Flexible Encoding of Mathematics on the Computer

Fairouz Kamareddine, Manuel Maarek and J. B. Wells

ULTRA Group – Heriot-Watt University
Edinburgh - Scotland

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Białowieża – Poland

Motivations
Our objectives

Our objectives in the design of MathLang are:

- to have a language close to the Common Mathematical Language
- to make use of the automation capability of computers to assist the mathematician
- to combine these two points to have a computerized language reflecting how mathematicians think about the mathematics

Motivations

The background, existing mathematical encodings

Theorem (Pythagoras' Theorem). $\sqrt{2}$ *is irrational.*

Proof. If $\sqrt{2}$ is rational, then the equation

$$a^2 = 2b^2$$

is soluble in integers a, b with $(a, b) = 1$. Hence a^2 is even, and therefore a is even. If $a = 2c$, then $4c^2 = 2b^2$, $2c^2 = b^2$, and b is also even, contrary to the hypothesis that $(a, b) = 1$. □

- *An introduction to the Theory of Numbers*, G.H. Hardy and E.M. Wright
- *The Fifteen Provers of the World*, Freek Wiedijk

Motivations

The background, existing mathematical encodings

- **LaTeX code**: printing oriented, automatic reasoning unfriendly

```
\begin{theorem}[Pythagoras' Theorem]
  $\sqrt{2}$ is irrational.
\end{theorem}
\begin{proof}
  If $\sqrt{2}$ is rational, then the equation
  $$a^2 = 2b^2$$
  is soluble in integers $a$, $b$ with $(a,b)=1$.  Hence
  $a^2$ is even, and therefore $a$ is even.  If $a=2c$, then
  $4c^2=2b^2$, $2c^2=b^2$, and $b$ is also even, contrary to the
  hypothesis that $(a,b)=1$.
\end{proof}
```

Motivations

The background, existing mathematical encodings

- **LaTeX code**: printing oriented, automatic reasoning unfriendly
- **Theorem provers' codes**: automatic reasoning oriented, do not facilitate human reading

```
Theorem irrationalRsqrt2: (irrational (sqrt (S (S O)))).
Red.
Intros p q H; Red; Intros H0; Case H.
Apply (main_thm p).
Replace (Div2.double (mult q q)) with (mult (S (S O)) (mult q q));
 [Idtac | Unfold Div2.double; Ring].
Case (Peano_dec.eq_nat_dec (mult p p) (mult (S (S O)) (mult q q))); Auto;
 Intros H1.
Case (not_nm_INR ? ? H1); Repeat Rewrite mult_INR.
Rewrite <- (sqrt_def (INR (S (S O)))); Auto with real.
Rewrite Rabsolu_right in H0; Auto with real.
Rewrite H0; Auto with real.
Cut ~ <R> q == R0; [Intros H2; Field | Idtac]; Auto with real.
Apply Rle_ge; Apply Rlt_le; Apply sqrt_lt_R0; Auto with real.
Qed.
```

Part of Laurent Théry's Coq proof.

Motivations

The background, existing mathematical encodings

- **LaTeX code**: printing oriented, automatic reasoning unfriendly
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- **OMDoc/OpenMath document**: printing and automatic reasoning oriented, mix formal content and natural language

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<assertion id="th" type="theorem">
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         is soluble in integers a, b with (a, b) = 1. Hence
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In these systems there is always the tradeoff between readability and computerization.

Motivations

Main features of MathLang

Flexible language The language allows a flexibility close to CML. MathLang's constructions are mimicking CML ones.

Cost effective encoding The MathLang weak type checking checks the well fondness of MathLang texts. mathlang's explicit encoding leads to this type analyses.

Natural language view To make the link between CML and our explicit encoding, we provide tools to get a CML view of MathLang's encodings.

The language MathLang

Constructions of the language, phrase level

Mathematical objects encoded as variables, constants, binders.
Some constants and binders describe statements.

$*$ is associative on E if $\forall x, y, z, (x * y) * z = x * (y * z)$.

The language MathLang

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The language MathLang

Constructions of the language, sentence level

A definition of a new symbol, or the establishment of new statement are atomic step of reasoning. They are usually stated in a specific context. We decompose mathematical texts into atomic steps.

$$\forall x, y\ \exists z \text{ such that } x + y = z$$

$$+ \text{ is associative on } \mathbb{N}.$$

$$\text{If } a \in \mathbb{N},\ \ a + 0 = a.$$

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The language MathLang

Constructions of the language, discourse level

Bigger reasoning steps like proofs are themselves composed by sub-sequences of reasoning steps.

Theorem. $\forall n \in \mathbb{N}, A(n)$

Proof. Proof of $A(0)$ [...] Proof of $\forall n, A(n) \implies A(n+1)$ [...]

$\square$

The language MathLang

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□

lines

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lines flags

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lines flags blocks

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lines flags blocks

The language MathLang

MathLang checking

- A weak type is attributed to each symbol.
  The typing depends on the belonging to one of these grammatical categories: `term`, `set`, `noun`, `adjective`, `statement`.
- A MathLang weak type system.
  To check the good formation of MathLang texts: good types for parameters, presence of definition, coherent steps.
- A cost effective encoding.
  The MathLang encoding makes things become explicit and leads to an automatic type analysis.

This low level typing validates the structure of the text without telling anything about its truthfulness.

MathLang's output view system

Natural Language annotation

Natural Language annotation
of the MathLang XML concrete syntax

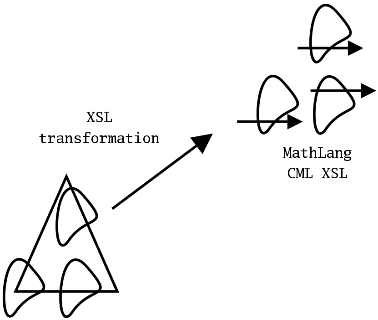
- Similarly to the language MathLang, the design of its output view system bring the framework nearer to CML.
- MathLang's constructions are to be annotated with representation information.
- With these annotations at each level of the text, we reconstruct a CML text faithful to its original version with the explicit MathLang encoding in addition.

MathLang's output view system
Local and global annotations

- Representation information is described locally for each element using XSL and with additional MathLang elements.
- The structure of the language assists in the writing down of these annotations.
- The representation of a symbol could be generalized to the entire document.
- Similar patterns of representation could be used for similar constructions.

MathLang's output view system

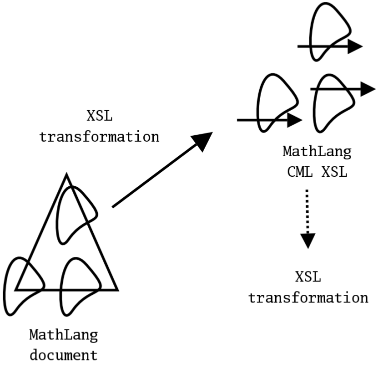
From coating to output view



- An engine collects the annotations

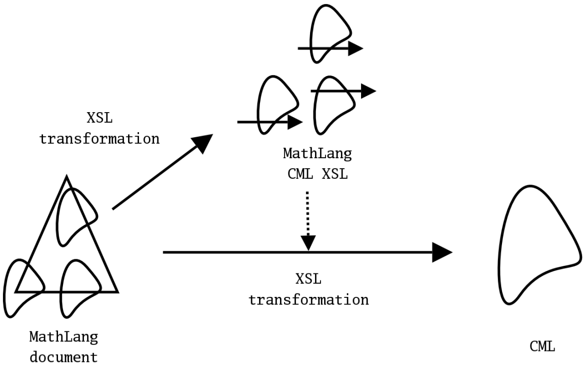
MathLang's output view system

From coating to output view



- An engine collects the annotations
- then produces one single XSL transformation file

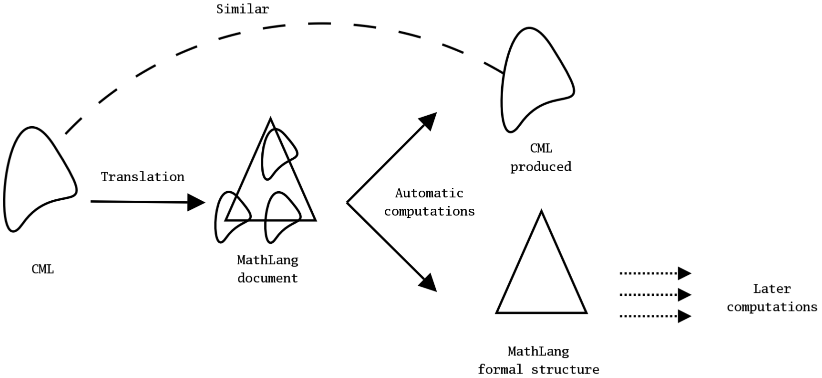
MathLang's output view system
From coating to output view



- An engine collects the annotations
- then produces one single XSL transformation file
- that generates an output view of the MathLang document

MathLang's output view system

Translation process



MathLang's output view system

Other systems' approaches, OMDoc presented to humans

- Elements structuring the entire document
- CML (with embedded OpenMath formulas) texts spread all around the document
- Formal content
- Customization system for the rendering of OpenMath symbols

```
<assertion id="th" type="theorem">
   <commonname> Pythagoras' Theorem
   <FMP> √2 ∉ ℚ
   <CMP> √2 is irrational.
<proof id="pr-th" for="th">
   <CMP>
    If √2 is rational, then the
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    integers a, b with (a, b) = 1. Hence
    a² is even, and therefore a is even.
    If a = 2c, then 4c² = 2b², 2c² = b², and
    b is also even, contrary to the
    hypothesis that (a, b) = 1.
```

MathLang's output view system

Other systems' approaches, theorem provers

- Fully formal code plus CML explanation related to piece of code.
- CML explanations are separately given to facilitate the navigation inside formal proofs and libraries.
- Comments/documentation more than CML view.

theory:/Coq/Init/Peano/index.theory [search]

Natural numbers nat built from o and s are defined in Datatypes.v
This module defines the following operations on natural numbers :

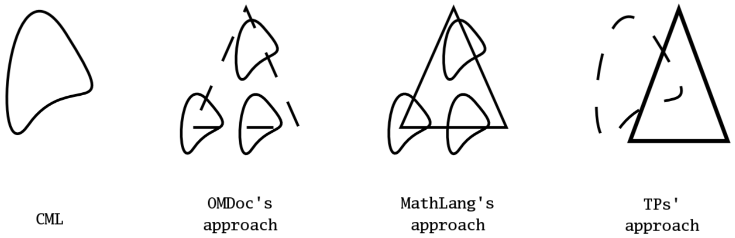
- predecessor pred
- addition plus
- multiplication mult
- less or equal order le
- less lt
- greater or equal ge
- greater gt

This module states various lemmas and theorems about natural numbers, including Peano's axioms of arithmetic (in Coq, these are in fact provable)
Case analysis on nat and induction on nat * nat are provided too
Require Coq.Init.Notations.
Require Coq.Init.Datatypes.
Require Coq.Init.Logic.
Definition eq_S:
∀x:IN.∀y:IN.x=y→1+x=1+y
:=
λx:IN.λy:IN.f_equal{A:=IN ; B:=IN ; f:=S ; x:=x ; y:=y}

The predecessor function
Definition pred:
IN→IN
:=
λn:IN.<λn0:IN.IN> CASE n OF O⇒O | S u⇒u

MathLang's output view system

Other systems' approaches, comparison



Different approaches for different aims. .

An example

From CML to MathLang

Theorem (Pythagoras' Theorem). $\sqrt{2}$ *is irrational.*

Proof. If $\sqrt{2}$ is rational, then the equation

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T Terms S Sets N Nouns A Adjectives P Statements Z Declarations Γ Contexts L Lines F Flags K Blocks B Books

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$Th := irrational(\sqrt{2})$ 1

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$\{1\}$

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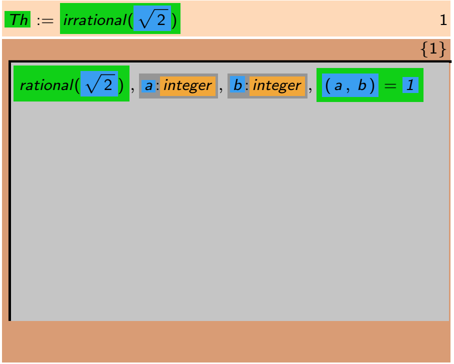

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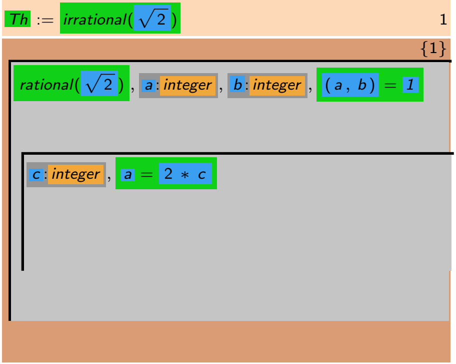

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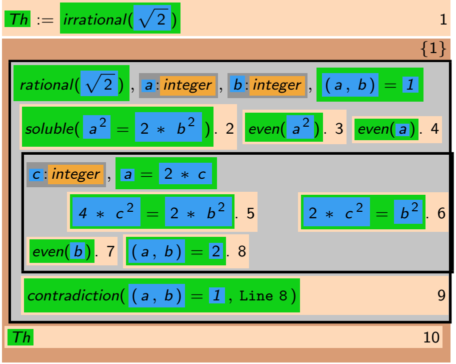

T Terms | S Sets | N Nouns | A Adjectives | P Statements | Z Declarations | Γ Contexts | L Lines | F Flags | K Blocks | B Books

An example

Representation annotations

T Terms S Sets N Nouns A Adjectives P Statements Z Declarations Γ Contexts L Lines F Flags K Blocks B Books

$Th() := irrational(\sqrt{2})$ 1

{1}

$rational(\sqrt{2})$, $a : integer$, $b : integer$, $(a, b) = 1$

$soluble(a^2 = 2 * b^2)$. 2

$even(a^2)$. 3 $even(a)$. 4

$c : integer$, $a = 2 * c$

$4 * c^2 = 2 * b^2$. 5 $2 * c^2 = b^2$. 6

$even(b)$. 7 $(a, b) = 2$. 8

$contradiction((a, b) = 1$, Line 8) 9

Th 10

- Symbolic view
- CML view of symbols
- CML view of the document

An example
Representation annotations

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Th() := $\sqrt{2}$ is irrational 1

{1}

$\sqrt{2}$ is rational, *a* : *integer*, *b* : *integer*, (*a* , *b*) = *1*

the equation a ^{2} = 2 b ^{2} is soluble. 2

a ^{2} is even. 3 a is even. 4

c : *integer*, a = 2 c

4 c ^{2} = 2 b ^{2}. 5 2 c ^{2} = b ^{2}. 6

b is also even. 7 (*a* , *b*) = 2. 8

contrary to the hypothesis that (a , b) = *1* 9

Th 10

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- **CML view of symbols**
- CML view of the document

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Proof.

If $\sqrt{2}$ is rational, then the equation $a^2 = 2b^2$ is soluble 2 in integers a, b with $(a, b) = 1$.

Hence a^2 is even, 3 and therefore a is even. 4

If $a = 2c$, then

$4c^2 = 2b^2$, 5 $2c^2 = b^2$, 6

and b is also even, 7 8

contrary to the hypothesis that $(a, b) = 1$. 9

10

□

- Symbolic view
- CML view of symbols
- **CML view of the document**

An example

MathLang views

From a single MathLang document we produce with the same procedure several views:

- Symbolic view of the MathLang document using LaTeX.
- CML view using LaTeX.
- CML view using MathML.
- An OpenMath/OMDoc view.

CML view without colors:

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Conclusion and future works

MathLang's assets

- Human oriented language for mathematics on computers.
  MathLang's constructions are mimicking CML.
- Cost effective encoding.
  The encoding effort results in the weak typing of the document.
- Separation of the explicit encoding and the CML layers.
  Both symbolic structure and CML view are available.

Conclusion and future works

Outlooks for MathLang's development

- Extending the input for MathLang with:
  - A user-friendly editor.
  - Inputs from other systems (eg. OpenMath/OMDoc).
- Moving to the next levels of MathLang.
  - Extending current MathLang with more logic and semantic content.
  - Identifying the logical rules to move from one step to another.