A refinement of de Bruijn's formal language of Mathematics

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24 September 2001

$\textsc{Aut}$-68

- $\mathcal{E} ::= \quad \mathcal{V} \quad | \quad a(\underbrace{\mathcal{E}, \ldots, \mathcal{E}}) \quad | \quad [\mathcal{V} : \mathcal{E}^+]\mathcal{E} \quad | \quad \langle\mathcal{E}\rangle\mathcal{E}.$

  $a \in \mathcal{C}, \qquad \mathcal{V} \cap \mathcal{C} = \varnothing \qquad \mathcal{E}^+ \stackrel{\text{def}}{=} \mathcal{E} \cup \{\texttt{type}\}.$

- Contexts $\Gamma ::= \quad \langle\rangle \quad | \quad \Gamma, \mathcal{V} : \mathcal{E}$ where variables are declared at most once.

- Lines $l ::= \quad \Gamma; \mathcal{V}; -; \mathcal{E}^+ \quad | \quad \Gamma; \mathcal{C}; \textsc{pn}\ ; \mathcal{E}^+ \quad | \quad \Gamma; \mathcal{C}; \mathcal{E}; \mathcal{E}^+$

- Books $\mathfrak{B} \quad ::= \quad \emptyset \quad | \quad \mathfrak{B}, l.$

Example of an AUTOMATH-book

∅	prop	PN	type	(1)
∅	x	—	prop	(2)
x	y	—	prop	(3)
x,y	and	PN	prop	(4)
x	proof	PN	type	(5)
x,y	px	—	proof(x)	(6)
x,y,px	py	—	proof(y)	(7)
x,y,px,py	and-I	PN	proof(and)	(8)
x,y	pxy	—	proof(and)	(9)
x,y,pxy	and-O1	PN	proof(x)	(10)
x,y,pxy	and-O2	PN	proof(y)	(11)
x	prx	—	proof(x)	(12)
x,prx	and-R	and-I(x,x,prx,prx)	proof(and(x,x))	(13)
x,y,pxy	and-S	and-I(y,x,and-O2,and-O1)	proof(and(y,x))	(14)

Notions of correctness and of typing

- See D. van Daalen 1980 [2].
- $\mathfrak{B}; \emptyset \vdash \textsc{ok}$ indicates that book $\mathfrak{B}$ is correct.
- $\mathfrak{B}; \Gamma \vdash \textsc{ok}$ indicates that the context Γ is correct with respect to the (correct) book $\mathfrak{B}$.
- $\mathfrak{B}; \Gamma \vdash \Sigma_1 : \Sigma_2$ indicates that Σ_1 is a correct expression of type Σ_2 with respect to $\mathfrak{B}$ and Γ.
- We also say: $\Sigma_1 : \Sigma_2$ is a correct *statement* with respect to $\mathfrak{B}$ and Γ.
- The Automath book given earlier is correct.

Correct books and contexts

(axiom)

$$\varnothing; \varnothing \vdash \text{OK}$$

(context ext.)

$$\frac{\mathfrak{B}_1, (\Gamma; x; —; \alpha), \mathfrak{B}_2; \Gamma \vdash \text{OK}}{\mathfrak{B}_1, (\Gamma; x; —; \alpha), \mathfrak{B}_2; \Gamma, x{:}\alpha \vdash \text{OK}}$$

(book ext.: var1)

$$\frac{\mathfrak{B}; \Gamma \vdash \text{OK}}{\mathfrak{B}, (\Gamma; x; —; \mathtt{type}); \varnothing \vdash \text{OK}}$$

(book ext.: var2)

$$\frac{\mathfrak{B}; \Gamma \vdash \Sigma_2 : \mathtt{type}}{\mathfrak{B}, (\Gamma; x; —; \Sigma_2); \varnothing \vdash \text{OK}}$$

(book ext.: pn1)

$$\frac{\mathfrak{B}; \Gamma \vdash \text{OK}}{\mathfrak{B}, (\Gamma; k; \text{PN}; \mathtt{type}); \varnothing \vdash \text{OK}}$$

(book ext.: pn2)

$$\frac{\mathfrak{B}; \Gamma \vdash \Sigma_2 : \mathtt{type}}{\mathfrak{B}, (\Gamma; k; \text{PN}; \Sigma_2); \varnothing \vdash \text{OK}}$$

(book ext.: def1)

$$\frac{\mathfrak{B}; \Gamma \vdash \Sigma_1 : \mathtt{type}}{\mathfrak{B}, (\Gamma; k; \Sigma_1; \mathtt{type}); \varnothing \vdash \text{OK}}$$

(book ext.: def2)

$$\frac{\mathfrak{B}; \Gamma \vdash \Sigma_2 : \mathtt{type} \quad \mathfrak{B}; \Gamma \vdash \Sigma_1 : \Sigma_2' \quad \mathfrak{B}; \Gamma \vdash \Sigma_2 =_{\beta\text{d}} \Sigma_2'}{\mathfrak{B}, (\Gamma; k; \Sigma_1; \Sigma_2); \varnothing \vdash \text{OK}}$$

For rules (book ext.) we assume $x \in \mathcal{V}$ and $k \in \mathcal{C}$ do not occur in $\mathfrak{B}$ or Γ.

Correct statements

(start)
$$\frac{\mathfrak{B};\Gamma_1, x{:}\alpha, \Gamma_2 \vdash \text{OK}}{\mathfrak{B};\Gamma_1, x{:}\alpha, \Gamma_2 \vdash x{:}\alpha}$$

(parameters)
$$\frac{\mathfrak{B} \equiv \mathfrak{B}_1, (x_1{:}\alpha_1, \ldots, x_n{:}\alpha_n; b; \Omega_1; \Omega_2), \mathfrak{B}_2 \quad \mathfrak{B};\Gamma \vdash \Sigma_i{:}\alpha_i[x_1, \ldots, x_{i-1}{:=}\Sigma_1, \ldots, \Sigma_{i-1}] (i = 1, \ldots, n)}{\mathfrak{B};\Gamma \vdash b(\Sigma_1, \ldots, \Sigma_n) : \Omega_2[x_1, \ldots, x_n{:=}\Sigma_1, \ldots, \Sigma_n]}$$

(abstr.1)
$$\frac{\mathfrak{B};\Gamma \vdash \Sigma_1{:}\mathtt{type} \quad \mathfrak{B};\Gamma, x{:}\Sigma_1 \vdash \Omega_1{:}\mathtt{type}}{\mathfrak{B};\Gamma \vdash [x{:}\Sigma_1]\Omega_1 : \mathtt{type}}$$

(abstr.2)
$$\frac{\mathfrak{B};\Gamma \vdash \Sigma_1{:}\mathtt{type} \quad \mathfrak{B};\Gamma, x{:}\Sigma_1 \vdash \Omega_1{:}\mathtt{type} \quad \mathfrak{B};\Gamma, x{:}\Sigma_1 \vdash \Sigma_2{:}\Omega_1}{\mathfrak{B};\Gamma \vdash [x{:}\Sigma_1]\Sigma_2 : [x{:}\Sigma_1]\Omega_1}$$

(application)
$$\frac{\mathfrak{B};\Gamma \vdash \Sigma_1 : [x{:}\Omega_1]\Omega_2 \quad \mathfrak{B};\Gamma \vdash \Sigma_2 : \Omega_1}{\mathfrak{B};\Gamma \vdash \langle\Sigma_2\rangle\Sigma_1 : \Omega_2[x{:=}\Sigma_2]}$$

(conversion)
$$\frac{\mathfrak{B};\Gamma \vdash \Sigma : \Omega_1 \quad \mathfrak{B};\Gamma \vdash \Omega_2{:}\mathtt{type} \quad \mathfrak{B};\Gamma \vdash \Omega_1 =_{\beta\mathrm{d}} \Omega_2}{\mathfrak{B};\Gamma \vdash \Sigma : \Omega_2}$$

When using the parameter rule, we assume that $\mathfrak{B}; \Gamma \vdash \textsc{ok}$, even if $n = 0$.

Definitional Equality

- (β) $\langle\Sigma\rangle[x{:}\Omega_2]\Omega_1 \to_\beta \Omega_1[x{:=}\Sigma]$.

- (δ) If $\Sigma = b(\Sigma_1, \ldots, \Sigma_n)$, and $\mathfrak{B}$ contains a line $(x_1{:}\alpha_1, \ldots, x_n{:}\alpha_n; b; \Xi_1; \Xi_2)$ where $\Xi_1 \in \mathcal{E}$, then $\Sigma \to_\delta \Xi_1[x_1, \ldots, x_n{:=}\Sigma_1, \ldots, \Sigma_n]$.

The refined language

- The famous mathematician Frege was frustrated by the informalities of the common mathematical language: *. . . I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain precision...*

- none of the logical languages of the 20th century satisfies the criteria expected of a language of mathematics. A logical language does not have mathematico-linguistic categories, is not universal to all users of mathematics, and is not a satisfactory communication medium.

- Logical languages make fixed choices (first versus higher order, predicative versus impredicative, constructive versus classical, types or sets, etc.). But different parts of mathematics need different choices and there is no universal agreement as to which is the best formalism.

- A logician writes in logic their understanding of a mathematical-text as a formal, complete text which is structured considerably unlike the original, and is of little use to the *ordinary* mathematician.

- Mathematicians do not want to use formal logic and have for centuries done mathematics without it.

- De Bruijn intended Automath *not just [...] as a technical system for verification of mathematical texts, it was rather a life style with its attitudes towards understanding, developing and teaching mathematics.* He added: *The way mathematical material is to be presented to the system should correspond to* *the usual way we write mathematics. The only things to be added should be details that are usually omitted in standard mathematics.*

- De Bruijn presented his Mathematical Vernacular in two rounds *In the first round we express the general framework of organization of mathematical texts. It is about books and lines, introduction of variables, assumptions, definitions, axioms and theorems [...]. In the second round we get the rules about validity.*

- De Bruijn added: *It is quite conceivable that MV, or variations of it, can have an impact on computing science. A thing that comes at once into mind, is the use of MV as an intermediate language in expert systems. Another possible use might be formal or informal specification language for computer programs*.

- The new language follows all the above ideas of de Bruijn, but attempts to remain as close as possible to the Mathematicians' language by avoiding any extra logical decisions that the matematician himself did not make.

References

[1] D.T. van Daalen. A description of Automath and some aspects of its language theory. In P. Braffort, editor, *Proceedings of the Symposium APLASM*, volume I, pages 48–77, 1973.

[2] D.T. van Daalen. *The Language Theory of Automath*. PhD thesis, Eindhoven University of Technology, 1980.