# The evolution of types and logic in the 20th century[*]

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## Summary

- *General definition of function 1879* [11] is key to Frege's *formalisation of logic*.
- *Self-application of functions* was at the heart of *Russell's paradox 1902* [31].
- To *avoid paradox* Russell controled function application via *type theory*.
- Russell [32] *1903* gives the first type theory: the *Ramified Type Theory* (RTT).
- But, *type theory* existed since the time of *Euclid* (325 B.C.).
- RTT is used in Russell and Whitehead's Principia Mathematica [36] 1910–1912.
- *Simple theory of types* (STT): Ramsey [29] *1926*, Hilbert and Ackermann [19] *1928*.

- Church's *simply typed $\lambda$-calculus* $\lambda\!\!\rightarrow$ [7] 1940 $=$ $\lambda$-calculus $+$ STT.
- Church's $\lambda$-calculus has been at the heart of *Montague's semantics* of natural language [10].
- The hierarchies of types (and orders) as found in RTT and STT are *unsatisfactory*.
- Frege's functions $\neq$ Principia's functions $\neq$ *$\lambda$-calculus* functions (1932).
- The *notion of function adopted in the $\lambda$-calculus* is *unsatisfactory* (cf. [24]).
- Hence, birth of *different systems of functions and types*, each with *different functional power*.
- We discuss the *evolution of functions and types* and their *influence on modern mathematics, logic, language and computation*.

## Prehistory of Types (Euclid)

- Euclid's *Elements* (circa 325 B.C.) begins with:

  1. A *point* is that which has no part;
  2. A *line* is breadthless length.

     ⋮

  15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

- 1..15 *define* points, lines, and circles which Euclid *distinguished* between.

- Euclid always mentioned to which *class* (points, lines, etc.) an object belonged.

## Prehistory of Types (Euclid)

- By distinguishing classes of objects, Euclid prevented *undesired/impossible* situations. E.g., whether two points (instead of two lines) are parallel.
- Intuition implicitly forced Euclid to think about the *type* of the objects.
- As intuition does not support the notion of parallel points, he did not even *try* to undertake such a construction.
- In this manner, types have always been present in mathematics, although they were not noticed explicitly until the late 1800s.
- If you studied geometry, then you have an (implicit) understanding of types.

## Prehistory of Types (Paradox Threats)

- From 1800, mathematical systems became less intuitive, for several reasons:
  - Very *complex* or abstract systems.
  - *Formal* systems.
  - Something with *less intuition* than a human using the systems: a *computer* or an *algorithm*.
- These situations are *paradox threats*. An example is Frege's Naive Set Theory.
- *Not enough intuition* to activate the (implicit) type theory *to warn against an impossible situation*.

## Prehistory of Types (formal systems in 19th century)

In the 19th century, the *need for a more precise* style in mathematics arose, *because controversial results* had appeared in *analysis*.

- 1821: Many of these controversies were solved by the work of Cauchy. E.g., he introduced *a precise definition of convergence* in his *Cours d'Analyse* [4].
- 1872: Due to the more *exact definition of real numbers* given by Dedekind [9], the rules for reasoning with real numbers became even more precise.
- 1895-1897: Cantor began formalizing *set theory* [2, 3] and made contributions to *number theory*.

## Prehistory of Types (formal systems in 19th century)

- 1889: *Peano* formalized *arithmetic* [26], but did not treat logic or quantification.
- 1879: *Frege* was not satisfied with the use of *natural language in mathematics*:

  "... I found *the inadequacy of language to be an obstacle*; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required."

  (*Begriffsschrift*, Preface)

  Frege therefore presented *Begriffsschrift* [11], the first formalisation of logic giving logical concepts via symbols rather than natural language.

## Prehistory of Types (formal systems in 19th century)

"[Begriffsschrift's] first purpose is to *provide us with the most reliable test of the validity of a chain of inferences* and to point out *every presupposition* that tries to sneak in unnoticed, so that its origin *can be investigated*."
(*Begriffsschrift*, Preface)

- 1892-1903 Frege's *Grundgesetze der Arithmetik* [13, 17], could handle elementary arithmetic, set theory, logic, and quantification.

## Prehistory of Types (Begriffsschrift's functions)

The introduction of a *very general definition of function* was the key to the formalisation of logic. Frege defined the Abstraction Principle.

**Abstraction Principle 1.**

> *"If in an expression, [. . . ] a simple or a compound* sign has one or more occurrences *and if we regard that sign as* replaceable *in all or some of these occurrences* by something else *(but everywhere by the same thing), then we call* the part that remains invariant in the expression *a* function, *and* the replaceable part *the* argument *of the function."*

*(Begriffsschrift,* Section 9*)*

## Prehistory of Types (Begriffsschrift's functions)

- Frege put *no restrictions* on what could play the role of *an argument*.
- An argument could be a *number* (as was the situation in analysis), but also a *proposition*, or a *function*.
- Similarly, the *result of applying* a function to an argument did not necessarily have to be a number.
- Functions of more than one argument were constructed by a method that is very close to the method presented by Schönfinkel [34] in 1924.

## Prehistory of Types (Begriffsschrift's functions)

*With this definition of function*, two of the three possible *paradox threats occurred*:

1. The generalisation of the concept of function made the system more abstract and *less intuitive*.

2. Frege introduced a *formal* system instead of the informal systems that were used up till then.

*Type theory*, that would be helpful in distinguishing between the different types of arguments that a function might take, *was left informal*.

So, *Frege had to proceed with caution*. And so he did, at this stage.

## Prehistory of Types (Begriffsschrift's functions)

Frege was *aware* of some typing rule that does *not* allow to *substitute functions for object variables or objects for function variables*:

> "if the [. . . ] letter [sign] occurs as a function sign, this circumstance [should] be taken into account."
>
> (*Begriffsschrift*, Section 11)

> " Now just as functions are fundamentally different from objects, so also *functions whose arguments are and must be functions* are fundamentally different from *functions whose arguments are objects and cannot be anything else*. I call the latter *first-level*, the former *second-level*."
>
> (*Function and Concept*, pp. 26–27)

## Prehistory of Types (Begriffsschrift's functions)

In *Function and Concept* he was aware of the fact that making a *difference between first-level and second-level objects is essential to prevent paradoxes*:

> "The ontological proof of God's existence suffers from the fallacy of treating existence as a first-level concept."
>
> (*Function and Concept*, p. 27, footnote)

The above discussion on functions and arguments shows that *Frege did indeed avoid the paradox in his Begriffsschrift*.

## Prehistory of Types (Grundgesetze's functions)

The *Begriffsschrift*, however, was only a prelude to Frege's writings.

- In Grundlagen der Arithmetik [12] he argued that mathematics can be seen as a branch of logic.

- In Grundgesetze der Arithmetik [13, 17] he described the elementary parts of arithmetic within an extension of the logical framework of *Begriffsschrift*.

- Frege approached the *paradox threats for a second time* at the end of Section 2 of his *Grundgesetze*.

- He did *not* want to *apply a function to itself*, but to its course-of-values.

## Prehistory of Types (Grundgesetze's functions)

"the function $\Phi(x)$ has the same *course-of-values* as the function $\Psi(x)$" if:

> "the functions $\Phi(x)$ and $\Psi(x)$ always have the same value for the same argument."
>
> (*Grundgesetze*, p. 7)

- Note that functions $\Phi(x)$ and $\Psi(x)$ may have equal courses-of-values even if they have different definitions. E.g., $x \wedge \neg x$, and $x \leftrightarrow \neg x$.

- Frege denoted the course-of-values of a function $\Phi(x)$ by $\grave{\varepsilon}\Phi(\varepsilon)$. The definition of equal courses-of-values could therefore be expressed as

$$\grave{\varepsilon}f(\varepsilon) = \grave{\varepsilon}g(\varepsilon) \longleftrightarrow \forall a[f(a) = g(a)]. \tag{1}$$

In modern terminology, we could say that the functions $\Phi(x)$ and $\Psi(x)$ have the same course-of-values if they have the same *graph*.

## Prehistory of Types (Grundgesetze's functions)

- The notation $\grave{\varepsilon}\Phi(\varepsilon)$ may be the *origin* of Russell's notation $\hat{x}\Phi(x)$ for the class of objects that have the property $\Phi$.

- According to a paper by Rosser [30], the notation $\hat{x}\Phi(x)$ has been at the *basis* of the current notation $\lambda x.\Phi(x)$.

- Church is supposed to have written $\wedge x\Phi(x)$ for the function $x \mapsto \Phi(x)$: the hat $\wedge$ in front of the $x$ distinguishes this function from the class $\hat{x}\Phi(x)$.

## Prehistory of Types (Grundgesetze's functions)

- Frege treated *courses-of-values* as *ordinary objects*.
- As a consequence, *a function that takes objects as arguments could have its own course-of-values as an argument*.
- In modern terminology: a function that takes objects as arguments can have its own graph as an argument.
- *BUT,* all essential information of a function is contained in its graph.
- A system in which a function can be applied to its own graph should have similar possibilities as a system in which a function can be applied to itself.
- Frege *excluded the paradox threats* by *forbidding self-application*
- but due to his *treatment of courses-of-values* these threats were able to *enter his system through a back door*.

## Prehistory of Types (Russell's paradox in *Grundgesetze*)

- In 1902, Russell wrote a letter to Frege [31], informing him that he had *discovered a paradox* in his *Begriffsschrift.*

- *WRONG: Begriffsschrift does not suffer from a paradox*.

- Russell gave his well-known argument, defining the propositional function

  $$f(x) \textit{ by } \neg x(x).$$

  In Russell's words: "*to be a predicate that cannot be predicated of itself.*"

- Russell assumed $f(f)$. Then by definition of $f$, $\neg f(f)$, *a contradiction*. Therefore: $\neg f(f)$ holds. But then (again by definition of $f$), $f(f)$ holds. Russell concluded that *both* $f(f)$ *and* $\neg f(f)$ hold, a *contradiction*.

## Prehistory of Types (Russell's paradox in *Grundgesetze*)

- *6 days later*, Frege wrote [16] that *Russell's derivation of paradox is incorrect*.
- Ferge explained that *self-application $f(f)$ is not possible in* Begriffsschrift.
- $f(x)$ *is a function, which requires an* object *as an argument.*
  *A function cannot be an object in the* Begriffsschrift.
- Frege explained that *Russell's argument could be amended to a paradox* in Grundgesetze, using the *course-of-values* of functions:

  *Let* $f(x) = \neg\forall\varphi[(\grave{\alpha}\varphi(\alpha) = x) \longrightarrow \varphi(x)]$
  *I.e.* $f(x) = \quad \exists\varphi[(\grave{\alpha}\varphi(\alpha) = x) \ \wedge \neg\varphi(x)]$ hence $\neg\varphi(\grave{\alpha}\varphi(\alpha))$
- *Both* $f(\grave{\varepsilon}f(\varepsilon))$ and $\neg f(\grave{\varepsilon}f(\varepsilon))$ hold.
- Frege added an appendix of 11 pages to the 2nd volume of *Grundgesetze* in which he gave a very detailed description of the paradox.

## Prehistory of Types (How wrong was Frege?)

- Due to Russell's Paradox, Frege is often depicted as the pitiful person whose system was inconsistent.
- This suggests that Frege's system was the only one that was inconsistent, and that Frege was very inaccurate in his writings.
- On these points, history does Frege an injustice.
- Frege's system was much more accurate than other systems of those days.
- Peano's work, for instance, was *less precise* on several points:
- Peano *hardly paid attention to logic* especially quantification theory;
- Peano *did not make a strict distinction* between his *symbolism* and the *objects underlying this symbolism*. Frege was much more accurate on this point (see Frege's paper *Über Sinn und Bedeutung* [14]);

## Prehistory of Types (How wrong was Frege?)

- Frege *made a strict distinction* between a *proposition* (as an object) and the assertion *of a proposition*. Frege denoted a *proposition*, by $-A$, and *its assertion* by $\vdash A$. Peano did not make this distinction and simply wrote $A$.

Nevertheless, Peano's work was very popular, for several reasons:

- Peano had *able collaborators*, and a *better eye for presentation and publicity*.

- Peano bought *his own press* to supervise the printing of his own journals Rivista di Matematica and Formulaire [27]

## Prehistory of Types (How wrong was Frege?)

- Peano used a *familiar symbolism* to the notations used in those days.
- Many of *Peano's notations*, like $\in$ for "is an element of", and $\supset$ for logical implication, are used in *Principia Mathematica*, and are actually still in use.
- *Frege's work* did not have these advantages and *was hardly read before 1902*
- When *Peano* published his formalisation of mathematics in 1889 [26] he clearly *did not know* Frege's *Begriffsschrift* as he did not mention the work, and *was not aware* of Frege's formalisation of quantification theory.

## Prehistory of Types (How wrong was Frege?)

- Peano considered quantification theory to be “abstruse” in [27]:

  “In this respect my *[Frege] conceptual notion of 1879 is superior to the Peano one*. Already, at that time, I specified all the laws necessary for my designation of generality, so that nothing fundamental remains to be examined. These laws are few in number, and *I do not know why they should be said to be abstruse*. If it is otherwise with the *Peano conceptual notation*, then this is due to the *unsuitable* notation.”

  ([15], p. 376)

## Prehistory of Types (How wrong was Frege?)

- In the last paragraph of [15], Frege concluded:

  "... I observe merely that the *Peano notation* is unquestionably *more convenient for the typesetter*, and in many cases *takes up less room* than mine, but that these advantages seem to me, due to the inferior perspicuity and *logical defectiveness*, to have been paid for too dearly — at any rate for the purposes I want to pursue."
  (*Ueber die Begriffschrift des Herrn Peano und meine eigene*, p. 378)

## Prehistory of Types (paradox in Peano and Cantor's systems)

- Frege's system was *not the only paradoxical* one.
- The Russell Paradox can be derived in *Peano's system* as well, by defining the class $K \stackrel{\mathrm{def}}{=} \{x \mid x \notin x\}$ and deriving $K \in K \longleftrightarrow K \notin K$.
- In *Cantor's Set Theory* one can derive the paradox via the same class (or *set*, in Cantor's terminology).

## Prehistory of Types (paradoxes)

- Paradoxes were already widely known in *antiquity*.
- The oldest logical paradox: the *Liar's Paradox* "This sentence is not true", also known as the Paradox of Epimenides. It is referred to in the Bible (Titus $1{:}12$) and is based on the confusion between language and meta-language.
- The *Burali-Forti paradox* ([1], 1897) is the first of the modern paradoxes. It is a paradox within Cantor's theory on ordinal numbers.
- Cantor was *aware* of the Burali-Forti paradox but *did not think* it would render his system incoherent.
- *Cantor's paradox* on the largest cardinal number occurs in the same field. It was discovered by Cantor around 1895, but was not published before 1932.

## Prehistory of Types (paradoxes)

- Logicians considered these paradoxes to be *out of the scope of logic*:
  - The *Liar's Paradox* can be regarded as a problem of *linguistics*.
  - The *paradoxes of Cantor and Burali-Forti* occurred in what was considered in those days a *highly questionable* part of mathematics: *Cantor's Set Theory*.
- The Russell Paradox, however, was *a paradox that could be formulated in all* the systems of the end of the 19th century (except for Frege's *Begriffsschrift*).
- Russell's Paradox was at the very basics of logic.
- It could not be disregarded, and a solution to it had to be found.
- In 1903-1908, Russell suggested the use of *types* to solve the problem [33].

## Prehistory of Types (vicious circle principle)

When Russell proved Frege's *Grundgesetze* to be inconsistent, Frege was not the only person in *trouble*. In Russell's letter to Frege (1902), we read:

"I am on the point of finishing a book on the principles of mathematics"

(*Letter to Frege*, [31])

Russell *had to find a solution* to the paradoxes, before finishing his book.

His paper Mathematical logic as based on the theory of types [33] *(1908)*, in which a first step is made towards the Ramified Theory of Types, started with a description of the most important contradictions that were known up till then, including Russell's own paradox. He then concluded:

## Prehistory of Types (vicious circle principle)

"In all the above contradictions there is a common characteristic, which we may describe as *self-reference* or *reflexiveness*. [. . . ] In each contradiction something is said about all cases of some kind, and from what is said a new case seems to be *generated*, which both *is and is not* of the same kind as the cases of which *all* were concerned in what was said."

(*Ibid.*)

Russell's plan was, *to avoid the paradoxes* by *avoiding all possible self-references*. He postulated the *"vicious circle principle"*:

## Ramified Type Theory

"*Whatever involves all of a collection* must not be one *of the collection.*"

(Mathematical logic as based on the theory of types)

- Russell applies this principle *very strictly*.
- He implemented it using *types*, in particular the so-called ramified *types*.
- The type theory of 1908 was elaborated in Chapter II of the Introduction to the famous *Principia Mathematica* [36] (1910-1912).

## Ramified Type Theory and Principia

- In the *Principia*, *mathematics was founded on logic*, as far as possible.
- The *logical part* of *Principia* was *based* on the works of *Frege* (acknowledged by Whitehead and Russell in the preface, and can be seen throughout the description of Type Theory).
- The notion of *function is based on Frege's Abstraction Principles*.
- The *Principia* *notation* $\hat{x}f(x)$ for a class looks very *similar to Frege's* $\grave{\varepsilon}f(\varepsilon)$ for course-of-values.
- An important difference is that Whitehead and Russell treated functions as first-class citizens. Frege used courses-of-values when speaking about functions.
- In the *Principia* a direct approach was possible.

## Ramified Type Theory and Principia

- The description of the Ramified Theory of Types (RTT) in *Principia* was extensive, *yet informal*. A *formalisation* was *not considered* in those days.

- Type Theory had *not yet* become an *independent subject*. The theory

  "only recommended itself to us in the first instance by its ability to solve certain contradictions. .......... it has also a certain consonance with common sense which makes it inherently credible"
  (*Principia Mathematica*, p. 37)

- Type Theory was not introduced because it was interesting on its own, but because it had to serve as a *tool* for logic and mathematics.

- Types in *Principia* have a double hierarchy: *(simple) types* and *orders*.

- RTT was *not mentioned very often*, but when necessary, Russell made a remark on types.

## Ramified Type Theory and Principia

- There is *no definition of "type"* in *Principia*, only a definition of *"being of the same type"*:

  "…. We say that $u$ and $v$ *are of the same type* if

  1. both are individuals,
  2. both are elementary [propositional] functions taking arguments of the same type,
  3. $u$ is a pf and $v$ is its negation,
  4. …..
  5. ….
  6. both are elementary propositions,
  7. ….
  8. $u$ is $(x).\varphi x$ and $v$ is $(y).\psi y$, where $\varphi\hat{x}$ and $\psi\hat{x}$ are of the same type."

  (*Principia Mathematica*, $*9{\cdot}131$, p. 133)

- There are some omissions in Russell and Whitehead's definition.

## Ramsey's Simple Types

- The idea behind simple types was already explained by Frege (see earlier quotes from *Function and Concept*).

- *Ramsey's Simple types*:

  1. 0 is a simple type, the type of *individuals*.
  2. If $t_1, \ldots, t_n$ are simple types, then also $(t_1, \ldots, t_n)$ is a simple type.[1]
     $n = 0$ is allowed: then we obtain the simple type $()$ of *propositions*.
  3. All simple types can be constructed using the rules 1 and 2.

- $\mathtt{R}(\mathtt{x})$ has type $(\mathtt{0})$, as it takes one individual as argument.

- $S(a)$ has type $()$.

- We conclude that in $\mathtt{z}(\mathtt{R}(\mathtt{x}), \mathtt{S}(\mathtt{a}))$, we must substitute pfs of type $((\mathtt{0}), ())$ for $\mathtt{z}$. Therefore, $\mathtt{z}(\mathtt{R}(\mathtt{x}), \mathtt{S}(\mathtt{a}))$ has type $(((\mathtt{0}), ()))$.

[1] $(t_1, \ldots, t_n)$ is the type of pfs that should take $n$ arguments, the $i$th argument having type $t_i$.

## Whitehead and Russell's Ramified Types

- With *simple types*, the type of a pf only depends on the *types of the arguments* that it can take.

- In the *Principia*, a *second hierarchy* is introduced by regarding also *the types of the variables that are bound by a quantifier* (see *Principia*, pp. 51–55).

- Whitehead and Russell consider, for instance, the propositions $\mathtt{R}(\mathtt{a})$ and $\forall \mathtt{z}{:}()[\mathtt{z}() \vee \neg\mathtt{z}()]$ to be of a *different level*.

- The first is an *atomic proposition*, while the latter is based on the pf $\mathtt{z}() \vee \neg\mathtt{z}()$.

## Whitehead and Russell's Ramified Types

- The pf $\mathtt{z}() \vee \neg\mathtt{z}()$ involves an arbitrary proposition $\mathtt{z}$, therefore $\forall\mathtt{z}{:}()[\mathtt{z}() \vee \neg\mathtt{z}()]$ *quantifies over all* propositions $\mathtt{z}$.

- According to the *vicious circle principle*, $\forall\mathtt{z}{:}()[\mathtt{z}() \vee \neg\mathtt{z}()]$ cannot belong to this collection of propositions.

- This problem is solved by dividing types into *orders* which are natural numbers.

- *Basic* propositions are of order 0. In $\forall\mathtt{z}{:}()[\mathtt{z}() \vee \neg\mathtt{z}()]$ we must mention the *order of the propositions over which is quantified*. The pf $\forall\mathtt{z}{:}()^n[\mathtt{z}() \vee \neg\mathtt{z}()]$ quantifies over *all propositions of order $n$*, and has order $n+1$.

## Whitehead and Russell's Ramified Types

1. $0^0$ is a ramified type of order 0;

2. If $t_1^{a_1}, \ldots, t_n^{a_n}$ are ramified types, and $a \in \mathbb{N}$, $a > \max(a_1, \ldots, a_n)$, then $(t_1^{a_1}, \ldots, t_n^{a_n})^a$ is a ramified type of order a (if $n = 0$ then take $a \geq 0$);

3. All ramified types can be constructed using the rules 1 and 2.

$0^0$; $(0^0)^1$; $\left((0^0)^1, (0^0)^4\right)^5$; and $\left(0^0, ()^2, \left(0^0, (0^0)^1\right)^2\right)^7$ are *all ramified types*.

$\left(0^0, \left(0^0, (0^0)^2\right)^2\right)^7$ is *not a ramified type*.

## Predicative Types

- In the type $(0^0)^1$, all *orders are "minimal"*, i.e., not higher than strictly necessary. Unlike $(0^0)^2$ where orders are not minimal.

- Types in which all orders are minimal are called predicative and play a special role in the Ramified Theory of Types.

1. $0^0$ is a predicative type;
2. If ${t_1}^{a_1}, \ldots, {t_n}^{a_n}$ are predicative types, and $a = 1 + \max(a_1, \ldots, a_n)$ (take $a = 0$ if $n = 0$), then $(t_1^{a_1}, \ldots, t_n^{a_n})^a$ is a predicative type;
3. All predicative types can be constructed using the rules 1 and 2 above.

## Problems of Ramified Type Theory

- The main part of the *Principia* is devoted to the development of logic and mathematics using the legal pfs of the ramified type theory.

- *ramification*/division of simple types into orders make $\mathrm{RTT}$ not easy to use.

- **(Equality)** $\mathtt{x} =_{\mathrm{L}} \mathtt{y} \overset{\mathrm{def}}{\leftrightarrow} \forall \mathtt{z}[\mathtt{z}(\mathtt{x}) \leftrightarrow \mathtt{z}(\mathtt{y})],$.
  In order to express this general notion in $\mathrm{RTT}$, we have to incorporate *all* pfs $\forall \mathtt{z} : (0^0)^n[\mathtt{z}(\mathtt{x}) \leftrightarrow \mathtt{z}(\mathtt{y})]$ for $n > 1$, and this cannot be expressed in one pf.

- Not possible to give a constructive proof of the theorem of the least upper bound within a ramified type theory.

## Axiom of Reducibility

- It is not possible in RTT to give a definition of an object that refers to the class to which this object belongs (because of the Vicious Circle Principle). Such a definition is called an *impredicative definition*.

- An object defined by an impredicative definition is of a higher order than the order of the elements of the class to which this object should belong. This means that the defined object has an *impredicative type*.

- But impredicativity is not allowed by the vicious circle principle.

- Russell and Whitehead tried to solve these problems with the so-called axiom of reducibility.

## Axiom of Reducibility

- *(Axiom of Reducibility)* For each formula $f$, there is a formula $g$ with a *predicative* type such that $f$ and $g$ are (logically) equivalent.

- The validity of the Axiom of Reducibility has been questioned from the moment it was introduced.

- In the 2nd edition of the *Principia*, Whitehead and Russell admit:

  "This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content."

  (*Principia Mathematica*, p. xiv)

## Axiom of Reducibility

- Though Weyl [35] made an effort to develop analysis within the Ramified Theory of Types (without the Axiom of Reducibility),
- and various parts of mathematics can be developed within RTT and without the Axiom,
- the general attitude towards RTT (without the axiom) was that the system was too restrictive, and that a *better solution* had to be found.

## Deramification

- Ramsey considers it essential to divide the paradoxes into two parts:

- One group of paradoxes is removed

  > "by pointing out that a propositional function cannot significantly take itself as argument, and by dividing functions and classes into a hierarchy of types according to their possible arguments."
  >
  > (The Foundations of Mathematics, p. 356)

  This means that a class can never be a member of itself. The paradoxes solved by introducing the hierarchy of types (but not orders), like the Russell paradox, and the Burali-Forti paradox, are logical or syntactical paradoxes;

## Deramification

- The second group of paradoxes is excluded by the hierarchy of orders. These paradoxes (like the Liar's paradox, and the Richard Paradox) are based on the confusion of language and meta-language. These paradoxes are, therefore, not of a purely mathematical or logical nature. When a proper distinction between object language and meta-language is made, these so-called semantical paradoxes disappear immediately.

- Ramsey agrees with the part of the theory that eliminates the syntactic paradoxes. I.e., RTT without the orders of the types.

- The second part, the hierarchy of orders, does not gain Ramsey's support.

## Deramification

- By accepting the hierarchy in its full extent one either has to accept the Axiom of Reducibility or reject ordinary real analysis.
- Ramsey is supported in his view by Hilbert and Ackermann [19].
- They all suggest a deramification of the theory, i.e. leaving out the orders of the types.
- When making a proper distinction between language and meta-language, the deramification will not lead to a re-introduction of the (semantic) paradoxes.

## Deramification

- *Deramification* and the *Axiom of Reducibility* are both *violations of the Vicious Circle Principle*. Gödel fills the gap why they can be *harmlessly made*

> "it seems that the vicious circle principle [. . . ] *applies* only if the entities involved are *constructed by ourselves*. In this case there must clearly exist a *definition* (namely the description of the construction) which does *not refer to a totality* to which the object defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If, however, it is a question of objects that *exist independently of our constructions*, there is *nothing in the least absurd in the existence of totalities* containing members, which can be described only by reference to this totality."
>
> (Russell's mathematical logic)

## Deramification

- This turns the Vicious Circle Principle into a *philosophical principle* that will be easily *accepted by intuitionists* but that will be *rejected*, at least in its full strength, by mathematicians with a more *platonic* point of view.

- Gödel is supported in his ideas by *Quine* [28], sections 34 and 35.

- Quine's *criticism on impredicative* definitions (for instance, the definition of the least upper bound of a nonempty subset of the real numbers with an upper bound) is *not on the definition* of a special symbol, but rather on the very assumption of the existence of such an object at all.

## Deramification

- Quine states that even for *Poincaré*, who was an *opponent of impredicative definitions and deramification*, one of the doctrines of *classes is that they are there "from the beginning"*. So, even for Poincaré *there should be no evident fallacy in impredicative definitions*.

- The *deramification* has played an *important role* in the development of *type theory*. In *1932 and 1933*, Church presented his (untyped) *$\lambda$-calculus* [5, 6]. *In 1940* he combined this theory with a deramified version of Russell's theory of types to the system that is known as the simply typed $\lambda$-calculus

## The Simple Theory of Types

- Ramsey [29], and Hilbert and Ackermann [19], *simplified* the Ramified Theory of Types RTT by removing the orders. The result is known as the Simple Theory of Types *(STT)*.

- Nowadays, STT is known via Church's formalisation in $\lambda$-calculus. However, STT *already existed (1926) before $\lambda$-calculus did (1932)*, and is therefore not inextricably bound up with $\lambda$-calculus.

- How to obtain STT from RTT? Just *leave out all the orders* and the references to orders (including the notions of predicative and impredicative types).

## Church's Simply Typed $\lambda$-calculus $\lambda\!\rightarrow$

- The *types* of $\lambda\!\rightarrow$ are defined as follows:
  - $\iota$ *individuals* and $o$ *propositions* are types;
  - If $\alpha$ and $\beta$ are types, then so is $\alpha \rightarrow \beta$.

- The *terms* of $\lambda\!\rightarrow$ are the following:
  - $\neg$, $\wedge$, $\forall_\alpha$ for each type $\alpha$, and $\imath_\alpha$ for each type $\alpha$, are terms;
  - A *variable* is a term;
  - If $A, B$ are terms, then so is $AB$;
  - If $A$ is a term, and $x$ a variable, then $\lambda x{:}\alpha.A$ is a term.

- *($\beta$)* $\qquad (\lambda x{:}\alpha.A)B \rightarrow_\beta A[x := B]$.

## Typing rules in Church's Simply Typed $\lambda$-calculus $\lambda\!\rightarrow$

- $\Gamma \vdash \neg : o \rightarrow o$;

  $\Gamma \vdash \wedge : o \rightarrow o \rightarrow o$;

  $\Gamma \vdash \forall_\alpha : (\alpha \rightarrow o) \rightarrow o$;

  $\Gamma \vdash \imath_\alpha : (\alpha \rightarrow o) \rightarrow \alpha$;

- $\Gamma \vdash x : \alpha$ if $x{:}\alpha \in \Gamma$;

- If $\Gamma, x{:}\alpha \vdash A : \beta$ then $\Gamma \vdash (\lambda x{:}\alpha.A) : \alpha \rightarrow \beta$;

- If $\Gamma \vdash A : \alpha \rightarrow \beta$ and $\Gamma \vdash B : \alpha$ then $\Gamma \vdash (AB) : \beta$.

## Limitation of the simply typed $\lambda$-calculus

- $\lambda\!\rightarrow$ is very restrictive.
- Numbers, booleans, the identity function have to be defined at every level.
- We can represent (and type) terms like $\lambda x : o.x$ and $\lambda x : \iota.x$.
- We cannot type $\lambda x : \alpha.x$, where $\alpha$ can be instantiated to any type.
- This led to new (modern) type theories that allow more general notions of functions (e.g, *polymorphic*).

## The evolution of functions with Frege and Church

- Historically, functions have long been treated as a kind of meta-objects.
- Function *values* were the important part, not abstract functions.
- In the *low level/operational approach* there are only function values.
- The sine-function, is always expressed with a value: $\sin(\pi)$, $\sin(x)$ and properties like: $\sin(2x) = 2\sin(x)\cos(x)$.
- In many mathematics courses, one calls $f(x)$—and not $f$—the function.
- Frege, Russell and Church wrote $x \mapsto x + 3$ resp. as $x + 3$, $\hat{x} + 3$ and $\lambda x.x + 3$.
- Church made every function a first-class citizen. This is rigid.
- Russell allowed both the low level approach and the first-class citizen approach.
- The low-level approach is still worthwhile for many exact disciplines.

## The Goal: Open borders between mathematics, logic and computation

- Ordinary mathematicians *avoid* formal mathematical logic.
- Ordinary mathematicians *avoid* proof checking (via a computer).
- Ordinary mathematicians *may use* a computer for computation: there are over 1 million people who use Mathematica (including linguists, engineers, etc.).
- Mathematicians may also use other computer forms like Maple, LaTeX, etc.
- But we are not interested in only *libraries* or *computation* or *text editing*.
- We want *freedeom of movement* between mathematics, logic and computation.
- At every stage, we must have *the choice* of the level of formalilty and the depth of computation.

## Common Mathematical Language of mathematicians: CML

+ CML is *expressive*: it has linguistic categories like *proofs* and *theorems*.

+ CML has been refined by intensive use and is rooted in *long traditions*.

+ CML is *approved* by most mathematicians as a communication medium.

+ CML *accommodates many branches* of mathematics, and is adaptable to new ones.

– Since CML is based on natural language, it is *informal* and *ambiguous*.

– CML is *incomplete:* Much is left implicit, appealing to the reader's intuition.

– CML is *poorly organised:* In a CML text, many structural aspects are omitted.

– CML is *automation-unfriendly:* A CML text is a plain text and cannot be easily automated.

## A CML-text

From chapter 1, § 2 of E. Landau's *Foundations of Analysis* [Lan51].

**Theorem 6. [Commutative Law of Addition]**

$$x + y = y + x.$$

**Proof** Fix $y$, and let $\mathfrak{M}$ be the set of all $x$ for which the assertion holds.
I) We have

$$y + 1 = y',$$

and furthermore, by the construction in the proof of Theorem 4,

$$1 + y = y',$$

so that

$$1 + y = y + 1$$

and $1$ belongs to $\mathfrak{M}$.
II) If $x$ belongs to $\mathfrak{M}$, then

$$x + y = y + x,$$

Therefore

$$(x + y)' = (y + x)' = y + x'.$$

By the construction in the proof of Theorem 4, we have

$$x' + y = (x + y)',$$

hence

$$x' + y = y + x',$$

so that $x'$ belongs to $\mathfrak{M}$. The assertion therefore holds for all $x$. □

## What are the options for computerization?

Computers can handle mathematical text at various levels:

- Images of pages may be stored. While useful, this is not a good representation of *language* or *knowledge*.
- Typesetting systems like LaTeX can be used.
- Document representations like OMDoc can be used.
- Formal logics used by theorem provers can be used.

We are gradually developing a system named MathLang which we hope will eventually allow building a bridge between the latter 3 levels.

This talk aims at discussing the motivations rather than the details.

## The issues with typesetting systems

+ A system like LaTeX provides good defaults for visual appearance, while allowing fine control when needed.

+ LaTeX supports commonly needed document structures, while allowing custom structures to be created.

– Unless the mathematician is amazingly disciplined, the logical structure of symbolic formulas is not represented at all.

– The logical structure of mathematics as embedded in natural language text is not represented. Automated discovery of the semantics of natural language text is still too primitive and requires human oversight.

## LaTeX example

| | |
|---|---|
| draft documents | ✓ |
| public documents | ✓ |
| computations and proofs | ✗ |

```
\begin{theorem}[Commutative Law of Addition]\label{theorem:6}
  $$x+y=y+x.$$
\end{theorem}
\begin{proof}
  Fix $y$, and $\mathfrak{M}$ be the set of all $x$ for which the
  assertion holds.
  \begin{enumerate}
  \item We have $$y+1=y',$$
    and furthermore, by the construction in
    the proof of Theorem~\ref{theorem:4}, $$1+y=y',$$
    so that $$1+y=y+1$$
    and $1$ belongs to $\mathfrak{M}$.
  \item If $x$ belongs to $\mathfrak{M}$, then $$x+y=y+x,$$
    Therefore
    $$(x+y)'=(y+x)'=y+x'.$$
    By the construction in the proof of
    Theorem~\ref{theorem:4}, we have $$x'+y=(x+y)',$$
    hence
    $$x'+y=y+x',$$
    so that $x'$ belongs to $\mathfrak{M}$.
  \end{enumerate}
  The assertion therefore holds for all $x$.
\end{proof}
```

## The differences of OMDoc

OMDoc attempts to solve some of the difficulties of typesetting systems.

+ Translation to LaTeX (still needed) or MathML can handle visual appearance.
– Precise appearance control must work *through* a translation (difficult!).
+ OMDoc supports commonly needed document structures.
+ The tree structure of symbolic formulas is represented.
– The semantics of symbolic formulas is not represented.
– Type checking symbolic formulas (beyond arity) must be outside OMDoc.
– The logical structure of mathematics as embedded in natural language text is still not represented. There are ways to associate symbolic formulas with natural language text, but no way to check their consistency.

## The beginnings of computerized formalization

- In 1967 the famous mathematician de Bruijn began work on logical languages for complete books of mathematics that can be *fully* checked by machine.
- People are prone to error, so if a machine can do proof checking, we expect fewer errors.
- Most mathematicians doubted de Bruijn could achieve success, and computer scientists had no interest at all.
- However, he persevered and built *Automath* (AUTOmated MATHematics).
- Today, there is much interest in many approaches to proof checking for verification of computer hardware and software.
- Many theorem provers have been built to mechanically check mathematics and computer science reasoning (e.g. Isabelle, HOL, Coq, etc.).

## The problem with formal logic

- No logical language has the criteria expected of a language of mathematics.
  - A logical language does not have *mathematico-linguistic* categories, is *not universal* to all mathematicians, and is *not a good communication medium*.
  - Logical languages make fixed choices (*first versus higher order, predicative versus impredicative, constructive versus classical, types or sets*, etc.). But different parts of mathematics need different choices and there is no universal agreement as to which is the best formalism.
  - A logician reformulates in logic their *formalization* of a mathematical-text as a formal, complete text which is structured considerably *unlike* the original, and is of little use to the *ordinary* mathematician.
  - Mathematicians do not want to use formal logic and have *for centuries* done mathematics without it.
- *So, mathematicians kept to* CML.
- We would like to find an alternative to CML which avoids some of the features of the logical languages which made them unattractive to mathematicians.

## Full formalization difficulties: choices

A CML-text is structured differently from a fully formalized text proving the same facts. *Making the latter involves extensive knowledge and many choices:*

- The choice of the *underlying logical system.*
- The choice of *how concepts are implemented* (equational reasoning, equivalences and classes, partial functions, induction, etc.).
- The choice of the *formal foundation*: a type theory (dependent?), a set theory (ZF? FM?), a category theory? etc.
- The choice of the *proof checker*: Automath, Isabelle, Coq, PVS, Mizar, ...

An issue is that one must in general commit to one set of choices.

## Full formalization difficulties: informality

Any informal reasoning in a CML-text will cause various problems when fully formalizing it:

- A single (big) step may need to expand into a (series of) syntactic proof expressions. Very long expressions can replace a clear CML-text.

- The entire CML-text may need *reformulation* in a fully *complete* syntactic formalism where every detail is spelled out. New details may need to be woven throughout the entire text. The text may need to be “turned inside out”.

- Reasoning may be obscured by *proof tactics*, whose meaning is often *ad hoc* and implementation-dependent.

Regardless, ordinary mathematicians do not find the new text useful.

## Coq example

|  |  |
|---|---|
| draft documents | ✗ |
| public documents | ✗ |
| computations and proofs | ✓ |

From Module `Arith.Plus` of Coq standard library (`http://coq.inria.fr/`).

```
Lemma plus_sym : (n,m:nat)(n+m)=(m+n).
Proof.
Intros n m ; Elim n ; Simpl_rew ; Auto with arith.
Intros y H ; Elim (plus_n_Sm m y) ; Simpl_rew ; Auto with arith.
Qed.
```

## Where do we start? de Bruijn's Mathematical Vernacular MV

- *De Bruijn's Automath not just [...] as a technical system for verification of mathematical texts, it was rather a life style with its attitudes towards understanding, developing and teaching mathematics....The way mathematical material is to be presented to the system should correspond to the usual way we write mathematics. The only things to be added should be details that are usually omitted in standard mathematics.*

- MV is faithful to CML yet is formal and avoids ambiguities.

- MV is close to the usual way in which mathematicians write.

- MV has a syntax based on linguistic categories not on set/type theory.

- MV is weak as regards correctness: the rules of MV mostly concern *linguistic* correctness, its types are mostly linguistic so that the formal translation into MV is satisfactory *as a readable, well-organized text*.

## Problems with MV

- MV makes many logical and mathematical choices which are best postponed.
- MV incorporates certain correctness requirements, there is for example a hierarchy of types corresponding with sets and subsets.
- MV is already *on its way* to a full formalization, while we want the option of remaining *closer to* a given informal mathematical content.
- We want a *formal* language MathLang which •has the advantages of CML but not its disadvantages and •respects CML content.
- *MV does not respect* CML *content.*

## What is the aim for MathLang?

Can we formalise a CML text, avoiding as much as possible the ambiguities of natural language, while still guaranteeing the following four goals?

1. The formalised text looks very much like the original CML text (and hence the content of the original CML text is respected).

2. The formalised text can be fully manipulated and searched in ways that respect its mathematical structure and meaning.

3. Steps can be made to do computation (via computer algebra systems) and proof checking (via proof checkers) on the formalised text.

4. This formalisation of text is not much harder for the ordinary mathematician than LaTeX. *Full formalization down to a foundation of mathematics is not required*, although allowing and supporting this is one goal.

(No theorem prover's language satisfies these goals.)

## Starting point for MathLang: MV and WTT

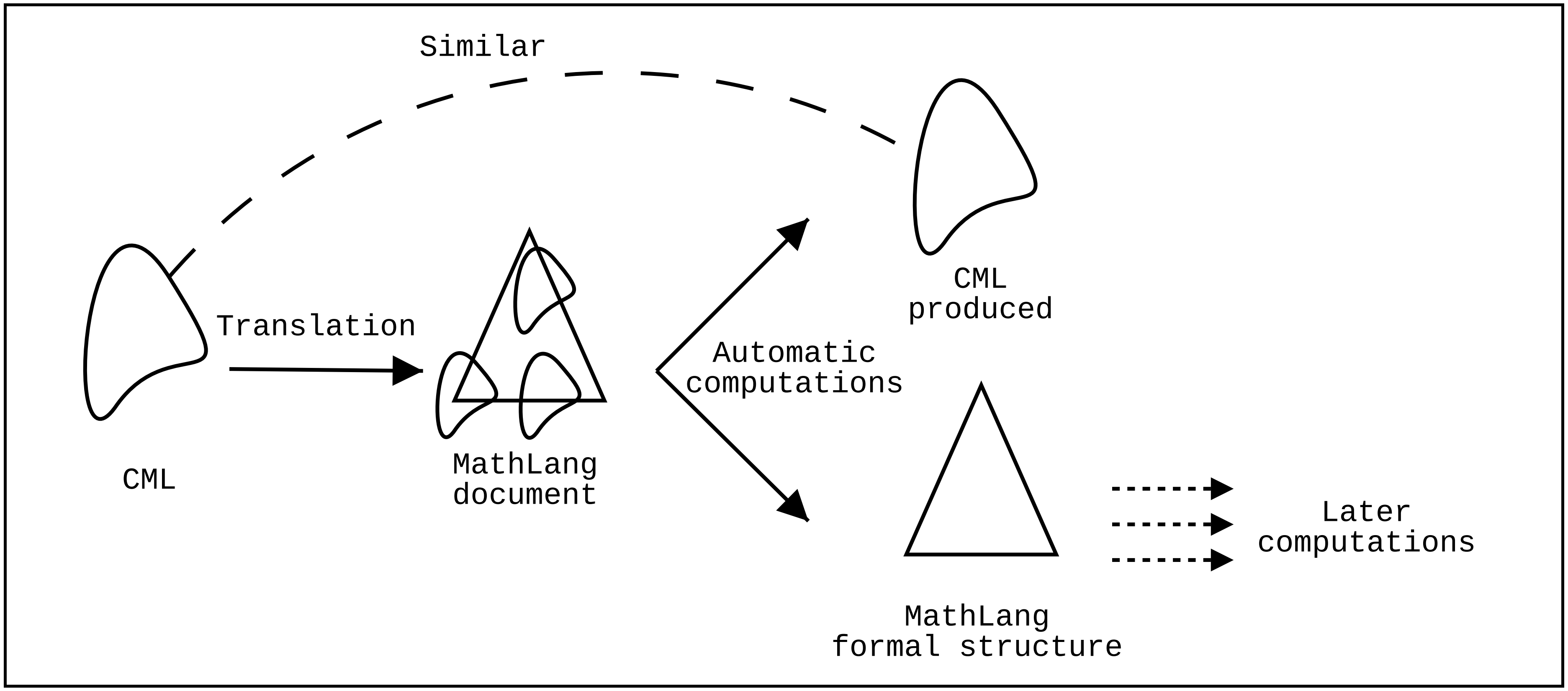
- MV was an initial inspiration for MathLang. But MV fails on goal 1.
- Weak Type Theory, WTT [21], is MV minus the added logic.
- Although in many ways WTT succeeds and improves on MV, it still fails on goal 1. A WTT text is not close to its CML original.
- With MathLang, we start from WTT, add some features, and investigate how to integrate it with natural language text.
- Our ongoing development of MathLang is driven by testing it in translating a set of sample texts chosen to cover a large portion of CML usages, both current and historical.

## MathLang

| | |
|---|---|
| draft documents | ✓ |
| public documents | ✓ |
| computations and proofs | ✓ |

- A MathLang text captures the grammatical and reasoning aspects of mathematical structure for further computer manipulation.
- A *weak type system* checks MathLang documents at a grammatical level.
- A MathLang text remains *close* to its CML original, allowing confidence that the CML has been captured correctly.
- We have been developing ways to weave natural language text into MathLang.
- MathLang aims to eventually support *all encoding uses*.

## Process of translation into MathLang



- The CML view of a MathLang text should match the mathematician's intentions.
- The formal structure should be suitable for various automated uses.

## Linguistic categories in WTT and MathLang

- At the *atomic* level, WTT has separate syntactic categories for *variables*, *constants*, and *binders*. The latest MathLang uses one syntactic category and instead distinguishes these roles via weak types.
- At the *phrase* level, there are *terms*, *sets*, *nouns*, and *adjectives*. (Manuel's talk will give details on how this is handled in the latest MathLang.)
- At the *sentence* level, there are *statements* and *definitions*.
- At the *discourse* level, WTT has *contexts*, *lines*, *books*, and *prefaces*. The latest MathLang replaces these by *blocks* and *scoping* operators.

Generally, each syntactic category has a corresponding *weak type*.

## Examples of linguistic categories

- Terms: the triangle $ABC$; the center of $\boxed{ABC}$; $d(\boxed{x}, \boxed{y})$.
- Nouns: a triangle; an edge of $\boxed{\text{ABC}}$; a group.
- Adjectives: equilateral $\boxed{\text{triangle}}$; prime $\boxed{\text{number}}$; Abelian $\boxed{\text{group}}$.
- Statements: $\boxed{P}$ lies between $\boxed{Q}$ and $\boxed{R}$; $\boxed{5} \geq \boxed{3}$; $\boxed{\text{AB}}$ is $\boxed{\text{an edge of ABC}}$.
- Definition: a number $p$ is prime whenever $\boxed{\cdots}$.

## MathLang example

**Definition 2.** A *Fermat-sum* is a natural number which is the sum of two squares of natural numbers.

**Lemma 3.** The product of a square and a Fermat-sum is a Fermat sum.

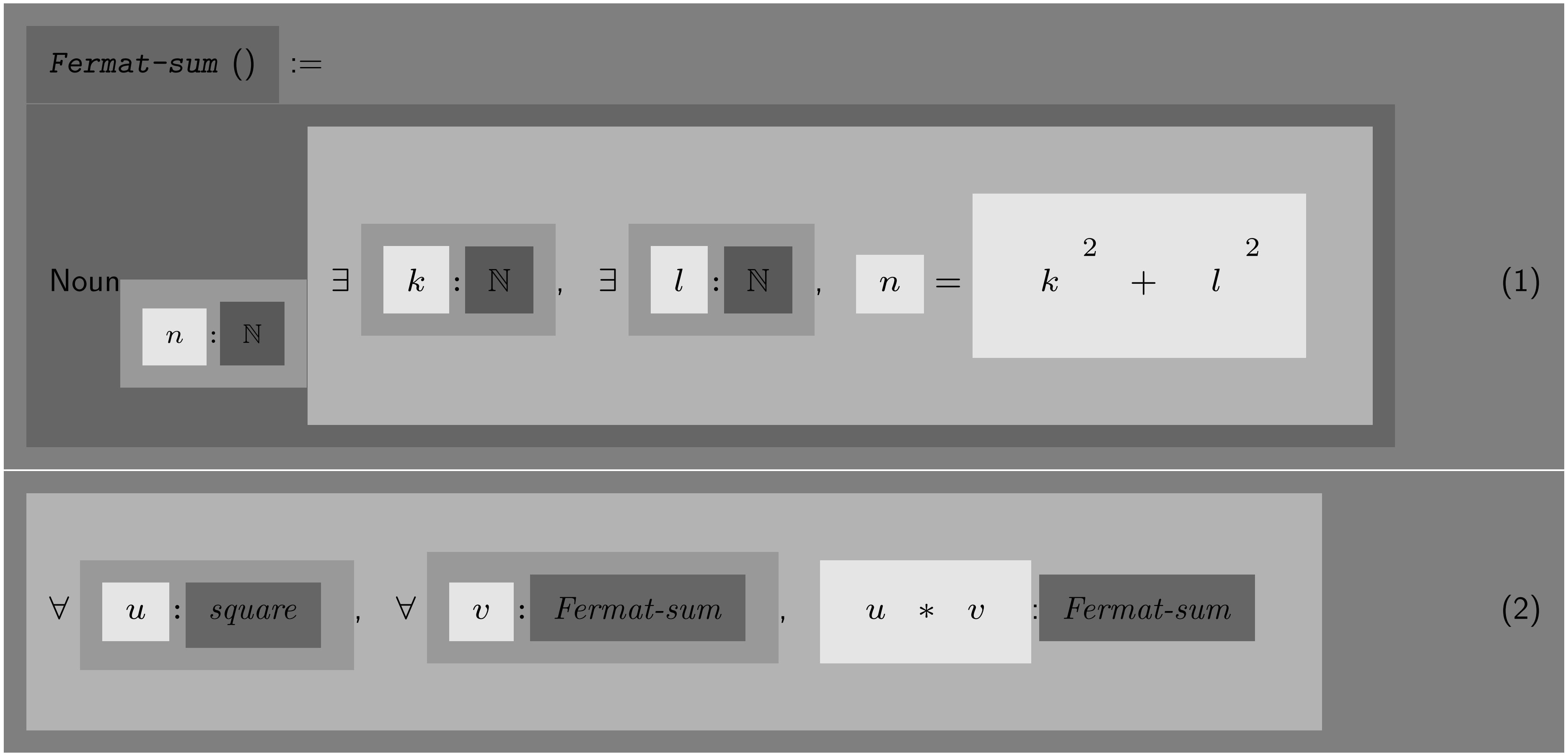
In an older MathLang version, the above text could be translated as the following two *lines*:

$$a\ \mathit{Fermat\text{-}sum}\ := \mathtt{Noun}_{n\in\mathbb{N}}\exists_{k\in\mathbb{N}}\exists_{l\in\mathbb{N}}(n = k^2 + l^2)$$

$$\forall_{u:\ a\ square}\forall_{v:\ a\ Fermat\text{-}sum}(uv:\ \ a\ \mathit{Fermat\text{-}sum})$$

We can also give the following interesting *views* of this example.

## MathLang example: Symbolic structure view

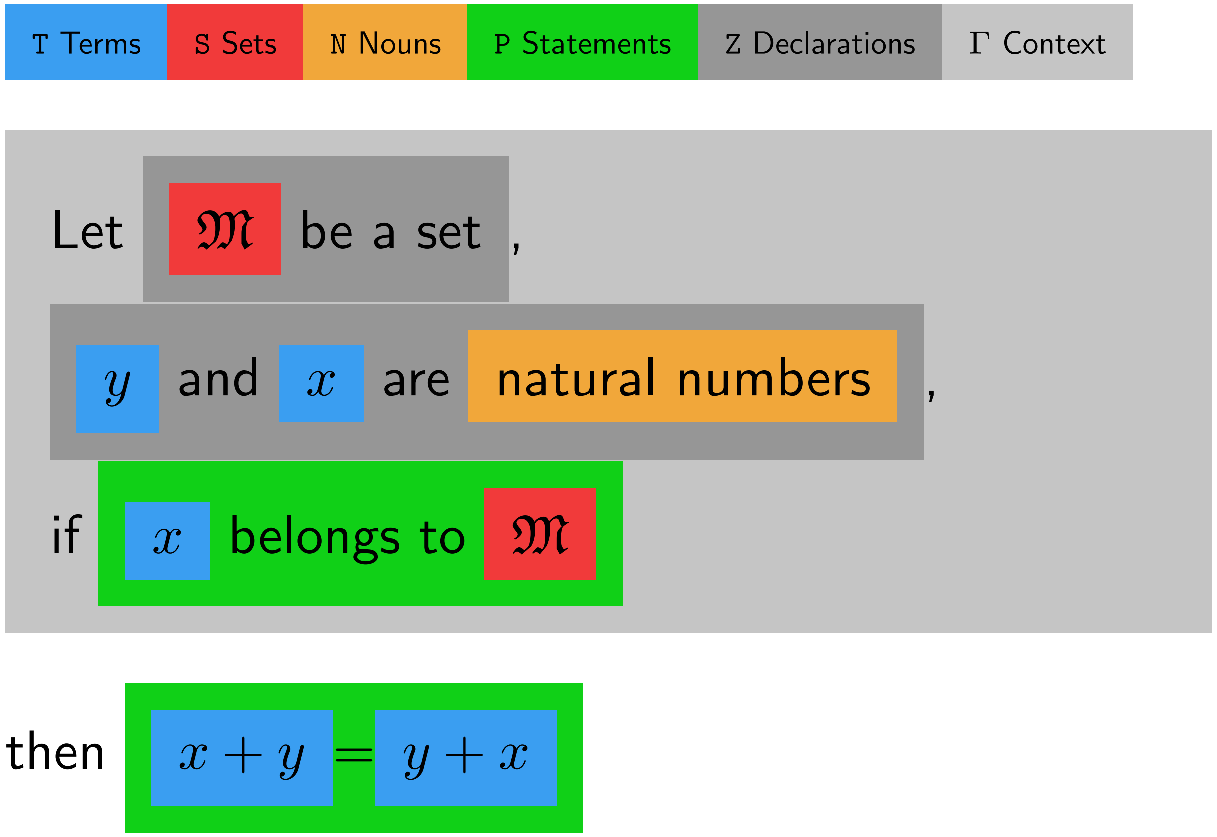

Fermat-sum () :=
Noun $n : \mathbb{N}$ $\exists$ $k : \mathbb{N}$ , $\exists$ $l : \mathbb{N}$ , $n = k^2 + l^2$
(1)
$\forall$ $u$ : $square$ , $\forall$ $v$ : $Fermat\text{-}sum$ , $u * v$ : $Fermat\text{-}sum$
(2)

## MathLang example: CML view

**Definition 4.** [Fermat-sum]
A *Fermat-sum* is a natural number which is the sum of two squares of natural numbers. 1

**Lemma 5.**
The product of a *square* and a *Fermat-sum* is a *Fermat-sum*. 2

## Another MathLang example



T Terms | S Sets | N Nouns | P Statements | Z Declarations | Γ Context

Let $\mathfrak{M}$ be a set ,

$y$ and $x$ are natural numbers ,

if $x$ belongs to $\mathfrak{M}$

then $x + y = y + x$

## Another MathLang example: Type checking

| | | | | | |
|---|---|---|---|---|---|
| T Terms | S Sets | N Nouns | P Statements | Z Declarations | $\Gamma$ Context |

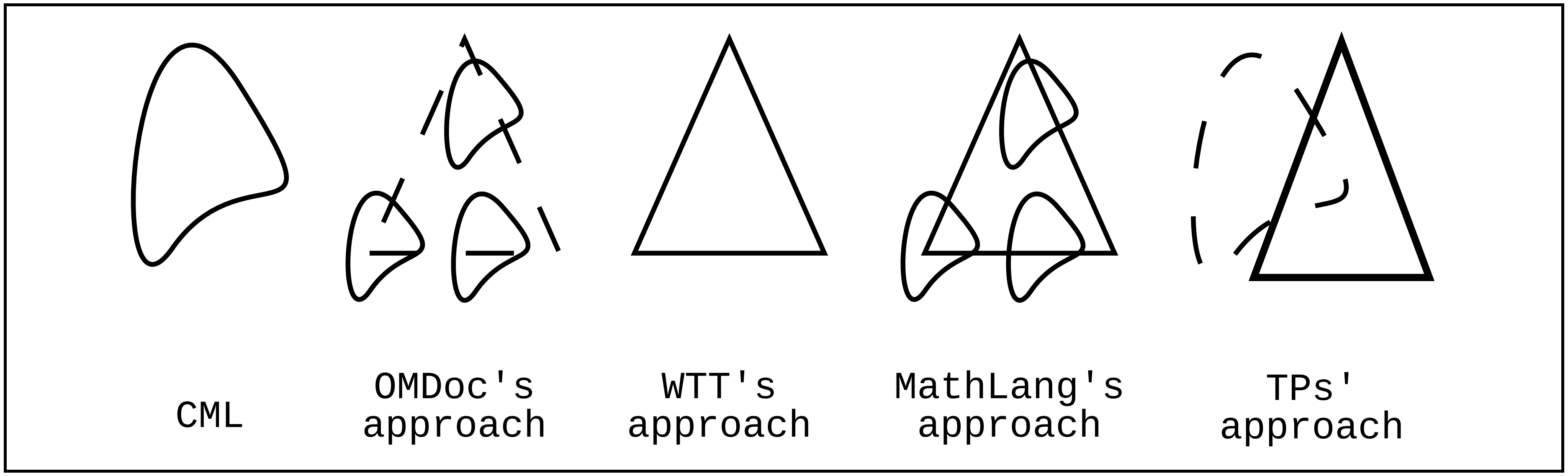
Let $\mathfrak{M}$ be a set ,

$y$ and $x$ are natural numbers ,

if $x$ belongs to $\mathfrak{M}$

then $x + y$ $\Leftarrow$ **error**

## Various approaches to representing mathematics



We visually summarize the approaches. Blobs represent natural language text whose structure is not understood by the computer. A broken blob is text maintained separately, not as part of the data structure. Triangles represent a tree-shaped structures understood by the computer. Solid triangles represent additional computer-checked well-formedness conditions. A heavy solid triangle represents full formalization.

## Additional comparison with other related work

- *Galina* serves as a *command language* for Coq, aimed at full formalization.
- The *mathematical vernacular* of *$\Omega$MEGA* gives CML-like *views* of fully/partially formalized proofs.
- The basic languages of *Mizar* and Isar preserve the mathematical content. They are aimed at full formalization. Their syntax does not give the same expressive freedom to the mathematician as CML.
- In the *Theorema project* computer algebra systems, the provers are designed to imitate the proof style humans employ in their proving attempts. The proofs can be produced in human-readable style. However, this is done by *post-processing* a fully formal proof.
- The typed functional programming language *GF* can define languages such as fragments of natural languages, programming languages, and formal calculi. GF is based on Martin-Löf's type theory.

## Some points to consider

- We do not at all assume/prefer one type/logical theory instead of another.
- The formalisation of a language of mathematics should separate the questions:
  - *which type/logical theory is necessary for which part of mathematics*
  - *which language should mathematics be written in*.
- Mathematicians don't usually know or work with type/logical theories.
- Mathematicians usually *do* mathematics (manipulations, calculations, etc), but are not interested in general in reasoning *about* mathematics.

## Conclusion

- MathLang aims to support non-fully-formalized mathematics practiced by the ordinary mathematician as well as work toward full formalization.
- MathLang aims to handle mathematics as expressed in natural language as well as symbolic formulas.
- MathLang aims to do some amount of type checking even for non-fully-formalized mathematics. This corresponds roughly to grammatical conditions.
- MathLang aims for a formal representation of CML texts that closely corresponds to the CML conceived by the ordinary mathematician.
- MathLang aims to support automated processing of mathematical knowledge.
- MathLang aims to be independent of any foundation of mathematics.

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