On automating the extraction of programs from proofs using product types

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- The ProPre system was designed as a prototype to show the feasibility of the theory.

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- The automated termination proofs $\neq$ techniques of rewriting systems.

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- The class of automated extracted programs are thus enlarged.

Overview

- The ProPre system
- Logical framework: AF2, TTR
- The rules and proofs in ProPre
- Analysis of the I-proofs
- The skeleton proofs
- The connection between skeleton proofs and I-proofs
- The product type
- The canonical proofs
- Conclusion

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- Automated strategies for proving termination of recursive functions.

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- Logical Interpretation coincides with the Algorithmic Interpretation of the formula.

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Data-Type : Formula of Second Order

$$\Downarrow$$

Programs for constructors (sucessor for integers, cons for lists, etc ...)

Intuitionistic rules

$$\frac{}{\Gamma,\ A \vdash\ A} \quad (ax) \qquad \frac{\Gamma \vdash\ A[u] \qquad \Gamma \vdash_{\mathcal{E}}\ u{=}v}{\Gamma \vdash\ A[v]} \quad (eq)$$

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Second Order Functional Arithmetic

$$\frac{}{\Gamma,\ x{:}A \vdash x{:}A}\ (ax) \qquad \frac{\Gamma \vdash t{:}A[u] \qquad \Gamma \vdash_{\mathcal{E}} u{=}v}{\Gamma \vdash t{:}A[v]}\ (eq)$$

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$$\vdash_{\mathcal{E}_f} t : \forall x_1, \ldots, \forall x_n \{D_1[x_1], \ldots, D_n[x_n] \rightarrow D[f(x_1, \ldots, x_n)]\}$$

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  - Then "t computes f"
- Let $f : nat \to nat$. If $\vdash_{\mathcal{E}_f} t : \forall x(N(x) \to N[f(x)])$ then
  $\vdash_{\mathcal{E}_f} f(s^n(0)) = s^m(0)$ iff $(t\ \underline{n}) \to_\beta \underline{m}$

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- It uses a logical operator of least fixed point allowing recursive definitions of data types.
- A logical hiding connective for hiding the algorithmic content of some part of the proofs.

Somes rules in TTR

- Rules of the hiding operator $\restriction$

  If A is a formula, u, v terms then $A \restriction (u \prec v)$ is a formula.

$$\frac{\Gamma \vdash_{\mathcal{E}} t{:}A \qquad \Gamma \vdash_{\mathcal{E}} e}{\Gamma \vdash_{\mathcal{E}} t{:}A{\restriction}e}\ (\restriction_1) \qquad \frac{\Gamma \vdash_{\mathcal{E}} t{:}A{\restriction}e}{\Gamma \vdash_{\mathcal{E}} t{:}A}\ (\restriction_2) \qquad \frac{\Gamma \vdash_{\mathcal{E}} t{:}A{\restriction}e}{\Gamma \vdash_{\mathcal{E}} e}\ (\restriction_3)$$

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- External induction rule

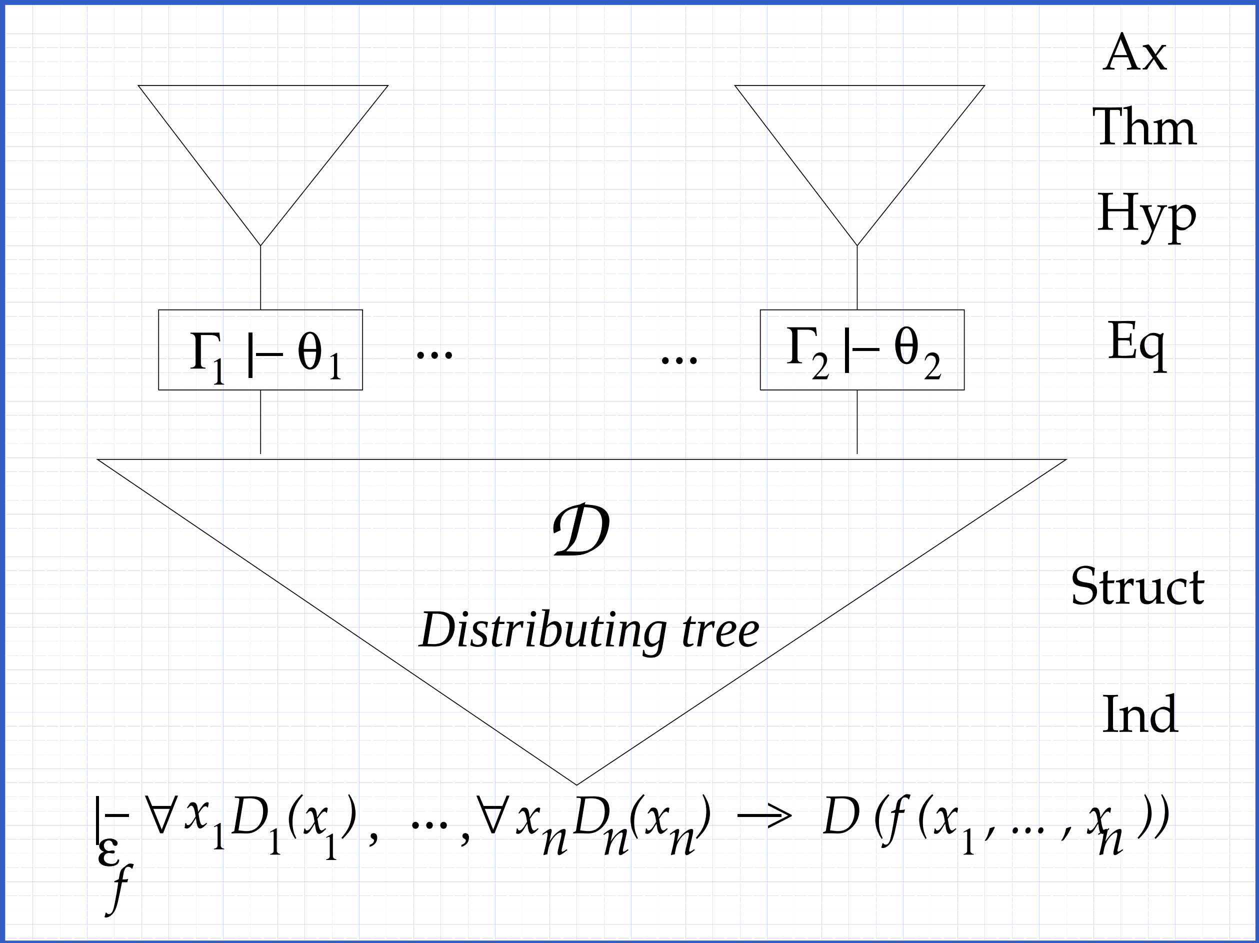
$$\frac{\Gamma \vdash_{\mathcal{E}} t : \forall x[\forall z[Dz_{\prec x} \rightarrow B[z/x]] \rightarrow [D(x) \rightarrow B]]}{\Gamma \vdash_{\mathcal{E}} (T\ t) : \forall x[D(x) \rightarrow B]}\ (Ext)$$

  T is a turing fixed-point operator,
  The relation $\prec$ is a well founded partial ordering on the terms of the algebra.

Macro Rules (tactics, derived rules)

- Thm : Application of an already proven termination statement (auxiliary functions)
- Hyp : Application of induction hypotheses
- Ax : Application of Axiom
- Eq : Application of an equational rule
- Struct : Use of structural rules + manipulations of formulas (Reasoning by cases)
- Ind : Use of induction rules + manipulations of formulas

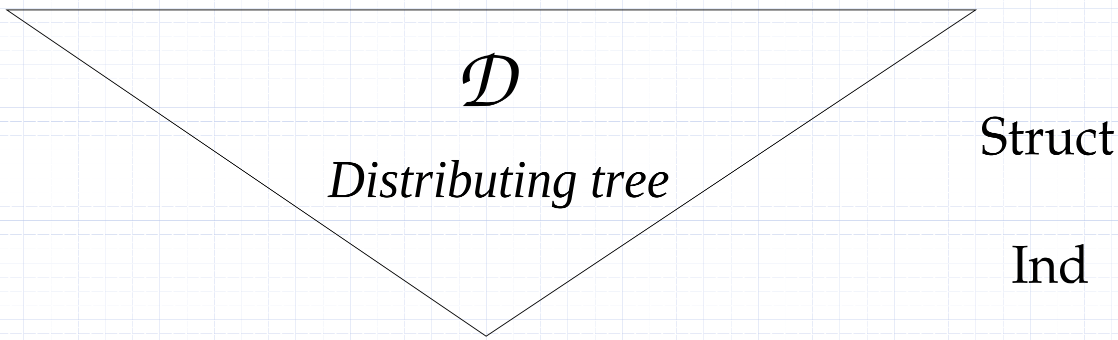
Shape of I-Proofs



Shape of I-Proofs

The Distributing tree must follow a property:

The formal terminal state property



$$\vdash_{\varepsilon_f} \forall x_1 D_1(x_1), \ \ldots, \forall x_n D_n(x_n) \Rightarrow D(f(x_1, \ldots, x_n))$$

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- We make simplification of Distributing Trees and Formulas.

The skeleton proofs

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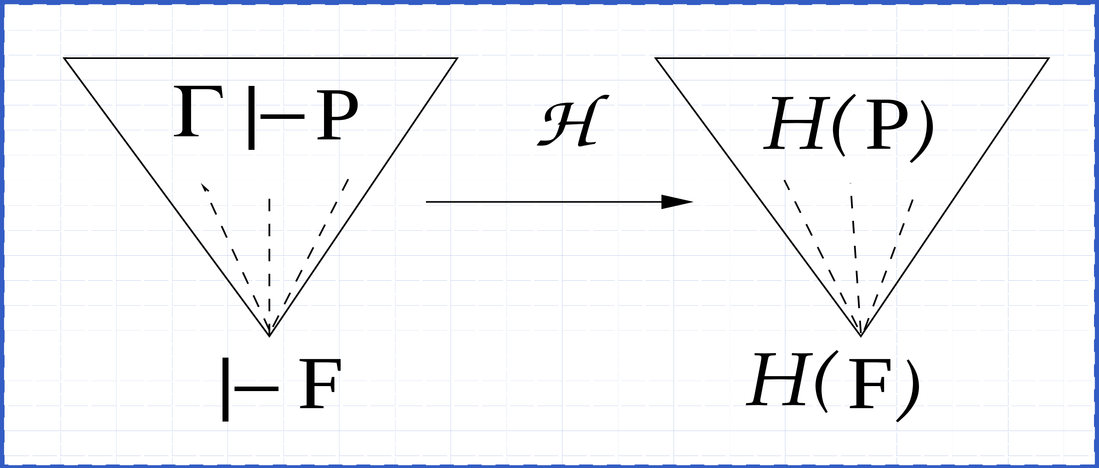
- 

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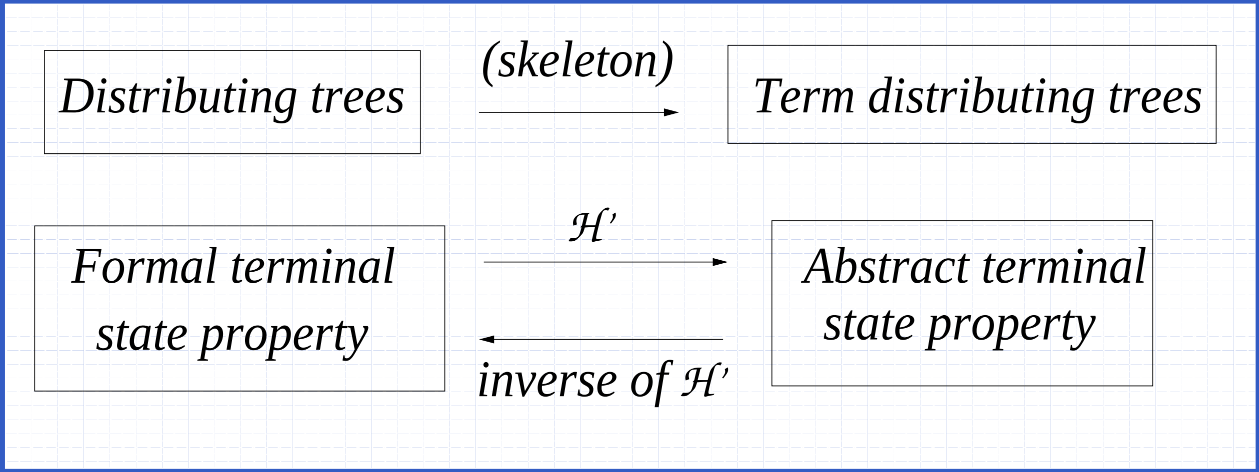
- 

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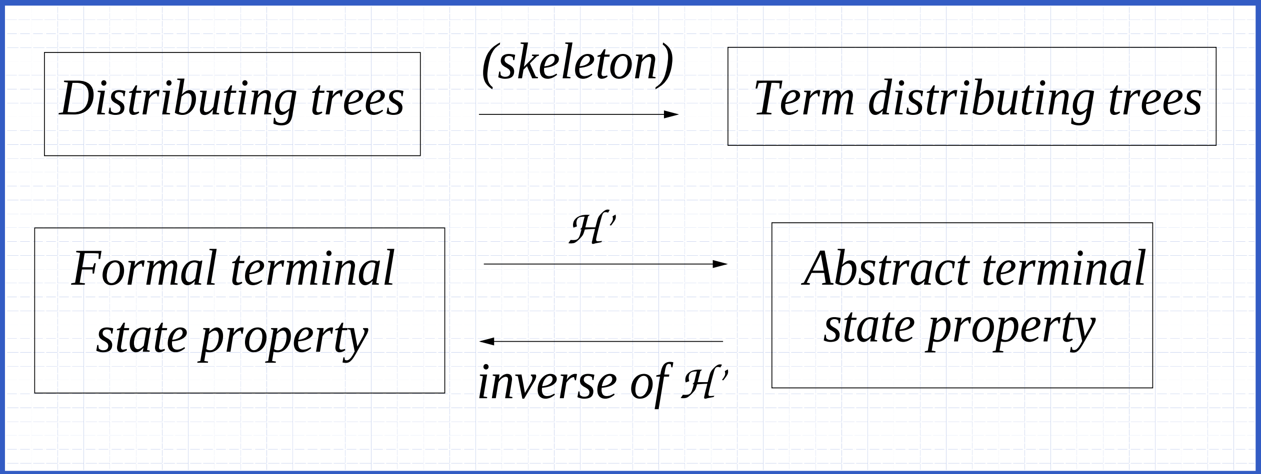
- 

- We can rebuild proofs from *Atsp*

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Termination proof of a function f

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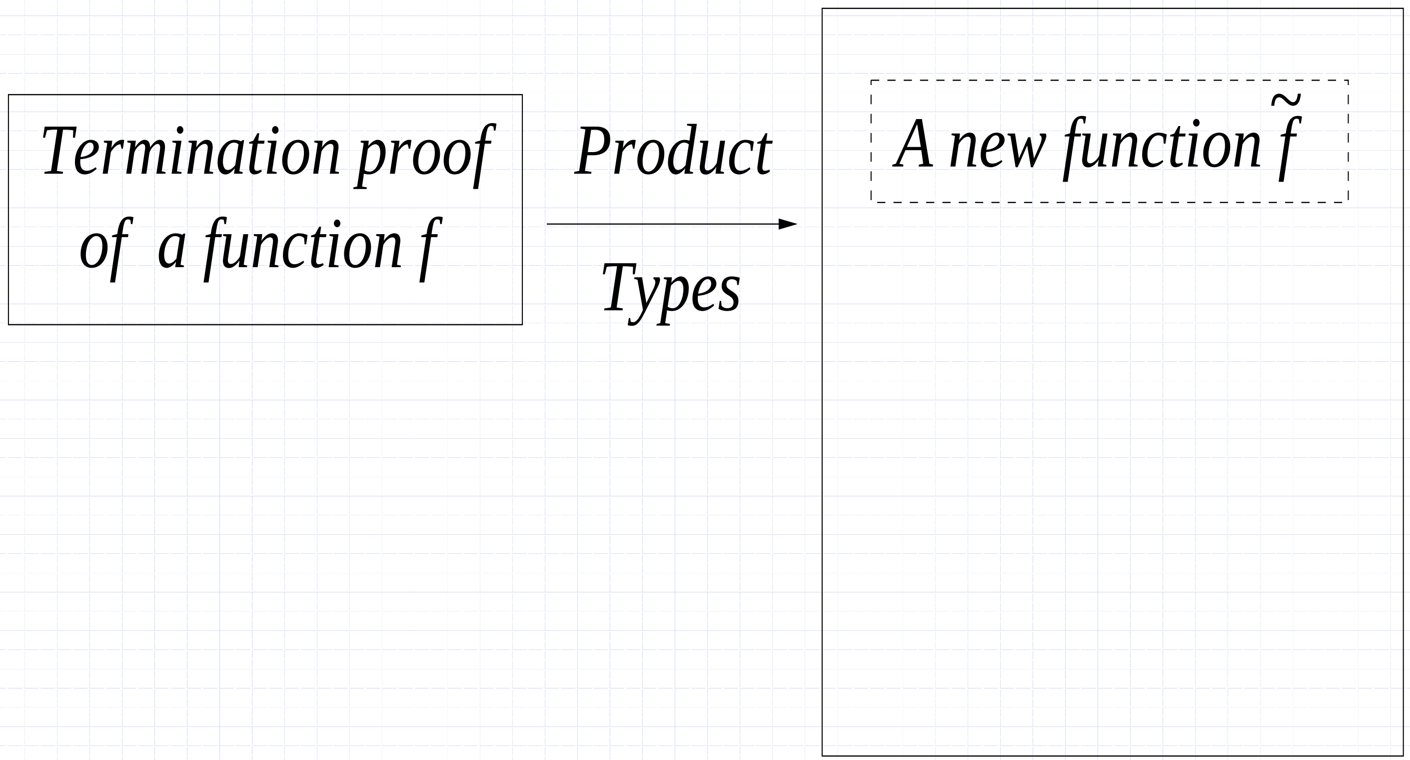

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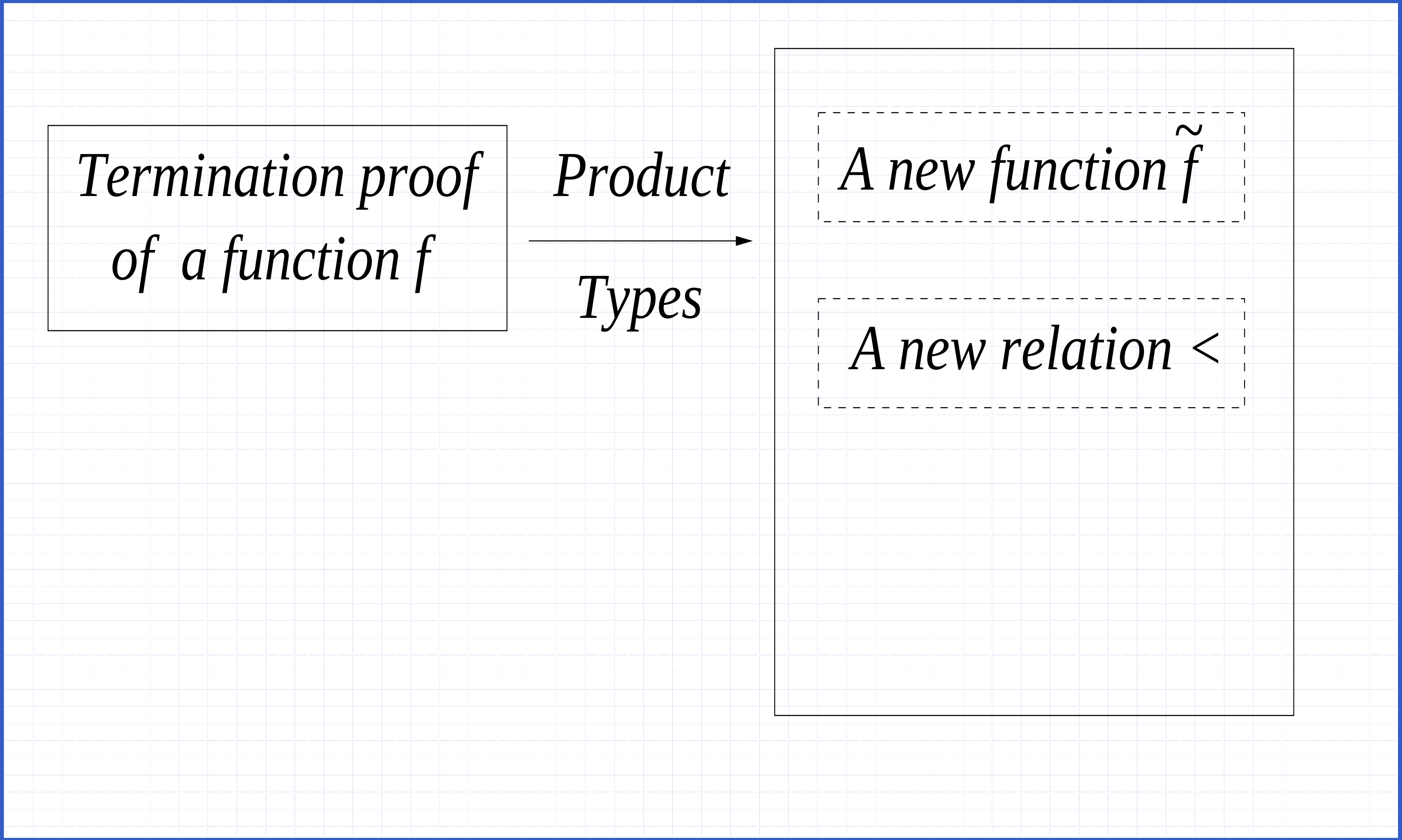

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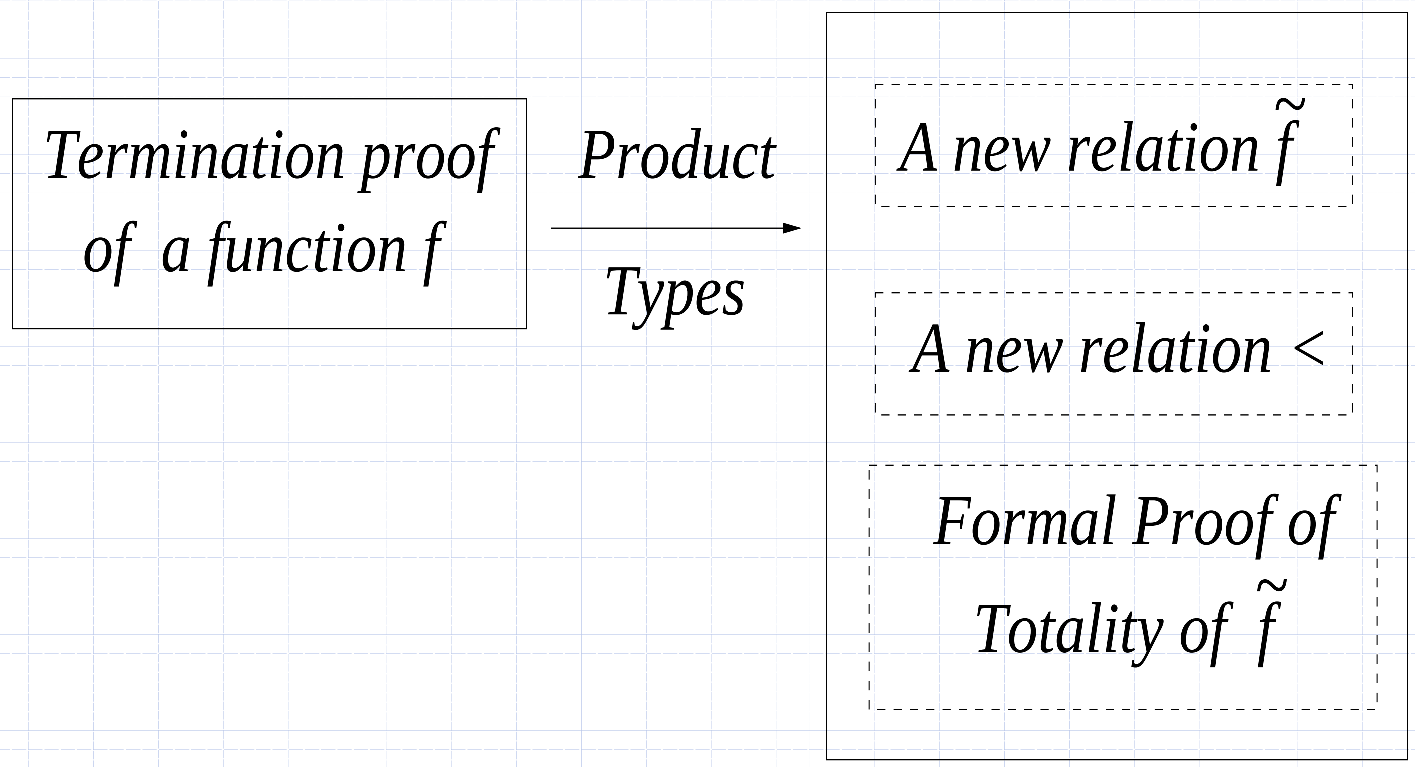

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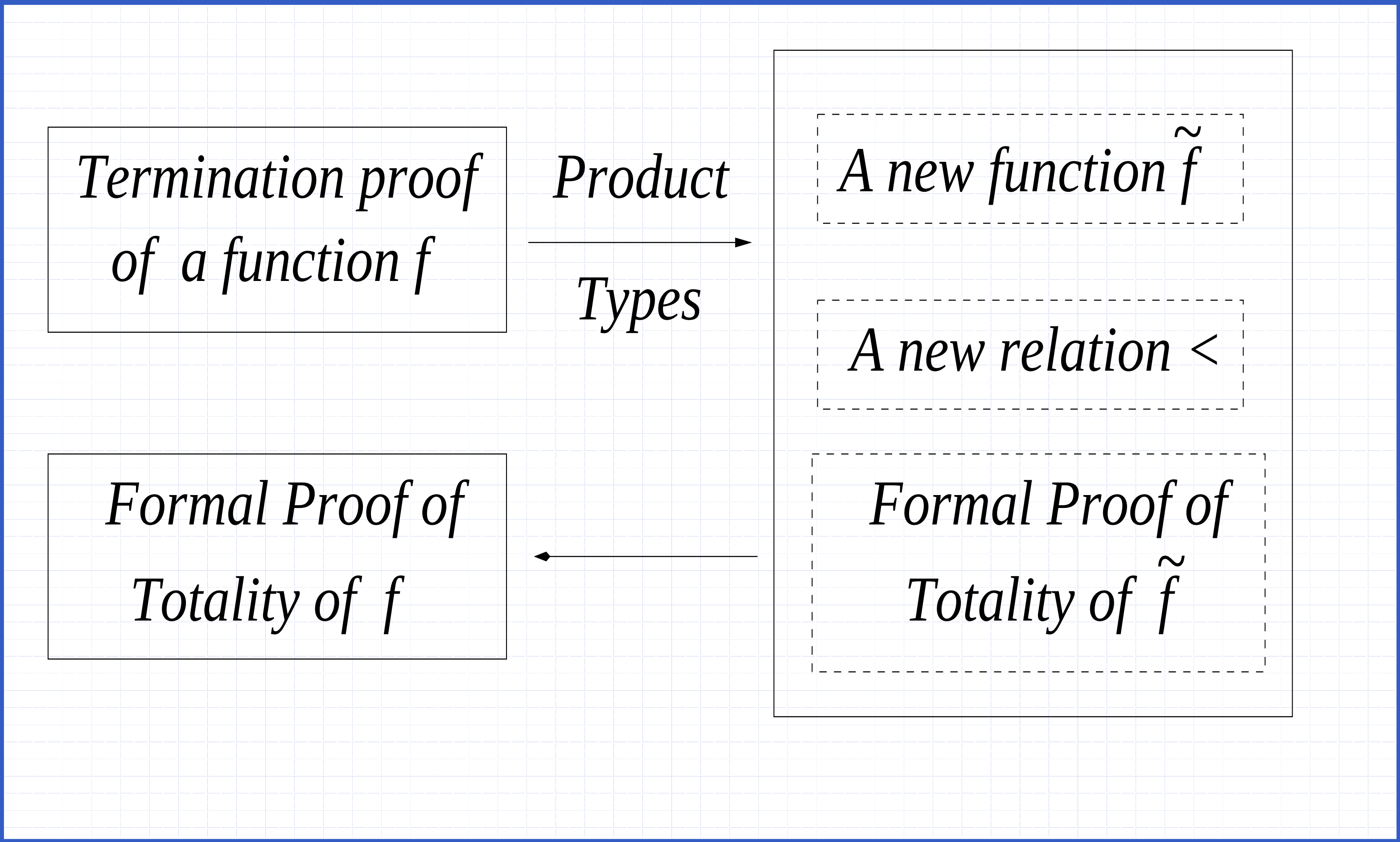

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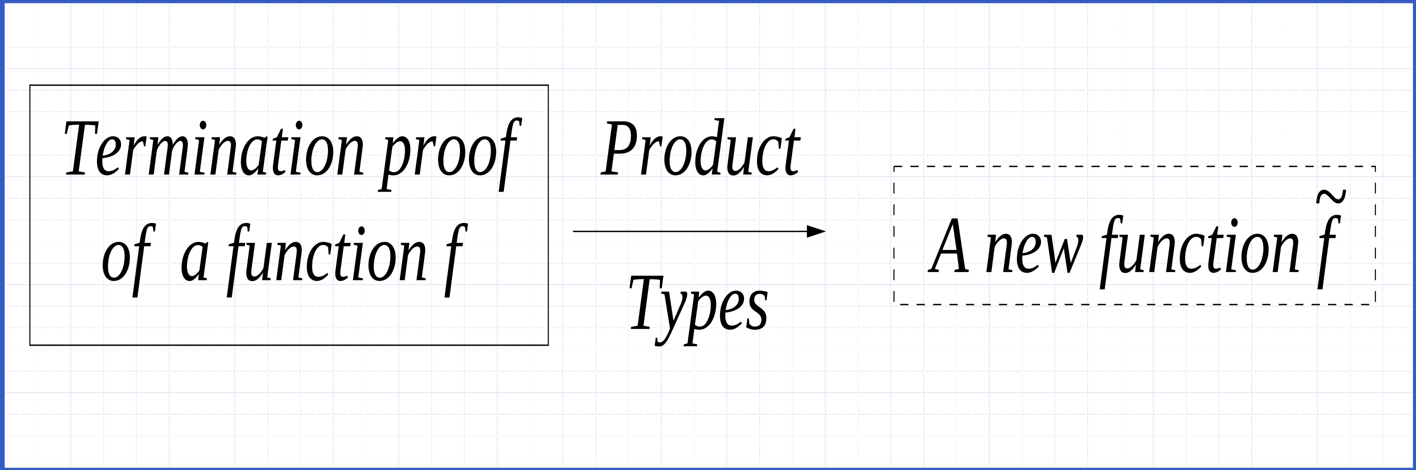
- 

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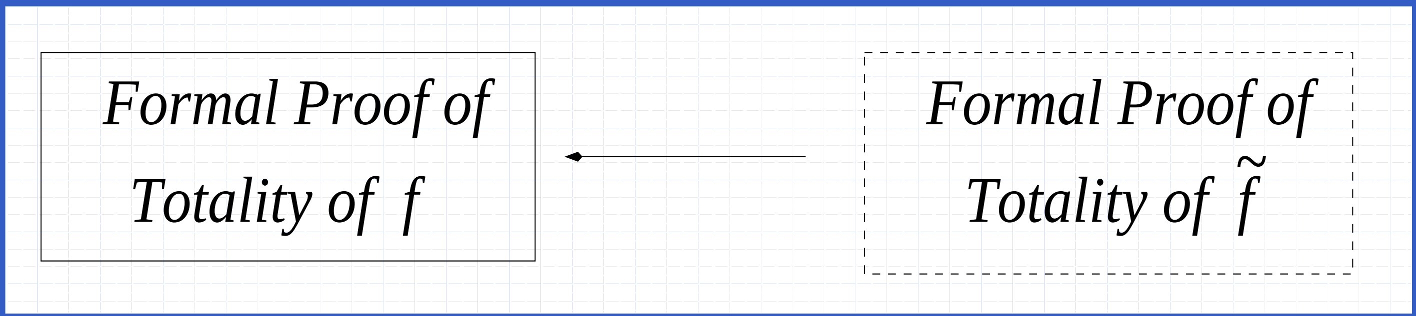
- 

Canonical I-proofs

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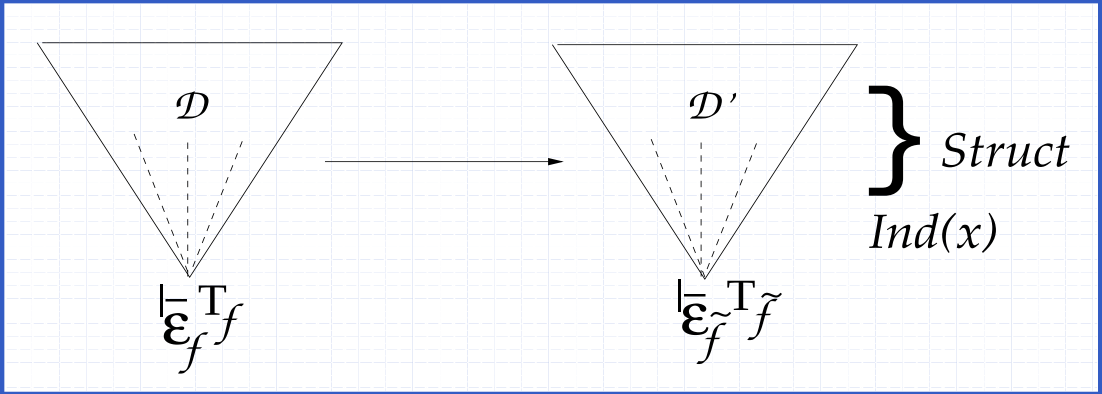
- 

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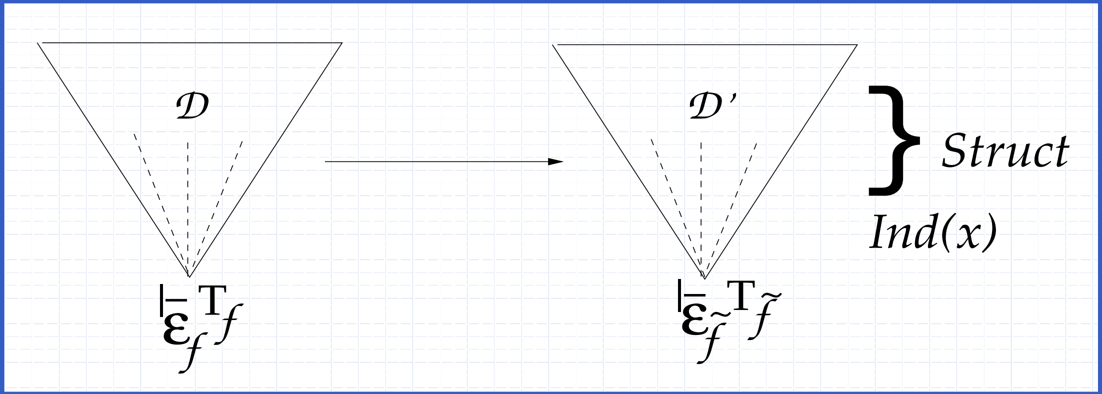
- 

- The hiding rules allow the formal proofs to be released from the termination part.

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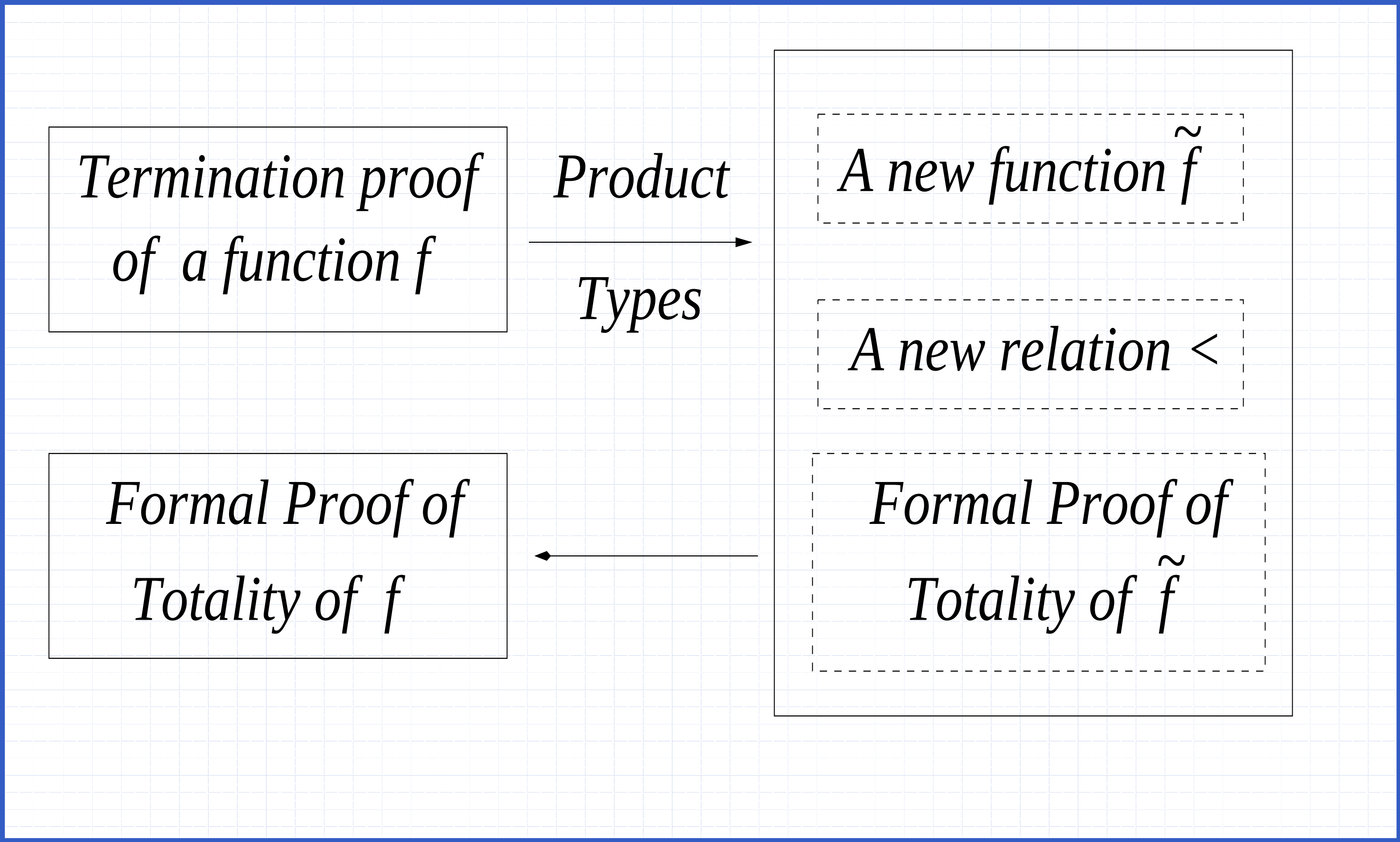

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- It remains the implementation.

The End