ has similar properties.
5. $(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_{<i}.(\Gamma^1 \wedge \cdots \wedge \Gamma^m)_{>i} = (\Gamma^1_{<i}.\Gamma^1_{>i}) \wedge \cdots \wedge (\Gamma^m_{<i}.\Gamma^m_{>i})$.
6. $|\Gamma^1 \wedge \cdots \wedge \Gamma^m| = max(|\Gamma^1|, \ldots, |\Gamma^m|)$.

NOTATION 3.3

- A list of typings denoted by either $M : \langle \Gamma^1 \vdash \sigma_1 \rangle \ldots M : \langle \Gamma^m \vdash \sigma_m \rangle$ or $\forall 1 \leq i \leq m, M : \langle \Gamma^i \vdash \sigma_i \rangle$.
- A list of typing derivations $\mathcal{D}_1 \cdots \mathcal{D}_m$ is denoted by $\mathcal{D}_i, \forall 1 \leq i \leq m$

4 Intersection type systems for λs and λs_e

In order to have an IT system for λs_e, we introduce a system for λs as an intermediate step. In both cases, we focus in two properties while developing the typing rules: relevance [21, 23], where the available indices play an essential role, and subject reduction. The latter is a basic property which any type assignment system should satisfy while the former was the way to obtain such a system as restricted as possible.

The alternative for relevance would be an IT system isomorphic modulo idempotency to an extension of a simple type assignment system as in [46, 17, 7]. While the λs-calculus has the preservation of strong normalisation property [30], PSN for short, the rules allowing the composition of substitutions in λs_e invalidate the property for the calculus. B. Guillaume presents in [26] a counter-example of some simply typed term in λs_e with an infinite reduction strategy. Therefore, any typing system extending the simple type system for λs_e (c.f. [3]) would automatically inherits the Guillaume's counter-example. We then consider whether the relevant systems presented in this paper are able to characterise SN for λs_e.

Moreover, we introduced the system λ^{SM}_{dB} in [58], proving its properties and then proposing the IT system for λs based on those results. Here, we first present the IT system λs^{SM}, deriving the very same properties of the system λ^{SM}_{dB} from it. We take the same approach with the system proposed for λs_e, denoted by $\lambda {s_e}^{\wedge}$, to prove for the first time the SR property for system $\lambda^{\wedge}_{dB}$, which is a variation of system λ^{SM}_{dB}. See Section 5 for more details on the IT systems for λ_{dB}.

Proofs of SR in the present section follows a standard procedure, where generation lemmas are stated beforehand. The SR property is then verified for each rule in the respective calculus. Since the system for λs is relevant, we introduce an appropriate notion of SR.

4.1 The system λs^{SM}

DEFINITION 4.1 (The system λs^{SM})
The typing rules of system λs^{SM} are given in Figure 1.

Compared with the simple type system for λs and λs_e, which introduces one type inference rule for each operator (cf. [3]), there are multiple rules introduced in Figure 1 for the σ and φ operators. Below an example is presented to illustrate the necessity of more rules than just ($\wedge$-σ) in our typing system.

$$\frac{}{\underline{1}:\langle\tau.nil\vdash\tau\rangle}\text{ var}\qquad\frac{\underline{n}:\langle\Gamma\vdash\tau\rangle}{\underline{n+1}:\langle\omega.\Gamma\vdash\tau\rangle}\text{ varn}\qquad\frac{M:\langle u.\Gamma\vdash\tau\rangle}{\lambda.M:\langle\Gamma\vdash u\to\tau\rangle}\to_i$$

$$\frac{M_1:\langle\Gamma\vdash\omega\to\tau\rangle\qquad M_2:\langle\Delta\vdash\sigma\rangle}{(M_1\ M_2):\langle\Gamma\wedge\Delta\vdash\tau\rangle}\to_e'\qquad\frac{M:\langle nil\vdash\tau\rangle}{\lambda.M:\langle nil\vdash\omega\to\tau\rangle}\to_i'$$

$$\frac{M_1:\langle\Gamma\vdash\wedge_{i=1}^n\sigma_i\to\tau\rangle\qquad M_2:\langle\Delta^1\vdash\sigma_1\rangle\ \dots\ M_2:\langle\Delta^n\vdash\sigma_n\rangle}{(M_1\ M_2):\langle\Gamma\wedge\Delta^1\wedge\cdots\wedge\Delta^n\vdash\tau\rangle}\to_e$$

$$(\omega\text{-}\varphi)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{\varphi_k^iM:\langle\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{>k}\vdash\tau\rangle},|\Gamma|>k\qquad(\omega\text{-}\sigma)\ \frac{N:\langle\Delta\vdash\rho\rangle\qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^iN:\langle\Gamma_{<i}.\Gamma_{>i}\vdash\tau\rangle},\Gamma_i=\omega$$

$$(nil\text{-}\varphi)\ \frac{M:\langle\Gamma\vdash\tau\rangle}{\varphi_k^iM:\langle\Gamma\vdash\tau\rangle},|\Gamma|\leq k\qquad(nil\text{-}\sigma)\ \frac{N:\langle\Delta\vdash\rho\rangle\qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^iN:\langle\Gamma\vdash\tau\rangle},|\Gamma|<i$$

$$(\wedge\text{-}nil\text{-}\sigma)\ \frac{N:\langle nil\vdash\sigma_1\rangle\dots N:\langle nil\vdash\sigma_m\rangle\qquad M:\langle\omega^{\underline{i-1}}.\wedge_{j=1}^m\sigma_j.nil\vdash\tau\rangle}{M\sigma^iN:\langle nil\vdash\tau\rangle}$$

$$(\wedge\text{-}\omega\text{-}\sigma)\ \frac{N:\langle nil\vdash\sigma_1\rangle\dots N:\langle nil\vdash\sigma_m\rangle\qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^iN:\langle\Gamma_{<(i-k)}.nil\vdash\tau\rangle},\ \Gamma_i=\wedge_{j=1}^m\sigma_j\ (*)$$

$$(\wedge\text{-}\sigma)\ \frac{N:\langle\Delta^1\vdash\sigma_1\rangle\dots N:\langle\Delta^m\vdash\sigma_m\rangle\qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^iN:\langle(\Gamma_{<i}.\Gamma_{>i})\wedge\omega^{\underline{i-1}}.(\Delta^1\wedge\cdots\wedge\Delta^m)\vdash\tau\rangle},\ \Gamma_i=\wedge_{j=1}^m\sigma_j\ (**)$$

(*) $\Gamma=\Gamma_{<(i-k)}.\omega^{\underline{k}}.\wedge_{j=1}^m\sigma_j.nil$ and $\Gamma_{(i-k-1)}\neq\omega$ (**) $\Delta^k\neq nil$, for some $1\leq k\leq m$, or $\Gamma_{>i}\neq nil$

FIG. 1. Typing rules of the system λs^{SM}

EXAMPLE 4.2
Let $\underline{3}:\langle\omega^{\underline{2}}.\alpha\to\alpha\vdash\alpha\to\alpha\rangle$ and $\lambda.\underline{1}:\langle nil\vdash\alpha\to\alpha\rangle$. Applying the rule ($\wedge$-$\sigma$), ignoring its side condition, one has $\underline{3}\sigma^3(\lambda.\underline{1}):\langle\omega^{\underline{2}}\vdash\alpha\to\alpha\rangle$.

Hence, we need the rules ($\wedge$-ω-σ) and ($\wedge$-*nil*-σ) to satisfy the relevance property. In fact, this multiplicity corresponds to the cases for the updating and substitution lemmas for λ_{dB}^{SM} (see Section 5 for further details).

Another important thing to note is that the typing information related to empty substitutions, handled by the rules (ω-σ) and (*nil*-σ), is discharged. In other words, the typing information related to an empty application is forgotten as soon as the substitution procedure is started. In a non-idempotent intersection type system this is necessary in order to have SR, as seen in the following example.

EXAMPLE 4.3
Suppose that the rule (ω-σ) is defined as

$$\frac{N:\langle\Delta\vdash\rho\rangle\qquad M:\langle\Gamma\vdash\tau\rangle}{M\sigma^iN:\langle(\Gamma_{<i}.\Gamma_{>i})\wedge\omega^{\underline{i-1}}.\Delta\vdash\tau\rangle},\Gamma_i=\omega$$

Let $M\equiv(\lambda.(\underline{2}\ \underline{3})\ \underline{3})$ and $M'\equiv((\underline{2}\sigma^1\underline{3})\ (\underline{3}\sigma^1\underline{3}))$ thus $M\to_{\lambda s}^+ M'$. If $M:\langle\alpha_1\to\alpha_2.\alpha_1.\beta.nil\vdash\alpha_2\rangle$ then $M':\langle\alpha_1\to\alpha_2.\alpha_1.(\beta\wedge\beta).nil\vdash\alpha_2\rangle$. Let Γ and Γ' be the contexts in the typings of M and M', respectively. Note that $\Gamma'=\Gamma\wedge(\omega^{\underline{2}}.\beta.nil)$.

Hence, although the typing information related to the empty substitution disappears in the s-nf, it is duplicated after each reduction with the (σ-app-transition) rewriting rule.

The system λs^{SM} is relevant w.r.t. available indices, as stated below.

LEMMA 4.4 (Relevance for λs^{SM})
If $M\!:\!\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$, then $|\Gamma| = sav(M)$ and $\forall 1 \le i \le |\Gamma|$, $\Gamma_i \neq \omega$ iff $\underline{i} \in AI(M)$.

PROOF. By induction on the derivation of $M : \langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$. We present the case for the application of the rule (nil-φ). Then, $\varphi^i_k M\!:\!\langle\Gamma \vdash \tau\rangle$ where $M\!:\!\langle\Gamma \vdash \tau\rangle$ and $|\Gamma| \le k$. By the induction hypothesis (IH) one has that $|\Gamma| = sav(M)$ and $\forall 1 \le j \le |\Gamma|$, $\Gamma_j \neq \omega$ iff $\underline{j} \in AI(M)$. Observe that $AI(\varphi^i_k M) = AI(M)_{\le k} \cup (AI(M)_{>k} + (i-1)) = AI(M)$ thus $sav(\varphi^i_k M) = sav(M)$. ∎

The relevance of λs^{SM} does not allow the system to satisfy SR in the usual sense. The following example in the λ_{dB}-calculus illustrates the issue.

EXAMPLE 4.5
For SR, we need to prove the statement: If $(\lambda.M\ N)\!:\!\langle\Gamma \vdash \tau\rangle$ then $\{\underline{1}/N\}M\!:\!\langle\Gamma \vdash \tau\rangle$.

Let $M \equiv \lambda.\underline{1}$ and $N \equiv \underline{3}$, hence $\{\underline{1}/\underline{3}\}\lambda.\underline{1} = \lambda.\underline{1}$. We have that $(\lambda.\lambda.\underline{1}\ \underline{3})$: $\langle\omega.\omega.\beta.nil \vdash \alpha\to\alpha\rangle$. Hence, $\lambda.\lambda.\underline{1} : \langle nil \vdash \omega\to\alpha\to\alpha\rangle$ and $\underline{3} : \langle\omega.\omega.\beta.nil \vdash \beta\rangle$ thus $\lambda.\underline{1}\!:\!\langle nil \vdash \alpha\to\alpha\rangle$.

In other words, one has a restriction on the original context after the β-reduction, since the typing information regarding $N \equiv \underline{3}$ vanishes.

Notions of expansion and restriction of contexts are an interesting way to talk about subject expansion and reduction in relevant typing systems. These concepts were presented in [29] for environments. We introduce the notion of restriction for sequential contexts related to available indices to prove SR for one step of the β-simulation in the λs-calculus. This approach of restriction/expansion for contexts is not sufficient to have the subject expansion property because the rule $\to'_e$ has the typeability of the argument as a premiss. Hence, for any non-typeable term N, $\{\underline{1}/N\}\,\underline{2}$ is typeable while $(\lambda.\underline{2}\ N)$ is not typeable in system λs^{SM}.

Although the relevant type system, SR holds for the full s-calculus. Some generation lemmas are stated in order to proof the SR property in λs^{SM}.

LEMMA 4.6 (Generation for λs^{SM})

1. If $\underline{n}\!:\!\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$ then $\Gamma_n = \tau$.
2. If $\lambda.M : \langle nil \vdash_{\lambda_s{}^{SM}} \tau\rangle$, then $\tau = \omega \to \sigma$ and $M : \langle nil \vdash_{\lambda_s{}^{SM}} \sigma\rangle$ or $\tau = \wedge^n_{i=1}\sigma_i \to \sigma$, $n > 0$, and $M\!:\!\langle\wedge^n_{i=1}\sigma_i.nil \vdash_{\lambda_s{}^{SM}} \sigma\rangle$ for $\sigma, \sigma_1, \ldots, \sigma_n \in \mathcal{T}$.
3. If $\lambda.M\!:\!\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$ and $|\Gamma| > 0$, then $\tau = u \to \sigma$ for some $u \in \mathcal{U}$ and $\sigma \in \mathcal{T}$, where $M\!:\!\langle u.\Gamma \vdash_{\lambda_s{}^{SM}} \sigma\rangle$.
4. If $(M\ N) : \langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$ then $\Gamma = \Gamma^1 \wedge \Gamma^2$ s.t. $M : \langle\Gamma^1 \vdash_{\lambda_s{}^{SM}} \omega\to\tau\rangle$ and N : $\langle\Gamma^2 \vdash_{\lambda_s{}^{SM}} \rho\rangle$ or $M : \langle\Gamma^1 \vdash_{\lambda_s{}^{SM}} (\wedge^m_{i=1}\sigma_i)\to\tau\rangle$ where $\Gamma^2 = \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \le i \le m$, $N\!:\!\langle\Delta^i \vdash_{\lambda_s{}^{SM}} \sigma_i\rangle$.

PROOF. 1. By induction on the derivation $\underline{n}\!:\!\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$ (note that $(\omega.\Gamma)_{n+1} = \Gamma_n$).
2, 3, 4. By case analysis on the respective derivation. ∎

Below, we present the generation lemmas for typings related to substitution and update operators in λs.

LEMMA 4.7 (Generation for operators in λs^{SM})

1. Let $\varphi_k^i N : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$. If $|\Gamma| \leq k$, then $N : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$. If $|\Gamma| > k$ then $N : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash_{\lambda s^{SM}} \tau \rangle$, where $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$.
2. If $M\sigma^i N : \langle nil \vdash_{\lambda s^{SM}} \tau \rangle$, then $M : \langle nil \vdash_{\lambda s^{SM}} \tau \rangle$ and $N : \langle \Delta \vdash_{\lambda s^{SM}} \rho \rangle$ or $M : \langle \omega^{\underline{i-1}}. \wedge_{j=1}^m \sigma_j.nil \vdash_{\lambda s^{SM}} \tau \rangle$ where $\forall 1 \leq j \leq m$, $N : \langle nil \vdash_{\lambda s^{SM}} \sigma_j \rangle$.
3. If $M\sigma^i N : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$ and $0 < |\Gamma| < i$, then $M : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$ and $N : \langle \Delta \vdash_{\lambda s^{SM}} \rho \rangle$ or $M : \langle \Gamma.\omega^{\underline{n}}. \wedge_{j=1}^m \sigma_j.nil \vdash_{\lambda s^{SM}} \tau \rangle$ where $|\Gamma.\omega^{\underline{n}}. \wedge_{j=1}^m \sigma_j.nil| = i$, $n \geq 0$ and $\forall 1 \leq j \leq m$, $N : \langle nil \vdash_{\lambda s^{SM}} \sigma_j \rangle$.
4. If $M\sigma^i N : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$ and $|\Gamma| \geq i$ then $M : \langle \Gamma_{<i}.\omega.\Gamma_{\geq i} \vdash_{\lambda s^{SM}} \tau \rangle$ and $N : \langle \Delta \vdash_{\lambda s^{SM}} \rho \rangle$ or $M : \langle \Gamma_{<i}. \wedge_{j=1}^m \sigma_j.\Gamma' \vdash_{\lambda s^{SM}} \tau \rangle$ where, for $|\Gamma_{\geq i}| > 0$, $\Gamma_{\geq i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \leq j \leq m$, $N : \langle \Delta^j \vdash_{\lambda s^{SM}} \sigma_j \rangle$.

PROOF. 1, 2, 3, 4. By case analysis on the respective derivation. ∎

REMARK 4.8

Possibilities considered in each item on Lemma 4.7 above are mutually exclusive by syntactic characteristics. For instance, let $M\sigma^i N : \langle \Gamma \vdash \tau \rangle$. The item to be applied is uniquely determined by the relation between i and $|\Gamma|$. Suppose that $0 < |\Gamma| < i$. Hence, by the item 3 above one has two possibilities. The proper one in system λs^{SM} is determined by the value of $sav(M)$. Therefore, if $sav(M) < i$, then $M : \langle \Gamma \vdash \tau \rangle$ and $N : \langle \Delta \vdash \rho \rangle$. Otherwise, the alternative described in item 3 is the one to be applied.

THEOREM 4.9 (SR for s in λs^{SM})

Let $M : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$. If $M \to_s M'$, then $M' : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$.

PROOF. By the verification of SR for each rewriting rule of the s-calculus.

We present the proof-sketch for the rewriting rule (σ-app-transition). Hence, if $(M_1\ M_2)\sigma^i N : \langle \Gamma \vdash \tau \rangle$ we might prove that $((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle \Gamma \vdash \tau \rangle$. Therefore we need to consider the three possibilities for Γ: $\Gamma = nil$, $0 < |\Gamma| < i$ or $|\Gamma| \geq i$. Each one of them is related to one generation lemma, Lemmas 4.7.2, 4.7.3 and 4.7.4 respectively. We present the case where $\Gamma = nil$ while the proofs for $0 \leq |\Gamma| < i$ and $|\Gamma| \geq i$ are analogous. Hence, by Lemma 4.7.2 the last rule applied is either (1) the (nil-σ) or (2) the ($\wedge$-nil-σ) rule.

(1) If the last rule applied is (nil-σ) then: $\dfrac{N : \langle \Delta \vdash \rho \rangle \qquad (M_1\ M_2) : \langle nil \vdash \tau \rangle}{(M_1\ M_2)\sigma^i N : \langle nil \vdash \tau \rangle}$

For $(M_1\ M_2) : \langle nil \vdash \tau \rangle$, one has by Lemma 4.6.4 that the last rule applied is either (a) the rule $\to_e'$ or (b) the rule $\to_e$. Hence:

(a)

$$\dfrac{N : \langle \Delta \vdash \rho \rangle \qquad \dfrac{M_1 : \langle nil \vdash \omega \to \tau \rangle \qquad M_2 : \langle nil \vdash \sigma \rangle}{(M_1\ M_2) : \langle nil \vdash \tau \rangle}}{(M_1\ M_2)\sigma^i N : \langle nil \vdash \tau \rangle}$$

thus

$$\dfrac{\dfrac{N : \langle \Delta \vdash \rho \rangle \qquad M_1 : \langle nil \vdash \omega \to \tau \rangle}{M_1\sigma^i N : \langle nil \vdash \omega \to \tau \rangle} \qquad \dfrac{N : \langle \Delta \vdash \rho \rangle \qquad M_2 : \langle nil \vdash \sigma \rangle}{M_2\sigma^i N : \langle nil \vdash \sigma \rangle}}{((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle nil \vdash \tau \rangle}$$

(b)

$$\dfrac{N:\langle\Delta\vdash\rho\rangle \qquad \dfrac{M_1:\langle nil\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle \qquad \forall 1\leq j\leq m, M_2:\langle nil\vdash\sigma_j\rangle}{(M_1\ M_2):\langle nil\vdash\tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

thus

$$\dfrac{\dfrac{N:\langle\Delta\vdash\rho\rangle \qquad M_1:\langle nil\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle}{M_1\sigma^i N:\langle nil\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle} \qquad \dfrac{N:\langle\Delta\vdash\rho\rangle \qquad M_2:\langle nil\vdash\sigma_j\rangle}{M_2\sigma^i N:\langle nil\vdash\sigma_j\rangle}, \forall 1\leq j\leq m}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle}$$

(2) If the last rule applied is ($\wedge$-nil-σ) then:

$$\dfrac{\forall 1\leq j\leq n, N:\langle nil\vdash\tau_j\rangle \qquad (M_1\ M_2):\langle\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil\vdash\tau\rangle}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

For $(M_1\ M_2):\langle\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil\vdash\tau\rangle$, one has by Lemma 4.6.4 that the last rule applied is either (a) the rule $\to_e'$ or (b) the rule $\to_e$. Hence:

(a) For some $\Gamma^1\wedge\Gamma^2=\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil$ one has:

$$\dfrac{\forall 1\leq j\leq n, N:\langle nil\vdash\tau_j\rangle \qquad \dfrac{M_1:\langle\Gamma^1\vdash\omega\to\tau\rangle \qquad M_2:\langle\Gamma^2\vdash\rho\rangle}{(M_1\ M_2):\langle\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil\vdash\tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

Note that Γ^1 and Γ^2 can be either a partition both with length i or one of them is nil. Suppose w.l.o.g. that $\Gamma^2=nil$ thus $\Gamma^1=\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil$ and

$$\dfrac{\dfrac{\forall 1\leq j\leq n, N:\langle nil\vdash\tau_j\rangle \qquad M_1:\langle\Gamma^1\vdash\omega\to\tau\rangle}{M_1\sigma^i N:\langle nil\vdash\omega\to\tau\rangle} \qquad \dfrac{N:\langle nil\vdash\tau_1\rangle \qquad M_2:\langle nil\vdash\rho\rangle}{M_2\sigma^i N:\langle nil\vdash\rho\rangle}}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle}$$

(b) For some $\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m=\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil$ one has:

$$\dfrac{\forall 1\leq j\leq n, N:\langle nil\vdash\tau_j\rangle \qquad \dfrac{M_1:\langle\Gamma'\vdash\wedge_{h=1}^m\sigma_h\to\tau\rangle \qquad \forall 1\leq h\leq m, M_2:\langle\Delta^h\vdash\sigma_h\rangle}{(M_1\ M_2):\langle\omega\,\underline{^{i-1}}.\wedge_{j=1}^n\tau_j.nil\vdash\tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

Suppose that $\Gamma'=nil$, $\Delta^1=\omega\,\underline{^{i-1}}.\wedge_{j=1}^{n_1}\tau_j.nil$, $\Delta^2=\omega\,\underline{^{i-1}}.\wedge_{j=(n_1+1)}^n\tau_j.nil$ and that $\Delta^j=nil,\forall n<j\leq m$. Any other possibility is handled similarly. Hence, one has

$$\mathcal{D}_1=\dfrac{\forall 1\leq j\leq n_1, N:\langle nil\vdash\tau_j\rangle \qquad M_2:\langle\Delta^1\vdash\sigma_1\rangle}{M_2\sigma^i N:\langle nil\vdash\sigma_1\rangle}$$

$$\mathcal{D}_2=\dfrac{\forall n_1<j\leq n, N:\langle nil\vdash\tau_j\rangle \qquad M_2:\langle\Delta^2\vdash\sigma_2\rangle}{M_2\sigma^i N:\langle nil\vdash\sigma_2\rangle}$$

and for each $n<j\leq m$ one has $\mathcal{D}_j=\dfrac{N:\langle nil\vdash\tau_1\rangle \qquad M_2:\langle nil\vdash\sigma_i\rangle}{M_2\sigma^i N:\langle nil\vdash\sigma_i\rangle}$ thus

$$\dfrac{\dfrac{N:\langle nil\vdash\tau_1\rangle \qquad M_1:\langle nil\vdash\wedge_{h=1}^m\sigma_h\to\tau\rangle}{M_1\sigma^i N:\langle nil\vdash\wedge_{h=1}^m\sigma_h\to\tau\rangle} \qquad \mathcal{D}_h, \forall 1\leq h\leq m}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle}$$

∎

Type information associated with the empty application disappears when becomes an empty substitution, since the rules (nil-σ) and (ω-σ) discard the corresponding contexts. Therefore, we need a restriction notion, related to available indices, to have an SR statement for the simulation of β-contraction.

Definition 4.10 (AI restriction)
Let $\Gamma\restriction_M$ be a $\Gamma' \sqsubseteq \Gamma$ s.t. $|\Gamma'| = sav(M)$ and that $\forall 1 \le i \le |\Gamma'|$, $\Gamma'_i \neq \omega$ iff $\underline{i} \in AI(M)$.

Remark 4.11
Since the manner in which intersection types are partitioned may vary, the restriction in Definition 4.10 above is not uniquely defined. For instance, $(\alpha \wedge \beta.nil)\restriction_{\underline{1}} = \alpha.nil, \beta.nil$ or the context itself.

Below, some properties of the AI restriction are presented.

Lemma 4.12
Let $M, M' \in \Lambda s$:

1. If $AI(M) = \emptyset$ then $\Gamma\restriction_M = nil$, for any context Γ.
2. If $M : \langle \Gamma \vdash_{\lambda_s{}^{SM}} \tau \rangle$ and $AI(M) = AI(M')$ then $(\Gamma \wedge \Delta)\restriction_{M'} = \Gamma$, for any context Δ.

Proof. Straightforward from Definition 4.10 in both cases. ∎

Theorem 4.13 (SR for simulation of β-contraction in λs^{SM})
If $(\lambda.M\ M') : \langle \Gamma \vdash_{\lambda_s{}^{SM}} \tau \rangle$ then $\{\underline{1}/M'\}M : \langle \Gamma\restriction_{\{\underline{1}/M'\}M} \vdash_{\lambda_s{}^{SM}} \tau \rangle$, for any $(\lambda.M\ M') \in \Lambda_{dB}$.

Proof. The proof consists in the verification of SR with context restriction for $(\lambda.M\ M') : \langle \Gamma \vdash_{\lambda_s{}^{SM}} \tau \rangle$ when the rule (σ-generation) is applied and then of SR for the s-calculus. Let $(\lambda.M\ M') : \langle \Gamma \vdash \tau \rangle$. By Lemma 4.6.4 one has two cases.

On the first case one has $\lambda.M : \langle \Gamma^1 \vdash \omega \to \tau \rangle$ and $M' : \langle \Gamma^2 \vdash \rho \rangle$, where $\Gamma = \Gamma^1 \wedge \Gamma^2$.

If $\Gamma^1 = nil$, then by Lemma 4.6.2 one has that $M : \langle nil \vdash \tau \rangle$ thus, by the rule (nil-σ), $M\sigma^1 M' : \langle nil \vdash \tau \rangle$. By Theorem 4.9 one has that $N : \langle nil \vdash \tau \rangle$ for any N s.t. $M\sigma^1 M' \to_s N$. Hence, by induction on the number of reduction steps in s one has that $s(M\sigma^1 M') : \langle nil \vdash \tau \rangle$, where $s(M\sigma^1 M') \equiv \{\underline{1}/M'\}M$. Note that, by Lemma 4.4, $AI(M) = \emptyset$ thus $AI(\{\underline{1}/M'\}M) = AI(M\sigma^1 M') = AI(M^{-1}) = \emptyset$. Hence, $(\Gamma^1 \wedge \Gamma^2)\restriction_{\{\underline{1}/M'\}M} = nil$.

If $|\Gamma^1| > 0$, then by Lemma 4.6.3 one has that $M : \langle \omega.\Gamma^1 \vdash \tau \rangle$ hence, by the rule (ω-σ), $M\sigma^1 M' : \langle \Gamma^1 \vdash \tau \rangle$. Hence, by Theorem 4.9, $s(M\sigma^1 M') : \langle \Gamma^1 \vdash \tau \rangle$. Note that, by Lemma 4.4, $\underline{1} \notin AI(M)$ thus $AI(\{\underline{1}/M'\}M) = AI(M\sigma^1 M') = AI(M^{-1}) = AI(\lambda.M)$. Hence, $(\Gamma^1 \wedge \Gamma^2)\restriction_{\{\underline{1}/M'\}M} = (\Gamma^1 \wedge \Gamma^2)\restriction_{\lambda.M} = \Gamma_1$.

On the second case one has $\lambda.M : \langle \Gamma^1 \vdash \wedge_{j=1}^m \sigma_j \to \tau \rangle$ and $\forall 1 \le j \le m$, $M' : \langle \Delta^j \vdash \sigma_j \rangle$, where $\Gamma = \Gamma^1 \wedge \Gamma^2$ for $\Gamma^2 = \Delta^1 \wedge \cdots \wedge \Delta^m$.

If $\Gamma^1 = nil$, then by Lemma 4.6.2 one has that $M_1 : \langle \wedge_{j=1}^m \sigma_j.nil \vdash \tau \rangle$. If $sav(M') = 0$ then by Lemma 4.4 one has that $\forall 1 \le j \le m$, $\Delta^j = nil$. Hence, by the rule ($\wedge$-nil-σ) one has that $M\sigma^1 M' : \langle nil \vdash \tau \rangle$ thus, by Theorem 4.9, $s(M\sigma^1 M') : \langle nil \vdash \tau \rangle$. Note that $\Gamma^1 \wedge \Gamma^2 = nil$. If $sav(M') > 0$, then by the rule ($\wedge$-σ) one has that $M\sigma^1 M' : \langle \Delta^1 \wedge \cdots \wedge \Delta^m \vdash \tau \rangle$. Hence, by Theorem 4.9, $s(M\sigma^1 M') : \langle \Delta^1 \wedge \cdots \wedge \Delta^m \vdash \tau \rangle$. Note that $AI(\{\underline{1}/M'\}M) = AI(M\sigma^1 M') = AI(\varphi_0^1 M') = AI(M')$. Hence, $(\Gamma^1 \wedge \Gamma^2)\restriction_{\{\underline{1}/M'\}M} = (\Gamma^1 \wedge \Gamma^2)\restriction_{M'} = \Gamma^2$.

If $|\Gamma^1| > 0$, then by Lemma 4.6.3 one has that $M : \langle \wedge_{j=1}^m \sigma_j.\Gamma^1 \vdash \tau\rangle$ thus, by the rule $(\wedge\text{-}\sigma)$, $M\sigma^1 M' : \langle \Gamma^1 \wedge (\Delta^1 \wedge \cdots \wedge \Delta^m) \vdash \tau\rangle$. Hence, by Theorem 4.9 one has that $s(M\sigma^1 M') : \langle \Gamma^1 \wedge \Gamma^2 \vdash \tau\rangle$. Note that $AI(\{\underline{1}/M'\}M) = AI(M\sigma^1 M') = AI(M^{-1}) \cup AI(\varphi_0^1 M') = AI(\lambda.M) \cup AI(M') = AI(\lambda.M\ M')$. ∎

Remark 4.14
Note that, w.r.t. β-reduction, the type information lost after β-contractions can affect the type as well. For instance, $\lambda.(\lambda.\underline{2}\ \underline{1}) : \langle nil \vdash (\alpha\wedge\beta)\to\alpha\rangle$, $\lambda.(\lambda.\underline{2}\ \underline{1}) \to_{\lambda s}^{+} \lambda.\underline{1}$ and $\lambda.\underline{1} : \langle nil \vdash \alpha\to\alpha\rangle$. Therefore, one would need a subtyping relation, and an associated inference rule, in order to obtain SR for β-reduction In the example above, $\alpha\wedge\beta \leq \alpha$ thus $\alpha\to\alpha \leq (\alpha\wedge\beta)\to\alpha$[6] and $\langle nil \vdash \alpha\to\alpha\rangle \leq \langle nil \vdash (\alpha\wedge\beta)\to\alpha\rangle$. Therefore $\lambda.\underline{1} : \langle nil \vdash (\alpha\wedge\beta)\to\alpha\rangle$ (cf. [11]).

4.2 The system $\lambda s_e{}^{\wedge}$

When applying λs^{SM} to the λs_e-calculus, the system does not satisfy SR due to the composition of operators. We present an example below, giving an intuition to how we changed the system λs^{SM}, and why, to obtain an IT system for λs_e with the SR property.

Example 4.15
Let $A \equiv (\underline{1}\ \underline{1})$, $M \equiv (\underline{3}\,\sigma^1 A)\,\sigma^1\lambda.A$, $M' \equiv (\underline{3}\,\sigma^2\lambda.A)\,\sigma^1(A\,\sigma^1\lambda.A)$. One has $M \to_{\lambda s_e} M'$, where M is typeable in λs^{SM} and M' is not typeable. One cannot obtain M' from M in λs while M is obtained from term $M_0 \equiv (\lambda.(\lambda.\underline{3}\ A)\ \lambda.A)$ in both calculi.

Remark 4.16
Non-typeability of the term M_0 above in λs^{SM} is due to the inclusion, by the rule $\to_e'$, of type information from the context of an argument to an empty application.

Typeability of both M_0 and $A\,\sigma^1\lambda.A$ in λs^{SM} reduces to typeability of $\Omega \equiv (\lambda.A\ \lambda.A)$ which has no type in systems like the Barendregt et al. [7] other than the universal ω type. Hence, we drop the typeability requirement on rules $\to_e'$, $(nil\text{-}\sigma)$ and $(\omega\text{-}\sigma)$, obtaining the system $\lambda s_e{}^{\wedge}$ below.

Definition 4.17 (The system $\lambda s_e{}^{\wedge}$)
The inference rules for $\lambda s_e{}^{\wedge}$ are given by the rules of the system λs^{SM} in Figure 1, where the inference rules $\to_e'$, $(nil\text{-}\sigma)$ and $(\omega\text{-}\sigma)$ are replaced by the rules below:

$$\frac{M : \langle\Gamma \vdash \omega\to\tau\rangle}{(M\ N) : \langle\Gamma\vdash\tau\rangle}\ \to_e^{\omega} \qquad (nil\text{-}\sigma)\ \frac{M : \langle\Gamma\vdash\tau\rangle}{M\sigma^i N : \langle\Gamma\vdash\tau\rangle},\ |\Gamma| \leq i$$

$$(\omega\text{-}\sigma)\ \frac{M : \langle\Gamma\vdash\tau\rangle}{M\sigma^i N : \langle\Gamma_{<i}.\Gamma_{>i}\vdash\tau\rangle},\ \Gamma_i = \omega$$

The system $\lambda s_e{}^{\wedge}$ is presented in the Figure 2.

The consequence of those changes is that the system $\lambda s_e{}^{\wedge}$ does not have a tight correspondence relating some syntactic characterisation and relevance. However, the system has a property related to relevance, stated below.

[6] counter-variant in the argument of functional types

$$\frac{}{\underline{1}:\langle \tau.nil \vdash \tau\rangle}\ \text{var} \qquad \frac{\underline{n}:\langle \Gamma \vdash \tau\rangle}{\underline{n+1}:\langle \omega.\Gamma \vdash \tau\rangle}\ \text{varn} \qquad \frac{M:\langle u.\Gamma \vdash \tau\rangle}{\lambda.M:\langle \Gamma \vdash u\to\tau\rangle}\ \to_i$$

$$\frac{M_1:\langle \Gamma \vdash \omega\to\tau\rangle}{(M_1\ M_2):\langle \Gamma \vdash \tau\rangle}\ \to_e^{\omega} \qquad \frac{M:\langle nil \vdash \tau\rangle}{\lambda.M:\langle nil \vdash \omega\to\tau\rangle}\ \to_i'$$

$$\frac{M_1:\langle \Gamma \vdash \wedge_{i=1}^{n}\sigma_i\to\tau\rangle \qquad M_2:\langle \Delta^1 \vdash \sigma_1\rangle\ \ldots\ M_2:\langle \Delta^n \vdash \sigma_n\rangle}{(M_1\ M_2):\langle \Gamma\wedge\Delta^1\wedge\cdots\wedge\Delta^n \vdash \tau\rangle}\ \to_e$$

$$(nil\text{-}\sigma)\ \frac{M:\langle \Gamma \vdash \tau\rangle}{M\sigma^i N:\langle \Gamma \vdash \tau\rangle},\ |\Gamma|<i \qquad (\omega\text{-}\sigma)\ \frac{M:\langle \Gamma \vdash \tau\rangle}{M\sigma^i N:\langle \Gamma_{<i}.\Gamma_{>i} \vdash \tau\rangle},\ \Gamma_i=\omega$$

$$(\wedge\text{-}nil\text{-}\sigma)\ \frac{N:\langle nil \vdash \sigma_1\rangle\ldots N:\langle nil \vdash \sigma_m\rangle \qquad M:\langle \omega^{\underline{i-1}}.\wedge_{j=1}^{m}\sigma_j.nil \vdash \tau\rangle}{M\sigma^i N:\langle nil \vdash \tau\rangle}$$

$$(\wedge\text{-}\omega\text{-}\sigma)\ \frac{N:\langle nil \vdash \sigma_1\rangle\ldots N:\langle nil \vdash \sigma_m\rangle \qquad M:\langle \Gamma \vdash \tau\rangle}{M\sigma^i N:\langle \Gamma_{<(i-k)}.nil \vdash \tau\rangle},\ \Gamma_i=\wedge_{j=1}^{m}\sigma_j\ (*)$$

$$(\wedge\text{-}\sigma)\ \frac{N:\langle \Delta^1 \vdash \sigma_1\rangle\ldots N:\langle \Delta^m \vdash \sigma_m\rangle \qquad M:\langle \Gamma \vdash \tau\rangle}{M\sigma^i N:\langle (\Gamma_{<i}.\Gamma_{>i})\wedge\omega^{\underline{i-1}}.(\Delta^1\wedge\cdots\wedge\Delta^m) \vdash \tau\rangle},\ \Gamma_i=\wedge_{j=1}^{m}\sigma_j\ (**)$$

$$(\omega\text{-}\varphi)\ \frac{M:\langle \Gamma \vdash \tau\rangle}{\varphi_k^i M:\langle \Gamma_{\le k}.\omega^{\underline{i-1}}.\Gamma_{>k} \vdash \tau\rangle},\ |\Gamma|>k \qquad (nil\text{-}\varphi)\ \frac{M:\langle \Gamma \vdash \tau\rangle}{\varphi_k^i M:\langle \Gamma \vdash \tau\rangle},\ |\Gamma|\le k$$

(*) $\Gamma=\Gamma_{<(i-k)}.\omega^{\underline{k}}.\wedge_{j=1}^{m}\sigma_j.nil$ and $\Gamma_{(i-k-1)}\neq\omega$ (**) $\Delta^k\neq nil$, for some $1\le k\le m$, or $\Gamma_{>i}\neq nil$

FIG. 2. Typing rules of the system $\lambda s_e{}^{\wedge}$

LEMMA 4.18
If $M:\langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ for $|\Gamma|=m>0$, then $m\le sav(M)$, $\Gamma_m\neq\omega$ and $\forall 1\le i\le m$, $\Gamma_i\neq\omega$ implies that $\underline{i}\in AI(M)$.

PROOF. By induction on the derivation of $M:\langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ when $\Gamma\neq nil$. We present the case for the application of the rule (ω-σ). Then, one has $M\sigma^i N:\langle \Gamma_{<i}.\Gamma_{>i} \vdash \tau\rangle$ where $M:\langle \Gamma \vdash \tau\rangle$ and $\Gamma_i=\omega$. Let $m=|\Gamma|$ and $\Gamma'=\Gamma_{<i}.\Gamma_{>i}$. By IH one has that $m\le sav(M)$, $\Gamma_m\neq\omega$ and $\forall 1\le j\le m$, $\Gamma_j\neq\omega$ implies that $\underline{j}\in AI(M)$. Hence, $\underline{m}\in AI(M)$ and $m>i$ thus $\underline{m}\in AI(M)_{>i}$. One has that $|\Gamma'|=m-1$ and that $\Gamma'_{m-1}=\Gamma_m\neq\omega$. By Definition 2.10 one has that $AI(M\sigma^i N)\supseteq AI(M^{-i})=AI(M)_{<i}\cup(AI(M)_{>i}\backslash 1)$ thus $sav(M\sigma^i N)\ge sav(M^{-i})\ge m-1$. For any $1\le j<i$ one has $\Gamma'_j=\Gamma_j\neq\omega$ implies that $\underline{j}\in AI(M)_{<i}\subseteq AI(M\sigma^i N)$ and for any $i\le j<(m-1)$ one has $\Gamma'_j=\Gamma_{j+1}\neq\omega$ implies that $\underline{j+1}\in AI(M)_{>i}$ thus $\underline{j}\in AI(M\sigma^i N)$. ∎

Since in the case of empty applications, handled by the rule $\to_e^{\omega}$, no type information about the argument is added to the context, the system satisfies the usual notion of the SR property. We prove the property in a standard way, proving some generation lemmas first, where only the $\Gamma_m\neq\omega$ piece of the Lemma 4.18 above is needed.

LEMMA 4.19 (Generation for $\lambda s_e{}^{\wedge}$)
1. If $\underline{n}:\langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ then $\Gamma=\omega^{\underline{n-1}}.\tau.nil$.

2. If $\lambda.M : \langle nil \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$, then $\tau = \omega \rightarrow \sigma$ and $M : \langle nil \vdash_{\lambda s_e{}^{\wedge}} \sigma\rangle$ or $\tau = \wedge_{i=1}^{n}\sigma_i \rightarrow \sigma$, $n > 0$, and $M : \langle \wedge_{i=1}^{n}\sigma_i.nil \vdash_{\lambda s_e{}^{\wedge}} \sigma\rangle$ for $\sigma, \sigma_1, \ldots, \sigma_n \in \mathcal{T}$.
3. If $\lambda.M : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ and $|\Gamma| > 0$, then $\tau = u \rightarrow \sigma$ for some $u \in \mathcal{U}$ and $\sigma \in \mathcal{T}$, where $M : \langle u.\Gamma \vdash_{\lambda s_e{}^{\wedge}} \sigma\rangle$.
4. If $(M\ N) : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ then $M : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \omega \rightarrow \tau\rangle$ or $\Gamma = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ s.t. $M : \langle \Gamma' \vdash_{\lambda s_e{}^{\wedge}} \wedge_{i=1}^{m}\sigma_i \rightarrow \tau\rangle$ and $\forall 1 \leq i \leq m$, $N : \langle \Delta^i \vdash_{\lambda s_e{}^{\wedge}} \sigma_i\rangle$.

Proof. 1. By induction on n. If $n = 1$, nothing to prove. Let $\underline{n{+}1} : \langle \Gamma \vdash \tau\rangle$. By the rule varn one has that $\Gamma = |\omega.\Gamma'|$, where $\underline{n} : \langle \Gamma' \vdash \tau\rangle$. Hence, by IH one has that $\Gamma' = \omega^{\underline{n-1}}.\tau.nil$ thus $\Gamma = \omega^{\underline{n}}.\tau.nil$. 2, 3, 4. By case analysis in the respective derivation. ∎

Remark 4.20

- Observe that the item 1 on Lemma 4.19 above is equivalent to Lemma 4.6.1 combined with Lemma 4.4 in system λs^{SM}.
- Even though items 2 and 3 are similar to Lemmas 4.6.2 and 4.6.3 for system λs^{SM}, the proper alternative in each case is linked to the sets $AI(\lambda.M)$ and $AI(M)$ on the latter ones, while we do not have this correspondence in the items above. The loss of this relation is a consequence of the rule $\rightarrow_e^{\omega}$, described by the property in item 4 above.

Lemma 4.21 (Generation for operators in $\lambda s_e{}^{\wedge}$)

1. Let $\varphi_k^i N : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$. If $|\Gamma| \leq k$, then $N : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$. If $|\Gamma| > k$ then $N : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$, where $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$.
2. If $M\sigma^i N : \langle nil \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$, then $M : \langle \omega^{\underline{i-1}}.\wedge_{j=1}^{m}\sigma_j.nil \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ where $\forall 1 \leq j \leq m$, $N : \langle nil \vdash_{\lambda s_e{}^{\wedge}} \sigma_j\rangle$ or $M : \langle nil \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$.
3. If $M\sigma^i N : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ and $0 < |\Gamma| < i$, then $M : \langle \Gamma' \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ where $\Gamma' = \Gamma.\omega^{\underline{n}}.\wedge_{j=1}^{m}\sigma_j.nil$ for $n \geq 0$ s.t. $|\Gamma'| = i$ and $\forall 1 \leq j \leq m$, $N : \langle nil \vdash_{\lambda s_e{}^{\wedge}} \sigma_j\rangle$ or $M : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$.
4. If $M\sigma^i N : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ and $|\Gamma| \geq i$ then $M : \langle \Gamma_{<i}.\omega.\Gamma_{\geq i} \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ or $\Gamma_{\geq i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ for $|\Gamma_{\geq i}| > 0$ s.t. $M : \langle \Gamma_{<i}.\wedge_{j=1}^{m}\sigma_j.\Gamma' \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ and $\forall 1 \leq j \leq m$, $N : \langle \Delta^j \vdash_{\lambda s_e{}^{\wedge}} \sigma_j\rangle$.

Proof. 1, 2, 3, 4. By case analysis on the respective derivation. ∎

Below, we present the subject reduction theorem for system $\lambda s_e{}^{\wedge}$.

Theorem 4.22 (SR for $\lambda s_e{}^{\wedge}$)
If $M : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$ and $M \rightarrow_{\lambda s_e} M'$, then $M' : \langle \Gamma \vdash_{\lambda s_e{}^{\wedge}} \tau\rangle$.

Proof. By the verification of SR for each λs_e rewriting rule.
We present the proof for the rule (σ-generation), to allow a comparison with the proof in Theorem 4.13 and for the rule (σ-app-transition), to compare with the proof for the same rule for system λs^{SM} presented in Theorem 4.9.
• (σ-generation): If $(\lambda.M\ N) : \langle \Gamma \vdash \tau\rangle$ we might prove that $M\sigma^1 N : \langle \Gamma \vdash \tau\rangle$. By Lemma 4.19.4 the last rule applied for $(\lambda.M\ N) : \langle \Gamma \vdash \tau\rangle$ is either $\rightarrow_e$ or $\rightarrow_e^{\omega}$. We present the latter, which represents the key to obtain SR for the simulation of β-reduction. Therefore, $\lambda.M : \langle \Gamma \vdash \omega \rightarrow \tau\rangle$ thus we need to consider the cases (1) $\Gamma = nil$ and (2) $|\Gamma| > 0$, related with Lemmas 4.19.2 and 4.19.3 respectively.

(1) If $\Gamma = nil$ then, by Lemma 4.19.2, the last rule applied is $\rightarrow'_i$ then

$$\dfrac{\dfrac{M:\langle nil \vdash \tau\rangle}{\lambda.M:\langle nil \vdash \omega\rightarrow\tau\rangle}}{(\lambda.M\ N):\langle nil \vdash \tau\rangle}$$

thus

$$\dfrac{M:\langle nil \vdash \tau\rangle}{M\sigma^1 N:\langle nil \vdash \tau\rangle}$$

(2) If $|\Gamma|>0$ then, by Lemma 4.19.3, the last rule applied is $\rightarrow_i$ then

$$\dfrac{\dfrac{M:\langle \omega.\Gamma \vdash \tau\rangle}{\lambda.M:\langle \Gamma \vdash \omega\rightarrow\tau\rangle}}{(\lambda.M\ N):\langle \Gamma \vdash \tau\rangle}$$

thus

$$\dfrac{M:\langle \omega.\Gamma \vdash \tau\rangle}{M\sigma^1 N:\langle \Gamma \vdash \tau\rangle}$$

• (σ-app-transition): We might prove that $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle\Gamma\vdash\tau\rangle$ whenever $(M_1\ M_2)\,\sigma^i N:\langle\Gamma\vdash\tau\rangle$. Hence, we need to consider the three possibilities for Γ, each one of them related with one generation lemma, the Lemmas 4.21.2, 4.21.3 and 4.21.4. Similarly to the proof in Theorem 4.9, we present here the case for $\Gamma = nil$. Then, by Lemma 4.21.2 the last rule applied is either (1) the (nil-σ) or (2) the ($\wedge$-nil-σ) rule.

(1) If the last rule applied is (nil-σ) then: $\dfrac{(M_1\ M_2):\langle nil\vdash\tau\rangle}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$

For $(M_1\ M_2):\langle nil\vdash\tau\rangle$, one has by Lemma 4.19.4 that the last rule applied is either (a) the rule $\rightarrow_e^\omega$ or (b) the rule $\rightarrow_e$. Hence:

(a)

$$\dfrac{\dfrac{M_1:\langle nil\vdash\omega\rightarrow\tau\rangle}{(M_1\ M_2):\langle nil\vdash\tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

thus

$$\dfrac{\dfrac{M_1:\langle nil\vdash\omega\rightarrow\tau\rangle}{M_1\sigma^i N:\langle nil\vdash\omega\rightarrow\tau\rangle}}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle}$$

(b)

$$\dfrac{\dfrac{M_1:\langle nil\vdash\wedge_{j=1}^m\sigma_j\rightarrow\tau\rangle \qquad \forall 1\leq j\leq m, M_2:\langle nil\vdash\sigma_j\rangle}{(M_1\ M_2):\langle nil\vdash\tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

thus

$$\dfrac{\dfrac{M_1:\langle nil\vdash\wedge_{j=1}^m\sigma_j\rightarrow\tau\rangle}{M_1\sigma^i N:\langle nil\vdash\wedge_{j=1}^m\sigma_j\rightarrow\tau\rangle} \qquad \dfrac{M_2:\langle nil\vdash\sigma_j\rangle}{M_2\sigma^i N:\langle nil\vdash\sigma_j\rangle},\forall 1\leq j\leq m}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle}$$

(2) If the last rule applied is ($\wedge$-nil-σ) then:

$$\dfrac{\forall 1\leq j\leq n, N:\langle nil\vdash\tau_j\rangle \qquad (M_1\ M_2):\langle \omega^{\underline{i-1}}.\wedge_{j=1}^n\tau_j.nil\vdash\tau\rangle}{(M_1\ M_2)\sigma^i N:\langle nil\vdash\tau\rangle}$$

For $(M_1\ M_2):\langle \omega \underline{\ i-1\ }.\wedge_{j=1}^{n} \tau_j.nil \vdash \tau\rangle$, one has by Lemma 4.19.4 that the last rule applied is either (a) the rule $\to_e^\omega$ or (b) the rule $\to_e$. Hence:

(a)

$$\dfrac{\forall 1\leq j\leq n, N:\langle nil \vdash \tau_j\rangle \qquad \dfrac{M_1:\langle \omega \underline{\ i-1\ }.\wedge_{j=1}^{n} \tau_j.nil \vdash \omega\to\tau\rangle}{(M_1\ M_2):\langle \omega \underline{\ i-1\ }.\wedge_{j=1}^{n} \tau_j.nil \vdash \tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil \vdash \tau\rangle}$$

thus

$$\dfrac{\dfrac{\forall 1\leq j\leq n, N:\langle nil \vdash \tau_j\rangle \qquad M_1:\langle \omega \underline{\ i-1\ }.\wedge_{j=1}^{n} \tau_j.nil \vdash \omega\to\tau\rangle}{M_1\sigma^i N:\langle nil \vdash \omega\to\tau\rangle}}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil \vdash \tau\rangle}$$

(b) For some $\Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m = \omega \underline{\ i-1\ }.\wedge_{j=1}^{n} \tau_j.nil$ one has:

$$\dfrac{\forall 1\leq j\leq n, N:\langle nil \vdash \tau_j\rangle \qquad \dfrac{M_1:\langle \Gamma' \vdash \wedge_{h=1}^{m}\sigma_h\to\tau\rangle \qquad \forall 1\leq h\leq m, M_2:\langle \Delta^h \vdash \sigma_h\rangle}{(M_1\ M_2):\langle \omega \underline{\ i-1\ }.\wedge_{j=1}^{n} \tau_j.nil \vdash \tau\rangle}}{(M_1\ M_2)\sigma^i N:\langle nil \vdash \tau\rangle}$$

As for the similar case in Theorem 4.9, suppose that $\Gamma' = nil$, $\Delta^1 = \omega \underline{\ i-1\ }.\wedge_{j=1}^{n_1} \tau_j.nil$, $\Delta^2 = \omega \underline{\ i-1\ }.\wedge_{j=(n_1+1)}^{n} \tau_j.nil$ and that $\forall n<j\leq m, \Delta^j = nil$. Then, one has

$$\mathcal{D}_1 = \dfrac{\forall 1\leq j\leq n_1, N:\langle nil \vdash \tau_j\rangle \qquad M_2:\langle \Delta^1 \vdash \sigma_1\rangle}{M_2\sigma^i N:\langle nil \vdash \sigma_1\rangle}$$

$$\mathcal{D}_2 = \dfrac{\forall n_1< j\leq n, N:\langle nil \vdash \tau_j\rangle \qquad M_2:\langle \Delta^2 \vdash \sigma_2\rangle}{M_2\sigma^i N:\langle nil \vdash \sigma_2\rangle}$$

and for each $n<j\leq m$ one has $\mathcal{D}_j = \dfrac{M_2:\langle nil \vdash \sigma_i\rangle}{M_2\sigma^i N:\langle nil \vdash \sigma_i\rangle}$ thus

$$\dfrac{\dfrac{M_1:\langle nil \vdash \wedge_{h=1}^{m}\sigma_h\to\tau\rangle}{M_1\sigma^i N:\langle nil \vdash \wedge_{h=1}^{m}\sigma_h\to\tau\rangle} \qquad \mathcal{D}_h, \forall 1\leq h\leq m}{((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil \vdash \tau\rangle}$$

∎

Since the proof of SR is stated inspecting each rule of the λs_e-calculus, this property holds when restricted to the λs-calculus. Therefore, we can state the SR for the system $\lambda s^\wedge$ defined below as a corollary from the Theorem 4.22 above.

Definition 4.23 (The system $\lambda s^\wedge$)
Let the system $\lambda s^\wedge$ be the system $\lambda s_e{}^\wedge$, introduced in Definition 4.17, restricted to the λs-calculus.

Corollary 4.24
The system $\lambda s^\wedge$ has the SR property.

Moreover, all generation lemmas stated in Lemmas 4.19 and 4.21 are satisfied by the system $\lambda s^\wedge$. In Subsection 5.2 we derive the properties for system $\lambda_{dB}^\wedge$ from the properties of system $\lambda s^\wedge$.

Remark 4.25
Differently from λs^{SM}, $\lambda s_e{}^{\wedge}$ (and consequently $\lambda s^{\wedge}$) owns the contextual closure for SR. Intuitively, for any reduction in a typeable term one needs to replace the subterm and its typing information by its reduct. Hence, either the corresponding typing tree, holding exactly the same conclusion as the original one since SR holds in the usual sense, or the term in the rules $\rightarrow_e^{\omega}$, (nil-σ) or (ω-σ) is replaced.

For instance, $\langle nil \vdash (\alpha \wedge \beta) \rightarrow \alpha \rangle$ is not a typing of $\lambda.(\lambda.\underline{2}\ \underline{1})$ in $\lambda s^{\wedge}$ but $\langle nil \vdash \alpha \rightarrow \alpha \rangle$ is a typing for both $\lambda.(\lambda.\underline{2}\ \underline{1})$ and $\lambda.\underline{1}$ in the system.

As a result, SR holds for the simulation of β-reduction in $\lambda_{dB}^{\wedge}$. See Subsection 5.2 for further discussion.

5 Intersection type systems for the λ_{dB}-calculus

In [57] we introduced an IT system for λ_{dB} called SM, based on the system of Sayag and Mauny [50], to characterise principal typings for β-nfs in the λ_{dB}-calculus. In [58] this system, then called λ_{dB}^{SM}, is the base for the system proposed for λs, called λs^{SM}. As presented in the previous section, when applying the system λs^{SM} to λs_e, the SR property is not satisfied. Hence, the system obtained satisfying SR is called $\lambda s_e{}^{\wedge}$ and its base is the system $\lambda_{dB}^{\wedge}$.

The system λ_{dB}^{SM} was proved to be relevant w.r.t. free indices thus a special notion of SR was proposed. Then, updating and substitution lemmas were proved in order to establish SR for the β-contraction in the λ_{dB}-calculus. Typing rules for operators in λs^{SM} were then based on the statements from those lemmas for λ_{dB}^{SM}. In the present work, we derive the properties of system λ_{dB}^{SM} from the system λs^{SM}. Although not necessary to prove SR here, we present both updating and substitution lemmas to show the correspondence with the typing rules of λs^{SM}.

The system $\lambda_{dB}^{\wedge}$ was introduced in [58] in a discussion regarding recovering the SR property in the usual sense. However, the property was not proved for $\lambda_{dB}^{\wedge}$ due to the missing of a strong relation between some syntactic characteristic and relevance. Hence, for the first time we prove SR for both β-contraction and β-reduction in the system. The proof is derived from system $\lambda s^{\wedge}$, the restriction of system $\lambda s_e{}^{\wedge}$ to the λs-calculus. The system $\lambda_{dB}^{\wedge}$ is a de Bruijn version of the IT system in [52] and although is claimed it owns SR, the proof presented in [51] has a mistake (cf. [57]).

5.1 The system λ_{dB}^{SM}

In contrast to the work in [58], the system λ_{dB}^{SM} is here presented as a restriction of system λs^{SM} to the λ_{dB}-calculus.

Definition 5.1 (The system λ_{dB}^{SM})
The system λ_{dB}^{SM} is formed by the rules var, varn, $\rightarrow_e$, $\rightarrow'_e$, $\rightarrow_i$ and $\rightarrow'_i$, introduced in Figure 1.

Hence, any typing of M in λ_{dB}^{SM} is a typing in λs^{SM}. The lemma below states that the inverse is also true whenever $M \in \Lambda_{dB}$.

Lemma 5.2
If $M \in \Lambda_{dB}$, then $M : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$ iff $M : \langle \Gamma \vdash_{\lambda_{dB}^{SM}} \tau \rangle$.

Proof. By induction on the derivation $M : \langle \Gamma \vdash_{\lambda s^{SM}} \tau \rangle$ for $M \in \Lambda_{dB}$. ∎

Therefore, one can obtain the properties of system λ^{SM}_{dB}, regarding the β-contraction as in Definition 2.6, from the properties of system λs^{SM}. Relevance [21, 23] w.r.t. free indices is the first one to be presented.

LEMMA 5.3 (Relevance for λ^{SM}_{dB} [57])
If $M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$, then $|\Gamma| = sup(M)$ and $\forall 1 \leq i \leq |\Gamma|$, $\Gamma_i \neq \omega$ iff $\underline{i} \in FI(M)$.

PROOF. If $M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$, then $\langle\Gamma \vdash \tau\rangle$ is a typing of M in λs^{SM}. Hence, by Lemma 4.4 one has that $|\Gamma| = sav(M)$ and $\forall 1 \leq i \leq |\Gamma|$, $\Gamma_i \neq \omega$ iff $\underline{i} \in AI(M)$. Since $M \in \Lambda_{dB}$ one has by Corollary 2.12 that $AI(M) = FI(M)$ thus $sav(M) = sup(M)$. Therefore, we have the relevance property for system λ^{SM}_{dB} regarding FI. ∎

LEMMA 5.4 (Updating)
Let $M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$, $k \in \mathbb{N}$ and $i \in \mathbb{N}^*$. If $k \geq |\Gamma|$ then $U^i_k(M):\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$. If $0 \leq k < |\Gamma|$ then $U^i_k(M):\langle\Gamma_{\leq k}.\underline{\omega^{i-1}}.\Gamma_{>k} \vdash_{\lambda^{SM}_{dB}} \tau\rangle$.

PROOF. If $M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$ then $M:\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$. By Lemma 2.9.3 one has that $s(\varphi^i_k M) = U^i_k(s(M))$ and since $M \in \Lambda_{dB}$ one has $s(M) = M$. If $k \geq |\Gamma|$ then by the rule (nil-φ) one has that $\varphi^i_k M:\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$ thus by SR of the s-calculus one has that $U^i_k(M):\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$. Observe that $U^i_k(M) \in \Lambda_{dB}$ thus $\langle\Gamma \vdash \tau\rangle$ is a typing of $U^i_k(M)$ in λ^{SM}_{dB}. The case when $0 \leq k < |\Gamma|$ is analogous, where the rule (nil-φ) is applied. ∎

The substitutions lemmas stated below are similar to the ones presented in [58].

LEMMA 5.5 (Substitution for λ^{SM}_{dB})
Let $M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$.

1. If $i > |\Gamma|$ then, for any $N \in \Lambda_{dB}$ typeable in λ^{SM}_{dB}, $\{\underline{i}/N\}M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$.
2. If $\Gamma_i = \omega$ where $0 < i < |\Gamma|$ then, for any $N \in \Lambda_{dB}$ typeable in λ^{SM}_{dB}, $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i} \vdash_{\lambda^{SM}_{dB}} \tau\rangle$.
3. Let $\Gamma_i = \wedge^m_{j=1}\sigma_j$, where $0 < i \leq |\Gamma|$, and $\forall 1 \leq j \leq m$, $N:\langle nil \vdash_{\lambda^{SM}_{dB}} \sigma_j\rangle$. If $sup(M) = i$ then $\{\underline{i}/N\}M:\langle\Gamma_{\leq k}.nil \vdash_{\lambda^{SM}_{dB}} \tau\rangle$ for $k = sup(\{\underline{i}/N\}M)$. Otherwise, $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i} \vdash_{\lambda^{SM}_{dB}} \tau\rangle$.
4. Let $\Gamma_i = \wedge^m_{j=1}\sigma_j$, where $0 < i \leq |\Gamma|$, and $N \in \Lambda_{dB}$ s.t. $sup(N) > 0$. If $\forall 1 \leq j \leq m$, $N:\langle\Delta^j \vdash_{\lambda^{SM}_{dB}} \sigma_j\rangle$ then for $\Delta' = \Delta^1 \wedge \cdots \wedge \Delta^m$ one has that $\{\underline{i}/N\}M:\langle(\Gamma_{<i}.\Gamma_{>i}) \wedge \underline{\omega^{i-1}}.\Delta' \vdash_{\lambda^{SM}_{dB}} \tau\rangle$.

PROOF. If $M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$ then $M:\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$.

1. If $N:\langle\Delta \vdash_{\lambda^{SM}_{dB}} \rho\rangle$ and $i > |\Gamma|$ then by the rule (nil-σ) one has $M\sigma^i N:\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$. Then, by SR for s one has that $s(M\sigma^i N):\langle\Gamma \vdash_{\lambda_s{}^{SM}} \tau\rangle$ where $s(M\sigma^i N) = \{\underline{i}/s(N)\}s(M) = \{\underline{i}/N\}M \in \Lambda_{dB}$. Hence, $\{\underline{i}/N\}M:\langle\Gamma \vdash_{\lambda^{SM}_{dB}} \tau\rangle$
2. Analogous to lemma 5.5.1, with the applications of the rule (ω-σ).
3. Note that $FI(\{\underline{i}/N\}M) = AI(M\sigma^i N)$ and by the relevance lemma 5.3 one has that $\underline{i} \in FI(M)$ and that $sup(N) = 0$.
   If $sup(M) = i$ then by relevance one has $|\Gamma| = i$. If $FI(M) = \{\underline{i}\}$ then also by the relevance of system λ^{SM}_{dB} one has that $\Gamma = \underline{\omega^{i-1}}.\wedge^m_{j=1}\sigma_j$. Hence, by the rule ($\wedge$-$nil$-$\sigma$) one has that $M\sigma^i N:\langle nil \vdash_{\lambda_s{}^{SM}} \tau\rangle$. Note that $FI(\{\underline{i}/N\}M) =$

$\emptyset$[7] thus $k = sup(\{\underline{i}/N\}M) = 0$. If $FI(M) \neq \{\underline{i}\}$ then let $k = sup(\{\underline{i}/N\}M)$. Observe that $FI(\{\underline{i}/N\}M) = FI(M)_{<i}$. Hence, by relevance one has that $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-k-1}}. \wedge_{j=1}^{m}\sigma_j.nil$ where $\Gamma_k \neq \omega$. Hence, by the rule ($\wedge$-ω-σ) one has that $M\sigma^i N : \langle \Gamma_{\leq k}.nil \vdash_{\lambda_s SM} \tau\rangle$.

If $sup(M) \neq i$ then $sup(M) > i$ thus by relevance $\Gamma_{>i} \neq nil$. Therefore, by the rule ($\wedge$-σ) one has that $M\sigma^i N : \langle (\Gamma_{<i}.\Gamma_{>i}) \wedge \omega^{\underline{i-1}}.\Delta' \vdash_{\lambda_s SM} \tau\rangle$ where $\Delta' = \Delta^1 \wedge \cdots \wedge \Delta^m$. Observe that $\Delta' = nil$ thus $M\sigma^i N : \langle \Gamma_{<i}.\Gamma_{>i} \vdash_{\lambda_s SM} \tau\rangle$.

In each case, if $M\sigma^i N : \langle \Gamma' \vdash_{\lambda_s SM} \tau\rangle$ then by SR for s, from the fact that $s(M\sigma^i N) = \{\underline{i}/N\}M$ and from Lemma 5.2 one has that $\{\underline{i}/N\}M : \langle \Gamma' \vdash_{\lambda_{dB}^{SM}} \tau\rangle$.

4. If $sup(N) > 0$ then by the relevance lemma 5.3 one has $\forall 1 \leq j \leq m$, $\Delta^k \neq nil$. Therefore, by the rule ($\wedge$-σ) one has for $\Delta' = \Delta^1 \wedge \cdots \wedge \Delta^m$ that $M\sigma^i N : \langle (\Gamma_{<i}.\Gamma_{>i}) \wedge \omega^{\underline{i-1}}.\Delta' \vdash_{\lambda_s SM} \tau\rangle$. Be Lemma 2.9.4 one has that $s(M\sigma^i N) = \{\underline{i}/N\}M$ thus by SR for the s-calculus $\{\underline{i}/N\}M : \langle (\Gamma_{<i}.\Gamma_{>i}) \wedge \omega^{\underline{i-1}}.\Delta' \vdash_{\lambda_s SM} \tau\rangle$. Therefore, by Lemma 5.2 this typing is in system λ_{dB}^{SM}.

∎

Among the differences from the substitution lemmas in [58], we have the ones related with empty substitutions in items 1 and 2 above. In [58] the property was stated for any $N \in \Lambda_{dB}$ while we only state them for some typeable N on items 1 and 2 above. This is due to the correspondence of each item to the rules (nil-σ) and (ω-σ), respectively. In fact, as in the case for lemma 5.4 above and the φ operator, each item in the substitution lemma 5.5 corresponds to some typing rule of λs^{SM} for the σ operator. Besides that, we have the β-contraction defined differently, using the updating function in Definition 2.4 instead of the i-lift as in [58]. The change is reflected on the statement of item 4 above.

Since N is typeable, items 1 and 2 represent the loss of its type information. Therefore, as for λs^{SM}, we need the restriction notion introduced below to establish SR.

Definition 5.6 (FI restriction)
Let Γ and M be a context and a term. The FI restriction of Γ to M, denoted by $\Gamma\downharpoonleft_M$, is a context $\Gamma' \sqsubseteq \Gamma$ s.t. $|\Gamma'| = sup(M)$ and that $\forall 1 \leq i \leq |\Gamma'|$, $\Gamma'_i \neq \omega$ iff $\underline{i} \in FI(M)$.

Although one has by Corollary 2.12 that AI and FI define the same set for terms in Λ_{dB}, we use a different notation to remark when the restriction is based on AI and on FI. The properties stated in Lemma 4.12 for $\Gamma\upharpoonright_M$ are inherited by $\Gamma\downharpoonleft_M$ defined above. Now, one can state SR for β-contraction.

Theorem 5.7 (SR for β-contraction in λ_{dB}^{SM})
If $(\lambda.M\ N) : \langle \Gamma \vdash_{\lambda_{dB}^{SM}} \tau\rangle$ then $\{\underline{1}/N\}M : \langle \Gamma\downharpoonleft_{\{\underline{1}/N\}M} \vdash_{\lambda_{dB}^{SM}} \tau\rangle$.

Proof. If $(\lambda.M\ N) : \langle \Gamma \vdash_{\lambda_{dB}^{SM}} \tau\rangle$ then $(\lambda.M\ N) : \langle \Gamma \vdash_{\lambda_s SM} \tau\rangle$. By Theorem 4.13 one has that $\{\underline{1}/N\}M : \langle \Gamma\upharpoonright_{\{\underline{1}/N\}M} \vdash_{\lambda_s SM} \tau\rangle$. One has that $\{\underline{1}/N\}M \in \Lambda_{dB}$ thus $\Gamma\upharpoonright_{\{\underline{1}/N\}M} = \Gamma\downharpoonleft_{\{\underline{1}/N\}M}$ and, by Lemma 5.2, $\{\underline{1}/N\}M : \langle \Gamma\downharpoonleft_{\{\underline{1}/N\}M} \vdash_{\lambda_{dB}^{SM}} \tau\rangle$. ∎

As remarked for system λs^{SM}, the type information lost during β-contraction demands a subtyping relation, and an associated inference rule, in order to obtain the SR property for the β-reduction.

[7] see Definition 2.10

5.2 The system $\lambda^{\wedge}_{dB}$

Analogous to the previous subsection, we now derive the properties of a system called $\lambda^{\wedge}_{dB}$ from the properties of $\lambda s_e{}^{\wedge}$. In fact, we only need the system $\lambda s^{\wedge}$, which is the restriction of system $\lambda s_e{}^{\wedge}$ to the λs-calculus.

DEFINITION 5.8 (The system $\lambda^{\wedge}_{dB}$)
The system $\lambda^{\wedge}_{dB}$ is obtained from system λ^{SM}_{dB}, replacing the rule $\to'_e$ by the rule $\to^{\omega}_e$ introduced in Figure 2.

Hence, $\lambda^{\wedge}_{dB}$ is defined to be a variation of system λ^{SM}_{dB}. Below we state that system $\lambda^{\wedge}_{dB}$ is a restriction of system $\lambda s^{\wedge}$ to the λ_{dB}-calculus.

LEMMA 5.9
If $M \in \Lambda_{dB}$ then $M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ iff $M : \langle \Gamma \vdash_{\lambda s^{\wedge}} \tau \rangle$.

Therefore, we can derive the properties for $\lambda^{\wedge}_{dB}$ from the system $\lambda s^{\wedge}$. The first property to be presented is related to relevance.

LEMMA 5.10
If $M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ and $|\Gamma| = m > 0$ then $m \leq sup(M)$, $\Gamma_m \neq \omega$ and $\forall 1 \leq i \leq |\Gamma|$, $\Gamma_i \neq \omega$ implies that $\underline{i} \in FI(M)$.

PROOF. If $M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ then $M : \langle \Gamma \vdash_{\lambda s^{\wedge}} \tau \rangle$. If $|\Gamma| = m > 0$ then by Lemma 4.18 one has that $m \leq sav(M)$, $\Gamma_m \neq \omega$ and $\forall 1 \leq i \leq |\Gamma|$, $\Gamma_i \neq \omega$ implies that $\underline{i} \in AI(M)$. Since $M \in \Lambda_{dB}$ one has that $AI(M) = FI(M)$ and $sav(M) = sup(M)$. ∎

THEOREM 5.11 (SR for β-contraction in $\lambda^{\wedge}_{dB}$)
If $(\lambda.M\ N) : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ then $\{\underline{1}/N\}M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$.

PROOF. The proof is similar to the one for Theorem 4.13. The main difference is that we have the SR property for the (σ-generation) rule in $\lambda s^{\wedge}$.

If $(\lambda.M\ M') : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ then $(\lambda.M\ M') : \langle \Gamma \vdash_{\lambda s^{\wedge}} \tau \rangle$. By Corollary 4.24 one has $M\sigma^1 M' : \langle \Gamma \vdash_{\lambda s^{\wedge}} \tau \rangle$. Note that $M, M' \in \Lambda_{dB}$ thus $s(M\sigma^1 M') = \{\underline{1}/M'\}M$. Hence, by Corollary 4.24, $\{\underline{1}/M'\}M : \langle \Gamma \vdash_{\lambda s^{\wedge}} \tau \rangle$. Since $\{\underline{1}/M'\}M \in \Lambda_{dB}$, by Lemma 5.9 one has that $\{\underline{1}/M'\}M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$. ∎

In contrast to λ^{SM}_{dB}, we can establish the SR property for the β-reduction in system $\lambda^{\wedge}_{dB}$ without a definition of some subtyping relation.

REMARK 5.12
All items in Lemma 4.19 are valid for system $\lambda^{\wedge}_{dB}$.

THEOREM 5.13 (SR for β-reduction in $\lambda^{\wedge}_{dB}$)
If $M : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$ and $M \to_{\beta} N$ then $N : \langle \Gamma \vdash_{\lambda^{\wedge}_{dB}} \tau \rangle$.

PROOF. By induction on the derivation of $M \to_{\beta} N$ (see Definition 2.7). ∎

6 An intersection type system for $\lambda\sigma$

P.-A. Melliès presented in [40] a counter-example in the $\lambda\sigma$-calculus for the PSN property where some term, corresponding to a simply typed term in the λ-calculus, has an infinity reduction strategy. Therefore, as in Section 4 we aim for a typing system satisfying SR which is as restricted as possible. We end up with an IT system, called $\lambda\sigma^{\wedge}$, with a property related to relevance (cf. Lemma 6.5). To begin with, the development process to obtain the system is presented.

$$\frac{S:\langle\Gamma \rhd \Gamma'\rangle \qquad M:\langle\Gamma' \vdash \tau\rangle}{M[S]:\langle\Gamma \vdash \tau\rangle}\text{ (clos)} \qquad \frac{M:\langle\Gamma \vdash \tau\rangle \qquad S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Gamma \wedge \Delta \rhd \tau.\Delta'\rangle}\text{ (cons)}$$

$$\frac{}{id:\langle\Gamma \rhd \Gamma\rangle}\text{ (id)} \qquad \frac{}{\uparrow:\langle\omega.\Gamma \rhd \Gamma\rangle}\text{ (ω-shift)} \qquad \frac{S:\langle\Gamma \rhd \Gamma''\rangle \qquad S':\langle\Gamma'' \rhd \Gamma'\rangle}{S' \circ S:\langle\Gamma \rhd \Gamma'\rangle}\text{ (comp)}$$

FIG. 3. Typing rules for system $\lambda\sigma_r^\wedge$

6.1 Towards an IT system for $\lambda\sigma$

The first approach to obtain an IT system for $\lambda\sigma$ is to extend the system λ_{dB}^{SM} to type the expressions of sort substitution and the closure term, presented in Figure 3, obtaining the system $\lambda\sigma_r^\wedge$.

The only difference from the typing rules for $\lambda\sigma^{\rightarrow}$, the simple type system for $\lambda\sigma$ (cf. [24]), is the rule (ω-shift): only ω's are shifted here in order to guarantee the context $\omega^{\underline{n}}.\tau.nil$ in a typing of $\underline{1}\,[\uparrow^n]$. Now, by the semantics of cons and the previous experience with the systems for $\lambda s/\lambda s_e$, we need typing rules for intersection types and empty substitutions. The first modification in system $\lambda\sigma_r^\wedge$ is to replace the rule (cons) by the following two rules:

$$\frac{M:\langle\Gamma \vdash \tau\rangle \qquad S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Gamma \wedge \Delta \rhd \omega.\Delta'\rangle}\text{ (ω-cons)}$$

$$\frac{M:\langle\Delta^1 \vdash \sigma_1\rangle \ \dots\ M:\langle\Delta^n \vdash \sigma_n\rangle \qquad S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Delta^1 \wedge \cdots \wedge \Delta^n \wedge \Delta \rhd (\wedge_{i=1}^n \sigma_i).\Delta'\rangle}\text{ ($\wedge$-cons)}$$

In fact, if the rule (ω-cons) is defined as above, one would have a problem of duplication of contexts analogous to the one discussed in Subsection 4.1. However, we present a counter-example below exploring the semantics of the sort substitution, which represents a list of substitutions.

EXAMPLE 6.1
Let $\Gamma = \alpha.nil$ and suppose that $\dfrac{\lambda.\underline{2}:\langle\Gamma \vdash \omega \to \alpha\rangle \qquad \underline{1}.id:\langle\Gamma \rhd \Gamma\rangle}{(\lambda.\underline{2})[\underline{1}\,.\,id]:\langle\Gamma \vdash \omega \to \alpha\rangle}$ where $\underline{2}$ denotes $\underline{1}\,[\uparrow]$. One has $(\lambda.\underline{2})[\underline{1}\,.\,id] \to_{(Abs)} \lambda.\underline{2}\,[\underline{1}.(\underline{1}\,.\,id)\circ\uparrow]$ and, by the rules (ω-shift) and (comp), $(\underline{1}\,.\,id)\circ\uparrow:\langle\omega.\Gamma \rhd \Gamma\rangle$. By the rule (var) one has $\underline{1}:\langle\sigma.nil \vdash \sigma\rangle$ for any $\sigma \in \mathcal{T}$. Hence, by the rule (ω-cons), $\underline{1}.(\underline{1}\,.\,id)\circ\uparrow:\langle\sigma.\Gamma \rhd \omega.\Gamma\rangle$. One also has $\underline{2}:\langle\omega.\Gamma \vdash \alpha\rangle$ thus, by the rules (clos) and $\to_i$, one has that $\lambda.\underline{2}\,[\underline{1}.(\underline{1}\,.\,id)\circ\uparrow]:\langle\Gamma \vdash \sigma \to \alpha\rangle$.

Consequently, besides that SR is not satisfied, a kind of weakening is introduced in the type system. As done before for the system λs^{SM}, the solution is to *"forget"* the context information of a term associated with ω thus related to an empty substitution. Therefore, we introduce the new (ω-cons) rule:

$$\frac{M:\langle\Gamma \vdash \tau\rangle \qquad S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Delta \rhd \omega.\Delta'\rangle}\text{ (ω-cons)}$$

One is still not able to type a closure when the typing of the corresponding term has context nil. At this point, there are two ways to change the system to allow this typing inference.

One way is to add the side condition $\Delta' \neq \omega^{\underline{n}}$, $\forall n \in \mathbb{N}$ to the rule (ω-cons), introducing the following rule:

$$\frac{M\,{:}\,\langle\Gamma\vdash\tau\rangle \qquad S\,{:}\,\langle\Delta\rhd nil\rangle}{M.S\,{:}\,\langle\Delta\rhd nil\rangle}\ (nil\text{-cons})$$

In this case, we need to change the rule (ω-shift) in a similar way, obtaining the two rules below:

$$\frac{\Gamma\neq\Delta.\omega^{\underline{n}}}{\uparrow\,{:}\,\langle\omega.\Gamma\rhd\Gamma\rangle}\ (\omega\text{-shift}) \qquad\qquad \uparrow\,{:}\,\langle nil\rhd nil\rangle\ (nil\text{-shift})$$

If we do not change the rule (ω-shift), we would have $\underline{1}\,.\!\uparrow\,{:}\,\langle\omega.nil\rhd nil\rangle$ even though $\underline{1}\,.\!\uparrow\,\to_{(VarShift)} id$.

The second way is to change both rules (clos) and (comp) as follows:

$$\frac{S\,{:}\,\langle\Gamma\rhd\Gamma'.\omega^{\underline{n}}\rangle \qquad M\,{:}\,\langle\Gamma'\vdash\tau\rangle}{M[S]\,{:}\,\langle\Gamma\vdash\tau\rangle}\ (\omega\text{-clos})$$

$$\frac{S\,{:}\,\langle\Gamma\rhd\Gamma'.\omega^{\underline{n}}\rangle \qquad S'\,{:}\,\langle\Gamma'\rhd\Gamma''\rangle}{S'\circ S\,{:}\,\langle\Gamma\rhd\Gamma''\rangle}\ (\omega\text{-comp})$$

Let $\lambda\sigma^{\wedge}_{SM}$ be the system with the nil rules and $\lambda\sigma^{\wedge}_{\omega}$ be the system with the rules (ω-comp) and (ω-clos). Both approaches have similar properties thus similar problems. The SR property is satisfied to all but the rule $(MapEnv)$[8] in the $\lambda\sigma$-calculus in both cases. Let $(M.S)\circ S'$ be such that, in either system $\lambda\sigma^{\wedge}_{SM}$ or system $\lambda\sigma^{\wedge}_{\omega}$:

$$\frac{M.S\,{:}\,\langle\Gamma'\rhd\omega.\Gamma''\rangle \qquad S'\,{:}\,\langle\Gamma\rhd\Gamma'\rangle}{(M.S)\circ S'\,{:}\,\langle\Gamma\rhd\omega.\Gamma''\rangle}$$

By the rule (ω-cons) one has $S\,{:}\,\langle\Gamma'\rhd\Gamma''\rangle$ and M typeable. Hence, $S\circ S'\,{:}\,\langle\Gamma\rhd\Gamma''\rangle$ but there is no guarantee that $M[S']$ is typeable in any of those two system. We present a counter-example in $\lambda\sigma^{\wedge}_{\omega}$ as follows.

Example 6.2

Let A be the self-application $(\underline{1}\ \underline{1})$, $S_1 \equiv A.id$ and $S_2 \equiv (\lambda.A).id$. We then have that $S_1 : \langle\omega.nil \rhd_{\lambda\sigma^{\wedge}_{\omega}} \omega^{\underline{2}}.nil\rangle$ and $S_2 : \langle nil \rhd_{\lambda\sigma^{\wedge}_{\omega}} \omega.nil\rangle$. Therefore, by the rule (ω-comp), $S_1\circ S_2\,{:}\,\langle nil\rhd\omega^{\underline{2}}.nil\rangle$. Observe that $S_1\circ S_2 \to_{(MapEnv)} A[S_2].(id\circ S_2)$ and $id\circ S_2\,{:}\,\langle nil\vdash_{\lambda\sigma^{\wedge}_{\omega}} nil\rangle$. Typeability of $A[S_2]$ depends on the unification of the context in the typing of term A with the type in the typing of substitution S_2, which reduces to the unification of $(\alpha\to\beta)\wedge\alpha$ with $(\alpha'\to\beta')\wedge\alpha'\to\beta'$. Therefore, the typeability problem for $A[S_2]$ reduces to the typeability of the self-replicator $\Omega\equiv(\lambda.A\ \lambda.A)$ in IT systems thus not typeable but with ω (cf. [19]).

The example above would not occur as a subexpression of any term derived by the $\lambda\sigma$ rewriting rules from a term corresponding to a typed term in the λ-calculus. Let M be any term such that $M : \langle nil \vdash_{\lambda\sigma^{\wedge}_{\omega}} \tau\rangle$. Hence, for S_1 and S_2 as in the Example 6.2 above, one has $(M[S_1])[S_2] : \langle nil \vdash_{\lambda\sigma^{\wedge}_{\omega}} \tau\rangle$ and $\lambda.M : \langle nil \vdash_{\lambda\sigma^{\wedge}_{\omega}} \omega\to\tau\rangle$. Analysing the rewriting rules in $\lambda\sigma$, the only way to obtain $(M[S_1])[S_2]$ is from term $M' \equiv \big(\lambda.\big(\lambda.M\ A\big)\ \lambda.A\big)$, where A is the self-application. Let

$$\cfrac{\cfrac{\lambda.M\,{:}\,\langle nil\vdash\omega\to\tau\rangle \qquad A\,{:}\,\langle(\alpha\to\beta)\wedge\alpha.nil\vdash\beta\rangle}{(\lambda.M\ A)\,{:}\,\langle(\alpha\to\beta)\wedge\alpha.nil\vdash\tau\rangle}}{\lambda.(\lambda.M\ A)\,{:}\,\langle nil\vdash((\alpha\to\beta)\wedge\alpha)\to\tau\rangle}$$

[8]The rule is pointed out by Melliès as the reason why $\lambda\sigma$ is not PSN (cf. [40]).

Typeability of M' is reduced to the typeability problem of the self-replicator in IT thus M' is not typeable. The same expressions in $\lambda\sigma$ compounds an analogous counter-example for system $\lambda\sigma^{\wedge}_{SM}$.

REMARK 6.3
Non-typeability of M' in both systems presented here is due to the definition of the rule $\to'_e$, which adds the type information from the context in a typing of an argument to an empty application.

We can then change the rules (ω-cons) and (nil-cons), dropping from the respective premises the requirement that M has to be typeable, obtaining:

$$(nil\text{-cons})\ \frac{S:\langle\Delta \rhd nil\rangle}{M.S:\langle\Delta \rhd nil\rangle} \qquad (\omega\text{-cons})\frac{S:\langle\Delta \rhd \Delta'\rangle}{M.S:\langle\Delta \rhd \omega.\Delta'\rangle}, \text{where } \Delta' \neq \omega^{\underline{n}}$$

Although WN expressions in $\lambda\sigma$, corresponding to terms in the λ-calculus, are not typeable neither in $\lambda\sigma^{\wedge}_{\omega}$ nor in $\lambda\sigma^{\wedge}_{SM}$, the Melliés example [40] is typeable in both systems.

Since a characterisation of SN terms in $\lambda\sigma$ is not possible, we replace the rule $\to'_e$ by $\to^{\omega}_e$ below:

$$\frac{M:\langle\Gamma \vdash \omega \to \tau\rangle}{(M\ N):\langle\Gamma \vdash \tau\rangle}\ \to^{\omega}_e$$

Similar to $\lambda s_e{}^{\wedge}$ in relation to λs^{SM}, there is no straightforward relation between typing contexts and syntactic properties of terms, because we have no type information about free indices of a term applied to ω. In Subsection 6.2 we present the system $\lambda\sigma^{\wedge}$, based on $\lambda\sigma^{\wedge}_{SM}$ with the rule $\to^{\omega}_e$ described above. Even though we do not have the exact notion of what should relevance be for a system such as $\lambda\sigma^{\wedge}$, we prove a property similar to Lemma 4.18 for $\lambda s_e{}^{\wedge}$, about the last element of a non-nil context.

6.2 The system $\lambda\sigma^{\wedge}$

Similar to the IT system proposed for λs_e, the system for $\lambda\sigma$ discards any type information from contexts of terms related to empty applications.

DEFINITION 6.4 (The system $\lambda\sigma^{\wedge}$)
The typing rules for $\lambda\sigma^{\wedge}$ are presented in Figure 4, where $m > 0$ and $n \geq 0$.

Note that both rules (id) and (ω-shift) include side conditions such that contexts ending with an omega context are precluded. Note also that the context nil is excluded from (ω-shift) but allowed on rule (id).

Below, a lemma stating a property of $\lambda\sigma^{\wedge}$ related to relevance.

LEMMA 6.5
If $M:\langle\Gamma \vdash_{\lambda\sigma^{\wedge}} \tau\rangle$ and $|\Gamma| = m > 0$, then $\Gamma_m \neq \omega$. In particular, if $S:\langle\Gamma \rhd_{\lambda\sigma^{\wedge}} \Gamma'\rangle$ and $|\Gamma| = m > 0$ then $\Gamma_m \neq \omega$ and if $|\Gamma'| = m' > 0$ then $\Gamma'_{m'} \neq \omega$.

PROOF. By induction on the derivation of $M:\langle\Gamma \vdash_{\lambda\sigma^{\wedge}} \tau\rangle$ when $\Gamma \neq nil$, with subinduction on the derivation of $S:\langle\Gamma \rhd_{\lambda\sigma^{\wedge}} \Gamma'\rangle$ when $\Gamma \neq nil$ or $\Gamma' \neq nil$. ■

COROLLARY 6.6
If $M:\langle\Gamma \vdash_{\lambda\sigma^{\wedge}} \tau\rangle$ then $\Gamma \neq \Delta.\omega^{\underline{m}}$, for any context Δ and $m > 0$. In particular, if $S:\langle\Gamma \rhd_{\lambda\sigma^{\wedge}} \Gamma'\rangle$ then $\Gamma \neq \Delta.\omega^{\underline{m}}$ and $\Gamma' \neq \Delta'.\omega^{\underline{m}}$, for any contexts Δ and Δ', and $m > 0$.

$$\frac{}{\underline{1}:\langle \tau.nil \vdash \tau\rangle}\ (\text{var}) \qquad \frac{M:\langle u.\Gamma \vdash \tau\rangle}{\lambda.M:\langle \Gamma \vdash u\to\tau\rangle}\ \to_i$$

$$\frac{M_1:\langle \Gamma \vdash \omega\to\tau\rangle}{(M_1\ M_2):\langle \Gamma \vdash \tau\rangle}\ \to_e^{\omega} \qquad \frac{M:\langle nil \vdash \tau\rangle}{\lambda.M:\langle nil \vdash \omega\to\tau\rangle}\ \to_i'$$

$$\frac{M_1:\langle \Gamma \vdash \wedge_{i=1}^{m}\sigma_i\to\tau\rangle \qquad M_2:\langle \Delta^1 \vdash \sigma_1\rangle\ \ldots\ M_2:\langle \Delta^m \vdash \sigma_m\rangle}{(M_1\ M_2):\langle \Gamma\wedge\Delta^1\wedge\cdots\wedge\Delta^m \vdash \tau\rangle}\ \to_e$$

$$(\text{clos})\ \frac{S:\langle \Gamma \rhd \Gamma'\rangle \qquad M:\langle \Gamma' \vdash \tau\rangle}{M[S]:\langle \Gamma \vdash \tau\rangle}$$

$$(\wedge\text{-cons})\ \frac{M:\langle \Delta^1 \vdash \sigma_1\rangle\ \ldots\ M:\langle \Delta^m \vdash \sigma_m\rangle \qquad S:\langle \Delta \rhd \Delta'\rangle}{M.S:\langle \Delta\wedge\Delta^1\wedge\cdots\wedge\Delta^m \rhd (\wedge_{i=1}^{m}\sigma_i).\Delta'\rangle}$$

$$(\text{id})\ \frac{\Gamma\neq\Delta.\omega^{\underline{m}}}{id:\langle \Gamma \rhd \Gamma\rangle} \qquad (\text{comp})\ \frac{S:\langle \Gamma \rhd \Gamma''\rangle \qquad S':\langle \Gamma'' \rhd \Gamma'\rangle}{S'\circ S:\langle \Gamma \rhd \Gamma'\rangle}$$

$$(nil\text{-shift})\ \frac{}{\uparrow:\langle nil \rhd nil\rangle} \qquad (nil\text{-cons})\ \frac{S:\langle \Delta \rhd nil\rangle}{M.S:\langle \Delta \rhd nil\rangle}$$

$$(\omega\text{-shift})\ \frac{\Gamma\neq\Delta.\omega^{\underline{n}}}{\uparrow:\langle \omega.\Gamma \rhd \Gamma\rangle} \qquad (\omega\text{-cons})\ \frac{S:\langle \Delta \rhd \Delta'\rangle}{M.S:\langle \Delta \rhd \omega.\Delta'\rangle},\ \Delta'\neq\omega^{\underline{n}}$$

FIG. 4. The inference rules for the system $\lambda\sigma^{\wedge}$

Side conditions for rules (id) and (ω-shift) guarantee the property described on Corollary 6.6 for every typeable substitution in $\lambda\sigma^{\wedge}$. This property is not necessary in order to obtain SR for the typing system, in which the side condition $\Gamma\neq\omega^{\underline{m}}$ for (id), where $m>0$, would be enough. In this case, Lemma 6.5, and consequently the corollary above, would be satisfied by terms and only by substitutions when applied to terms.

The proof of SR for system $\lambda\sigma^{\wedge}$ is standard, where some generation lemmas are stated before. The lemmas for substitutions are presented first and then the ones for terms.

LEMMA 6.7 (Generation for substitutions in $\lambda\sigma^{\wedge}$)
1. $S:\langle nil \rhd nil\rangle$ for any substitution S.
2. If $M.S:\langle \Gamma \rhd nil\rangle$ then $S:\langle \Gamma \rhd nil\rangle$.
3. If $M.S:\langle \Gamma \rhd \omega.\Gamma'\rangle$ then $S:\langle \Gamma \rhd \Gamma'\rangle$ and $\Gamma'\neq\omega^{\underline{n}}$.
4. If $M.S:\langle \Gamma \rhd \Gamma'\rangle$ for $\Gamma'=\wedge_{i=1}^{m}\sigma_i.\Gamma''$ then $S:\langle \Gamma''' \rhd \Gamma''\rangle$ and $\forall 1\leq i\leq m$, $M:\langle \Gamma^i \vdash \sigma_i\rangle$ such that $\Gamma=\Gamma'''\wedge\Gamma^1\wedge\cdots\wedge\Gamma^m$.
5. If $S:\langle \Gamma \rhd nil\rangle$ then $\Gamma=nil$.
6. $\uparrow^m:\langle \Gamma \rhd \Gamma'\rangle$ iff either $\Gamma=\Gamma'=nil$ or $\Gamma=\omega^{\underline{m}}.\Gamma'$, where $\Gamma'\neq\Delta.\omega^{\underline{n}}$.
7. If $S:\langle \Gamma \rhd \Gamma'\rangle$ and $S:\langle \Delta \rhd \Delta'\rangle$ then $S:\langle \Gamma\wedge\Delta \rhd \Gamma'\wedge\Delta'\rangle$.
8. If $S:\langle \Gamma \rhd \Delta^1\wedge\Delta^2\rangle$ for $\Delta^1\neq\Delta'.\omega^{\underline{m}}$ and $\Delta^2\neq\Delta''.\omega^{\underline{m}}$, then $\Gamma=\Gamma^1\wedge\Gamma^2$ such that $S:\langle \Gamma^1 \rhd \Delta^1\rangle$ and $S:\langle \Gamma^2 \rhd \Delta^2\rangle$.

PROOF. 1,5,7,8. By induction on the structure of S. 2,3,4. By case analysis on the respective derivation. 6. By induction on m taking item 5 as the induction base. ∎

LEMMA 6.8 (Generation for terms in $\lambda\sigma^{\wedge}$)

1. $\underline{1}\,[\uparrow^m]:\langle\Gamma \vdash_{\lambda\sigma^\wedge} \tau\rangle$ iff $\Gamma = \omega^{\,\underline{m}}.\tau.nil$.
2. If $\lambda.M:\langle nil \vdash_{\lambda\sigma^\wedge} \tau\rangle$, then $\tau=\omega\to\sigma$ and $M:\langle nil \vdash_{\lambda\sigma^\wedge} \sigma\rangle$ or $\tau=\wedge_{i=1}^n\sigma_i\to\sigma$, $n>0$, and $M:\langle\wedge_{i=1}^n\sigma_i.nil \vdash_{\lambda\sigma^\wedge} \sigma\rangle$ where $\sigma,\sigma_1,\dots,\sigma_n\in\mathcal{T}$.
3. If $\lambda.M:\langle\Gamma \vdash_{\lambda\sigma^\wedge} \tau\rangle$ and $|\Gamma|>0$, then $\tau=u\to\sigma$ for some $u\in\mathcal{U}$ and $\sigma\in\mathcal{T}$, where $M:\langle u.\Gamma \vdash_{\lambda\sigma^\wedge} \sigma\rangle$.
4. If $(M\ N):\langle\Gamma \vdash_{\lambda\sigma^\wedge} \tau\rangle$ then $M:\langle\Gamma \vdash_{\lambda\sigma^\wedge} \omega\to\tau\rangle$ or $M:\langle\Gamma' \vdash_{\lambda\sigma^\wedge} \wedge_{i=1}^m\sigma_i\to\tau\rangle$ and $\forall 1\le i\le m$, $N:\langle\Gamma^i \vdash_{\lambda\sigma^\wedge} \sigma_i\rangle$ where $\Gamma=\Gamma'\wedge\Gamma^1\wedge\cdots\wedge\Gamma^m$.

PROOF. 1. Suppose that $\underline{1}\,[\uparrow^m]:\langle\Gamma\vdash\tau\rangle$. By the rule (comp) one has $\uparrow:\langle\Gamma\rhd\Gamma'\rangle$ and $\underline{1}:\langle\Gamma'\vdash\tau\rangle$. Therefore, by the rule (var) one has $\Gamma'=\tau.nil$ thus, by Lemma 6.7.6, $\Gamma=\omega^{\,\underline{m}}.\tau.nil$. 2,3,4. By case analysis on the respective derivation. ∎

Now the SR property can be established for $\lambda\sigma^{\wedge}$.

THEOREM 6.9 (SR for $\lambda\sigma^{\wedge}$)
If $M:\langle\Gamma \vdash_{\lambda\sigma^\wedge} \tau\rangle$ and $M\to_{\lambda\sigma} M'$ then $M':\langle\Gamma \vdash_{\lambda\sigma^\wedge} \tau\rangle$. In particular, if $S:\langle\Gamma \rhd_{\lambda\sigma^\wedge} \Gamma'\rangle$ and $S\to_{\lambda\sigma} S'$ then $S':\langle\Gamma \rhd_{\lambda\sigma^\wedge} \Gamma'\rangle$.

PROOF. By the verification of SR for each $\lambda\sigma$ rewriting rule. We present the proof-sketch for the (*Beta*) and (*App*) rules.
• *Beta*: If $(\lambda.M\ N):\langle\Gamma\vdash\tau\rangle$ we might prove that $M[N.id]:\langle\Gamma\vdash\tau\rangle$. By Lemma 6.8.4 the last rule applied is either $\to_e$ or $\to_e^\omega$. We present the proof for the latter one thus $\lambda.M:\langle\Gamma\vdash\omega\to\tau\rangle$. We need to consider the cases (1) $\Gamma=nil$ and (2) $|\Gamma|>0$, related with Lemmas 6.8.2 and 6.8.3 respectively.

(1) If $\Gamma=nil$ then, by Lemma 6.8.2, the last rule applied is $\to_i'$ then

$$\dfrac{\dfrac{M:\langle nil\vdash\tau\rangle}{\lambda.M:\langle nil\vdash\omega\to\tau\rangle}}{(\lambda.M\ N):\langle nil\vdash\tau\rangle}$$

thus

$$\dfrac{\dfrac{id:\langle nil\rhd nil\rangle}{N.id:\langle nil\rhd nil\rangle}\qquad M:\langle nil\vdash\tau\rangle}{M[N.id]:\langle nil\vdash\tau\rangle}$$

(2) If $|\Gamma|>0$ then, by Lemma 6.8.3, the last rule applied is $\to_i$ then

$$\dfrac{\dfrac{M:\langle\omega.\Gamma\vdash\tau\rangle}{\lambda.M:\langle\Gamma\vdash\omega\to\tau\rangle}}{(\lambda.M\ N):\langle\Gamma\vdash\tau\rangle}$$

thus

$$\dfrac{\dfrac{id:\langle\Gamma\rhd\Gamma\rangle}{N.id:\langle\Gamma\rhd\omega.\Gamma\rangle}\qquad M:\langle\omega.\Gamma\vdash\tau\rangle}{M[N.id]:\langle\Gamma\vdash\tau\rangle}$$

• *App*: If $(M_1\ M_2)[S]:\langle\Gamma\vdash\tau\rangle$ we might prove that $(M_1[S]\ M_2[S]):\langle\Gamma\vdash\tau\rangle$. By the rule (clos): $\dfrac{S:\langle\Gamma\rhd\Delta\rangle\qquad (M_1\ M_2):\langle\Delta\vdash\tau\rangle}{(M_1\ M_2)[S]:\langle\Gamma\vdash\tau\rangle}$. Hence, by Lemma 6.8.4 the last rule applied for $(M_1\ M_2):\langle\Delta\vdash\tau\rangle$ is either (1) $\to_e^\omega$ or (2) $\to_e$.

(1) If the last rule applied is $\to_e^\omega$ then

$$\frac{S:\langle\Gamma\rhd\Delta\rangle \qquad \dfrac{M_1:\langle\Delta\vdash\omega\to\tau\rangle}{(M_1\ M_2):\langle\Delta\vdash\tau\rangle}}{(M_1\ M_2)[S]:\langle\Gamma\vdash\tau\rangle}$$

thus

$$\frac{\dfrac{S:\langle\Gamma\rhd\Delta\rangle \qquad M_1:\langle\Delta\vdash\omega\to\tau\rangle}{M_1[S]:\langle\Gamma\vdash\omega\to\tau\rangle}}{(M_1[S]\ M_2[S]):\langle\Gamma\vdash\tau\rangle}$$

(2) If the last rule applied is $\to_e$ then, for $\Delta=\Delta'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$, one has

$$\frac{S:\langle\Gamma\rhd\Delta\rangle \qquad \dfrac{M_1:\langle\Delta'\vdash\wedge_{i=1}^m\sigma_i\to\tau\rangle \qquad \forall 1\leq i\leq m, M_2:\langle\Delta^i\vdash\sigma_i\rangle}{(M_1\ M_2):\langle\Delta\vdash\tau\rangle}}{(M_1\ M_2)[S]:\langle\Gamma\vdash\tau\rangle}$$

By Corollary 6.6 one has that $\Delta',\Delta^1,\ldots,\Delta^m\neq\Delta'''.\omega^{\underline{n}}$, for any context Δ''' and $n>0$. Hence, by induction on m and Lemma 6.7.8 one has $\Gamma=\Gamma'\wedge\Gamma^1\wedge\cdots\wedge\Gamma^m$ s.t. $S:\langle\Gamma'\rhd\Delta'\rangle$ and $\forall 1\leq i\leq m$, $S:\langle\Gamma^i\rhd\Delta^i\rangle$. Hence,

$$\frac{\dfrac{S:\langle\Gamma'\rhd\Delta'\rangle \qquad M_1:\langle\Delta'\vdash\wedge_{i=1}^m\sigma_i\to\tau\rangle}{M_1[S]:\langle\Gamma'\vdash\wedge_{i=1}^m\sigma_i\to\tau\rangle} \qquad \dfrac{S:\langle\Gamma^i\rhd\Delta^i\rangle \qquad M_2:\langle\Delta^i\vdash\sigma_i\rangle}{M_2[S]:\langle\Gamma^i\vdash\sigma_i\rangle},\forall 1\leq i\leq m}{(M_1[S]\ M_2[S]):\langle\Gamma\vdash\tau\rangle}$$

∎

7 Some considerations about intersection types and $\lambda\upsilon$

Although in [10] the relation of the $\lambda\upsilon$-calculus and de Bruijn's $C\lambda\xi\phi$ calculus ([13]) was investigated, in counterposition with what was called $\lambda\sigma$ family, $\lambda\upsilon$ can be seen as a $\lambda\sigma$ without composition of substitutions. Therefore, as for the systems presented in Section 4, we would like to investigate a relevant IT system for $\lambda\upsilon$, and compare with both systems λs^{SM} and $\lambda\sigma^\wedge$.

However, as in $\lambda\sigma$, is not easy to define some syntactic characteristic similar to the available indices for λs. Even though we could prove the SR property for the υ-calculus in an IT system called $\lambda\upsilon^{SM}$, it was not possible to present an appropriate notion of relevance and thus an appropriate notion of SR for it. Instead, we present the system $\lambda\upsilon^\wedge$, very much similar to $\lambda\sigma^\wedge$, and prove SR for the full $\lambda\upsilon$-calculus.

7.1 The system $\lambda\upsilon^\wedge$

Definition 7.1 (The system $\lambda\upsilon^\wedge$)
The typing rules for $\lambda\upsilon^\wedge$ are presented in Figure 5, where $m>0$ and $n\geq 0$.

Lemma 7.2
If $M:\langle\Gamma\vdash_{\lambda\upsilon^\wedge}\tau\rangle$ and $|\Gamma|=m>0$, then $\Gamma_m\neq\omega$. In particular, if $S:\langle\Gamma\rhd_{\lambda\upsilon^\wedge}\Gamma'\rangle$ and $|\Gamma|=m>0$ then $\Gamma_m\neq\omega$ and if $|\Gamma'|=m'>0$ then $\Gamma'_{m'}\neq\omega$.

$$\frac{}{\underline{1}:\langle \tau.nil \vdash \tau\rangle}\ \text{var} \qquad \frac{\underline{n}:\langle \Gamma \vdash \tau\rangle}{\underline{n+1}:\langle \omega.\Gamma \vdash \tau\rangle}\ \text{varn} \qquad \frac{M:\langle u.\Gamma \vdash \tau\rangle}{\lambda.M:\langle \Gamma \vdash u\to\tau\rangle}\ \to_i$$

$$\frac{M_1:\langle \Gamma \vdash \omega\to\tau\rangle}{(M_1\ M_2):\langle \Gamma \vdash \tau\rangle}\ \to_e^{\omega} \qquad \frac{M:\langle nil \vdash \tau\rangle}{\lambda.M:\langle nil \vdash \omega\to\tau\rangle}\ \to_i'$$

$$\frac{M_1:\langle \Gamma \vdash \wedge_{i=1}^{n}\sigma_i\to\tau\rangle \qquad M_2:\langle \Delta^1 \vdash \sigma_1\rangle\ \dots\ M_2:\langle \Delta^n \vdash \sigma_n\rangle}{(M_1\ M_2):\langle \Gamma\wedge\Delta^1\wedge\cdots\wedge\Delta^n \vdash \tau\rangle}\ \to_e$$

$$(\text{clos})\ \frac{S:\langle \Gamma \rhd \Gamma'\rangle \qquad M:\langle \Gamma' \vdash \tau\rangle}{M[S]:\langle \Gamma \vdash \tau\rangle}$$

$$(\wedge\text{-B})\ \frac{M:\langle \Gamma^1 \vdash \sigma_1\rangle\ \dots\ M:\langle \Gamma^m \vdash \sigma_m\rangle}{M/:\langle \Gamma\wedge\Gamma^1\wedge\cdots\wedge\Gamma^m \rhd (\wedge_{i=1}^{m}\sigma_i).\Gamma\rangle},\Gamma \neq \Delta.\omega^{\underline{m}}$$

$$(nil\text{-B})\ \frac{}{M/:\langle nil \vdash nil\rangle} \qquad (\omega\text{-B})\ \frac{\Gamma \neq \Delta.\omega^{\underline{n}}}{M/:\langle \Gamma \vdash \omega.\Gamma\rangle}$$

$$(nil\text{-shift})\ \frac{}{\uparrow:\langle nil \rhd nil\rangle} \qquad (\omega\text{-shift})\ \frac{\Gamma \neq \Delta.\omega^{\underline{n}}}{\uparrow:\langle \omega.\Gamma \rhd \Gamma\rangle}$$

$$(nil\text{-lift})\ \frac{S:\langle nil \rhd nil\rangle}{\Uparrow(S):\langle nil \rhd nil\rangle} \qquad (u\text{-lift})\ \frac{S:\langle \Gamma \rhd \Gamma'\rangle}{\Uparrow(S):\langle u.\Gamma \rhd u.\Gamma'\rangle}(*)$$

$(*)$ either $\Gamma' \neq nil$ or $u \neq \omega$.

FIG. 5. The inference rules for the system $\lambda\upsilon^{\wedge}$

PROOF. By induction on the derivation of $M:\langle \Gamma \vdash_{\lambda\upsilon^{\wedge}} \tau\rangle$ when $\Gamma \neq nil$, with subinduction on the derivation of $S:\langle \Gamma \rhd_{\lambda\upsilon^{\wedge}} \Gamma'\rangle$ when $\Gamma \neq nil$ or $\Gamma' \neq nil$. ∎

COROLLARY 7.3
If $M:\langle \Gamma \vdash_{\lambda\upsilon^{\wedge}} \tau\rangle$ then $\Gamma \neq \Delta.\omega^{\underline{m}}$, for any context Δ and $m > 0$. In particular, if $S:\langle \Gamma \rhd_{\lambda\upsilon^{\wedge}} \Gamma'\rangle$ then $\Gamma \neq \Delta.\omega^{\underline{m}}$ and $\Gamma' \neq \Delta'.\omega^{\underline{m}}$, for any contexts Δ and Δ', and $m > 0$.

LEMMA 7.4 (Generation for substitutions in $\lambda\upsilon^{\wedge}$)
1. $S:\langle nil \rhd nil\rangle$ for any $S \in \Lambda\upsilon^{s}$.
2. If $S:\langle \Gamma \rhd nil\rangle$ then $\Gamma = nil$.
3. $M/:\langle nil \rhd nil\rangle$ and $M/:\langle \Gamma \rhd \omega.\Gamma\rangle$ for any $M \in \Lambda\upsilon^{t}$ and $\Gamma \neq \Delta.\omega^{\underline{n}}$.
4. If $S:\langle \Gamma \rhd \Delta^1 \wedge \Delta^2\rangle$ for $\Delta^1 \neq \Delta'.\omega^{\underline{m}}$ and $\Delta^2 \neq \Delta''.\omega^{\underline{m}}$, then $\Gamma = \Gamma^1 \wedge \Gamma^2$ such that $S:\langle \Gamma^1 \rhd \Delta^1\rangle$ and $S:\langle \Gamma^2 \rhd \Delta^2\rangle$.

PROOF. 1. By induction on the structure of S.
2. By case analysis. Note that for each $S \equiv \uparrow, \Uparrow(S')$ and $M/$ the result is straightforward.
3. By case analysis one the derivation of $M/:\langle \Gamma \rhd \Gamma'\rangle$.
4. By induction on the structure of S. Let $S:\langle \Gamma \rhd \Delta^1 \wedge \Delta^2\rangle$. Note that if $\Delta^j \equiv nil$ for $j \in \{1,2\}$ them, by the item 1 above, $S:\langle \Gamma^j \vdash \Delta^j\rangle$ for $\Gamma^j = nil$ thus the result holds trivially. Below, we consider only the cases where $\Delta^1, \Delta^2 \neq nil$.

- Let $S \equiv \uparrow$. By the rule (ω-shift) one has that $\Gamma = \omega.(\Delta^1 \wedge \Delta^2)$. By hypothesis one has $\Delta^1 \neq \Delta'.\omega^{\underline{m}}$ and $\Delta^2 \neq \Delta''.\omega^{\underline{m}}$. Hence, for $\Gamma^j = \omega.\Delta^j$ where $j \in \{1,2\}$ one has by the rule (ω-shift) that $\uparrow : \langle \Gamma^j \rhd \Delta^j \rangle$.
- Let $S \equiv \Uparrow(S')$. By the rule (u-lift) one has that $\Gamma = u.\Gamma'$ and that $\Delta^1 \wedge \Delta^2 = u.\Gamma''$ s.t. $S' : \langle \Gamma' \rhd \Gamma'' \rangle$. Hence, $\Delta^1 = u_1.\Delta^3$ and $\Delta^2 = u_2.\Delta^4$ where $\Gamma'' = \Delta^3 \wedge \Delta^4$ and $u = u_1 \wedge u_2$. Observe that $\Delta^3, \Delta^4 \neq \Delta'.\omega^{\underline{m}}$. Therefore, by IH one has $S' : \langle \Gamma^3 \rhd \Delta^3 \rangle$ and $S' : \langle \Gamma^4 \rhd \Delta^4 \rangle$ s.t. $\Gamma' = \Gamma^3 \wedge \Gamma^4$ thus, by the rule (u-lift), $\Uparrow(S') : \langle u_1.\Gamma^3 \rhd u_1.\Delta^3 \rangle$ and $\Uparrow(S') : \langle u_2.\Gamma^4 \rhd u_2.\Delta^4 \rangle$ and $(u_1.\Gamma^3) \wedge (u_2.\Gamma^4) = (u_1 \wedge u_2).(\Gamma^3 \wedge \Gamma^4) = u.\Gamma' = \Gamma$
- Let $S \equiv M/$. If $\Delta^1 \wedge \Delta^2 = (\wedge_{i=1}^m \sigma_i).\Delta'$ then by the rule ($\wedge$-B) one has that $\forall 1 \leq i \leq m$, $M : \langle \Gamma^i \vdash \sigma^i \rangle$ where $\Gamma = \Delta' \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m$. Suppose w.l.o.g. that $\Delta^1 = (\wedge_{i=1}^m \sigma_i).\Delta^3$ and that $\Delta^2 = \omega.\Delta^4$ thus $\Delta' = \Delta^3 \wedge \Delta^4$. Note that $\Delta^3, \Delta^4 \neq \Delta''.\omega^{\underline{m}}$. Hence, by the rule ($\wedge$-B) one has $M/ : \langle \Delta^3 \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m \rhd (\wedge_{i=1}^m \sigma_i).\Delta^3 \rangle$ and by the rule (ω-B) one has $M/ : \langle \Delta^4 \rhd \omega.\Delta^4 \rangle$ where $(\Delta^3 \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m) \wedge \Delta^4 = \Gamma$. If $\Delta^1 \wedge \Delta^2 = \omega.\Delta'$ the proof is analogous.

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LEMMA 7.5 (Generation for terms in $\lambda\upsilon^\wedge$)

1. $\underline{n} : \langle \Gamma \vdash_{\lambda\upsilon^\wedge} \tau \rangle$ iff $\Gamma = \omega^{\underline{n-1}}.\tau.nil$.
2. If $\lambda.M : \langle nil \vdash_{\lambda\upsilon^\wedge} \tau \rangle$, then $\tau = \omega \to \sigma$ and $M : \langle nil \vdash_{\lambda\upsilon^\wedge} \sigma \rangle$ or $\tau = \wedge_{i=1}^n \sigma_i \to \sigma$, $n > 0$, and $M : \langle \wedge_{i=1}^n \sigma_i.nil \vdash_{\lambda\upsilon^\wedge} \sigma \rangle$ where $\sigma, \sigma_1, \ldots, \sigma_n \in \mathcal{T}$.
3. If $\lambda.M : \langle \Gamma \vdash_{\lambda\upsilon^\wedge} \tau \rangle$ and $|\Gamma| > 0$, then $\tau = u \to \sigma$ for some $u \in \mathcal{U}$ and $\sigma \in \mathcal{T}$, where $M : \langle u.\Gamma \vdash_{\lambda\upsilon^\wedge} \sigma \rangle$.
4. If $(M\ N) : \langle \Gamma \vdash_{\lambda\upsilon^\wedge} \tau \rangle$ then $M : \langle \Gamma \vdash_{\lambda\upsilon^\wedge} \omega \to \tau \rangle$ or $M : \langle \Gamma' \vdash_{\lambda\upsilon^\wedge} \wedge_{i=1}^m \sigma_i \to \tau \rangle$ and $\forall 1 \leq i \leq m$, $N : \langle \Gamma^i \vdash_{\lambda\upsilon^\wedge} \sigma_i \rangle$ where $\Gamma = \Gamma' \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m$.

PROOF. 1. By induction on n. 2,3,4. By case analysis on the respective derivation.

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THEOREM 7.6 (SR for $\lambda\upsilon^\wedge$)
If $M : \langle \Gamma \vdash_{\lambda\upsilon^\wedge} \tau \rangle$ and $M \to_{\lambda\upsilon} M'$ then $M' : \langle \Gamma \vdash_{\lambda\upsilon^\wedge} \tau \rangle$. Similarly, if $S : \langle \Gamma \rhd_{\lambda\upsilon^\wedge} \Gamma' \rangle$ and $S \to_{\lambda\upsilon} S'$ then $S' : \langle \Gamma \rhd_{\lambda\upsilon^\wedge} \Gamma' \rangle$.

PROOF. We proof the property for each rule of the $\lambda\upsilon$-calculus.

- *B*: Let $(\lambda.M\ N) : \langle \Gamma \vdash \tau \rangle$. By Lemma 7.5.4 one has two cases.

  In the first case, $\lambda.M : \langle \Gamma \vdash \omega \to \tau \rangle$. If $\Gamma = nil$ then by Lemma 7.5.2 one has $M : \langle nil \vdash \tau \rangle$. Hence, by the rule ($nil$-B) one has that $N/ : \langle nil \rhd nil \rangle$ thus by the rule (clos) one has $M[N/] : \langle nil \vdash \tau \rangle$. If $|\Gamma| > 0$ then by Lemma 7.5.3 one has $M : \langle \omega.\Gamma \vdash \tau \rangle$. By Corollary 7.3 one has that $\Gamma \neq \Delta.\omega^{\underline{m}}$. Hence, by the rule ($\omega$-B) one has that $N/ : \langle \Gamma \rhd \omega.\Gamma \rangle$ thus by the rule (clos) one has $M[N/] : \langle \Gamma \vdash \tau \rangle$.

  In the second case, $\Gamma = \Gamma' \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m$ where $\lambda.M : \langle \Gamma' \vdash \wedge_{i=1}^m \sigma_i \to \tau \rangle$ and $\forall 1 \leq i \leq m$, $N : \langle \Gamma^i \vdash \sigma_i \rangle$. By Lemma 7.5.2 or 7.5.3 one has that $M : \langle \wedge_{i=1}^m \sigma_i.\Gamma' \vdash \tau \rangle$. By Corollary 7.3 one has that $\Gamma' \neq \Delta.\omega^{\underline{m}}$. Therefore, by the rule ($\wedge$-B) one has $N/ : \langle \Gamma \rhd \wedge_{i=1}^m \sigma_i.\Gamma' \rangle$ thus, by the rule (clos), $M[N/] : \langle \Gamma \vdash \tau \rangle$.
- *App*: Let $(M\ N)[S] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $(M\ N) : \langle \Gamma' \vdash \tau \rangle$ and $S : \langle \Gamma \rhd \Gamma' \rangle$. By Lemma 7.5.4 one has two cases.

  In the first case, $M : \langle \Gamma' \vdash \omega \to \tau \rangle$. Hence, by the rule (clos) one has $M[S] : \langle \Gamma \vdash \omega \to \tau \rangle$ thus by the rule $\to_e^\omega$ one has $(M[S]\ N[S]) : \langle \Gamma \vdash \tau \rangle$.

In the second case, $M : \langle \Gamma'' \vdash \wedge_{i=1}^{m} \sigma_i \to \tau \rangle$ and $\forall 1 \le i \le m,\ N : \langle \Gamma^i \vdash \sigma_i \rangle$ where $\Gamma' = \Gamma'' \wedge \Gamma^1 \wedge \cdots \wedge \Gamma^m$. By Corollary 7.3 one has that $\Gamma'' \neq \Delta''.\omega^{\underline{m}}$. and $\forall 1 \le i \le m$, $\Gamma^i \neq (\Delta')^i.\omega^{\underline{m_i}}$. Hence, by Lemma 7.4.4 with an induction on $m+1$ one has that $\Gamma = \Delta \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ s.t. $S : \langle \Delta \rhd \Gamma'' \rangle$ and $\forall 1 \le i \le m,\ S : \langle \Delta^i \rhd \Gamma^i \rangle$. Then, by the rule (clos) one has $M[S] : \langle \Delta \vdash \wedge_{i=1}^{m} \sigma_i \to \tau \rangle$ and $\forall 1 \le i \le m,\ N[S] : \langle \Delta^i \vdash \sigma_i \rangle$. Hence, by the rule $\to_e$ one has $(M[S]\ N[S]) : \langle \Gamma \vdash \tau \rangle$.

- *Lambda*: Let $(\lambda.M)[S] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $\lambda.M : \langle \Gamma' \vdash \tau \rangle$ and $S : \langle \Gamma \rhd \Gamma' \rangle$.

  If $\Gamma' = nil$ then by Lemma 7.4.2 one has that $\Gamma = nil$ and, by Lemma 7.5.2, either $\tau = \omega \to \sigma$ and $M : \langle nil \vdash \sigma \rangle$ or $\tau = \wedge_{i=1}^{m} \sigma_i \to \sigma$ and $M : \langle \wedge_{i=1}^{m} \sigma_i.nil \vdash \sigma \rangle$. If $M : \langle nil \vdash \sigma \rangle$ then by the rule (nil-$lift$) one has $\Uparrow(S) : \langle nil \rhd nil \rangle$. Therefore, by the rules (cons) and $\to_i'$ one has that $\lambda.(M[\Uparrow(S)]) : \langle nil \vdash \omega \to \sigma \rangle$. If $M : \langle \wedge_{i=1}^{m} \sigma_i.nil \vdash \sigma \rangle$ then by the rule (u-$lift$) one has $\Uparrow(S) : \langle \wedge_{i=1}^{m} \sigma_i.nil \rhd \wedge_{i=1}^{m} \sigma_i.nil \rangle$. Therefore, by the rules (cons) and $\to_i$ one has that $\lambda.(M[\Uparrow(S)]) : \langle nil \vdash \wedge_{i=1}^{m} \sigma_i \to \sigma \rangle$.

  If $|\Gamma'| > 0$ then, by Lemma 7.5.3, $\tau = u \to \sigma$ and $M : \langle u.\Gamma' \vdash \sigma \rangle$. By the rule ($u$-$lift$) one has that $\Uparrow(S) : \langle u.\Gamma \rhd u.\Gamma' \rangle$ thus, by the rules (cons) and $\to_i$, $\lambda.(M[\Uparrow(S)]) : \langle \Gamma \vdash u \to \sigma \rangle$.

- *FVar*: Let $\underline{1}\,[M/] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $\underline{1} : \langle \Gamma' \vdash \tau \rangle$ and $M/ : \langle \Gamma \rhd \Gamma' \rangle$. By Lemma 7.5.1 one has $\Gamma' = \tau.nil$. Hence, by the rule ($\wedge$-B) one has that $M : \langle \Gamma \vdash \tau \rangle$.

- *RVar*: Let $\underline{n+1}\,[M/] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $\underline{n+1} : \langle \Gamma' \vdash \tau \rangle$ and $M/ : \langle \Gamma \rhd \Gamma' \rangle$. By Lemma 7.5.1 one has $\Gamma' = \omega^{\underline{n}}.\tau.nil$. Hence, by the rule ($\omega$-$B$) one has that $\Gamma' = \omega.\Gamma$ thus $\Gamma = \omega^{\underline{n-1}}.\tau.nil$. Therefore, $\underline{n} : \langle \Gamma \vdash \tau \rangle$.

- *FVarLift*: Let $\underline{1}\,[\Uparrow(S)] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $\underline{1} : \langle \Gamma' \vdash \tau \rangle$ and $\Uparrow(S) : \langle \Gamma \rhd \Gamma' \rangle$. By Lemma 7.5.1 one has $\Gamma' = \tau.nil$. Hence, by the rule (u-lift) one has that $S : \langle \Gamma'' \rhd nil \rangle$, where $\Gamma = \tau.\Gamma''$. By Lemma 7.4.2 one has that $\Gamma'' = nil$ thus $\underline{1} : \langle \Gamma \vdash \tau \rangle$.

- *RVarLift*: Let $\underline{n+1}\,[\Uparrow(S)] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $\underline{n+1} : \langle \Gamma' \vdash \tau \rangle$ and $\Uparrow(S) : \langle \Gamma \rhd \Gamma' \rangle$. By Lemma 7.5.1 one has $\Gamma' = \omega^{\underline{n}}.\tau.nil$. By the rule ($u$-lift) one has that $S : \langle \Gamma'' \rhd \omega^{\underline{n-1}}.\tau.nil \rangle$, where $\Gamma = \omega.\Gamma''$. By Lemma 7.5.1 one has that $\underline{n} : \langle \omega^{\underline{n-1}}.\tau.nil \vdash \tau \rangle$ thus by the rule (cons) one has $\underline{n}\,[S] : \langle \Gamma'' \vdash \tau \rangle$. By Corollary 7.3 one has that $\Gamma = \omega.\Gamma'' \neq \Delta.\omega^{\underline{m}}$ thus $\Gamma'' \neq \Delta.\omega^{\underline{n}}$. Therefore, by the rule (ω-shift) one has that $\uparrow : \langle \Gamma \rhd \Gamma'' \rangle$ and, by the rule (cons), $\underline{n}\,[S][\uparrow] : \langle \Gamma \vdash \tau \rangle$.

- *VarShift*: Let $\underline{n}\,[\uparrow] : \langle \Gamma \vdash \tau \rangle$. By the rule (clos) one has $\underline{n} : \langle \Gamma' \vdash \tau \rangle$ and $\uparrow : \langle \Gamma \rhd \Gamma' \rangle$. By Lemma 7.5.1 one has $\Gamma' = \omega^{\underline{n-1}}.\tau.nil$. Hence, by the rule ($\omega$-shift) one has that $\Gamma = \omega^{\underline{n}}.\tau.nil$ thus $\underline{n+1} : \langle \Gamma \vdash \tau \rangle$.

∎

8 Conclusion

In this paper, we presented the first intersection type (IT) systems for the λs-, λs_e-, $\lambda\sigma$- and $\lambda\upsilon$-calculi. We aimed for relevant IT systems [21, 23] satisfying subject reduction (SR) in order to obtain a typing system "as restricted as possible". Our interest in IT systems for these explicit substitution calculi is in the investigation of termination properties. Already the characterisation of strong normalisation (SN) through IT systems is a successful venture and it is already known neither λs_e nor $\lambda\sigma$ preserve SN. To emphasise this, we give examples to show that our IT systems are not able to characterise SN neither in λs_e nor in $\lambda\sigma$. However, a characterisation of SN in λs thus λ_{dB} might be obtained through either an extension of the present system with a subtyping relation or a Klop-like version of the calculus [36]. In addition, a non-idempotent IT system allows one to consider termination properties by combinatorial arguments similar to the systems of [11, 22], with a complexity result as a consequence. In doing so, a relevant type system is very convenient to obtain tight upper bounds.

Our interest in IT systems for explicit substitutions is related to the need of polymorphism in programming languages and computational systems. The simply typed version of the λs_e- and $\lambda\sigma$-calculi have applications on the HOU problem [24, 3] and, to the best of our knowledge, the IT systems presented here are the first polymorphic type systems proposed for them. The IT system for λs was originally based on the system λ_{dB}^{SM} [58] and thus called λs^{SM}. Similarly, the systems for λs_e and $\lambda\sigma$ were based on the system $\lambda_{dB}^{\wedge}$ and hence called $\lambda {s_e}^{\wedge}$ and $\lambda\sigma^{\wedge}$, respectively.

We proved the SR property for the simulation of β-contraction in the system λs^{SM}, using an adaptation for sequential contexts of the restricted environments, introduced in [29] to prove SR in a relevant IT system for the λ-calculus. The concept of available indices, needed in the definition of relevance for λs, was introduced based on the available variables in [38] and proved to be the correct generalisation of free indices for the λ_{dB}-calculus. We then obtained the SR for relevant type systems of β-contraction in λ_{dB}^{SM} from the property for λs^{SM}. The IT system $\lambda {s_e}^{\wedge}$ was proposed after some considerations and we proved SR in the usual sense for the full λs_e-calculus. Although not relevant, the system $\lambda {s_e}^{\wedge}$ has a property related to relevance. We then proved SR for the system $\lambda_{dB}^{\wedge}$, deriving it from the system $\lambda s^{\wedge}$, the restriction of system $\lambda {s_e}^{\wedge}$ to the λs-calculus.

We presented the process of developing the system $\lambda\sigma^{\wedge}$ from an extension of λ_{dB}^{SM} to infer typings for closures and substitutions. The system obtained is actually an extension of system $\lambda_{dB}^{\wedge}$. We proved a property related to relevance and the SR property for the full $\lambda\sigma$-calculus. The system $\lambda\upsilon^{\wedge}$ is very similar to the IT system for $\lambda\sigma$, as can be noted by its properties.

We intend to use the systems presented here as the basic system for studying the PT property in IT systems in each calculus. The PT property allows one to support features in a computational type system which include separate compilation, as in the *smartest recompilation*, and *recursive definitions* [28]. Partial typing inference algorithms can also be proposed once PT is established for the systems along with an IT version of the HOU problem. The system $\lambda_{dB}^{\wedge}$ is a de Bruijn version of the system in [52], were the PT property for β-nfs described in [50] is extended for any normalisable term. Hence, as a first step towards the PT for explicit substitutions, we need to extend the results presented in [57] to normalisable terms in λ_{dB}. Besides, we believe the systems λ_{dB}^{SM} and λs^{SM} are able to provide a characterisation for SN in λ_{dB} and λs, respectively. On the other hand, it seems that $\lambda_{dB}^{\wedge}$, $\lambda {s_e}^{\wedge}$ and $\lambda\sigma^{\wedge}$ can provide a characterisation of weak normalisation (WN) for λ_{dB}, λs_e and $\lambda\sigma$, respectively. Another interesting line of investigation is to propose a relevant IT system for a de Bruijn version of the λex-calculus [33].

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A Subject reduction proofs

Below, we present the complete proofs of SR omitted in Section 4.

A.1 SR for the s-calculus in system λs^{SM}

PROOF. [Theorem 4.9]

- (σ-λ-transition): Let $(\lambda.M)\sigma^i N : \langle\Gamma \vdash \tau\rangle$.

  If $\Gamma = nil$, then by Lemma 4.7.2 one has two possibilities.

  In the first case one has $\lambda.M : \langle nil \vdash \tau\rangle$ and $N : \langle\Delta \vdash \rho\rangle$ thus, by Lemma 4.6.2, $M : \langle nil \vdash \tau'\rangle$ where $\tau = \omega \to \tau'$ or $M : \langle \wedge_{j=1}^m \sigma_j.nil \vdash \tau'\rangle$ where $\tau = \wedge_{j=1}^m \sigma_j \to \tau'$. Note that $i \geq 1$ thus $i+1 \geq 2$. Hence, by the rule (nil-σ) one has $M\sigma^{i+1}N : \langle nil \vdash \tau'\rangle$ or $M\sigma^{i+1}N : \langle \wedge_{j=1}^m \sigma_j.nil \vdash \tau'\rangle$ thus $\lambda.(M\sigma^{i+1}N) : \langle nil \vdash \omega \to \tau'\rangle$ by the rule $\to_i'$ or $\lambda.(M\sigma^{i+1}N) : \langle nil \vdash \wedge_{j=1}^m \sigma_j \to \tau'\rangle$ by the rule $\to_i$.

  In the second case, $\lambda.M : \langle \omega \underline{\,i-1\,}. \wedge_{j=1}^m \sigma_j.nil \vdash_{\lambda s^{SM}} \tau\rangle$ where $\forall 1 \leq j \leq m$, $N : \langle nil \vdash \sigma_j\rangle$. Hence, by Lemma 4.6.3 one has $M : \langle u.\omega \underline{\,i-1\,}. \wedge_{j=1}^m \sigma_j.nil \vdash \tau'\rangle$ where $\tau = u \to \tau'$. Therefore, if $u \neq \omega$ then by the rule ($\wedge$-ω-σ) one has that $M\sigma^{i+1}N : \langle u.nil \vdash \tau'\rangle$ and if $u = \omega$ then by the rule ($\wedge$-nil-σ) one has that $M\sigma^{i+1}N : \langle nil \vdash \tau'\rangle$. Hence, $\lambda.(M\sigma^{i+1}N) : \langle nil \vdash u \to \tau'\rangle$ by the rule $\to_i$ or $\lambda.(M\sigma^{i+1}N) : \langle nil \vdash \omega \to \tau'\rangle$ by the rule $\to_i'$.

  If $0 < |\Gamma| < i$, then by Lemma 4.7.3 one has two cases.

  In the first case one has that $\lambda.M : \langle\Gamma \vdash \tau\rangle$ and that $N : \langle\Delta \vdash \rho\rangle$ thus, by Lemma 4.6.3, $M : \langle u.\Gamma \vdash \tau'\rangle$ where $\tau = u \to \tau'$. Note that $|\Gamma|+1 < i+1$. Hence, by the rule (nil-σ) one has $M\sigma^{i+1}N : \langle u.\Gamma \vdash \tau'\rangle$ thus, by the rule $\to_i$, $\lambda.(M\sigma^{i+1}N) : \langle\Gamma \vdash u \to \tau'\rangle$.

  In the second case one has that $\lambda.M : \langle \Gamma.\omega \underline{\,n\,}. \wedge_{j=1}^m \sigma_j.nil \vdash \tau\rangle$ where $n \geq 0$, $|\Gamma.\omega \underline{\,n\,}. \wedge_{j=1}^m \sigma_j.nil| = i$ and $\forall 1 \leq j \leq m$, $N : \langle nil \vdash \sigma_j\rangle$. By Lemma 4.6.3 one has $M : \langle u.\Gamma.\omega \underline{\,n\,}. \wedge_{j=1}^m \sigma_j.nil \vdash \tau'\rangle$ where $\tau = u \to \tau'$. Hence, by the rule ($\wedge$-ω-σ) one has that $M\sigma^{i+1}N : \langle u.\Gamma \vdash \tau'\rangle$ and, by the rule $\to_i$, $\lambda.(M\sigma^{i+1}N) : \langle\Gamma \vdash u \to \tau'\rangle$.

  If $|\Gamma| \geq i$, then by Lemma 4.7.4 one has two cases.

  In the first case one has that $\lambda.M : \langle \Gamma_{<i}.\omega.\Gamma_{\geq i} \vdash \tau\rangle$ and that $N : \langle\Delta \vdash \rho\rangle$ thus, by Lemma 4.6.3, $M : \langle u.\Gamma_{<i}.\omega.\Gamma_{\geq i} \vdash \tau'\rangle$ where $\tau = u \to \tau'$. Hence, by the rule (ω-σ), $M\sigma^{i+1}N : \langle u.\Gamma_{<i}.\Gamma_{\geq i} \vdash \tau'\rangle$ and, by the rule $\to_i$, $\lambda.(M\sigma^{i+1}N) : \langle \Gamma_{<i}.\Gamma_{\geq i} \vdash u \to \tau'\rangle$.

  In the second case one has $\lambda.M : \langle \Gamma_{<i}. \wedge_{j=1}^m \sigma_j.\Gamma' \vdash \tau\rangle$ where $\Gamma_{\geq i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$, for $|\Gamma_{\geq i}| > 0$, and $\forall 1 \leq j \leq m$, $N : \langle \Delta^j \vdash \sigma_j\rangle$. Hence, by Lemma 4.6.3, $M : \langle u.\Gamma_{<i}. \wedge_{j=1}^m \sigma_j.\Gamma' \vdash \tau'\rangle$ where $\tau = u \to \tau'$. Therefore, by the rule ($\wedge$-σ), $M\sigma^{i+1}N : \langle (u.\Gamma_{<i}.\Gamma') \wedge \omega \underline{\,i\,}.(\Delta^1 \wedge \cdots \wedge \Delta^m) \vdash \tau'\rangle$. Observe that $(u.\Gamma_{<i}.\Gamma') \wedge \omega \underline{\,i\,}.(\Delta^1 \wedge \cdots \wedge \Delta^m) = u.\Gamma$ thus, by the rule $\to_i$, $\lambda.(M\sigma^{i+1}N) : \langle\Gamma \vdash u \to \tau'\rangle$.

- (σ-app-transition): Let $(M_1\ M_2)\sigma^i N : \langle\Gamma \vdash \tau\rangle$.

  If $\Gamma = nil$, then by Lemma 4.7.2 one has two cases.

In the first case, $(M_1\ M_2) : \langle nil \vdash \tau\rangle$ and $N : \langle \Delta \vdash \rho\rangle$. By Lemma 4.6.4 one has that either $M_1 : \langle nil \vdash \omega \to \tau\rangle$ and $M_2 : \langle nil \vdash \rho\rangle$ or $M_1 : \langle nil \vdash \wedge_{j=1}^m \sigma_j \to \tau\rangle$ and $\forall 1 \le j \le m,\ M_2 : \langle nil \vdash \sigma_j\rangle$. For the former one has, by the rule (nil-σ), that $M_1\sigma^i N : \langle nil \vdash \omega \to \tau\rangle$ and $M_2\sigma^i N : \langle nil \vdash \rho\rangle$ thus, by the rule $\to'_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle nil \vdash \tau\rangle$. The proof for the latter one is analogous.

In the second case, $(M_1\ M_2) : \langle \underline{\omega^{\,i-1}}. \wedge_{k=1}^l \tau_k.nil \vdash \tau\rangle$ where $\forall 1 \le k \le l,\ N : \langle nil \vdash \tau_k\rangle$. By Lemma 4.6.4 one has that $\underline{\omega^{\,i-1}}. \wedge_{k=1}^l \tau_k.nil = \Gamma^1 \wedge \Gamma^2$ s.t. $M_1 : \langle \Gamma^1 \vdash \omega \to \tau\rangle$ and $M_2 : \langle \Gamma^2 \vdash \rho\rangle$ or $M_1 : \langle \Gamma^1 \vdash \wedge_{j=1}^m \sigma_j \to \tau\rangle$ where $\Gamma^2 = \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \le j \le m,\ M_2 : \langle \Delta^j \vdash \sigma_j\rangle$. Note that Γ^1 and Γ^2 are a partition where both have the same length of the original context or one of then is nil. Suppose w.l.o.g. that $\Gamma_2 = nil$ thus $\Gamma^1 = \underline{\omega^{\,i-1}}. \wedge_{k=1}^l \tau_k.nil$. Hence, for the first possibility regarding Lemma 4.6.4 one has by the rule ($\wedge$-nil-σ) that $M_1\sigma^i N : \langle nil \vdash \omega \to \tau\rangle$ and by the rule (nil-σ) that $M_2\sigma^i N : \langle nil \vdash \rho\rangle$. Therefore, by the rule $\to'_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle nil \vdash \tau\rangle$. The proof for the second possibility regarding Lemma 4.6.4 is analogous.

If $0 \le |\Gamma| < i$, then by Lemma 4.7.3 one has two cases.

In the first case one has that $(M_1\ M_2) : \langle \Gamma \vdash \tau\rangle$ and $N : \langle \Delta \vdash \rho\rangle$. By Lemma 4.6.4 one has that $\Gamma = \Gamma^1 \wedge \Gamma^2$ s.t. $M_1 : \langle \Gamma^1 \vdash \omega \to \tau\rangle$ and $M_2 : \langle \Gamma^2 \vdash \rho\rangle$ or $M_1 : \langle \Gamma^1 \vdash \wedge_{j=1}^m \sigma_j \to \tau\rangle$ where $\Gamma^2 = \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \le j \le m,\ M_2 : \langle \Delta^j \vdash \sigma_j\rangle$. Observe that $max(|\Gamma^1|, |\Gamma^2|) = |\Gamma| < i$. Hence, for the former one has by the rule (nil-σ) that $M_1\sigma^i N : \langle \Gamma^1 \vdash \omega \to \tau\rangle$ and that $M_2\sigma^i N : \langle \Gamma^2 \vdash \rho\rangle$ thus, by the rule $\to'_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle \Gamma^1 \wedge \Gamma^2 \vdash \tau\rangle$. The proof for the latter one is analogous.

In the second case one has that $(M_1\ M_2) : \langle \Gamma.\underline{\omega^{\,n}}. \wedge_{k=1}^l \tau_k.nil \vdash \tau\rangle$ where $n \ge 0$, $|\Gamma.\underline{\omega^{\,n}}. \wedge_{k=1}^l \tau_k.nil| = i$ and $\forall 1 \le k \le l,\ N : \langle nil \vdash \tau_k\rangle$. By Lemma 4.6.4 one has that $\Gamma.\underline{\omega^{\,n}}. \wedge_{k=1}^l \tau_k.nil = \Gamma^1 \wedge \Gamma^2$ s.t. $M_1 : \langle \Gamma^1 \vdash \omega \to \tau\rangle$ and $M_2 : \langle \Gamma^2 \vdash \rho\rangle$ or $M_1 : \langle \Gamma^1 \vdash \wedge_{j=1}^m \sigma_j \to \tau\rangle$ where $\Gamma^2 = \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \le j \le m,\ M_2 : \langle \Delta^j \vdash \sigma_j\rangle$. Suppose w.l.o.g. that $\Gamma^1 = \Gamma$ and that $\Gamma^2 = \underline{\omega^{\,i-1}}. \wedge_{k=1}^l \tau_k.nil$ thus $\forall 1 \le j \le m,\ \Delta^j = \underline{\omega^{\,i-1}}.u_j.nil$, where $u_j \sqsubseteq \wedge_{k=1}^l \tau_k$ and $u_1 \wedge \cdots \wedge u_m = \wedge_{k=1}^l \tau_k$. Hence, for the second possibility regarding Lemma 4.6.4 one has by the rule (nil-σ) that $M_1\sigma^i N : \langle \Gamma \vdash \wedge_{j=1}^m \sigma_j \to \tau\rangle$ and by the rule ($\wedge$-nil-σ) that $\forall 1 \le j \le m,\ M_2\sigma^i N : \langle nil \vdash \sigma_j\rangle$. Therefore, by the rule $\to_e$, $((M_1\sigma^i N)\ (M_2\,\sigma^i N)) : \langle \Gamma \vdash \tau\rangle$. The proof for the first possibility regarding Lemma 4.6.4 is analogous.

If $|\Gamma| \ge i$, then by Lemma 4.7.4 one has two cases.

In the first case one has that $(M_1\ M_2) : \langle \Gamma_{<i}.\omega.\Gamma_{\ge i} \vdash \tau\rangle$ and $N : \langle \Delta \vdash \rho\rangle$. By Lemma 4.6.4 one has that $\Gamma_{<i}.\omega.\Gamma_{\ge i} = \Gamma^1 \wedge \Gamma^2$ s.t. $M_1 : \langle \Gamma^1 \vdash \omega \to \tau\rangle$ and $M_2 : \langle \Gamma^2 \vdash \rho\rangle$ or $M_1 : \langle \Gamma^1 \vdash \wedge_{j=1}^m \sigma_j \to \tau\rangle$ where $\Gamma^2 = \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \le j \le m,\ M_2 : \langle \Delta^j \vdash \sigma_j\rangle$. Suppose w.l.o.g. that $|\Gamma^2| < i$ thus $\Gamma^1 = \Gamma'.\omega.\Gamma_{\ge i}$ for $|\Gamma'| = i-1$ s.t. $\Gamma^1 \wedge \Gamma^2 = (\Gamma' \wedge \Gamma^2).\omega.\Gamma_{\ge i}$. Hence, for the first possibility regarding Lemma 4.6.4 one has by the rule (ω-σ) that $M_1\sigma^i N : \langle \Gamma'.\Gamma_{\ge i} \vdash \omega \to \tau\rangle$ and by the rule (nil-σ) that $M_2\sigma^i N : \langle \Gamma^2 \vdash \rho\rangle$. Therefore, by the rule $\to'_e$ one has that $((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle (\Gamma' \wedge \Gamma^2).\Gamma_{\ge i} \vdash \tau\rangle$, where $\Gamma' \wedge \Gamma^2 = \Gamma_{<i}$. The proof for the second possibility regarding Lemma 4.6.4 is analogous.

In the second case one has $(M_1\ M_2) : \langle \Gamma_{<i}. \wedge_{j=1}^m \sigma_j.\Gamma' \vdash \tau\rangle$ where $\Gamma_{\ge i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$, for $|\Gamma_{\ge i}| > 0$, and $\forall 1 \le j \le m,\ N : \langle \Delta^j \vdash \sigma_j\rangle$. By Lemma 4.6.4 one has $\Gamma^1 \wedge \Gamma^2 = \Gamma_{<i}. \wedge_{j=1}^m \sigma_j.\Gamma'$ s.t. $M_1 : \langle \Gamma^1 \vdash \omega \to \tau\rangle$ and $M_2 : \langle \Gamma^2 \vdash \rho\rangle$ or $M_1 : \langle \Gamma^1 \vdash \wedge_{k=1}^l \sigma'_k \to \tau\rangle$ where $\Gamma^2 = (\Delta')^1 \wedge \cdots \wedge (\Delta')^l$ and $\forall 1 \le k \le l,\ M_2 : \langle (\Delta')^k \vdash \sigma'_k\rangle$. Suppose w.l.o.g. that $|\Gamma^2| < i$ thus $\Gamma^1 = \Gamma^1_{<i}. \wedge_{j=1}^m \sigma_j.\Gamma'$ s.t $\Gamma^1_{<i} \wedge \Gamma^2 = \Gamma_{<i}$. Hence, for the first possibility regarding Lemma 4.6.4 one has by the rule ($\wedge$-σ) that $M_1\sigma^i N : \langle (\Gamma^1_{<i}.\Gamma') \wedge \underline{\omega^{\,i-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m) \vdash \omega \to \tau\rangle$ and by the rule (nil-σ) that $M_2\sigma^i N : \langle \Gamma^2 \vdash \rho\rangle$. Observe that $(\Gamma^1_{<i}.\Gamma') \wedge \underline{\omega^{\,i-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m) = \Gamma^1_{<i}.(\Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m) = \Gamma^1_{<i}.\Gamma_{\ge i}$. Therefore, by the rule $\to'_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)) : \langle (\Gamma^1_{<i}.\Gamma_{\ge i}) \wedge \Gamma^2 \vdash \tau\rangle$, where $(\Gamma^1_{<i}.\Gamma_{\ge i}) \wedge \Gamma^2 = (\Gamma^1_{<i} \wedge \Gamma^2).\Gamma_{\ge i} = \Gamma$. The proof for the second possibility regarding Lemma 4.6.4 is analogous.

- (σ-destruction): Let $\underline{n}\,\sigma^i N : \langle \Gamma \vdash \tau\rangle$.

  If $n < i$, then $\underline{n}\,\sigma^i N \to \underline{n}$ and $AI(\underline{n}\,\sigma^i N) = \{\underline{n}\}$ thus, by Lemma 4.4, $|\Gamma| = n$. Observe that for any typing $\langle \Gamma' \vdash \tau'\rangle$ of $\underline{n}$, $|\Gamma'| = n$. Hence, by Lemma 4.7.3 one has that $\underline{n} : \langle \Gamma \vdash \tau\rangle$ and $N : \langle \Delta \vdash \rho\rangle$.

  If $n = i$, then $\underline{n}\,\sigma^i N \to \varphi_0^i N$ and $AI(\underline{n}\,\sigma^i N) = AI(\varphi_0^i N)$ thus, by Lemma 4.4, $|\Gamma| = sav(\varphi_0^i N)$. By Lemmas 4.4 and 4.6.1 one has that $\underline{i} : \langle \underline{\omega^{\,i-1}}.\tau.nil \vdash \tau\rangle$. If $\Gamma = nil$, then by Lemma 4.7.2 one has that $N : \langle nil \vdash \tau\rangle$ thus, by the rule (nil-φ), $\varphi_0^i N : \langle nil \vdash \tau\rangle$. If $|\Gamma| > 0$, then by Lemma 2.13.4 one has that $sav(\varphi_0^i N) = sav(N) + (i-1)$, where $sav(N) > 0$, thus $|\Gamma| = sav(\varphi_0^i N) \ge i$. Hence, by Lemma 4.7.4 one has that $N : \langle \Gamma_{\ge i} \vdash \tau\rangle$, where $|\Gamma_{\ge i}| > 0$. Therefore, by the rule (ω-φ), $\varphi_0^i N : \langle \underline{\omega^{\,i-1}}.\Gamma_{\ge i} \vdash \tau\rangle$. Observe that $\Gamma_{<i} = \underline{\omega^{\,i-1}}$ thus $\underline{\omega^{\,i-1}}.\Gamma_{\ge i} = \Gamma$.

If $n>i$, then $\underline{n}\,\sigma^i N \to \underline{n-1}$ and $AI(\underline{n}\,\sigma^i N)=\{\,\underline{n-1}\,\}$ thus, by Lemma 4.4, $|\Gamma|=n-1\geq i$. One has that $\Gamma'_i=\omega$, for any context Γ' from a typing of $\underline{n}$, where $n>i$. Therefore, by Lemma 4.7.4, $\underline{n}:\langle\Gamma_{<i}.\omega.\Gamma_{\geq i}\vdash\tau\rangle$ and $N:\langle\Delta\vdash\rho\rangle$. By Lemmas 4.4 and 4.6.1 one has that $\Gamma_{<i}.\omega.\Gamma_{\geq i}=\omega^{\underline{n-1}}.\tau.nil$ thus $\Gamma_{<i}.\Gamma_{\geq i}=\omega^{\underline{n-2}}.\tau.nil$. Therefore, by the rules var and varn one has that $\underline{n-1}:\langle\Gamma\vdash\tau\rangle$.

- (φ-λ-transition): Let $\varphi^i_k(\lambda.M):\langle\Gamma\vdash\tau\rangle$.
  If $|\Gamma|\leq k$ then by Lemma 4.7.1 one has $\lambda.M:\langle\Gamma\vdash\tau\rangle$. If $\Gamma=nil$, then by Lemma 4.6.2 one has that $M:\langle nil\vdash\tau'\rangle$ where $\tau=\omega\to\tau'$ or that $M:\langle\wedge^m_{j=1}\sigma_j.nil\vdash\tau'\rangle$ where $\tau=\wedge^m_{j=1}\sigma_j\to\tau'$. Note that $1\leq k+1$. Hence, by the rule (nil-φ), $\varphi^i_{k+1}M:\langle nil\vdash\tau'\rangle$ or $\varphi^i_{k+1}M:\langle\wedge^m_{j=1}\sigma_j.nil\vdash\tau'\rangle$. Therefore, one has that $\lambda.(\varphi^i_{k+1}M):\langle nil\vdash\omega\to\tau'\rangle$ by the rule $\to'_i$ or that $\lambda.(\varphi^i_{k+1}M):\langle nil\vdash\wedge^m_{j=1}\sigma_j\to\tau\rangle$ by the rule $\to_i$. The proof for $|\Gamma|>0$ is analogous.
  If $|\Gamma|>k$ then by Lemma 4.7.1 one has that $\lambda.M:\langle\Gamma_{\leq k}.\Gamma_{\geq k+i}\vdash\tau\rangle$ where $\Gamma=\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. Hence, by Lemma 4.6.3, $M:\langle u.\Gamma_{\leq k}.\Gamma_{\geq k+i}\vdash\tau'\rangle$ where $u\to\tau'$ and, by the rule (ω-φ), $\varphi^i_{k+1}M:\langle u.\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}\vdash\tau\rangle$. Therefore, by the rule $\to_i$ one has $\lambda.(\varphi^i_{k+1}M):\langle\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}\vdash u\to\tau'\rangle$.
- (φ-app-transition): Let $\varphi^i_k(M_1\ M_2):\langle\Gamma\vdash\tau\rangle$.
  If $|\Gamma|\leq k$ then by Lemma 4.7.1 one has $(M_1\ M_2):\langle\Gamma\vdash\tau\rangle$. By Lemma 4.6.4 one has that $\Gamma=\Gamma^1\wedge\Gamma^2$ s.t. $M_1:\langle\Gamma^1\vdash\omega\to\tau\rangle$ and $M_2:\langle\Gamma^2\vdash\rho\rangle$ or that $M_1:\langle\Gamma^1\vdash\wedge^m_{j=1}\sigma_j\to\tau\rangle$ where $\Gamma^2=\Delta^1\wedge\cdots\wedge\Delta^m$ and $\forall 1\leq j\leq m$, $M_2:\langle\Delta^j\vdash\sigma_j\rangle$. One has that $max(|\Gamma^1|,|\Gamma^2|)=|\Gamma|\leq k$. Hence, for the first possibility regarding Lemma 4.6.4 one has, by the rule (nil-φ), that $\varphi^i_kM_1:\langle\Gamma^1\vdash\omega\to\tau\rangle$ and that $\varphi^i_kM_2:\langle\Gamma^2\vdash\rho\rangle$. Therefore, by the rule $\to'_e$, $((\varphi^i_kM_1)\ (\varphi^i_kM_2)):\langle\Gamma^1\wedge\Gamma^2\vdash\tau\rangle$. The proof for the second possibility regarding Lemma 4.6.4 is analogous.
  If $|\Gamma|>k$ then by Lemma 4.7.1 one has $(M_1\ M_2):\langle\Gamma_{\leq k}.\Gamma_{\geq k+i}\vdash\tau\rangle$ where $\Gamma=\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. By Lemma 4.6.4 one has $\Gamma_{\leq k}.\Gamma_{\geq k+i}=\Gamma^1\wedge\Gamma^2$ s.t. $M_1:\langle\Gamma^1\vdash\omega\to\tau\rangle$ and $M_2:\langle\Gamma^2\vdash\rho\rangle$ or $M_1:\langle\Gamma^1\vdash\wedge^m_{j=1}\sigma_j\to\tau\rangle$ where $\Gamma^2=\Delta^1\wedge\cdots\wedge\Delta^m$ and $\forall 1\leq j\leq m$, $M_2:\langle\Delta^j\vdash\sigma_j\rangle$. Suppose w.l.o.g. that $|\Gamma^2|\leq k$ thus $\Gamma^1=\Gamma^1_{\leq k}.\Gamma_{\geq k+i}$ where $\Gamma^1_{\leq k}\wedge\Gamma^2=\Gamma_{\leq k}$. Hence, for the first possibility regarding Lemma 4.6.4 one has by the rule (ω-φ) that $\varphi^i_kM_1:\langle\Gamma^1_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}\vdash\omega\to\tau\rangle$ and by the rule (nil-φ) that $\varphi^i_kM_2:\langle\Gamma^2\vdash\rho\rangle$. Therefore, by the rule $\to'_e$, $(\varphi^i_kM_1\ \varphi^i_kM_2):\langle(\Gamma^1_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i})\wedge\Gamma^2\vdash\tau\rangle$, where $(\Gamma^1_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i})\wedge\Gamma^2=(\Gamma^1_{\leq k}\wedge\Gamma^2).\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. The proof for the second possibility regarding Lemma 4.6.4 is analogous.
- (φ-destruction): Let $\varphi^i_k\,\underline{n}:\langle\Gamma\vdash\tau\rangle$.
  If $n\leq k$, then $\varphi^i_k\,\underline{n}\to\underline{n}$ and $AI(\varphi^i_k\,\underline{n})=\{\,\underline{n}\,\}$. Hence, $|\Gamma|=sav(\varphi^i_k\,\underline{n})=sav(\underline{n})\leq k$. Therefore, by Lemma 4.7.1 one has that $\underline{n}:\langle\Gamma\vdash\tau\rangle$.
  If $n>k$, then $\varphi^i_k\,\underline{n}\to\underline{n+i-1}$ and $AI(\varphi^i_k\,\underline{n})=\{\,\underline{n+i-1}\,\}$. Hence, $|\Gamma|=sav(\varphi^i_k\,\underline{n})=n+(i-1)>k$. Therefore, by Lemma 4.7.1 one has that $\underline{n}:\langle\Gamma_{\leq k}.\Gamma_{\geq k+i}\vdash\tau\rangle$, where $\Gamma=\Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. By Lemmas 4.4 and 4.6.1 one has that $\Gamma_{\leq k}.\Gamma_{\geq k+i}=\omega^{\underline{n-1}}.\tau.nil$. Therefore, $\Gamma=\omega^{\underline{n+i-2}}.\tau.nil$ thus, by the rules var and varn, one has that $\underline{n+i-1}:\langle\Gamma\vdash\tau\rangle$.

∎

A.2 SR for system $\lambda s_e{}^{\wedge}$

Proof. [Theorem 4.22]

- (σ-generation): Let $(\lambda.M\ N):\langle\Gamma\vdash\tau\rangle$. By Lemma 4.19.4 there exist two possibilities.
  Suppose that $\lambda.M:\langle\Gamma\vdash\omega\to\tau\rangle$.
  If $\Gamma=nil$, then by Lemma 4.19.2 one has that $M:\langle nil\vdash\tau\rangle$. Therefore, by the rule (nil-σ) one has that $M\sigma^1N:\langle nil\vdash\tau\rangle$.
  If $|\Gamma|>0$, then by Lemma 4.19.3 one has that $M:\langle\omega.\Gamma\vdash\tau\rangle$. Therefore, by the rule (ω-σ) one has that $M\sigma^1N:\langle\Gamma\vdash\tau\rangle$.
  Suppose that $\Gamma=\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$ s.t. $\lambda.M:\langle\Gamma'\vdash\wedge^m_{j=1}\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m$, $N:\langle\Delta^j\vdash\sigma_j\rangle$.
  If $\Gamma'=nil$, then by Lemma 4.19.2 one has $M:\langle\wedge^m_{j=1}\sigma_j.nil\vdash\tau\rangle$. If $\forall 1\leq j\leq m$, $\Delta^j=nil$, then by the rule ($\wedge$-nil-σ) one has that $M\sigma^1N:\langle nil\vdash\tau\rangle$. Otherwise, by the rule ($\wedge$-σ) one has that $M\sigma^1N:\langle\Delta^1\wedge\cdots\wedge\Delta^m\vdash\tau\rangle$, where $\Gamma=\Delta^1\wedge\cdots\wedge\Delta^m$.
  If $|\Gamma'|>0$, then by Lemma 4.19.3 one has that $M:\langle\wedge^m_{j=1}\sigma_j.\Gamma'\vdash\tau\rangle$. Therefore, by the rule ($\wedge$-σ) one has that $M\sigma^1N:\langle\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m\vdash\tau\rangle$.

- (σ-λ-transition): analogous to the proof of SR for the same rule in λs^{SM} (Theorem 4.9).
- (σ-app-transition): Let $(M_1\ M_2)\,\sigma^i N:\langle\Gamma\vdash\tau\rangle$.
  If $\Gamma=nil$, then by Lemma 4.21.2 there exist two possibilities.
  Suppose that $(M_1\ M_2):\langle nil\vdash\tau\rangle$. Hence, by Lemma 4.19.4 one has that $M_1:\langle nil\vdash\omega\to\tau\rangle$ or $M_1:\langle nil\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2:\langle nil\vdash\sigma_j\rangle$. For the latter one has, by the rule (nil-σ), that $M_1\sigma^i N:\langle nil\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2\sigma^i N:\langle nil\vdash\sigma_j\rangle$ thus, by the rule $\to_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle$. The proof for the former one is analogous.
  Suppose that $(M_1\ M_2):\langle\omega\,\underline{i-1}.\wedge_{k=1}^l\tau_k.nil\vdash\tau\rangle$ where $\forall 1\leq k\leq l,\ N:\langle nil\vdash\tau_k\rangle$. By Lemma 4.19.4 one has that $M_1:\langle\omega\,\underline{i-1}.\wedge_{k=1}^l\tau_k.nil\vdash\omega\to\tau\rangle$ or $\omega\,\underline{i-1}.\wedge_{k=1}^l\tau_k.nil=\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$ s.t. $M_1:\langle\Gamma'\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2:\langle\Delta^j\vdash\sigma_j\rangle$. Note that $\Gamma',\Delta^1,\dots,\Delta^m$ is a partition where each context has the same length as the original context or it is nil. Suppose w.l.o.g. that $\Gamma'=nil$. Hence, for the the second possibility w.r.t. Lemma 4.19.4 one has by the rule (nil-σ) that $M_1\sigma^i N:\langle nil\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m$, if $\Delta^j=nil$ then $M_2\sigma^i N:\langle nil\vdash\sigma_j\rangle$ by the rule (nil-σ) and if $\Delta^j\neq nil$ then $M_2\sigma^i N:\langle nil\vdash\sigma_j\rangle$ by the rule ($\wedge$-nil-σ). Therefore, by the rule $\to_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle nil\vdash\tau\rangle$. The proof is analogous for the first possibility regarding Lemma 4.19.4.
  If $0\leq|\Gamma|<i$, then by Lemma 4.21.3 there exist two possibilities.
  Suppose that $(M_1\ M_2):\langle\Gamma\vdash\tau\rangle$. By Lemma 4.19.4 one has that $M_1:\langle\Gamma\vdash\omega\to\tau\rangle$ or $\Gamma=\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$ s.t. $M_1:\langle\Gamma'\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2:\langle\Delta^j\vdash\sigma_j\rangle$. Observe that $max(|\Gamma'|,|\Delta^1|,\dots,|\Delta^m|)=|\Gamma|<i$. Hence, for the second possibility w.r.t. Lemma 4.19.4 one has by the rule (nil-σ) that $M_1\sigma^i N:\langle\Gamma'\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2\sigma^i N:\langle\Delta^j\vdash\sigma_j\rangle$ thus, by the rule $\to_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m\vdash\tau\rangle$. The proof is analogous for the first possibility regarding Lemma 4.19.4.
  Suppose that $(M_1\ M_2):\langle\Gamma'\vdash\tau\rangle$ where $\Gamma'=\Gamma.\omega\,\underline{n}.\wedge_{k=1}^l\tau_k.nil$ for $n\geq 0$ s.t. $|\Gamma'|=i$ and $\forall 1\leq k\leq l$, $N:\langle nil\vdash\tau_k\rangle$. By Lemma 4.19.4 one has that $M_1:\langle\Gamma'\vdash\omega\to\tau\rangle$ or $\Gamma'=\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$ s.t. $M_1:\langle\Gamma'\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2:\langle\Delta^j\vdash\sigma_j\rangle$. Suppose w.l.o.g. that $\Gamma'=\Gamma$. Hence, for the second possibility w.r.t. Lemma 4.19.4 one has by the rule (nil-σ) that $M_1\sigma^i N:\langle\Gamma\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m$, by either the rule ($\wedge$-nil-σ) or the rule (nil-σ) one has that $M_2\sigma^i N:\langle nil\vdash\sigma_j\rangle$. Therefore, by the rule $\to_e$, $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle\Gamma\vdash\tau\rangle$. The proof is analogous for the first possibility regarding Lemma 4.19.4.
  If $|\Gamma|\geq i$, then by Lemma 4.21.4 there exist two possibilities.
  Suppose that $(M_1\ M_2):\langle\Gamma_{<i}.\omega.\Gamma_{\geq i}\vdash\tau\rangle$. By Lemma 4.19.4 one has that $M_1:\langle\Gamma_{<i}.\omega.\Gamma_{\geq i}\vdash\omega\to\tau\rangle$ or $\Gamma_{<i}.\omega.\Gamma_{\geq i}=\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$ s.t. $M_1:\langle\Gamma'\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m,\ M_2:\langle\Delta^j\vdash\sigma_j\rangle$. Suppose w.l.o.g. that $|\Gamma'|<i$ thus $\forall 1\leq j\leq m$, if $|\Delta^j|>i$ then $\Delta^j_i=\omega$. Observe that, by Lemma 4.18, $\forall 1\leq j\leq m,\ |\Delta^j|\neq i$. Hence, for the second possibility w.r.t. Lemma 4.19.4 one has by the rule (nil-σ) that $M_1\sigma^i N:\langle\Gamma'\vdash\wedge_{j=1}^m\sigma_j\to\tau\rangle$ and $\forall 1\leq j\leq m$, if $|\Delta^j|>i$ then $M_2\sigma^i N:\langle\Delta^j_{<i}.\Delta^j_{>i}\vdash\sigma_j\rangle$ by the rule (ω-σ) and if $|\Delta^j|<i$ then $M_2\sigma^i N:\langle\Delta^j\vdash\sigma_j\rangle$ by the rule (nil-σ). Note that $\Gamma'\wedge(\Delta^1_{<i}.\Delta^1_{>i})\wedge\cdots\wedge(\Delta^m_{<i}.\Delta^m_{>i})=\Gamma_{<i}.\Gamma_{\geq i}$. Therefore, by the rule $\to_e$ one has that $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle\Gamma\vdash\tau\rangle$. The proof is analogous for the first possibility regarding Lemma 4.19.4.
  Suppose that $\Gamma_{\geq i}=\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m$, for $|\Gamma_{\geq i}|>0$, s.t. $(M_1\ M_2):\langle\Gamma_{<i}.\wedge_{j=1}^m\sigma_j.\Gamma'\vdash\tau\rangle$ and $\forall 1\leq j\leq m,\ N:\langle\Delta^j\vdash\sigma_j\rangle$. By Lemma 4.19.4 one has that $M_1:\langle\Gamma_{<i}.\wedge_{j=1}^m\sigma_j.\Gamma'\vdash\omega\to\tau\rangle$ or $\Gamma_{<i}.\wedge_{j=1}^m\sigma_j.\Gamma'=\Gamma''\wedge(\Delta')^1\wedge\cdots\wedge(\Delta')^l$ s.t. $M_1:\langle\Gamma''\vdash\wedge_{k=1}^l\sigma'_k\to\tau\rangle$ and $\forall 1\leq k\leq l$, $M_2:\langle(\Delta')^k\vdash\sigma'_k\rangle$. Hence, for the first possibility w.r.t. Lemma 4.19.4 one has by the rule ($\wedge$-σ) that $M_1\sigma^i N:\langle(\Gamma_{<i}.\Gamma')\wedge\omega\,\underline{i-1}.(\Delta^1\wedge\cdots\wedge\Delta^m)\vdash\omega\to\tau\rangle$ thus, by the rule $\to_e^\omega$, $((M_1\sigma^i N)\ (M_2\sigma^i N)):\langle(\Gamma_{<i}.\Gamma')\wedge\omega\,\underline{i-1}.(\Delta^1\wedge\cdots\wedge\Delta^m)\vdash\tau\rangle$. Note that $(\Gamma_{<i}.\Gamma')\wedge\omega\,\underline{i-1}.(\Delta^1\wedge\cdots\wedge\Delta^m)=\Gamma_{<i}.(\Gamma'\wedge\Delta^1\wedge\cdots\wedge\Delta^m)$. The proof is analogous for the second possibility regarding Lemma 4.19.4.
- (σ-destruction): By Lemma 4.19.1 one has that $\underline{n}:\langle\omega\,\underline{n-1}.\tau.nil\vdash\tau\rangle$. Below, the possible typings for $\underline{n}\,\sigma^i N$ are analysed.
  If $n<i$ then by the rule (nil-σ) one has that $\underline{n}\,\sigma^i N:\langle\omega\,\underline{n-1}.\tau.nil\vdash\tau\rangle$ and $\underline{n}\,\sigma^i N\to\underline{n}$.
  If $n=i$ then $\underline{n}\,\sigma^i N\to\varphi^i_0 N$. Note that N must have type τ in order to $\underline{i}\,\sigma^i N$ be typeable. If $N:\langle nil\vdash\tau\rangle$, then by the rule ($\wedge$-$nil$-$\sigma$) one has that $\underline{i}\,\sigma^i N:\langle nil\vdash\tau\rangle$ and, by the rule (nil-φ), $\varphi^i_0 N:\langle nil\vdash\tau\rangle$. If $N:\langle\Gamma\vdash\tau\rangle$, for $|\Gamma|>0$, then by the rule ($\wedge$-σ) one has that $\underline{i}\,\sigma^i N:\langle\omega\,\underline{i-1}.\Gamma\vdash\tau\rangle$ and, by the rule (ω-φ), $\varphi^i_0 N:\langle\omega\,\underline{i-1}.\Gamma\vdash\tau\rangle$.

If $n > i$ then by the rule (ω-σ) one has that $\underline{n}\,\sigma^i N : \langle \omega^{\underline{n-2}}.\tau.nil \vdash \tau \rangle$ and $\underline{n}\,\sigma^i N \to \underline{n-1}$. Therefore, by the rules var and varn one has that $\underline{n-1} : \langle \omega^{\underline{n-2}}.\tau.nil \vdash \tau \rangle$.

- (φ-λ-transition): analogous to the proof of SR for the same rule in λs^{SM} (Theorem 4.9).
- (φ-app-transition): Let $\varphi^i_k(M_1\ M_2) : \langle \Gamma \vdash \tau \rangle$.

If $|\Gamma| \leq k$ then by Lemma 4.21.1 one has $(M_1\ M_2) : \langle \Gamma \vdash \tau \rangle$. By Lemma 4.19.4 one has $M_1 : \langle \Gamma \vdash \omega \to \tau \rangle$ or $\Gamma = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ s.t. $M_1 : \langle \Gamma' \vdash \wedge^m_{j=1} \sigma_j \to \tau \rangle$ and $\forall 1 \leq j \leq m$, $M_2 : \langle \Delta^j \vdash \sigma_j \rangle$. One has that $max(|\Gamma'|, |\Delta^1|, \ldots, |\Delta^m|) = |\Gamma| \leq k$. Hence, for the second possibility w.r.t. Lemma 4.19.4 one has, by the rule (nil-φ), that $\varphi^i_k M_1 : \langle \Gamma' \vdash \wedge^m_{j=1} \sigma_j \to \tau \rangle$ and $\forall 1 \leq j \leq m$, $\varphi^i_k M_2 : \langle \Delta^j \vdash \sigma_j \rangle$. Therefore, by the rule $\to_e$ one has that $((\varphi^i_k M_1)\ (\varphi^i_k M_2)) : \langle \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m \vdash \tau \rangle$. The proof is analogous for the first possibility regarding Lemma 4.19.4.

If $|\Gamma| > k$ then by Lemma 4.21.1 one has $(M_1\ M_2) : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau \rangle$ where $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. By Lemma 4.19.4 one has that $M_1 : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \omega \to \tau \rangle$ or $\Gamma_{\leq k}.\Gamma_{\geq k+i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ s.t. $M_1 : \langle \Gamma' \vdash \wedge^m_{j=1} \sigma_j \to \tau \rangle$ and $\forall 1 \leq j \leq m$, $M_2 : \langle \Delta^j \vdash \sigma_j \rangle$. Let $\Delta' = \Delta^1 \wedge \cdots \wedge \Delta^m$ and suppose w.l.o.g. that $|\Gamma'| \leq k$, thus $\Delta' = \Delta'_{\leq k}.\Gamma_{\geq k+i}$ where $\Gamma' \wedge \Delta'_{\leq k} = \Gamma_{\leq k}$. Hence, for the second possibility w.r.t. Lemma 4.19.4 one has by the rule (nil-φ) that $\varphi^i_k M_1 : \langle \Gamma' \vdash \wedge^m_{j=1} \sigma_j \to \tau \rangle$ and $\forall 1 \leq j \leq m$, if $|\Delta^j| > k$ then $\varphi^i_k M_2 : \langle \Delta^j_{\leq k}.\omega^{\underline{i-1}}.\Delta^j_{>k} \vdash \sigma_j \rangle$ by the rule (ω-φ) and if $|\Delta^j| \leq k$ then $\varphi^i_k M_2 : \langle \Delta^j \vdash \sigma_j \rangle$ by the rule (nil-φ). Note that $\Delta^1_{>k} \wedge \cdots \wedge \Delta^m_{>k} = \Delta'_{>k} = \Gamma_{\geq k+i}$. Therefore, by the rule $\to_e$, $(\varphi^i_k M_1\ \varphi^i_k M_2) : \langle \Gamma' \wedge (\Delta'_{\leq k}.\omega^{\underline{i-1}}.\Delta'_{>k}) \vdash \tau \rangle$, where $\Gamma' \wedge (\Delta'_{\leq k}.\omega^{\underline{i-1}}.\Delta'_{>k}) = (\Gamma' \wedge \Delta'_{\leq k}).\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. The proof is analogous for the first possibility regarding Lemma 4.19.4.

- (φ-destruction): The possible typings for $\varphi^i_k\,\underline{n}$ are analysed below, similar to the analysis for (σ-destruction) above. Let $\underline{n} : \langle \omega^{\underline{n-1}}.\tau.nil \vdash \tau \rangle$.

If $n \leq k$ then by the rule (nil-φ) one has that $\varphi^i_k\,\underline{n} : \langle \omega^{\underline{n-1}}.\tau.nil \vdash \tau \rangle$ and $\varphi^i_k\,\underline{n} \to \underline{n}$.

If $n > k$ then by the rule (ω-φ) one has that $\varphi^i_k\,\underline{n} : \langle \omega^{\underline{n+i-2}}.\tau.nil \vdash \tau \rangle$ and $\varphi^i_k\,\underline{n} \to \underline{n+i-1}$. Therefore, by the rules var and varn one has that $\underline{n+i-1} : \langle \omega^{\underline{n+i-2}}.\tau.nil \vdash \tau \rangle$.

- (σ-σ-transition): Let $(M_1 \sigma^i M_2)\,\sigma^j M_3 : \langle \Gamma \vdash \tau \rangle$, for $i \leq j$.

If $\Gamma = nil$, then by Lemma 4.21.2 one has two possibilities.

Suppose that $(M_1 \sigma^i M_2) : \langle nil \vdash \tau \rangle$. Hence, by Lemma 4.21.2 one has two subcases.

In the first subcase one has that $M_1 : \langle nil \vdash \tau \rangle$. Hence, by the rule ($nil$-$\sigma$) one has that $M_1 \sigma^{j+1} M_3 : \langle nil \vdash \tau \rangle$ and $(M_1 \sigma^{j+1} M_3)\,\sigma^i (M_2\,\sigma^{j-i+1} M_3) : \langle nil \vdash \tau \rangle$.

In the second subcase one has that $M_1 : \langle \omega^{\underline{i-1}}.\wedge^m_{k=1} \sigma_k.nil \vdash \tau \rangle$ where $\forall 1 \leq k \leq m$, $M_2 : \langle nil \vdash \sigma_k \rangle$. One has that $i \leq j$ thus $i < j+1$. Hence, by the rule (nil-σ) one has that $M_1 \sigma^{j+1} M_3 : \langle \omega^{\underline{i-1}}.\wedge^m_{k=1} \sigma_k.nil \vdash \tau \rangle$ and, since $j-i+1 > 0$, $\forall 1 \leq k \leq m$, $M_2\,\sigma^{j-i+1} M_3 : \langle nil \vdash \sigma_k \rangle$. Therefore, by the rule ($\wedge$-nil-σ) one has that $(M_1 \sigma^{j+1} M_3)\,\sigma^i (M_2\,\sigma^{j-i+1} M_3) : \langle nil \vdash \tau \rangle$.

Suppose that $(M_1 \sigma^i M_2) : \langle \omega^{\underline{j-1}}.\wedge^m_{k=1} \sigma_k.nil \vdash \tau \rangle$ where $\forall 1 \leq k \leq m$, $M_3 : \langle nil \vdash \sigma_k \rangle$. One has that $i \leq j = |\omega^{\underline{j-1}}.\wedge^m_{k=1} \sigma_k.nil|$ thus by Lemma 4.21.4 one has two subcases.

In the first subcase one has that $M_1 : \langle \omega^{\underline{j}}.\wedge^m_{k=1} \sigma_k.nil \vdash \tau \rangle$. Hence, by the rule ($\wedge$-$nil$-$\sigma$) one has that $M_1 \sigma^{j+1} M_3 : \langle nil \vdash \tau \rangle$ thus, by the rule (nil-σ), $(M_1 \sigma^{j+1} M_3)\,\sigma^i (M_2\,\sigma^{j-i+1} M_3) : \langle nil \vdash \tau \rangle$.

In the second subcase one has that $(\omega^{\underline{j-1}}.\wedge^m_{k=1} \sigma_k.nil)_{\geq i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^{m'}$ such that $M_1 : \langle (\omega^{\underline{j-1}}.\wedge^m_{k=1} \sigma_k.nil)_{<i}.\wedge^{m'}_{l=1} \sigma'_l.\Gamma' \vdash \tau \rangle$ and $\forall 1 \leq l \leq m'$, $M_2 : \langle \Delta^l \vdash \sigma'_l \rangle$. Observe that $\omega^{\underline{i-1}} = (\omega^{\underline{j-1}}.\wedge^m_{k=1} \sigma_k.nil)_{<i}$ and suppose w.l.o.g. that $\Gamma' = nil$ thus $\forall 1 \leq l \leq m'$, $\Delta^l = nil$ or $|\Delta^l| = j-(i-1)$. Hence, by the rule (nil-σ) one has $M_1 \sigma^{j+1} M_3 : \langle \omega^{\underline{i-1}}.\wedge^{m'}_{l=1} \sigma'_l.nil \vdash \tau \rangle$ and $\forall 1 \leq l \leq m'$, if $|\Delta^l| = 0$ then $M_2\,\sigma^{j-i+1} M_3 : \langle nil \vdash \sigma'_l \rangle$ by the rule (nil-σ) and if $|\Delta^l| > 0$ then $M_2\,\sigma^{j-i+1} M_3 : \langle nil \vdash \sigma'_l \rangle$ by the rule ($\wedge$-nil-σ). Hence, by the rule ($\wedge$-nil-σ), $(M_1 \sigma^{j+1} M_3)\,\sigma^i (M_2\,\sigma^{j-i+1} M_3) : \langle nil \vdash \tau \rangle$.

If $0 < |\Gamma| < j$, then by Lemma 4.21.3 one has two possibilities.

Suppose that $(M_1 \sigma^i M_2) : \langle \Gamma \vdash \tau \rangle$. Note that either $|\Gamma| < i$ or $|\Gamma| \geq i$.

If $|\Gamma| < i$ then by Lemma 4.21.3 one has two subcases.

In the first subcase one has that $M_1 : \langle \Gamma \vdash \tau \rangle$. Observe that $|\Gamma| < i \leq j < j+1$. Therefore, by the rule (nil-σ) one has that $M_1 \sigma^{j+1} M_3 : \langle \Gamma \vdash \tau \rangle$ and $(M_1 \sigma^{j+1} M_3)\,\sigma^i (M_2\,\sigma^{j-i+1} M_3) : \langle \Gamma \vdash \tau \rangle$.

In the second subcase one has $M_1 : \langle \Gamma' \vdash \tau \rangle$ where $\Gamma' = \Gamma.\omega^{\underline{n}}.\wedge^m_{k=1} \sigma_k.nil$ for $n \geq 0$ s.t. $|\Gamma'| = i$ and $\forall 1 \leq k \leq m$, $M_2 : \langle nil \vdash \sigma_k \rangle$. One has that $i \leq j < j+1$ thus by the rule (nil-σ) one has that $M_1 \sigma^{j+1} M_3 : \langle \Gamma' \vdash \tau \rangle$ and $\forall 1 \leq k \leq m$, $M_2\,\sigma^{j-i+1} M_3 : \langle nil \vdash \sigma_k \rangle$. Therefore, by the rule ($\wedge$-ω-σ), $(M_1 \sigma^{j+1} M_3)\,\sigma^i (M_2\,\sigma^{j-i+1} M_3) : \langle \Gamma \vdash \tau \rangle$.

If $|\Gamma| \geq i$ then by Lemma 4.21.4 one has two subcases.

In the first subcase one has $M_1 : \langle \Gamma_{<i}.\omega.\Gamma_{\geq i} \vdash \tau \rangle$. Note that $|\Gamma_{<i}.\omega.\Gamma_{\geq i}| = |\Gamma|+1 < j+1$. Hence, by the rule ($nil$-$\sigma$) one has that $M_1\sigma^{j+1}M_3 : \langle \Gamma_{<i}.\omega.\Gamma_{\geq i} \vdash \tau \rangle$ thus, by the rule (ω-σ), $(M_1\sigma^{j+1}M_3)\,\sigma^i(M_2\,\sigma^{j-i+1}M_3) : \langle \Gamma \vdash \tau \rangle$.

In the second subcase one has $\Gamma_{\geq i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ for $|\Gamma_{\geq i}| > 0$ s.t. $M_1 : \langle \Gamma_{<i}.\wedge_{k=1}^m \sigma_k.\Gamma' \vdash \tau \rangle$ and $\forall 1 \leq k \leq m$, $M_2 : \langle \Delta^k \vdash \sigma_k \rangle$. Note that $|\Gamma_{<i}.\wedge_{k=1}^m \sigma_k.\Gamma'| \leq |\Gamma|+1 < j+1$ and that $\forall 1 \leq k \leq m$, $|\Delta^k| \leq |\Gamma| - (i-1) < j-(i-1)$. Hence, by the rule ($nil$-$\sigma$), $M_1\sigma^{j+1}M_3 : \langle \Gamma_{<i}.\wedge_{k=1}^m \sigma_k.\Gamma' \vdash \tau \rangle$ and $\forall 1 \leq k \leq m$, $M_2\,\sigma^{j-i+1}M_3 : \langle \Delta^k \vdash \sigma_k \rangle$. Hence, by the rule ($\wedge$-$\sigma$), $(M_1\sigma^{j+1}M_3)\,\sigma^i(M_2\,\sigma^{j-i+1}M_3) : \langle (\Gamma_{<i}.\Gamma') \wedge \omega^{\underline{i-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m) \vdash \tau \rangle$ where $(\Gamma_{<i}.\Gamma') \wedge \omega^{\underline{i-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m) = \Gamma_{<i}.(\Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m)$.

Suppose that $(M_1\sigma^i M_2) : \langle \Gamma' \vdash \tau \rangle$, where $\Gamma' = \Gamma.\omega^{\underline{n}}.\wedge_{k=1}^m \sigma_k.nil$ for $n \geq 0$ s.t. $|\Gamma'| = j$ and $\forall 1 \leq k \leq m$, $M_3 : \langle nil \vdash \sigma_k \rangle$. Note that $|\Gamma'| = j \geq i$. Hence, by Lemma 4.21.4 one has two subcases.

In the first subcase one has that $M_1 : \langle \Gamma'_{<i}.\omega.\Gamma'_{\geq i} \vdash \tau \rangle$. Suppose w.l.o.g. that $|\Gamma| < i$ thus $\Gamma'_{<i}.\omega.\Gamma'_{\geq i} = \Gamma.\omega^{\underline{n+1}}.\wedge_{k=1}^m \sigma_k.nil$. Hence, by the rule ($\wedge$-$\omega$-$\sigma$), $M_1\sigma^{j+1}M_3 : \langle \Gamma \vdash \tau \rangle$. Therefore, by the rule (nil-σ) one has $(M_1\sigma^{j+1}M_3)\,\sigma^i(M_2\,\sigma^{j-i+1}M_3) : \langle \Gamma \vdash \tau \rangle$.

In the second subcase one has that $\Gamma'_{\geq i} = \Gamma'' \wedge (\Delta')^1 \wedge \cdots \wedge (\Delta')^{m'}$ for $|\Gamma'_{\geq i}| > 0$ s.t. $M_1 : \langle \Gamma'_{<i}.\wedge_{l=1}^{m'} \sigma'_l.\Gamma'' \vdash \tau \rangle$ and $\forall 1 \leq l \leq m'$, $M_2 : \langle (\Delta')^l \vdash \sigma'_l \rangle$. Suppose w.l.o.g. that $|\Gamma| \geq i$ and $\Gamma'' = \Gamma_{\geq i}$ thus $\Gamma'_{<i}.\wedge_{l=1}^{m'} \sigma'_l.\Gamma'' = \Gamma_{<i}.\wedge_{l=1}^{m'} \sigma'_l.\Gamma_{\geq i}$ and $\forall 1 \leq l \leq m'$, $(\Delta')^l = nil$ or $(\Delta')^l = \omega^{\underline{j-i+1}}.u_l.nil$ where $\omega \neq u_l \sqsubseteq \wedge_{k=1}^m \sigma_k$ and $u_1 \wedge \cdots \wedge u_{m'} = \wedge_{k=1}^m \sigma_k$. Hence, by the rule ($nil$-$\sigma$) one has that $M_1\sigma^{j+1}M_3 : \langle \Gamma_{<i}.\wedge_{l=1}^{m'} \sigma'_l.\Gamma_{\geq i} \vdash \tau \rangle$ and $\forall 1 \leq l \leq m'$, if $(\Delta')^l = nil$ then $M_2\,\sigma^{j-i+1}M_3 : \langle nil \vdash \sigma'_l \rangle$ by the rule (nil-σ) and if $|(\Delta')^l| = j-i+1$ then $M_2\,\sigma^{j-i+1}M_3 : \langle nil \vdash \sigma'_l \rangle$ by the rule ($\wedge$-nil-σ). Note that $(\Gamma_{<i}.\Gamma_{\geq i}) \wedge \omega^{\underline{i-1}}.(nil \wedge \cdots \wedge nil) = \Gamma_{<i}.(\Gamma_{\geq i} \wedge nil) = \Gamma$. Therefore, by the rule ($\wedge$-σ), $(M_1\sigma^{j+1}M_3)\,\sigma^i(M_2\,\sigma^{j-i+1}M_3) : \langle \Gamma \vdash \tau \rangle$.

If $|\Gamma| \geq j$, then by Lemma 4.21.4 one has two possibilities.

Suppose that $(M_1\sigma^i M_2) : \langle \Gamma_{<j}.\omega.\Gamma_{\geq j} \vdash \tau \rangle$. Note that $|\Gamma_{<j}.\omega.\Gamma_{\geq j}| = |\Gamma|+1 \geq j+1 > j \geq i$. Hence, by Lemma 4.21.4 one has two subcases.

In the first subcase one has $M_1 : \langle (\Gamma_{<j}.\omega)_{<i}.\omega.(\Gamma_{<j}.\omega)_{\geq i}.\Gamma_{\geq j} \vdash \tau \rangle$. Note that $(\Gamma_{<j}.\omega)_{<i} = \Gamma_{<i}$ and $(\Gamma_{<j}.\omega)_{\geq i} = (\Gamma_{<j})_{\geq i}.\omega$. Hence, by the rule ($\omega$-$\sigma$) one has $M_1\sigma^{j+1}M_3 : \langle \Gamma_{<i}.\omega.(\Gamma_{<j})_{\geq i}.\Gamma_{\geq j} \vdash \tau \rangle$ where $(\Gamma_{<j})_{\geq i}.\Gamma_{\geq j} = \Gamma_{\geq i}$. Hence, by the rule ($\omega$-$\sigma$), $(M_1\sigma^{j+1}M_3)\,\sigma^i(M_2\,\sigma^{j-i+1}M_3) : \langle \Gamma_{<i}.\Gamma_{\geq i} \vdash \tau \rangle$.

In the second subcase one has that $(\Gamma_{<j})_{\geq i}.\omega.\Gamma_{\geq j} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ s.t. $M_1 : \langle \Gamma_{<i}.\wedge_{k=1}^m \sigma_k.\Gamma' \vdash \tau \rangle$ and $\forall 1 \leq k \leq m$, $M_2 : \langle \Delta^k \vdash \sigma_k \rangle$. Suppose w.l.o.g. that $\Gamma' = \Gamma''.\omega.\Gamma_{\geq j}$ thus $\Gamma'' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m = (\Gamma_{<j})_{\geq i}$. Hence, by the rule ($\omega$-$\sigma$) one has $M_1\sigma^{j+1}M_3 : \langle \Gamma_{<i}.\wedge_{k=1}^m \sigma_k.\Gamma''.\Gamma_{\geq j} \vdash \tau \rangle$ and by the rule (nil-σ) one has that $\forall 1 \leq k \leq m$, $M_2\,\sigma^{j-i+1}M_3 : \langle \Delta^k \vdash \sigma_k \rangle$. Let $\Delta' = \Delta^1 \wedge \cdots \wedge \Delta^m$. Therefore, by the rule ($\wedge$-σ) one has $(M_1\sigma^{j+1}M_3)\,\sigma^i(M_2\,\sigma^{j-i+1}M_3) : \langle (\Gamma_{<i}.\Gamma''.\Gamma_{\geq j}) \wedge \omega^{\underline{i-1}}.\Delta' \vdash \tau \rangle$, where $(\Gamma_{<i}.\Gamma''.\Gamma_{\geq j}) \wedge \omega^{\underline{i-1}}.\Delta' = \Gamma_{<i}.((\Gamma''.\Gamma_{\geq j}) \wedge \Delta') = \Gamma_{<i}.(\Gamma'' \wedge \Delta').\Gamma_{\geq j}$.

Suppose that $\Gamma_{\geq j} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ for $|\Gamma_{\geq j}| > 0$ s.t. $(M_1\sigma^i M_2) : \langle \Gamma_{<j}.\wedge_{k=1}^m \sigma_k.\Gamma' \vdash \tau \rangle$ and $\forall 1 \leq k \leq m$, $M_3 : \langle \Delta^k \vdash \sigma_k \rangle$. Hence, by Lemma 4.21.4 one has two subcases, analogous to the subcases presented right above.

- (σ-φ-transition 1): Let $(\varphi_k^i M)\,\sigma^j N : \langle \Gamma \vdash \tau \rangle$, for $k < j < k+i$.

If $\Gamma = nil$, then by Lemma 4.21.2 one has that $\varphi_k^i M : \langle nil \vdash \tau \rangle$. Hence, by Lemma 4.21.1 one has that $M : \langle nil \vdash \tau \rangle$. Therefore, by the rule (nil-φ) one has that $\varphi_k^{i-1} M : \langle nil \vdash \tau \rangle$. Observe that the other possibility regarding Lemma 4.21.2 is s.t. $\varphi_k^i M : \langle \omega^{\underline{j-1}}.\wedge_{l=1}^m \sigma_l.nil \vdash \tau \rangle$ and $\forall 1 \leq l \leq m$, $N : \langle nil \vdash \sigma_l \rangle$. However, by Lemma 4.21.1 one has that $k+(i-1) \leq j-1$ thus $k+i \leq j$ and by hypothesis one has $j < k+i$.

If $0 < |\Gamma| < j$, then by Lemma 4.21.3 one has that $\varphi_k^i M : \langle \Gamma \vdash \tau \rangle$. Hence, by Lemma 4.21.1 one has that $|\Gamma| \leq k$ thus $M : \langle \Gamma \vdash \tau \rangle$. Therefore, by the rule (nil-φ), $\varphi_k^{i-1} M : \langle \Gamma \vdash \tau \rangle$. Note that if $|\Gamma| > k$ then, by Lemma 4.21.1, $k+i \leq |\Gamma| < j < k+i$. Observe that the other possibility regarding Lemma 4.21.3 is s.t. $\varphi_k^i M : \langle \Gamma' \vdash \tau \rangle$, where $\Gamma' = \Gamma.\omega^{\underline{n}}.\wedge_{l=1}^m \sigma_l.nil$ for $n \geq 0$ s.t. $|\Gamma'| = j$ and $\forall 1 \leq l \leq m$, $N : \langle nil \vdash \sigma_l \rangle$. However, by Lemma 4.21.1 one has that $k+i \leq |\Gamma'| = j$ and by hypothesis one has $j < k+i$.

If $|\Gamma| \geq j$, then by Lemma 4.21.4 one has that $\varphi_k^i M : \langle \Gamma_{<j}.\omega.\Gamma_{\geq j} \vdash \tau \rangle$. Hence, by Lemma 4.21.1 and by the hypothesis that $k < j < k+i$, one has that $M : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau \rangle$ where $(\Gamma_{<j})_{>k}.\omega.(\Gamma_{<k+i})_{\geq j} = \omega^{\underline{i-1}}$. Therefore, $(\Gamma_{<j})_{>k}.(\Gamma_{<k+i})_{\geq j} = (\Gamma_{<k+i})_{>k} = \omega^{\underline{i-2}}$. Hence, by the rule ($\omega$-$\varphi$) one has that $M\varphi_k^{i-1}M : \langle \Gamma_{\leq k}.\underline{\omega}^{i-2}.\Gamma_{\geq k+i} \vdash \tau\rangle$, where $\Gamma_{\leq k}.\underline{\omega}^{i-2}.\Gamma_{\geq k+i} = \Gamma_{\leq k}.(\Gamma_{<k+i})_{>k}.\Gamma_{\geq k+i} = \Gamma$. Note that the other possibility in Lemma 4.21.4 is s.t. $\Gamma_{\geq j} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ for $|\Gamma_{\geq j}|>0$ s.t. $\varphi_k^i M : \langle \Gamma_{<j}.\wedge_{l=1}^m \sigma_l.\Gamma' \vdash \tau\rangle$ and $\forall 1 \leq l \leq m,\ N : \langle \Delta^l \vdash \sigma_l\rangle$. However, by Lemma 4.21.1 and by the hypothesis that $k<j<k+i$, one has that $(\Gamma_{<j}.\wedge_{l=1}^m \sigma_l.\Gamma')_j = \omega$.

- (σ-φ-transition 2): Let $(\varphi_k^i M)\,\sigma^j N : \langle \Gamma \vdash \tau\rangle$, for $k+i \leq j$.
  If $\Gamma = nil$, then by Lemma 4.21.2 one has two possibilities.
  Suppose that $\varphi_k^i M : \langle nil \vdash \tau\rangle$. Hence, by Lemma 4.21.2 one has that $M : \langle nil \vdash \tau\rangle$. Therefore, by the rule (nil-σ) one has that $M\sigma^{j-i+1}N : \langle nil \vdash \tau\rangle$ and, by the rule (nil-φ), that $\varphi_k^i(M\sigma^{j-i+1}N) : \langle nil \vdash \tau\rangle$.
  Suppose that $\varphi_k^i M : \langle \underline{\omega}^{j-1}.\wedge_{l=1}^m \sigma_l.nil \vdash \tau\rangle$ and $\forall 1 \leq l \leq m,\ N : \langle nil \vdash \sigma_l\rangle$. Hence, by Lemma 4.21.1 and by the hypothesis that $k+i \leq j$ one has that $M : \langle \underline{\omega}^{(j-1)-(i-1)}.\wedge_{l=1}^m \sigma_l.nil \vdash \tau\rangle$. Therefore, by the rule ($\wedge$-nil-σ) one has that $M\sigma^{j-i+1}N : \langle nil \vdash \tau\rangle$ and by the rule (nil-φ) one has that $\varphi_k^i(M\sigma^{j-i+1}N) : \langle nil \vdash \tau\rangle$.
  If $0<|\Gamma|<j$, then by Lemma 4.21.3 one has two possibilities.
  Suppose that $\varphi_k^i M : \langle \Gamma \vdash \tau\rangle$. Note that either $|\Gamma| \leq k$ or $|\Gamma|>k$.
  If $|\Gamma| \leq k$, then by Lemma 4.21.1 one has that $M : \langle \Gamma \vdash \tau\rangle$. One has that $k+i \leq j$ thus $|\Gamma| \leq k \leq j-i < j-i+1$. Therefore, by the rule (nil-σ) one has that $M\sigma^{j-i+1}N : \langle \Gamma \vdash \tau\rangle$ and by the rule (nil-φ) one has that $\varphi_k^i(M\sigma^{j-i+1}N) : \langle \Gamma \vdash \tau\rangle$.
  If $|\Gamma|>k$, then by Lemma 4.21.1 one has that $M : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau\rangle$, where $\Gamma = \Gamma_{\leq k}.\underline{\omega}^{i-1}.\Gamma_{\geq k+i}$. One has that $|\Gamma_{\leq k}.\Gamma_{\geq k+i}| = |\Gamma| - (i-1) < j - (i-1)$. Therefore, by the rule (nil-σ) one has that $M\sigma^{j-i+1}N : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau\rangle$ and by the rule (ω-φ) one has that $\varphi_k^i(M\sigma^{j-i+1}N) : \langle \Gamma_{\leq k}.\underline{\omega}^{i-1}.\Gamma_{\geq k+i} \vdash \tau\rangle$.
  Suppose that $\varphi_k^i M : \langle \Gamma' \vdash \tau\rangle$, where $\Gamma' = \Gamma.\underline{\omega}^{n}.\wedge_{l=1}^m \sigma_l.nil$ for $n \geq 0$ s.t. $|\Gamma'| = j$ and $\forall 1 \leq l \leq m$, $N : \langle nil \vdash \sigma_l\rangle$. Note that, by Lemma 4.18, $\Gamma_{|\Gamma|} \neq \omega$ thus by Lemma 4.21.1 one has that $k+i \leq |\Gamma|$ or $|\Gamma|<k$. Suppose w.l.o.g. that $k+i \leq |\Gamma|$. Hence, $M : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i}.\underline{\omega}^{n}.\wedge_{l=1}^m \sigma_l.nil \vdash \tau\rangle$, where $\Gamma = \Gamma_{\leq k}.\underline{\omega}^{i-1}.\Gamma_{\geq k+i}$. Therefore, by the rule ($\wedge$-ω-σ) one has that $M\sigma^{j-i+1}N : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau\rangle$ and, by the rule (ω-φ), $\varphi_k^i(M\sigma^{j-i+1}N) : \langle \Gamma_{\leq k}.\underline{\omega}^{i-1}.\Gamma_{\geq k+i} \vdash \tau\rangle$.
  If $|\Gamma| \geq j$, then by Lemma 4.21.4 one has two possibilities.
  Suppose that $\varphi_k^i M : \langle \Gamma_{<j}.\omega.\Gamma_{\geq j} \vdash \tau\rangle$. Observe that $k+i \leq j$. Hence, by Lemma 4.21.1 one has that $M : \langle \Gamma_{\leq k}.(\Gamma_{<j})_{\geq k+i}.\omega.\Gamma_{\geq j} \vdash \tau\rangle$ where $\Gamma_{<j} = \Gamma_{\leq k}.\underline{\omega}^{i-1}.(\Gamma_{<j})_{\geq k+i}$. Therefore, by the rule (ω-σ) one has $M\sigma^{j-i+1}N : \langle \Gamma_{\leq k}.(\Gamma_{<j})_{\geq k+i}.\Gamma_{\geq j} \vdash \tau\rangle$ and, by the rule (ω-φ), $\varphi_k^i(M\sigma^{j-i+1}N) : \langle \Gamma_{\leq k}.\underline{\omega}^{i-1}.(\Gamma_{<j})_{\geq k+i}.\Gamma_{\geq j} \vdash \tau\rangle$, where $\Gamma_{\leq k}.\underline{\omega}^{i-1}.(\Gamma_{<j})_{\geq k+i}.\Gamma_{\geq j} = \Gamma$.
  Suppose that $\varphi_k^i M : \langle \Gamma_{<j}.\wedge_{l=1}^m \sigma_l.\Gamma' \vdash \tau\rangle$ where, for $|\Gamma_{\geq j}|>0$, $\Gamma_{\geq j} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ and $\forall 1 \leq l \leq m$, $N : \langle \Delta^l \vdash \sigma_l\rangle$. The proof that $\varphi_k^i(M\sigma^{j-i+1}N) : \langle \Gamma \vdash \tau\rangle$ is similar to the proof for the first possibility regarding Lemma 4.21.4, presented right above.
- (φ-σ-transition): Let $\varphi_k^i(M\sigma^j N) : \langle \Gamma \vdash \tau\rangle$, for $j \leq k+1$.
  If $|\Gamma| \leq k$, then by Lemma 4.21.1 one has that $M\sigma^j N : \langle \Gamma \vdash \tau\rangle$.
  If $\Gamma = nil$ then by Lemma 4.21.2 one has two subcases.
  In the first subcase one has that $M : \langle nil \vdash \tau\rangle$. Hence, by the rule ($nil$-$\varphi$) one has $\varphi_{k+1}^i M : \langle nil \vdash \tau\rangle$ and, by the rule (nil-σ), $(\varphi_{k+1}^i M)\,\sigma^j(\varphi_{k+1-j}^i N) : \langle nil \vdash \tau\rangle$.
  In the second subcase one has that $M : \langle \underline{\omega}^{j-1}.\wedge_{l=1}^m \sigma_l.nil \vdash \tau\rangle$ where $\forall 1 \leq l \leq m$, $N : \langle nil \vdash \sigma_l\rangle$. Hence, by the rule ($nil$-$\varphi$) one has $\varphi_{k+1}^i M : \langle \underline{\omega}^{j-1}.\wedge_{l=1}^m \sigma_l.nil \vdash \tau\rangle$ and $\forall 1 \leq l \leq m$, $\varphi_{k+1-j}^i N : \langle nil \vdash \sigma_l\rangle$. Therefore, by the rule ($\wedge$-nil-σ), $(\varphi_{k+1}^i M)\,\sigma^j(\varphi_{k+1-j}^i N) : \langle nil \vdash \tau\rangle$.
  If $0<|\Gamma|<j$ then by Lemma 4.21.3 one has two subcases.
  In the first subcase one has that $M : \langle \Gamma \vdash \tau\rangle$. Note that $|\Gamma| \leq k < k+1$ thus, by the rule (nil-φ), $\varphi_{k+1}^i M : \langle \Gamma \vdash \tau\rangle$. Therefore, by the rule (nil-σ), $(\varphi_{k+1}^i M)\,\sigma^j(\varphi_{k+1-j}^i N) : \langle \Gamma \vdash \tau\rangle$.
  In the second subcase one has $M : \langle \Gamma' \vdash \tau\rangle$ where $\Gamma' = \Gamma.\underline{\omega}^{n}.\wedge_{l=1}^m \sigma_l.nil$ for $n \geq 0$ s.t. $|\Gamma'| = j$ and $\forall 1 \leq l \leq m$, $N : \langle nil \vdash \sigma_l\rangle$. Note that $|\Gamma'| = j \leq k+1$. Hence, by the rule (nil-φ) one has $\varphi_{k+1}^i M : \langle \Gamma' \vdash \tau\rangle$ and $\forall 1 \leq l \leq m$, $\varphi_{k+1-j}^i N : \langle nil \vdash \sigma_l\rangle$. Therefore, by the rule ($\wedge$-ω-σ), $(\varphi_{k+1}^i M)\,\sigma^j(\varphi_{k+1-j}^i N) : \langle \Gamma \vdash \tau\rangle$.
  If $|\Gamma| \geq j$ then by Lemma 4.21.4 one has two subcases.
  In the first subcase one has $M : \langle \Gamma_{<j}.\omega.\Gamma_{\geq j} \vdash \tau\rangle$. Note that $|\Gamma_{<j}.\omega.\Gamma_{\geq j}| = |\Gamma|+1 \leq k+1$. Hence, by the rule ($nil$-$\varphi$) one has $\varphi_{k+1}^i M : \langle \Gamma_{<j}.\omega.\Gamma_{\geq j} \vdash \tau\rangle$. Therefore, by the rule (ω-σ), $(\varphi_{k+1}^i M)\,\sigma^j(\varphi_{k+1-j}^i N) : \langle \Gamma_{<j}.\Gamma_{\geq j} \vdash \tau\rangle$.

In the second subcase one has that $\Gamma_{\geq j} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ for $|\Gamma_{\geq j}| > 0$ s.t. $M : \langle \Gamma_{<j}. \wedge_{l=1}^{m} \sigma_l.\Gamma' \vdash \tau\rangle$ and $\forall 1 \leq l \leq m$, $N : \langle \Delta^l \vdash \sigma_l\rangle$. Observe that $\forall 1 \leq l \leq m$, $|\Delta^l| \leq |\Gamma| - (j-1) \leq k - (j-1) = k{+}1 - j$. Hence, by the rule ($nil$-$\varphi$) one has $\varphi^i_{k+1} M : \langle \Gamma_{<j}. \wedge_{l=1}^{m} \sigma_l.\Gamma' \vdash \tau\rangle$ and $\forall 1 \leq l \leq m$, $\varphi^i_{k+1-j} N : \langle \Delta^l \vdash \sigma_l\rangle$. Therefore, by the rule ($\wedge$-σ) one has that $(\varphi^i_{k+1} M)\, \sigma^j (\varphi^i_{k+1-j} N) : \langle (\Gamma_{<j}.\Gamma') \wedge \omega^{\underline{j-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m) \vdash \tau\rangle$, where $(\Gamma_{<j}.\Gamma') \wedge \omega^{\underline{j-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m) = \Gamma_{<j}.(\Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m)$.

If $|\Gamma| > k$, then by Lemma 4.21.1 one has that $M \sigma^j N : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau\rangle$, where $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. Note that $|\Gamma_{\leq k}.\Gamma_{\geq k+i}| \geq k{+}1 \geq j$ thus, by Lemma 4.21.4, one has two possibilities.

Suppose that $M : \langle (\Gamma_{\leq k}.\Gamma_{\geq k+i})_{<j}.\omega.(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\geq j} \vdash \tau\rangle$. Observe that $(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{<j} = \Gamma_{<j}$, that $(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\geq j} = (\Gamma_{\leq k})_{\geq j}.\Gamma_{\geq k+i}$ and that $|\Gamma_{<j}.\omega.(\Gamma_{\leq k})_{\geq j}.| = k{+}1$. Hence, by the rule ($\omega$-$\varphi$) one has that $\varphi^i_{k+1} M : \langle \Gamma_{<j}.\omega.(\Gamma_{\leq k})_{\geq j}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i} \vdash \tau\rangle$. Therefore, by the rule (ω-σ) one has $(\varphi^i_{k+1} M)\, \sigma^j (\varphi^i_{k+1-j} N) : \langle \Gamma_{<j}.(\Gamma_{\leq k})_{\geq j}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i} \vdash \tau\rangle$. Observe that $\Gamma_{<j}.(\Gamma_{\leq k})_{\geq j} = \Gamma_{\leq k}$.

Suppose that $(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\geq j} = (\Gamma_{\leq k})_{\geq j}.\Gamma_{\geq k+i} = \Gamma' \wedge \Delta^1 \wedge \cdots \wedge \Delta^m$ for $|(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\geq j}| > 0$ s.t. $M : \langle \Gamma_{<j}. \wedge_{l=1}^{m} \sigma_l.\Gamma' \vdash \tau\rangle$ and $\forall 1 \leq l \leq m$, $N : \langle \Delta^l \vdash \sigma_l\rangle$. Suppose w.l.o.g. that $\Gamma' = nil$ thus $(\Gamma_{\leq k})_{\geq j} = (\Delta^1 \wedge \cdots \wedge \Delta^m)_{\leq k+1-j}$ and $\Gamma_{\geq k+i} = (\Delta^1 \wedge \cdots \wedge \Delta^m)_{>k+1-j}$. Hence, by the rule ($nil$-$\varphi$) one has that $\varphi^i_{k+1} M : \langle \Gamma_{<j}. \wedge_{l=1}^{m} \sigma_l.nil \vdash \tau\rangle$ and $\forall 1 \leq l \leq m$, if $|\Delta^l| \leq k{+}1 - j$ then $\varphi^i_{k+1-j} N : \langle \Delta^l \vdash \sigma_l\rangle$ by the rule (nil-φ) and if $|\Delta^l| > k{+}1 - j$ then $\varphi^i_{k+1-j} N : \langle \Delta^l_{\leq k+1-j}.\omega^{\underline{i-1}}.\Delta^l_{>k+1-j} \vdash \sigma_l\rangle$ by the rule (ω-φ). Since $\forall 1 \leq l \leq m$, if $|\Delta^l| \leq k{+}1 - j$ then $\Delta^l_{>k+1-j} = nil$, one has that the intersection Δ' of contexts for $\varphi^i_{k+1-j} N$ can be described as $\Delta' = (\Delta^1 \wedge \cdots \wedge \Delta^m)_{\leq k+1-j}.\omega^{\underline{i-1}}.(\Delta^1 \wedge \cdots \wedge \Delta^m)_{>k+1-j}$. Hence, $\Delta' = (\Gamma_{\leq k})_{\geq j}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. Therefore, by the rule ($\wedge$-σ) one has that $(\varphi^i_{k+1} M)\, \sigma^j (\varphi^i_{k+1-j} N) : \langle (\Gamma_{<j}.nil) \wedge \omega^{\underline{j-1}}.\Delta' \vdash \tau\rangle$ where $(\Gamma_{<j}.nil) \wedge \omega^{\underline{j-1}}.\Delta' = \Gamma_{<j}.\Delta' = \Gamma_{<j}.(\Gamma_{\leq k})_{\geq j}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i} = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i} = \Gamma$.

- (φ-φ-transition 1): Let $\varphi^i_k(\varphi^j_l M) : \langle \Gamma \vdash \tau\rangle$, for $l{+}j \leq k$.

  If $|\Gamma| \leq k$, then by Lemma 4.21.1 one has that $\varphi^j_l M : \langle \Gamma \vdash \tau\rangle$. Hence, by Lemma 4.21.1 one has two possibilities.

  If $|\Gamma| \leq l$, then by Lemma 4.21.1 one has $M : \langle \Gamma \vdash \tau\rangle$. Note that $k{+}1{-}j \geq (l{+}j){+}1{-}j = l{+}1 > l > |\Gamma|$. Therefore, by the rule (nil-φ) one has that $\varphi^i_{k+1-j} M : \langle \Gamma \vdash \tau\rangle$ and $\varphi^j_l(\varphi^i_{k+1-j} M) : \langle \Gamma \vdash \tau\rangle$.

  If $|\Gamma| > l$ then, by Lemma 4.21.1, one has $M : \langle \Gamma_{\leq l}.\Gamma_{\geq l+j} \vdash \tau\rangle$ where $\Gamma_{\leq l}.\omega^{\underline{j-1}}.\Gamma_{\geq l+j} = \Gamma$. One has that $|\Gamma_{\leq l}.\Gamma_{\geq l+j}| = |\Gamma| - (j-1) \leq k - (j-1)$. Therefore, by the rule (nil-φ) one has that $\varphi^i_{k+1-j} M : \langle \Gamma_{\leq l}.\Gamma_{\geq l+j} \vdash \tau\rangle$ and, by the rule (ω-φ), $\varphi^j_l(\varphi^i_{k+1-j} M) : \langle \Gamma_{\leq l}.\omega^{\underline{j-1}}.\Gamma_{\geq l+j} \vdash \tau\rangle$.

  If $|\Gamma| > k$ then, by Lemma 4.21.1, $\varphi^j_l M : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau\rangle$ where $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. One has that $|\Gamma| > k \geq l{+}j > l$. Hence, by Lemma 4.21.1 one has that $M : \langle (\Gamma')_{\leq l}.(\Gamma')_{\geq l+j} \vdash \tau\rangle$ where $\Gamma' = (\Gamma')_{\leq l}.\omega^{\underline{j-1}}.(\Gamma')_{\geq l+j} = \Gamma_{\leq k}.\Gamma_{\geq k+i}$. Note that $(\Gamma')_{\leq l} = \Gamma_{\leq l}$ and that $(\Gamma')_{\geq l+j} = (\Gamma_{\leq k})_{\geq l+j}.\Gamma_{\geq k+i}$. Hence, $(\Gamma')_{\leq l}.(\Gamma')_{\geq l+j} = \Gamma_{\leq l}.(\Gamma_{\leq k})_{\geq l+j}.\Gamma_{\geq k+i}$ and $|\Gamma_{\leq l}.(\Gamma_{\leq k})_{\geq l+j}| = l + (k-(l{+}j-1)) = k{+}1-j$. Hence, $\varphi^i_{k+1-j} M : \langle \Gamma_{\leq l}.(\Gamma_{\leq k})_{\geq l+j}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i} \vdash \tau\rangle$ and $\varphi^j_l(\varphi^i_{k+1-j} M) : \langle \Gamma_{\leq l}.\omega^{\underline{j-1}}.(\Gamma_{\leq k})_{\geq l+j}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i} \vdash \tau\rangle$, where $\Gamma_{\leq k} = \Gamma_{\leq l}.\omega^{\underline{j-1}}.(\Gamma_{\leq k})_{\geq l+j}$, by the rule ($\omega$-$\varphi$).

- (φ-φ-transition 2): Let $\varphi^i_k(\varphi^j_l M) : \langle \Gamma \vdash \tau\rangle$, for $l \leq k < l{+}j$.

  If $|\Gamma| \leq k$, then by Lemma 4.21.1 one has that $\varphi^j_l M : \langle \Gamma \vdash \tau\rangle$. Observe that if $|\Gamma| > l$ then by Lemma 4.21.1 one has that $|\Gamma| > l{+}(j{-}1)$ thus $|\Gamma| \geq l{+}j > k$. Hence, for $|\Gamma| \leq l$ one has, by Lemma 4.21.1, that $M : \langle \Gamma \vdash \tau\rangle$. Therefore, by the rule (nil-φ), $\varphi^{j+i-1}_l M : \langle \Gamma \vdash \tau\rangle$.

  If $|\Gamma| > k$ then, by Lemma 4.21.1, $\varphi^j_l M : \langle \Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash \tau\rangle$ where $\Gamma = \Gamma_{\leq k}.\omega^{\underline{i-1}}.\Gamma_{\geq k+i}$. One has that $|\Gamma_{\leq k}.\Gamma_{\geq k+i}| > k \geq l$. Hence, by Lemma 4.21.1, $M : \langle (\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\leq l}.(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\geq l+j} \vdash \tau\rangle$ where $(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\leq l} = \Gamma_{\leq l}$, $(\Gamma_{\leq k})_{>l} = \omega^{\underline{k-l}}$, $(\Gamma_{\leq k}.\Gamma_{\geq k+i})_{\geq l+j} = (\Gamma_{\geq k+i})_{\geq l+j-k} = \Gamma_{\geq l+(j+i)-1}$ and $(\Gamma_{<l+(j+i)-1})_{\geq k+i} = \omega^{\underline{(j-1)-(k-l)}}$. Hence $\varphi^{j+i-1}_l M : \langle \Gamma_{\leq l}.\omega^{\underline{j+i-2}}.\Gamma_{\geq l+(j+i)-1} \vdash \tau\rangle$, by the rule ($\omega$-$\varphi$), where $\Gamma_{\leq l}.\omega^{\underline{j+i-2}}.\Gamma_{\geq l+(j+i)-1} = \Gamma_{\leq l}.(\Gamma_{\leq k})_{>l}.\omega^{\underline{i-1}}.(\Gamma_{<l+(j+i)-1})_{\geq k+i}.\Gamma_{\geq l+(j+i)-1}$. Observe that $\Gamma_{\leq k} = \Gamma_{\leq l}.(\Gamma_{\leq k})_{>l}$ and that $(\Gamma_{<l+(j+i)-1})_{\geq k+i}.\Gamma_{\geq l+(j+i)-1} = \Gamma_{\geq k+i}$. Therefore, $\Gamma_{\leq l}.\omega^{\underline{j+i-2}}.\Gamma_{\geq l+(j+i)-1} = \Gamma$.

$\blacksquare$

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