Three themes in the formalisation and automation of logic and mathematics

Fairouz Kamareddine (Universidade de Heriot-Watt, Edimburgo)

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Overview

- Deduction modulo versus reduction modulo
- Nuprl as a PTS
- MathLang

Le lambda calcul

- Rappelons l'essence du λ-calcul:
  Syntax: $\Lambda ::= \mathcal{V}|(\Lambda\Lambda)|\lambda\mathcal{V}.\Lambda.$
  Règle: $(\lambda x.A)B \rightarrow_\beta A[x := B]$

Le lambda Calcul à la de Bruijn (item notation) [Kamareddine and Nederpelt, 1995, 1996]

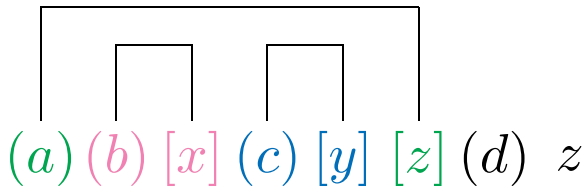
$'$: Notation classique	$\mapsto$	Notation de de Bruijn
x	$\mapsto$	x
$\lambda x.B$	$\mapsto$	$[x]B'$
AB	$\mapsto$	$(B')A'$

Example: $(\lambda x.\lambda y.xy)z \mapsto (z)[x][y](y)x$

- Dans le *train* $(z)[x][y](y)$, les *wagons* sont (z), $[x]$, $[y]$ and (y).
- Le dernier x dans $(z)[x][y](y)x$ est le *coeur* du terme.
- Le *wagon d'application* (z) et le *wagon d'abstraction* $[x]$ se trouvent l'un *à coté* de l'autre.
- La règle β $\quad (\lambda x.A)B \to_\beta A[x := B]$ devient: $(B)[x]A \to_\beta [x := B]A$

Réduction dans la notation de de Bruijn

Notation classique	Notation de de Bruijn
$\underline{((\lambda_x.(\lambda_y.\lambda_z.zd)c)b)a}$	$(a)\underline{(b)[x]}(c)[y][z](d)z$
$\downarrow_\beta$	$\downarrow_\beta$
$\underline{((\lambda_y.\lambda_z.zd)c)a}$	$(a)\underline{(c)[y]}[z](d)z$
$\downarrow_\beta$	$\downarrow_\beta$
$\underline{(\lambda_z.zd)a}$	$\underline{(a)[z]}(d)z$
$\downarrow_\beta$	$\downarrow_\beta$
ad	$(d)a$



Chaque wagon a un *partenaire* sauf (d) qui est *célibataire*.
Les crochets de $((\lambda_x.(\lambda_y.\lambda_z.\ -\ -)c)b)a)$, sont '$[_1\ [_2\ [_3\ ]_2\ ]_1\ ]_3$', où '$[_i$' and '$]_i$' vont ensemble.
Les crochets de $(a)(b)[x](c)[y][z]$ sont plus simple: $[[\,][\,]]$.

Réductions generalisées

- $((\lambda_x.(\underline{\lambda_y.\lambda_z.zd)c})b)a \rightarrow_\beta ((\lambda_x.\{\lambda_z.zd\}[y := c])b)a$

  $(a)(b)[x](c)[y][z](d)z \rightarrow_\beta (a)(b)[x][y := c][z](d)z$

- $(\underline{(\lambda_x.(\lambda_y.\lambda_z.zd)c)b})a \rightarrow_\beta \{(\lambda_y.\lambda_z.zd)c\}[x := b]a$

  $(a)(b)[x](c)[y][z](d)z \rightarrow_\beta (\mathsf{a})[\mathsf{x{:=}b}](\mathsf{c})[\mathsf{y}][\mathsf{z}](\mathsf{d})\mathsf{z}$

- $(a)(b)[x](c)[y][z](d)z \hookrightarrow_\beta (b)[x](c)[y][z := a](d)z$

Quelques notions de réduction dans la literature

Name	In Classical Notation	In de Bruijn's notation
(θ)	$((\lambda_x.N)P)Q$ $\downarrow$ $(\lambda_x.NQ)P$	$(Q)(P)[x]N$ $\downarrow$ $(P)[x](Q)N$
(γ)	$(\lambda_x.\lambda_y.N)P$ $\downarrow$ $\lambda_y.(\lambda_x.N)P$	$(P)[x][y]N$ $\downarrow$ $[y](P)[x]N$
(γ_C)	$((\lambda_x.\lambda_y.N)P)Q$ $\downarrow$ $(\lambda_y.(\lambda_x.N)P)Q$	$(Q)(P)[x][y]N$ $\downarrow$ $(Q)[y](P)[x]N$
(g)	$((\lambda_x.\lambda_y.N)P)Q$ $\downarrow$ $(\lambda_x.N[y := Q])P$	$(Q)(P)[x][y]N$ $\downarrow$ $(P)[x][y := Q]N$

Quelques usages de ces réductions/"term reshuffling"

- Regnier [1992] introduit θ and γ et les utilise pour analyzer la strategy perpetuelle de réduction.
- [Kfoury et al., 1994; Kfoury and Wells, 1994] utilisent term reshuffling pour analyzer des problèmes de typages (comme (non)décidibilité).
- [Nederpelt, 1973; de Groote, 1993; Kfoury and Wells, 1995] les utitlisent pour réduire le problème de normalisation forte à celui de normalisation faible.
- [Ariola et al., 1995] montre que term-reshuffling peut bien representer l'évaluation fonctionnelle paresseuse.
- [Kamareddine and Nederpelt, 1995; Kamareddine et al., 1999, 1998; Bloo et al., 1996] montrent que les réductions géneralisés sont plus efficaces (espace et temps).
- [Kamareddine, 2000] établit le théorème de conservation pour les réductions generalisées.

Reduction Modulo versus Deduction Modulo

- [Dowek et al., 2003] introduced a very neat and useful idea.
- Deduction modulo is a way to remove computational arguments from proofs by reasoning modulo a congruence on propositions.
- Separate computations and deductions in a clean way.
- Instead of $\dfrac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta}$
- They write $\dfrac{\Gamma \vdash_c P, \Delta \quad \Gamma \vdash_c Q, \Delta}{\Gamma \vdash_c R, \Delta}$ if R and $P \wedge Q$ are congruent ($R =_c P \wedge Q$)

Deduction modulo

- If $P =_c Q$ then $\Gamma \vdash_c P, \Delta$ iff $\Gamma \vdash_c Q, \Delta$ and $\Gamma, P \vdash_c \Delta$ iff $\Gamma, Q \vdash_c \Delta$ and the proofs have the same size.
- For each congruence c, there is a set of axioms $\mathcal{T}$ such that $\mathcal{T}, \Gamma \vdash \Delta$ iff $\Gamma \vdash_c \Delta$
- Hence we have the same theorems in both formalisms but the proofs modulo are shorter because the standard deduction steps performing computations described by c are eliminated.

Reduction modulo

- To do reduction modulo, we need the notation of de Bruijn
- To give you an example why this notation is needed, we take the description of normal forms in a substitution calculus.
- The set of terms, noted Λs, of the λs-calculus is given as follows:

$$\Lambda s ::= \mathbb{N} \mid \Lambda s \Lambda s \mid \lambda \Lambda s \mid \Lambda s \, \sigma^i \Lambda s \mid \varphi_k^i \Lambda s \quad where \quad i \geq 1 \,, \quad k \geq 0 \,.$$

The set of open terms, noted Λs_{op} is given as follows:

$$\Lambda s_{op} ::= \mathbf{V} | \mathbb{N} | \Lambda s_{op} \Lambda s_{op} | \lambda \Lambda s_{op} | \Lambda s_{op} \sigma \Lambda s_{op} | \varphi_k^i \Lambda s_{op} \quad where \quad i \geq 1 \,, \quad k \geq 0$$

s_e-normal forms in classical notation

It is cumbersome to describe s_e-normal forms of open terms. But this description is needed to show the weak normalisation of the s_e-calculus. In classical notation, an open term is an s_e-normal form iff one of the following holds:

- $a \in \mathbf{V} \cup \mathbb{N}$, i.e. a is a variable or a de Bruijn number.
- $a = b\,c$, where b and c are s_e-normal forms.
- $a = \lambda b$, where b is an s_e-normal form.
- $a = b\,\sigma^j c$, where c is an s_e-nf and b is an s_e-nf of the form X, or $d\,\sigma^i e$ with $j < i$, or $\varphi^i_k d$ with $j \leq k$.
- $a = \varphi^i_k b$, where b is an s_e-nf of the form X, or $c\,\sigma^j d$ with $j > k+1$, or $\varphi^j_l c$ with $k < l$.

s_e-normal forms in item notation

- Write $a\,\sigma^i b = (b\,\sigma^i)a$ and $\varphi^i_k a = (\varphi^i_k)a$.
- We call a and b the bodies of these items.
- A *normal $\sigma\varphi$-segment* $\overline{s}$ is a sequence of σ- and φ-items such that every pair of adjacent items in $\overline{s}$ are of the form:

  $(\varphi^i_k)(\varphi^j_l)$ and $k < l$ — $(\varphi^i_k)(b\,\sigma^j)$ and $k < j-1$

  $(b\,\sigma^i)(c\,\sigma^j)$ and $i < j$ — $(b\,\sigma^j)(\varphi^i_k)$ and $j \leq k$.

- The s_e-nf's can be described in item notation by the following syntax:

$$NF ::= \mathbf{V} \mid \mathsf{IN} \mid (NF\,\delta)NF \mid (\lambda)NF \mid \overline{s}\,\mathbf{V}$$

  where $\overline{s}$ is a normal $\sigma\varphi$-segment whose bodies belong to NF.

Formes Canoniques [Kamareddine et al., 2001]

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des [ ] célibataires	des ()[ ]-paires	des ()s célibataires	coeur
$[x_1]\ldots[x_n]$	$(A_1)[y_1]\ldots(A_m)[y_m]$	$(B_1)\ldots(B_p)$	x

- Dans [Regnier, 1994] et [Kfoury and Wells, 1995]

$$\lambda x_1 \cdots \lambda x_n.(\lambda y_1.(\lambda y_2.\cdots(\lambda y_m.xB_p\cdots B_1)A_m\cdots)A_2)A_1$$

- Par example, la forme canonique de:

$$[x][y](a)[z][x'](b)(c)(d)[y'][z'](e)x$$

est

$$[x][y][x'](a)[z](d)[y'](c)[z'](b)(e)x$$

Comment arriver aux formes canoniques?

For $M \equiv [x][y](a)[z][x'](b)(c)(d)[y'][z'](e)x$:

$\theta(M)$:	bach. $[\,]$s $[x][y]$	$()[\,]$-pairs mixed with bach. $[\,]$s $(a)[z][x'](d)[y'](c)[z']$	bach. $()$s $(b)(e)$	end var x
$\gamma(M)$:	bach. $[\,]$s $[x][y][x']$	$()[\,]$-pairs mixed with bach. $()$s $(a)[z](b)(c)[z'](d)[y']$	bach. $()$s (e)	end var x
$\theta(\gamma(M))$:	bach. $[\,]$s $[x][y][x']$	$()[\,]$-pairs $(a)[z](c)[z'](d)[y']$	bach. $()$s $(b)(e)$	end var x
$\gamma(\theta(M))$:	bach. $[\,]$s $[x][y][x']$	$()[\,]$-pairs $(a)[z](d)[y'](c)[z']$	bach. $()$s $(b)(e)$	end var x

$\to_\theta$ and $\to_\gamma$ sont SN et CR. Alors les θ-nf and γ-nf sont unique.

$\theta(\gamma(A))$ et $\gamma(\theta(A))$ sont les deux en *forme canonique*

Notons que: $\theta(\gamma(A)) =_p \gamma(\theta(A))$ où $\to_p$ est la règle

$$(A_1)[y_1](A_2)[y_2]B \to_p (A_2)[y_2](A_1)[y_1]B \qquad \text{si } y_1 \notin \mathrm{FV}(A_2)$$

Réduction basée sur les classes de formes canoniques, Reduction Modulo [Kamareddine and Bloo, 2002]

- Définissons $[A] = \{B \mid \theta(\gamma(A)) =_p \theta(\gamma(B))\}$.
- Quand $B \in [A]$,on écrit $B \approx_{\mathtt{equi}} A$.
- $\rightarrow_\theta, \rightarrow_\gamma, =_\gamma, =_\theta, =_p \subset \approx_{\mathtt{equi}} \subset =_\beta$ (inclusions strictes).
- $A \rightsquigarrow_\beta B$ iff $\exists A' \in [A].\exists B' \in [B].\ A' \rightarrow_\beta B'$ (avec compatibilité)
- Si $A \rightsquigarrow_\beta B$ alors $\forall A' \in [A].\forall B' \in [B].\ A' \rightsquigarrow_\beta B'$.
- $\rightarrow_\beta \subset \rightarrow_g \subset \rightsquigarrow_\beta \subset =_\beta = \approx_\beta$.
- $\twoheadrightsquigarrow_\beta$ *est CR:* Si $A \twoheadrightsquigarrow_\beta B$ et $A \twoheadrightsquigarrow_\beta C$, alors $\exists D$: $B \twoheadrightsquigarrow_\beta D$ et $C \twoheadrightsquigarrow_\beta D$.
- Soit $r \in \{\rightarrow_\beta, \rightsquigarrow_\beta\}$. Si $A \in SN_r$ et $A' \in [A]$ alors $A' \in SN_r$.
- $A \in SN_{\rightsquigarrow_\beta}$ ssi $A \in SN_{\rightarrow_\beta}$.

Properties of reduction modulo classes

- *$\rightsquigarrow_\beta$ generalises $\rightarrow_g$ and $\rightarrow_\beta$:* $\rightarrow_\beta \subset \rightarrow_g \subset \rightsquigarrow_\beta \subset =_\beta$.
- *$\approx_\beta$ and $=_\beta$ are equivalent:* $A \approx_\beta B$ iff $A =_\beta B$.
- *$\rightsquigarrow\!\!\!\!\rightarrow_\beta$ is Church Rosser:*
  If $A \rightsquigarrow\!\!\!\!\rightarrow_\beta B$ and $A \rightsquigarrow\!\!\!\!\rightarrow_\beta C$, then for some D: $B \rightsquigarrow\!\!\!\!\rightarrow_\beta D$ and $C \rightsquigarrow\!\!\!\!\rightarrow_\beta D$.
- *Classes preserve $SN_{\rightarrow_\beta}$:* If $A \in SN_{\rightarrow_\beta}$ and $A' \in [A]$ then $A' \in SN_{\rightarrow_\beta}$.
- *Classes preserve $SN_{\rightsquigarrow_\beta}$:* If $A \in SN_{\rightsquigarrow_\beta}$ and $A' \in [A]$ then $A' \in SN_{\rightsquigarrow_\beta}$.
- *$SN_{\rightarrow_\beta}$ and $SN_{\rightsquigarrow_\beta}$ are equivalent:* $A \in SN_{\rightsquigarrow_\beta}$ iff $A \in SN_{\rightarrow_\beta}$.
- *Classes preserve normal forms:* If $A' \in [A]$ and $nf(A)$ exists then $nf(A')$ exists and $nf(A) = nf(A')$.

Les types simples

- Soient $f : \mathbb{N} \to \mathbb{N}$ et $g_f : \mathbb{N} \to \mathbb{N}$ tel que $g_f(x) = f(f(x))$.
  Soit $F_{\mathbb{N}} : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$ tel que $F_{\mathbb{N}}(f)(x) = g_f(x) = f(f(x))$.

- Dans le lambda calcul typé de Church on écrit la fonction $F_{\mathbb{N}}$ comme suit:

$$\lambda_{f:\mathbb{N}\to\mathbb{N}}.\lambda_{x:\mathbb{N}}.f(f(x))$$

- Si on veut la même fonction sur les Booléens $\mathcal{B}$, on écrit:

la fonction $F_{\mathcal{B}}$ est	$\lambda_{f:\mathcal{B}\to\mathcal{B}}.\lambda_{x:\mathcal{B}}.f(f(x))$
le type de la fonction $F_{\mathcal{B}}$ est	$(\mathcal{B} \to \mathcal{B}) \to (\mathcal{B} \to \mathcal{B})$

- *Problèmes* dans RTT et STT. Alors la naissance des *systèmes de typages differents*, chacun avec son *pouvoir d'abstraire des fonctions*.

- *8 λ-calculs typés importants* 1940–1988 ont été unifié dans le *cube de Barendregt*.

Polymorphisme: le λ-calcul typé après Church

- A la place de répéter le travail, on prend $\alpha : *$ (α est un type quelconque) et on définit une fonction polymorphique F comme suit:

  $$\lambda_{\alpha:*}.\lambda_{f:\alpha\to\alpha}.\lambda_{x:\alpha}.f(f(x))$$

  On donne à F le type:

  $$\Pi_{\alpha:*}.(\alpha \to \alpha) \to (\alpha \to \alpha)$$

- Comme ça, $F(\alpha) = \lambda_{f:\alpha\to\alpha}.\lambda_{x:\alpha}.f(f(x)) : (\alpha \to \alpha) \to (\alpha \to \alpha)$

- On peut instancier α selon notre besoin:
  - $F(\mathbb{N}) = \lambda_{f:\mathbb{N}\to\mathbb{N}}.\lambda_{x:\mathbb{N}}.f(f(x)) : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$
  - $F(\mathcal{B}) = \lambda_{f:\mathcal{B}\to\mathcal{B}}.\lambda_{x:\mathcal{B}}.f(f(x)) : (\mathcal{B} \to \mathcal{B}) \to (\mathcal{B} \to \mathcal{B})$
  - $F(\mathcal{B} \to \mathcal{B}) = \lambda_{f:(\mathcal{B}\to\mathcal{B})\to(\mathcal{B}\to\mathcal{B})}.\lambda_{x:(\mathcal{B}\to\mathcal{B})}.f(f(x)) :$
    $((\mathcal{B} \to \mathcal{B}) \to (\mathcal{B} \to \mathcal{B})) \to ((\mathcal{B} \to \mathcal{B}) \to (\mathcal{B} \to \mathcal{B}))$

Le cube de Barendregt

- Syntax: $A ::= x \mid * \mid \Box \mid AB \mid \lambda x{:}A.B \mid \Pi x{:}A.B$
- Formation rule: $$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2} \quad \text{si} (s_1, s_2) \in \boldsymbol{R}$$

	Simple	Poly-morphic	Depend-ent	Constr-uctors	Related system	Refs.
$\lambda\!\rightarrow$	$(*,*)$				λ^{τ}	[Church, 1940; B
$\lambda 2$	$(*,*)$	$(\Box,*)$			F	[Girard, 1972; Re
λP	$(*,*)$		$(*,\Box)$		AUT-QE, LF	[Bruijn, 1968; Ha
$\lambda\underline{\omega}$	$(*,*)$			$(\Box,\Box)$	POLYREC	[Renardel de Lava
λP2	$(*,*)$	$(\Box,*)$	$(*,\Box)$			[Longo and Mogg
$\lambda\omega$	$(*,*)$	$(\Box,*)$		$(\Box,\Box)$	Fω	[Girard, 1972]
λP$\underline{\omega}$	$(*,*)$		$(*,\Box)$	$(\Box,\Box)$		
λC	$(*,*)$	$(\Box,*)$	$(*,\Box)$	$(\Box,\Box)$	CC	[Coquand and Hu

Les règles de typage

(axiom) $$\langle\rangle \vdash * : \Box$$

(start) $$\frac{\Gamma \vdash A : s \quad x \notin \mathrm{DOM}(\Gamma)}{\Gamma, x{:}A \vdash x : A}$$

(weak) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s \quad x \notin \mathrm{DOM}(\Gamma)}{\Gamma, x{:}C \vdash A : B}$$

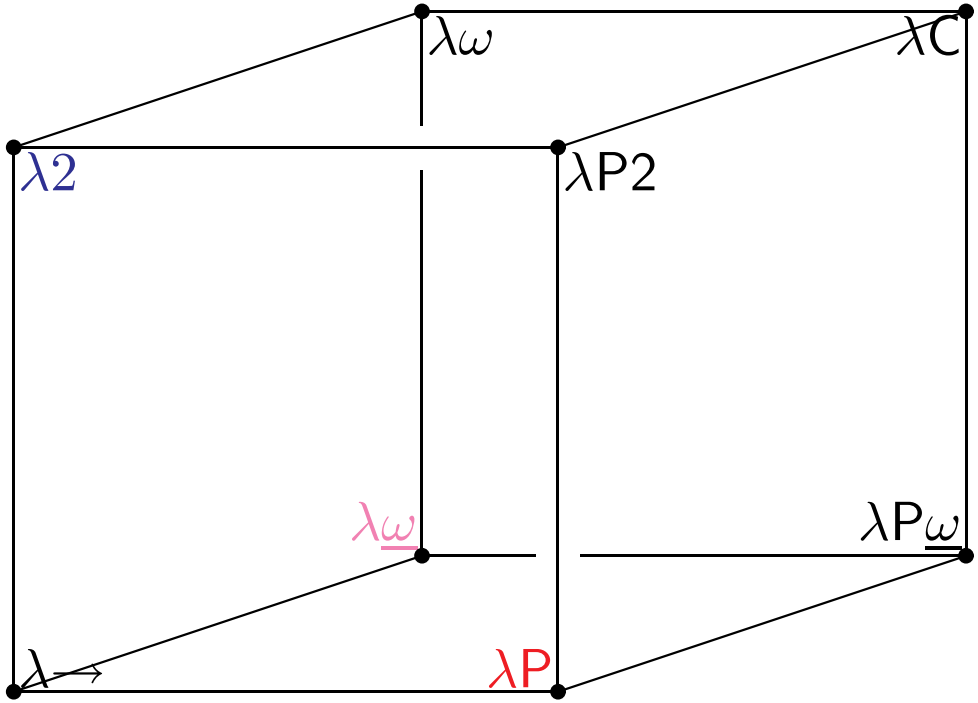
(Π) $$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2 \quad (s_1, s_2) \in \boldsymbol{R}}{\Gamma \vdash \Pi_{x:A}.B : s_2}$$

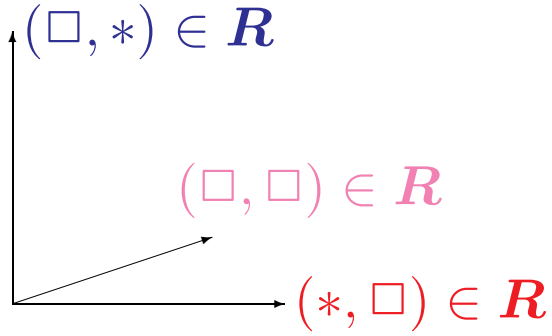
(λ) $$\frac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash \Pi_{x:A}.B : s}{\Gamma \vdash \lambda_{x:A}.b : \Pi_{x:A}.B}$$

(conv$_\beta$) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B =_\beta B'}{\Gamma \vdash A : B'}$$

(appl) $$\frac{\Gamma \vdash F : \Pi_{x:A}.B \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x{:=}a]}$$

Le cube de Barendregt





Notre example dans le système F de Girard

- Si $x \notin FV(B)$ on écrit $A \to B$ à la place de $\Pi_{x:A}.B$.
- $\alpha : *, f : \alpha \to \alpha \vdash \lambda_{x:\alpha}.f(f(x)) : \alpha \to \alpha : *$
  (besoin de la règle $(*, *)$).
- $\alpha : * \vdash \lambda_{f:\alpha\to\alpha}.\lambda_{x:\alpha}.f(f(x)) : (\alpha \to \alpha) \to (\alpha \to \alpha) : *$
  (besoin de la règle $(*, *)$).
- $\vdash \lambda_{\alpha:*}.\lambda_{f:\alpha\to\alpha}.\lambda_{x:\alpha}.f(f(x)) : \Pi_{\alpha:*}.(\alpha \to \alpha) \to (\alpha \to \alpha) : *$
  (besoin de la règle $(\Box, *)$).

Le cube de Barendregt en notation item avec réduction de classe

- Même formulation sauf que les termes sont écrits en notation item:
- $\mathcal{T} = * \mid \Box \mid V \mid (\mathcal{T}\,\delta)\mathcal{T} \mid (\mathcal{T}\,\lambda_V)\mathcal{T} \mid (\mathcal{T}\,\Pi_V)\mathcal{T}$.
- Les règles de typages ne changent même si on utilise $\rightsquigarrow_\beta$ en place de $\rightarrow_\beta$ (devinez pourquoi).

Le "Subject Reduction" n'ait plus valable

- La plupart des propriétés (inclusif SN) sont valable pour le cube étendu avec la réduction modulo les classes.
- *MAIS* SR n'est pas valable que dans $\lambda_{\rightarrow}$ $(*,*)$ et $\lambda\underline{\omega}$ $(\Box,\Box)$.
- SR ne marche plus dans λP $(*,\Box)$ (et alors dans $\lambda P2, \lambda P\underline{\omega}$ et λC).
- SR ne marche plus dans $\lambda 2$ $(\Box,*)$ (et alors dans $\lambda P2, \lambda\omega$ et λC):

Pourquoi on perd le SR?

- $(y'\delta)(\beta\delta)(*\lambda_\alpha)(\alpha\lambda_y)(y\delta)(\alpha\lambda_x)x \rightsquigarrow_\beta (\beta\delta)(*\lambda_\alpha)(y'\delta)(\alpha\lambda_x)x$.
- $(\lambda_{\alpha:*}.\lambda_{y:\alpha}.(\lambda_{x:\alpha}.x)y)\beta y' \rightsquigarrow_\beta (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y')\beta$
- $\beta : *, y' : \beta \vdash_{\lambda 2} (\lambda_{\alpha:*}.\lambda_{y:\alpha}.(\lambda_{x:\alpha}.x)y)\beta y' : \beta$
- Mais, $\beta : *, y' : \beta \not\vdash_{\lambda 2} (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y')\beta : \tau$ pour tout τ.
- On a perdu l'information que $y' : \beta$ a remplacé $y' : \alpha$ $(\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y')\beta$.
- On a besoin de $y' : \alpha$ pour typer le sousterm $(\lambda_{x:\alpha}.x)y'$ de $(\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y')\beta$ et alors pour typer $\beta : *, y' : \beta \vdash (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y')\beta : \beta$.

Solution de SR: Utilise "let expressions/définitions"

- Définitions/let expressions ont la forme: let $x : A = B$. On les ajoute aux contextes exactement comme les déclarations $y : C$.
- (def rule) $$\frac{\Gamma, \text{let } x : A = B \vdash^{\mathsf{c}} C : D}{\Gamma \vdash^{\mathsf{c}} (\lambda_{x:A}.C)B : D[x := A]}$$
- On définit $\Gamma \vdash^{\mathsf{c}} \cdot =_{\mathtt{def}} \cdot$ comme la relation d' equivalence generée par
  - si $A =_\beta B$ alors $\Gamma \vdash^{\mathsf{c}} A =_{\mathtt{def}} B$
  - si let $x : M = N$ est dans Γ et si B vient de A en substituant une occurrence de x dans A par N, alors $\Gamma \vdash^{\mathsf{c}} A =_{\mathtt{def}} B$.

Le Cube (simplifiée) avec définitions et réduction de classes [Kamareddine and Bloo, 2002]

(axiom) (app) (abs) et (form) ne changent pas.

(start) $$\frac{\Gamma \vdash^{c} A : s}{\Gamma, x{:}A \vdash^{c} x : A} \qquad \frac{\Gamma \vdash^{c} A : s \quad \Gamma \vdash^{c} B : A}{\Gamma,\ \mathsf{let}\ x : A\ = B \vdash^{c} x : A}$$ x fraic

(weak) $$\frac{\Gamma \vdash^{c} D : E \quad \Gamma \vdash^{c} A : s}{\Gamma, x{:}A \vdash^{c} D : E} \qquad \frac{\Gamma \vdash^{c} A : s \quad \Gamma \vdash^{c} B : A \quad \Gamma \vdash^{c} D : E}{\Gamma,\ \mathsf{let}\ x : A\ = B \vdash^{c} D : E}$$ x fraic

(conv) $$\frac{\Gamma \vdash^{c} A : B \qquad \Gamma \vdash^{c} B' : S \qquad \Gamma \vdash^{c} B =_{\mathtt{def}} B'}{\Gamma \vdash^{c} A : B'}$$

(def) $$\frac{\Gamma, \mathsf{let}\ x : A\ = B \vdash^{c} C : D}{\Gamma \vdash^{c} (\lambda_{x:A}.C)B : D[x := A]}$$

l'usage des définitions résoud le problème de subject reduction

1. $\beta : *, y' : \beta,$ let $\alpha : * = \beta$ $\quad \vdash^{\mathsf{c}} y' : \beta$
2. $\beta : *, y' : \beta,$ let $\alpha : * = \beta$ $\quad \vdash^{\mathsf{c}} \alpha =_{\mathtt{def}} \beta$
3. $\beta : *, y' : \beta,$ let $\alpha : * = \beta$ $\quad \vdash^{\mathsf{c}} y' : \alpha$ (de 1 and 2)
4. $\beta : *, y' : \beta,$ let $\alpha : * = \beta,$ let $x : \alpha = y'$ $\quad \vdash^{\mathsf{c}} x : \alpha$
5. $\beta : *, y' : \beta,$ let $\alpha : * = \beta$ $\quad \vdash^{\mathsf{c}} (\lambda_{x:\alpha}.x)y' : \alpha[x := y'] = \alpha$

$$\beta : *, y' : \beta \quad \vdash^{\mathsf{c}} \quad (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y')\beta : \alpha[\alpha := \beta] = \beta$$

Les propriétés du Cube avec définitions et réduction de classes

- $\vdash^{\mathsf{c}}$ est une generalisation of $\vdash$: Si $\Gamma \vdash A : B$ alors $\Gamma \vdash^{\mathsf{c}} A : B$.
- Equivalent terms have same types:
  Si $\Gamma \vdash^{\mathsf{c}} A : B$ et $A' \in [A]$, $B' \in [B]$ alors $\Gamma \vdash^{\mathsf{c}} A' : B'$.
- Subject Reduction for $\vdash^{\mathsf{c}}$ and $\twoheadrightarrow_\beta$:
  Si $\Gamma \vdash^{\mathsf{c}} A : B$ et $A \twoheadrightarrow_\beta A'$ alors $\Gamma \vdash^{\mathsf{c}} A' : B$.
- Unicity of Types for $\vdash^{\mathsf{c}}$:
  - Si $\Gamma \vdash^{\mathsf{c}} A : B$ et $\Gamma \vdash^{\mathsf{c}} A : B'$ alors $\Gamma \vdash^{\mathsf{c}} B =_{\mathtt{def}} B'$
  - Si $\Gamma \vdash^{\mathsf{c}} A : B$ et $\Gamma \vdash^{\mathsf{c}} A' : B'$ et $\Gamma \vdash^{\mathsf{c}} A =_\beta A'$ alors $\Gamma \vdash^{\mathsf{c}} B =_{\mathtt{def}} B'$.
- Strong Normalisation of $\twoheadrightarrow_\beta$:
  Chaque terme légal est fortement normalisable par rapport à $\twoheadrightarrow_\beta$.

In progress: connection of the cube with definitions and class reduction with a logical cube modulo

- The literature already has connections of the Barendregt cube with a logical cube.
- The PAT principle is also very well explained for the systems of these connections.
- Can we construct a logical cube modulo (based on the deduction modulo relation).
- Can we establish a generalisation of PAT which connects deduction modulo and reduction modulo?

Nuprl as a PTS

- $\mathbb{S} = \{*_1, *_2, \ldots\}$
- $$\mathbb{T} ::= \mathbb{S} \mid \mathbb{V} \mid \bot \mid \mathbb{Z} \mid \mathbb{TT} \mid \lambda \mathbb{V}{:}\mathbb{T}.\mathbb{T} \mid \Pi \mathbb{V}{:}\mathbb{T}.\mathbb{T} \mid \Sigma \mathbb{V}{:}\mathbb{T}.\mathbb{T} \mid \langle \mathbb{T}, \mathbb{T} \rangle \mid \pi_1(\mathbb{T}) \mid \pi_2(\mathbb{T})$$
- For each term A we define a term $|A|$ (where we collapse orders) as follows:

$$\begin{array}{ll} |*_m| = *_1 & |\Pi x{:}A.B| = \Pi x{:}|A|.|B| \\ |x| = x & |\Sigma x{:}A.B| = \Sigma x{:}|A|.|B| \\ |\bot| = \bot & |\langle A, B \rangle| = \langle |A|, |B| \rangle \\ |\mathbb{Z}| = \mathbb{Z} & |\pi_1(M)| = \pi_1(|M|) \\ |MN| = |M||N| & |\pi_2(M)| = \pi_2(|M|) \\ |\lambda x{:}A.b| = \lambda x{:}|A|.|b| & \end{array}$$

- We take the axioms:

$$(\rightarrow_\beta) : (\lambda x{:}T.A)B \rightarrow_\beta A[x{:=}B]$$

$$(\rightarrow_\sigma) : \pi_1(\langle A, B \rangle) \rightarrow_\sigma A \text{ and } \pi_2(\langle A, B \rangle) \rightarrow_\sigma B.$$

(Axioms) $\quad \vdash \bot : *_1 \qquad \vdash *_n : *_{n+1} \quad (n \in \mathbb{N})$
$\vdash \mathbb{z} : *_1 \qquad \vdash n : \mathbb{z} \quad (n \in \mathbb{z})$

(Start) $$\frac{\Gamma \vdash A : *_n}{\Gamma, x{:}A \vdash x{:}A} \qquad (x \text{ is } \Gamma\text{-fresh})$$

(Weak) $$\frac{\Gamma \vdash M : N \quad \Gamma \vdash A : *_n}{\Gamma, x{:}A \vdash M : N} \qquad (x \text{ is } \Gamma\text{-fresh})$$

(Π-form) $$\frac{\Gamma \vdash A : *_n \quad \Gamma, x{:}A \vdash B : *_n}{\Gamma \vdash (\Pi x{:}A.B) : *_n}$$

(λ) $$\frac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash (\Pi x{:}A.B) : *_n}{\Gamma \vdash (\lambda x{:}A.b) : (\Pi x{:}A.B)}$$

(App) $$\frac{\Gamma \vdash M : (\Pi x{:}A.B) \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x{:=}N]}$$

(Σ) $$\frac{\Gamma \vdash A : *_n \quad \Gamma, x{:}A \vdash B : *_n}{\Gamma \vdash (\Sigma x{:}A.B) : *_n}$$

(Pairs) $$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[x := a] \quad \Gamma \vdash \Sigma x{:}A.B : *_m}{\Gamma \vdash \langle a, b \rangle : \Sigma x{:}A.B}$$

(Left) $$\frac{\Gamma \vdash M : \Sigma x{:}A.B}{\Gamma \vdash \pi_1(M) : A}$$

(Right) $$\frac{\Gamma \vdash M : \Sigma x{:}A.B}{\Gamma \vdash \pi_2(M) : B[x := \pi_1(M)]}$$

$$\text{(Conv)}\quad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : *_n \qquad A =_{\beta\sigma} B}{\Gamma \vdash M : B}$$

$$(\subseteq)\qquad \frac{\Gamma \vdash A : *_n}{\Gamma \vdash A : *_{n+1}}$$

For the Nuprl formulation without Σ we have:

1. **(Church-Rosser Theorem for $\rightarrow_\beta$, $\rightarrow_\sigma$ and $\rightarrow_{\beta\sigma}$)** Let $r \in \{\beta, \sigma, \beta\sigma\}$. If $A \twoheadrightarrow_r B_1$ and $A \twoheadrightarrow_r B_2$ then there is a C such that $B_1 \twoheadrightarrow_r C$ and $B_2 \twoheadrightarrow_r C$.

2. **(Correctness of Types)** If $\Gamma \vdash A : B$ then there is an $n \geq 1$ such that $\Gamma \vdash B : *_n$.

3. **(Subject Reduction)** If $\Gamma \vdash A : B$ and $A \rightarrow_{\beta\sigma} A'$ then $\Gamma \vdash A' : B$.

4. **($\twoheadrightarrow_{\beta\sigma}$ Preserves Γ-terms)** If A is a Γ-term and $A \twoheadrightarrow_{\beta\sigma} A'$ then A' is a Γ-term.

5. **(Weak Unicity of Types)** If $\Gamma \vdash A : B_1$ and $\Gamma \vdash A : B_2$ then $|B_1| =_{\beta\sigma} |B_2|$.

6. If A is a Γ-term then there is a Γ-term B and an $n \geq 1$ such that $\Gamma \vdash A : B : *_n$.

Orders in Nuprl (Kamareddine and Laan 2001)

- The order of a term A is the smallest natural number n such that the type of A is of type $*_{n+1}$.
- By $(\subseteq)$, this means that for any $m > n$, the type of A is also of type $*_m$.
- This captures the notion of orders à la Russell.
- Here is the formal definition of orders.
  - **(Γ-terms modulo A)** We define $[A]_\Gamma = \{A' \mid A' \text{ is } \Gamma\text{-term and } A =_{\beta\sigma} A'\}$.
  - **(Order of a Term)** Assume A is a Γ-term. We define $\mathrm{ord}_\Gamma(A)$, the *order* of A in Γ, as the smallest natural number a (i.e. $a \geq 0$) for which there are $A' \in [A]_\Gamma$ and B such that $\Gamma \vdash A' : B : *_{a+1}$.

Lemma

Let A be a Γ-term and $\mathrm{ord}_\Gamma(A) = a$. The following holds:

1. If $A' \in [A]_\Gamma$ then $\mathrm{ord}_\Gamma(A) = \mathrm{ord}_\Gamma(A')$.
2. There are A' and B such that $\Gamma \vdash A' : B : *_{a+1}$ and $A \twoheadrightarrow_{\beta\sigma} A'$.
3. **(A type B reduces to a type B' of type $*_{\mathrm{ord}(B)}$)** Let B be a Γ-type and $b = \mathrm{ord}_\Gamma(B)$. $\exists B'$ such that $\Gamma \vdash B' : *_b$ and $B \twoheadrightarrow_{\beta\sigma} B'$.
4. **($*_a$ is the type of types of order $\leq a$)** If P is a Γ-type in $\beta\sigma$-normal form, then $\Gamma \vdash P : *_a \Leftrightarrow \mathrm{ord}_\Gamma(P) \leq a$.
5. **(A term is of a lower order than its type)** If $\Gamma \vdash A : B$ then $\mathrm{ord}_\Gamma(A) < \mathrm{ord}_\Gamma(B)$.

1. **(Order of Projections)** For a Γ-term $\langle A, B\rangle$, $\mathrm{ord}_\Gamma(\pi_1(\langle A, B\rangle)) = \mathrm{ord}_\Gamma(A)$ and $\mathrm{ord}_\Gamma(\pi_2(\langle A, B\rangle)) = \mathrm{ord}_\Gamma(B)$.

2. **(Orders of Constants and Sorts)** Let Γ be a legal context. Then $\mathrm{ord}_\Gamma(*_a) = a + 1$, $\mathrm{ord}_\Gamma(\bot) = 1$, $\mathrm{ord}_\Gamma(\mathrm{Z}) = 1$, and $\mathrm{ord}_\Gamma(n) = 0$.

3. Let C be a Γ-term. The following holds:
   (a) If $C \equiv x$ where $x{:}A \in \Gamma$ then $\mathrm{ord}_\Gamma(x) = \mathrm{ord}_\Gamma(A) - 1$.
   (b) If $C \equiv \Pi x{:}A.B$ then $\mathrm{ord}_\Gamma(\Pi x{:}A.B) = \max(\mathrm{ord}_\Gamma(A), \mathrm{ord}_{\Gamma,x:A}(B))$.
   (c) If $C \equiv \lambda x{:}A.b$ then $\mathrm{ord}_\Gamma(\lambda x{:}A.b) = \max(\mathrm{ord}_\Gamma(A) - 1, \mathrm{ord}_{\Gamma,x:A}(b))$.
   (d) If $C \equiv A \times B$ or $C \equiv \langle A, B\rangle$ then $\mathrm{ord}_\Gamma(C) = \max(\mathrm{ord}_\Gamma(A), \mathrm{ord}_\Gamma(B))$.

4. **(Order of an Application)** If $\Gamma \vdash M : \Pi x{:}P.Q$ and $\Gamma \vdash N : P$ then $\mathrm{ord}_\Gamma(N), \mathrm{ord}_\Gamma(MN) \leq \mathrm{ord}_\Gamma(M)$.

Extending Nuprl-PTS with definitions

(start-def) $$\frac{\Gamma \vdash_e A : s \quad \Gamma \vdash_e B : A}{\Gamma, x = B{:}A \vdash_e x : A} \qquad x \notin \mathrm{DOM}(\Gamma)$$

(weak-def) $$\frac{\Gamma \vdash_e A : B \quad \Gamma \vdash_e C : s \quad \Gamma \vdash_e D : C}{\Gamma, x = D{:}C \vdash_e A : B} \qquad x \notin \mathrm{DOM}(\Gamma)$$

(def) $$\frac{\Gamma, x = B{:}A \vdash_e C : D}{\Gamma \vdash_e (\pi x : A.C)B : D[x := B]} \qquad \text{for } \pi \in \{\lambda, \Pi\}$$

(new-conv) $$\frac{\Gamma \vdash_e A : B \quad \Gamma \vdash_e B' : s \quad \Gamma \vdash_e B =_{def} B'}{\Gamma \vdash_e A : B'}$$

(new-appl) $$\frac{\Gamma \vdash_e F : (\Pi x{:}A.B) \quad \Gamma \vdash_e a : A}{\Gamma \vdash_e Fa : (\Pi x{:}A.B)a}$$

Definitions make type derivations (hence proofs) shorter

without (def) we have the following type derivation:

0.		$\vdash$	$* : \Box$	axiom
1.	$\alpha : *$	$\vdash$	$\alpha : *$	0, (start)
2.	$\alpha : *, x : \alpha$	$\vdash$	$\alpha : *$	1, 1, (weak)
3.	$\alpha : *$	$\vdash$	$\Pi x : \alpha.\alpha : *$	1, 2, (Π), (*,*)
4.	$\alpha : *, x : \alpha$	$\vdash$	$x : \alpha$ 1, (start)	
5.	$\alpha : *$	$\vdash$	$\lambda x : \alpha.x : \Pi x : \alpha.\alpha$	4, 3, (λ)
6.	$\alpha : *, y : \alpha$	$\vdash$	$\lambda x : \alpha.x : \Pi x : \alpha.\alpha$	5, 1, (weak)
7.	$\alpha : *, y : \alpha$	$\vdash$	$y : \alpha$	1, (start)
8.	$\alpha : *, y : \alpha$	$\vdash$	$(\lambda x : \alpha.x)y : (\Pi x : \alpha.\alpha)y$	6, 7, (app)
9.	$\alpha : *, y : \alpha$	$\vdash$	$(\Pi x : \alpha.\alpha)y =_{def} \alpha$	
10.	$\alpha : *, y : \alpha$	$\vdash$	$(\lambda x : \alpha.x)y : \alpha$	

Using the (def) rule this type derivation can be shortened as follows:

0.		$\vdash$	$* : \Box$	axiom
1.	$\alpha : *$	$\vdash$	$\alpha : *$	0, (start)
2.	$\alpha : *, y : \alpha$	$\vdash$	$\alpha : *$	1, 1, (weak)
3.	$\alpha : *, y : \alpha$	$\vdash$	$y : \alpha$	1, (start)
4.	$\alpha : *, y : \alpha, x = y : \alpha$	$\vdash$	$x : \alpha$	2, 3, (start-def)
5.	$\alpha : *, y : \alpha$	$\vdash$	$(\lambda x : \alpha . x) y : \alpha[x := y] \equiv \alpha$	4, (def)

Le language de Mathématique

D'habitude, le mathématicien ignore la logique formelle. Les mathématiciens écrivent la mathématique avec un language (style) commun qu'on appelle le CML. Les avantages de CML:

- *Expressivité:* On peut exprimer toutes genres de notions.
- *Acceptabilité:* CML est accepté par la plupart des mathématicien.
- *traditionalité:* CML existe depuis très longtemps et a été raffiné avec le temps.
- *Universalité:* CML est utilisé partout dans le monde.
- *Flexibilité:* Avec CML on peut décrire plusieures branches de mathématiques.

Les désavantages de CML:

- *Informel et ambigu:* CML est basé sur le language naturelle.
- *Incomplet:* De choses implicites, l'écrivain compte sur l'intuition du lecteur.
- *Pas facile à automatiser* CML
- Au 19ème ciècle, les problèmes en Analyse créaient le besoin d'un style *précis*.
- Plusieures de ces problèmes ont été resolu par le travail de Cauchy (par example par sa définition précise de convergence dans son Cours d'Analyse).
- Les systèmes des nombres sont devenus plus précis avec la définition exacte des nombres réel de Dedekind.
- Cantor commençait la formalisation de la théorie des ensembles et contribuait à la théorie des nombres.

A CML-text

From chapter 1, § 2 of E. Landau's *Foundations of Analysis* (Landau 1930, 1951).

Theorem 6. [Commutative Law of Addition]

$$x + y = y + x.$$

Proof Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.
I) We have

$$y + 1 = y',$$

and furthermore, by the construction in the proof of Theorem 4,

$$1 + y = y',$$

so that

$$1 + y = y + 1$$

and 1 belongs to $\mathfrak{M}$.
II) If x belongs to $\mathfrak{M}$, then

$$x + y = y + x,$$

Therefore

$$(x + y)' = (y + x)' = y + x'.$$

By the construction in the proof of Theorem 4, we have

$$x' + y = (x + y)',$$

hence

$$x' + y = y + x',$$

so that x' belongs to $\mathfrak{M}$. The assertion therefore holds for all x. □

The problem with formal logic

- No logical language is an alternative to CML
  - A logical language does not have *mathematico-linguistic* categories, is *not universal* to all mathematicians, and is *not a good communication medium*.
  - Logical languages make fixed choices (*first versus higher order, predicative versus impredicative, constructive versus classical, types or sets*, etc.). But different parts of mathematics need different choices and there is no universal agreement as to which is the best formalism.
  - A logician reformulates in logic their *formalization* of a mathematical-text as a formal, complete text which is structured considerably *unlike* the original, and is of little use to the *ordinary* mathematician.
  - Mathematicians do not want to use formal logic and have *for centuries* done mathematics without it.

- *So, mathematicians kept to* CML*.*

- We would like to find an alternative to CML which avoids some of the features of the logical languages which made them unattractive to mathematicians.

What are the options for computerization?

Computers can handle mathematical text at various levels:

- Images of pages may be stored. While useful, this is not a good representation of *language* or *knowledge*.
- Typesetting systems like LaTeX, TeXmacs, can be used.
- Document representations like OpenMath, OMDoc, MathML, can be used.
- Formal logics used by theorem provers (Coq, Isabelle, Mizar, Isar, etc.) can be used.

We are gradually developing a system named Mathlang which we hope will eventually allow building a bridge between the latter 3 levels.

This talk aims at discussing the motivations rather than the details.

The issues with typesetting systems

+ A system like LaTeX, TeXmacs, provides good defaults for visual appearance, while allowing fine control when needed.

+ LaTeX and TeXmacs support commonly needed document structures, while allowing custom structures to be created.

– Unless the mathematician is amazingly disciplined, the *logical structure of symbolic formulas is not represented* at all.

– The *logical structure of mathematics as embedded in natural language text is not represented.* Automated discovery of the semantics of natural language text is still too primitive and requires human oversight.

LaTeX example

draft documents	✓
public documents	✓
computations and proofs	✗

```
\begin{theorem}[Commutative Law of Addition]\label{theorem:6}

  $$x+y=y+x.$$

\end {theorem}

\begin{proof}

  Fix $y$, and $\mathfrak{M}$ be the set of all $x$ for which
  the assertion holds.

  \begin{enumerate}

  \item We have $$y+1=y',$$
    and furthermore, by the construction in
    the proof of Theorem~\ref{theorem:4}, $$1+y=y',$$

    so that $$1+y=y+1$$
    and $1$ belongs to $\mathfrak{M}$.
```

```
  \item If $x$ belongs to $\mathfrak{M}$, then $$x+y=y+x,$$

    Therefore
    $$(x+y)'=(y+x)'=y+x'.$$

    By the construction in the proof of
    Theorem~\ref{theorem:4}, we have $$x'+y=(x+y)',$$

    hence
    $$x'+y=y+x',$$
    so that $x'$ belongs to $\mathfrak{M}$.

  \end{enumerate}

  The assertion therefore holds for all $x$.

\end{proof}
```

The beginnings of computerized formalization

- In 1967 the famous mathematician de Bruijn began work on logical languages for complete books of mathematics that can be *fully* checked by machine.
- People are prone to error, so if a machine can do proof checking, we expect fewer errors.
- Most mathematicians doubted de Bruijn could achieve success, and computer scientists had no interest at all.
- However, he persevered and built *Automath* (AUTOmated MATHematics).
- Today, there is much interest in many approaches to proof checking for verification of computer hardware and software.
- Many theorem provers have been built to mechanically check mathematics and computer science reasoning (e.g. Isabelle, HOL, Coq, Focal, etc.).

Full formalization difficulties: choices

A CML-text is structured differently from a fully formalized text proving the same facts. *Making the latter involves extensive knowledge and many choices:*

- The choice of the *underlying logical system.*
- The choice of *how concepts are implemented* (equational reasoning, equivalences and classes, partial functions, induction, etc.).
- The choice of the *formal foundation*: a type theory (dependent?), a set theory (ZF? FM?), a category theory? etc.
- The choice of the *proof checker*: Automath, Isabelle, Coq, PVS, Mizar, ...

An issue is that one must in general commit to one set of choices.

Full formalization difficulties: informality

Any informal reasoning in a CML-text will cause various problems when fully formalizing it:

- A single (big) step may need to expand into a (series of) syntactic proof expressions. *Very long expressions can replace a clear* CML*-text.*
- The entire CML-text may need *reformulation* in a fully *complete* syntactic formalism where every detail is spelled out. New details may need to be woven throughout the entire text. The text may need to be *turned inside out.*
- Reasoning may be obscured by *proof tactics*, whose meaning is often *ad hoc* and implementation-dependent.

Regardless, ordinary mathematicians do not find the new text useful.

Coq example

draft documents	✗
public documents	✗
computations and proofs	✓

From Module `Arith.Plus` of Coq standard library (`http://coq.inria.fr/`).

```
Lemma plus_sym: (n,m:nat)(n+m)=(m+n).

Proof.

Intros n m ; Elim n ; Simpl_rew ; Auto with arith.

Intros y H ; Elim (plus_n_-Sm m y) ; Simpl_rew ; Auto with arith.

Qed.
```

Mathlang's Goal: Open borders between mathematics, logic and computation

- Ordinary mathematicians *avoid* formal mathematical logic.
- Ordinary mathematicians *avoid* proof checking (via a computer).
- Ordinary mathematicians *may use* a computer for computation: there are over 1 million people who use Mathematica (including linguists, engineers, etc.).
- Mathematicians may also use other computer forms like Maple, LaTeX, etc.
- But we are not interested in only *libraries* or *computation* or *text editing*.
- We want *freedeom of movement* between mathematics, logic and computation.
- At every stage, we must have *the choice* of the level of formalilty and the depth of computation.

Aim for Mathlang? (Kamareddine and Wells 2001, 2002)

Can we formalise a mathematical text, avoiding as much as possible the ambiguities of natural language, while still guaranteeing the following four goals?

1. The formalised text looks very much like the original mathematical text (and hence the content of the original mathematical text is respected).

2. The formalised text can be fully manipulated and searched in ways that respect its mathematical structure and meaning.

3. Steps can be made to do computation (via computer algebra systems) and proof checking (via proof checkers) on the formalised text.

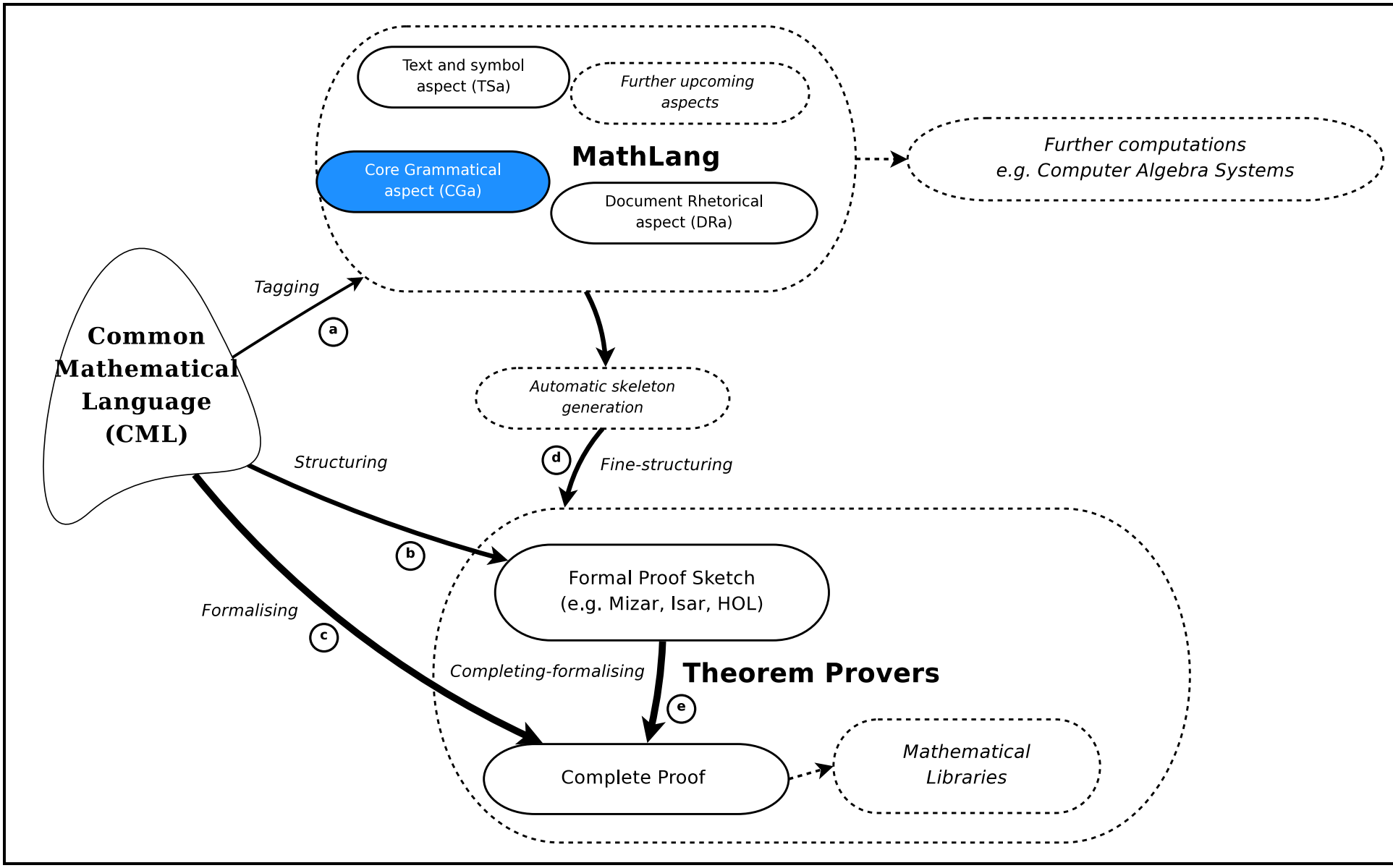
4. This formalisation of text is not much harder for the ordinary mathematician than LaTeX. *Full formalization down to a foundation of mathematics is not required*, although allowing and supporting this is one goal.

(No theorem prover's language satisfies these goals.)

Mathlang

draft documents	✓
public documents	✓
computations and proofs	✓

- A Mathlang text captures the grammatical and reasoning aspects of mathematical structure for further computer manipulation.
- A *weak type system* checks Mathlang documents at a grammatical level.
- A Mathlang text remains *close* to its CML original, allowing confidence that the CML has been captured correctly.
- We have been developing ways to weave natural language text into Mathlang.
- Mathlang aims to eventually support *all encoding uses*.
- The CML view of a Mathlang text should match the mathematician's intentions.
- The formal structure should be suitable for various automated uses.


Text and symbol aspect (TSa)
Further upcoming aspects
MathLang
Core Grammatical aspect (CGa)
Document Rhetorical aspect (DRa)
Further computations e.g. Computer Algebra Systems
Tagging
a
Common Mathematical Language (CML)
Automatic skeleton generation
d
Fine-structuring
Structuring
b
Formal Proof Sketch (e.g. Mizar, Isar, HOL)
Formalising
c
Completing-formalising
Theorem Provers
e
Complete Proof
Mathematical Libraries

What is CGa? (Maarek's PhD thesis)

- CGa is a formal language derived from MV (N.G. de Bruijn 1987) and WTT (Kamareddine and Nederpelt 2004) which aims at expliciting the grammatical role played by the elements of a CML text.
- The structures and common concepts used in CML are captured by CGa with a finite set of grammatical/linguistic/syntactic categories: *Term* "$\sqrt{2}$", *set* "$\mathbb{Q}$", *noun* "number", *adjective* "even", *statement* "$a = b$", *declaration* "Let a be a number", *definition* "An even number is..", *step* "a is odd, hence $a \neq 0$", *context* "Assume a is even".
- Generally, each syntactic category has a corresponding *weak type*.
- CGa's type system derives typing judgments to check whether the reasoning parts of a document are coherently built.

Examples of linguistic categories

- Terms: the triangle ABC; the center of ABC; $d($x, y$)$.
- Nouns: a triangle; an edge of ABC; a group.
- Adjectives: equilateral triangle; prime number; Abelian group.
- Statements: P lies between Q and R; 5$>$3; AB is an edge of ABC.
- Definition: a number p is prime whenever ....

Weak Type Theory

In Weak Type Theory (or WTT) we have the following linguistic categories:

- On the *atomic* level: *variables*, *constants* and *binders*,
- On the *phrase* level: *terms* $\mathcal{T}$, *sets* $\mathbb{S}$, *nouns* $\mathcal{N}$ and *adjectives* $\mathcal{A}$,
- On the *sentence* level: *statements* P and *definitions* $\mathcal{D}$,
- On the *discourse* level: *contexts* $\mathbb{\Gamma}$, *lines* $\mathbf{l}$ and *books* $\mathbf{B}$.

Categories of syntax of WTT

Other category	abstract syntax	symbol
expressions	$\mathcal{E} = T \vert \mathbb{S} \vert \mathcal{N} \vert P$	E
parameters	$\mathcal{P} = T \vert \mathbb{S} \vert P$ (note: $\vec{\mathcal{P}}$ is a list of $\mathcal{P}$s)	P
typings	$\mathbf{T} = \mathbb{S} :$ SET $\vert \mathcal{S} :$ STAT $\vert T : \mathbb{S} \vert T : \mathcal{N} \vert T : \mathcal{A}$	T
declarations	$\mathcal{Z} = \mathtt{V}^S :$ SET $\vert \mathtt{V}^P :$ STAT $\vert \mathtt{V}^T : \mathbb{S} \vert \mathtt{V}^T : \mathcal{N}$	Z

level	category	abstract syntax	symbol
atomic	*variables*	$\mathtt{V} = \mathtt{V}^T \vert \mathtt{V}^S \vert \mathtt{V}^P$	x
	constants	$\mathtt{C} = \mathtt{C}^T \vert \mathtt{C}^S \vert \mathtt{C}^N \vert \mathtt{C}^A \vert \mathtt{C}^P$	c
	binders	$\mathtt{B} = \mathtt{B}^T \vert \mathtt{B}^S \vert \mathtt{B}^N \vert \mathtt{B}^A \vert \mathtt{B}^P$	b
phrase	*terms*	$T = \mathtt{C}^T(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^T_{\mathcal{Z}}(\mathcal{E}) \vert \mathtt{V}^T$	t
	sets	$\mathbb{S} = \mathtt{C}^S(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^S_{\mathcal{Z}}(\mathcal{E}) \vert \mathtt{V}^S$	s
	nouns	$\mathcal{N} = \mathtt{C}^N(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^N_{\mathcal{Z}}(\mathcal{E}) \vert \mathcal{AN}$	n
	adjectives	$\mathcal{A} = \mathtt{C}^A(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^A_{\mathcal{Z}}(\mathcal{E})$	a
sentence	*statements*	$P = \mathtt{C}^P(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^P_{\mathcal{Z}}(\mathcal{E}) \vert \mathtt{V}^P$	S
	definitions	$\mathcal{D} = \mathcal{D}^\varphi \vert \mathcal{D}^P$	D
		$\mathcal{D}^\varphi = \mathtt{C}^T(\overrightarrow{V}) := T \vert \mathtt{C}^S(\overrightarrow{V}) := \mathbb{S} \vert$ $\mathtt{C}^N(\overrightarrow{V}) := \mathcal{N} \vert \mathtt{C}^A(\overrightarrow{V}) := \mathcal{A}$	
		$\mathcal{D}^P = \mathtt{C}^P(\overrightarrow{V}) := P$	
discourse	*contexts*	$\mathbb{\Gamma} = \emptyset \mid \mathbb{\Gamma}, \mathcal{Z} \mid \mathbb{\Gamma}, P$	Γ
	lines	$\mathbf{l} = \mathbb{\Gamma} \triangleright P \mid \mathbb{\Gamma} \triangleright \mathcal{D}$	l
	books	$\mathbf{B} = \emptyset \mid \mathbf{B} \circ \mathbf{l}$	B

Derivation rules

(1) B is a weakly well-typed book: $\vdash\ B :: \mathbf{B}.$

(2) Γ is a weakly well-typed context relative to book B: $B\ \vdash\ \Gamma :: \mathbb{\Gamma}.$

(3) t is a weakly well-typed term, etc., relative to book B and context Γ:

$$\begin{array}{lll} B;\Gamma\ \vdash\ t :: T, & B;\Gamma\ \vdash\ s :: S, & B;\Gamma\ \vdash\ n :: N, \\ B;\Gamma\ \vdash\ a :: A, & B;\Gamma\ \vdash\ p :: P, & B;\Gamma\ \vdash\ d :: D \end{array}$$

$OK(B;\Gamma).$ stands for: $\vdash\ B :: \mathbf{B}$, *and* $B\ \vdash\ \Gamma :: \mathbb{\Gamma}$

Examples of derivation rules

- $\texttt{dvar}(\emptyset) = \emptyset \qquad \texttt{dvar}(\Gamma', x : W) = \texttt{dvar}(\Gamma'), x \qquad \texttt{dvar}(\Gamma', P) = \texttt{dvar}(\Gamma')$

$$\frac{OK(B;\Gamma), \quad x \in \mathtt{V}^{\mathtt{T/S/P}}, \quad x \in \texttt{dvar}(\Gamma)}{B;\Gamma \ \vdash \ x :: T/S/P} \ (var)$$

$$\frac{B;\Gamma \ \vdash \ n :: N, \quad B;\Gamma \ \vdash \ a :: A}{B;\Gamma \ \vdash \ an :: N} \ (adj{-}noun)$$

$$\frac{}{\vdash \ \emptyset :: \mathbf{B}} \ (emp{-}book)$$

$$\frac{B;\Gamma \ \vdash \ p :: P}{\vdash \ B \circ \Gamma \triangleright p :: \mathbf{B}} \qquad \frac{B;\Gamma \ \vdash \ d :: D}{\vdash \ B \circ \Gamma \triangleright d :: \mathbf{B}} \ (book{-}ext)$$

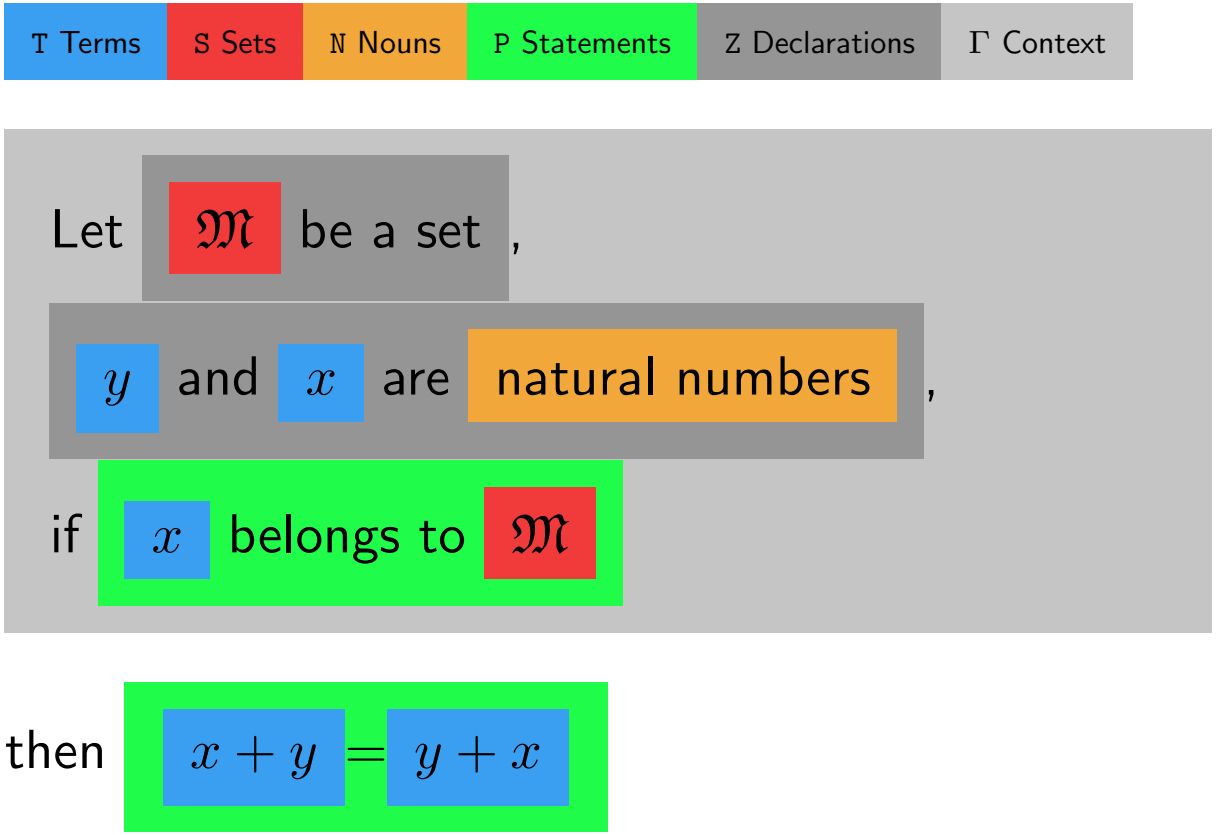
Properties of WTT

- *Every variable is declared* If $B;\Gamma \ \vdash \ \Phi :: \mathbf{W}$ then $FV(\Phi) \subseteq \mathtt{dvar}(\Gamma)$.
- *Correct subcontexts* If $B \ \vdash \ \Gamma :: \mathbb{T}$ and $\Gamma' \subseteq \Gamma$ then $B \ \vdash \ \Gamma' :: \mathbb{T}$.
- *Correct subbooks* If $\ \vdash \ B :: \mathbf{B}$ and $B' \subseteq B$ then $\ \vdash \ B' :: \mathbf{B}$.
- *Free constants are either declared in book or in contexts* If $B;\Gamma \ \vdash \ \Phi :: \mathbf{W}$, then $FC(\Phi) \subseteq \mathtt{prefcons}(B) \cup \mathtt{defcons}(B)$.
- *Types are unique* If $B;\Gamma \ \vdash \ A :: \mathbf{W_1}$ and $B;\Gamma \ \vdash \ A :: \mathbf{W_2}$, then $\mathbf{W_1} \equiv \mathbf{W_2}$.
- *Weak type checking is decidable* there is a decision procedure for the question $B;\Gamma \ \vdash \ \Phi :: \mathbf{W} \ ?$.
- *Weak typability is computable* there is a procedure deciding whether an answer exists for $B;\Gamma \ \vdash \ \Phi :: ?$ and if so, delivering the answer.

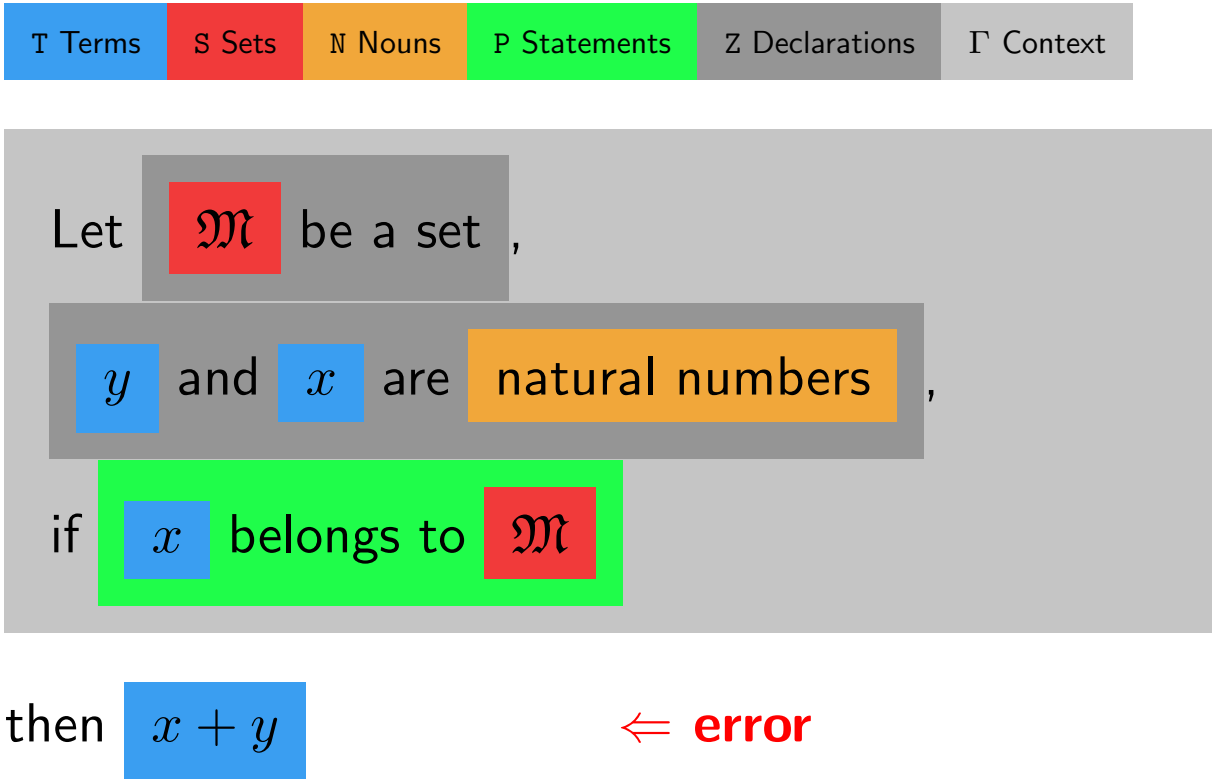
Definition unfolding

- Let $\vdash B :: \mathbf{B}$ and $\Gamma \triangleright c(x_1, \ldots, x_n) := \Phi$ a line in B.
- We write $B \vdash c(P_1, \ldots, P_n) \xrightarrow{\delta} \Phi[x_i := P_i]$.
- *Church-Rosser* If $B \vdash \Phi \xrightarrow{\delta}\!\!\!\!\rightarrow \Phi_1$ and $B \vdash \Phi \xrightarrow{\delta}\!\!\!\!\rightarrow \Phi_2$ then there exists Φ_3 such that $B \vdash \Phi_1 \xrightarrow{\delta}\!\!\!\!\rightarrow \Phi_3$ andf $B \vdash \Phi_2 \xrightarrow{\delta}\!\!\!\!\rightarrow \Phi_3$.
- *Strong Normalisation* Let $\vdash B :: \mathbf{B}$. For all subformulas Ψ occurring in B, relation $\xrightarrow{\delta}$ is strongly normalizing (i.e., definition unfolding inside a well-typed book is a well-founded procedure).

CGa Weak Type Checking



CGa Weak Type checking detects grammatical errors

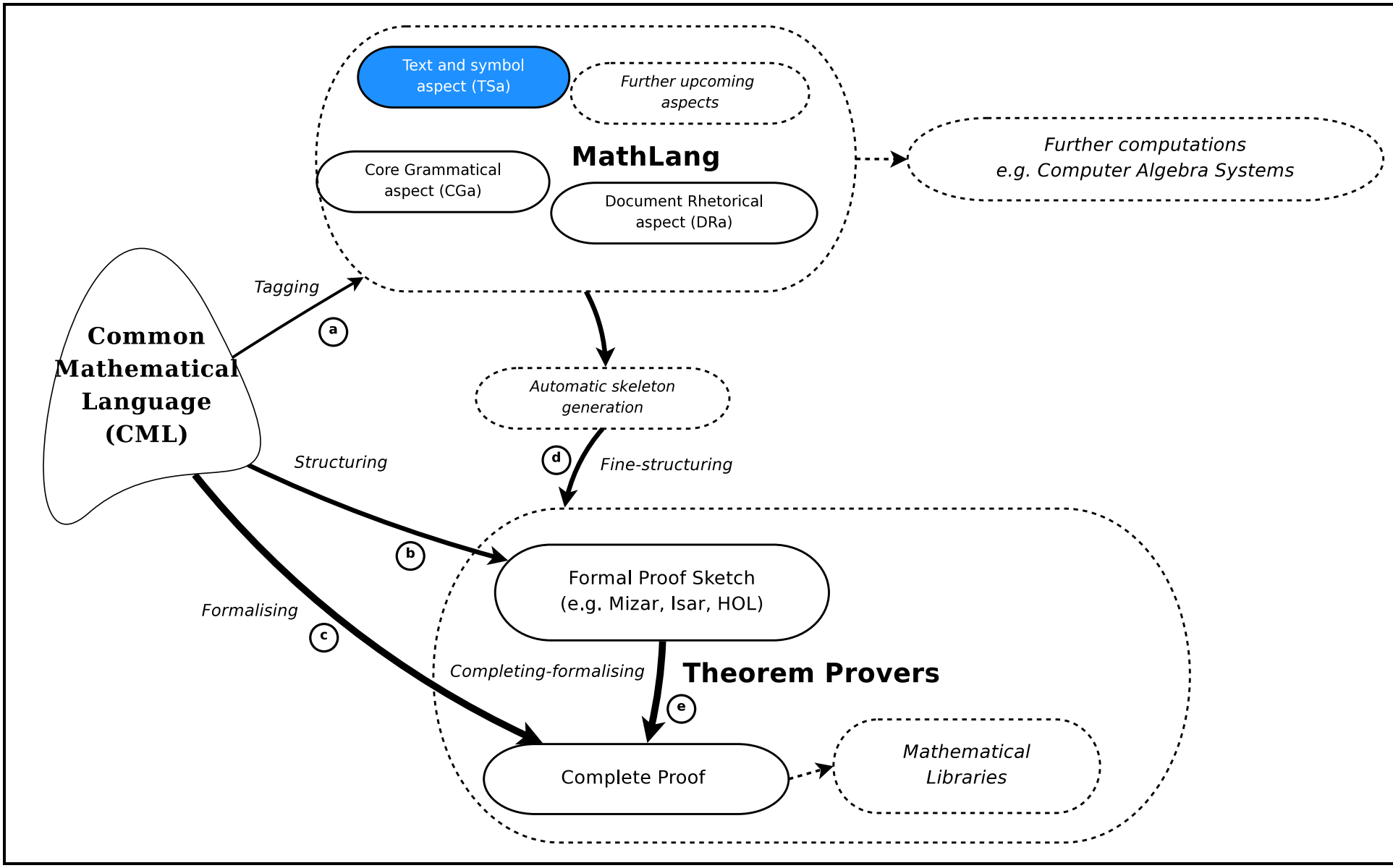


How complete is the CGa?

- CGa is quite advanced but remains under development according to new translations of mathematical texts. Are the current CGa categories sufficient?

- The metatheory of WTT has been established in (Kamareddine and Nederepelt 2004). That of CGa remains to be established. However, since CGa is quite similar to WTT, its metatheory might be similar to that of WTT.

- The type checker for CGa works well and gives some useful error messages. Error messages should be improved.


Text and symbol aspect (TSa)
Further upcoming aspects
MathLang
Core Grammatical aspect (CGa)
Document Rhetorical aspect (DRa)
Further computations e.g. Computer Algebra Systems
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Common Mathematical Language (CML)
Automatic skeleton generation
d
Fine-structuring
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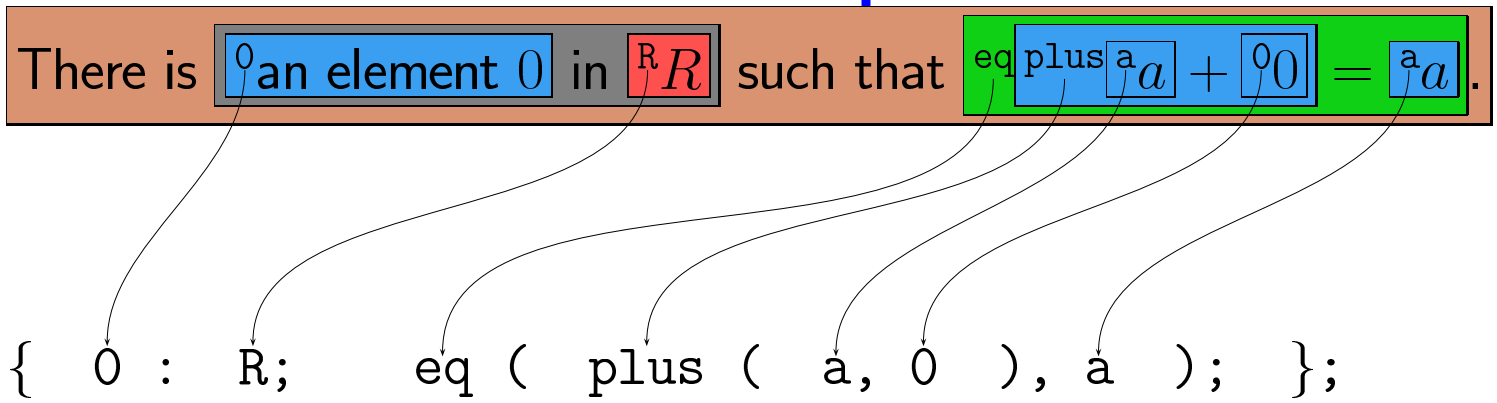
Formal Proof Sketch (e.g. Mizar, Isar, HOL)
Formalising
c
Completing-formalising
Theorem Provers
e
Complete Proof
Mathematical Libraries

What is TSa? Lamar's PhD thesis

- TSa builds the bridge between a CML text and its grammatical interpretation and adjoins to each CGa expression a string of words and/or symbols which aims to act as its CML representation.
- TSa plays the role of a user interface
- TSa can flexibly represent natural language mathematics.
- The author wraps the natural language text with boxes representing the grammatical categories (as we saw before).
- The author can also give interpretations to the parts of the text.

Interpretations

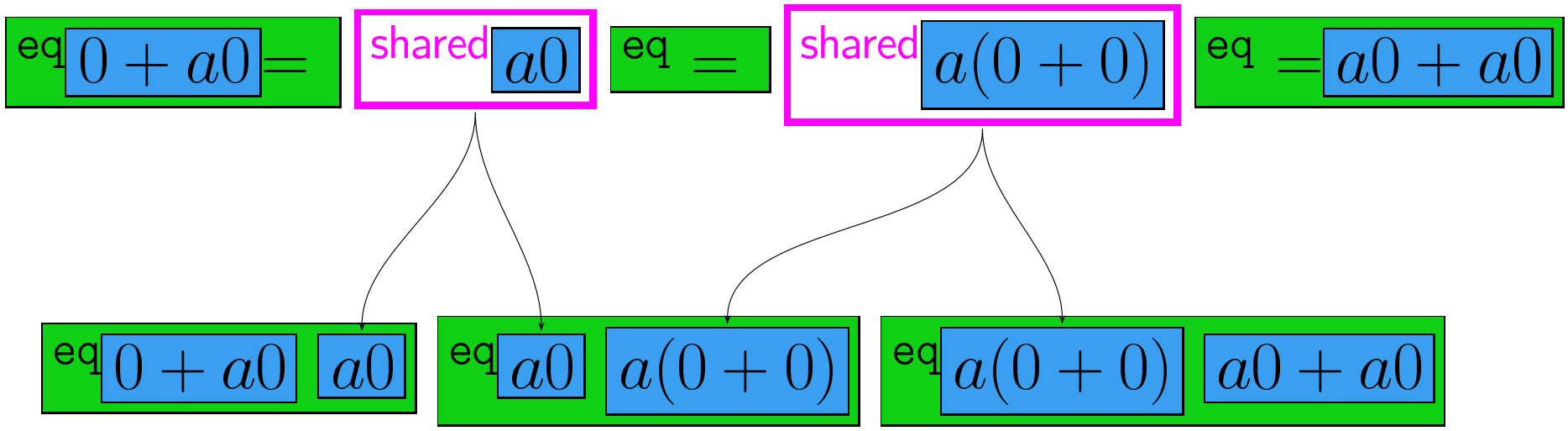


There is ^{0}an element 0 in ^{R}R such that $^{eq}\,^{plus}\,^{a}a + {}^{0}0 = {}^{a}a$.

There is ^{0}an element 0 in ^{R}R such that $^{eq}\,^{plus}\,^{a}a + {}^{0}0$ equals ^{a}a.

$^{0}0 \in {}^{R}R$, $^{eq}\,^{plus}\,^{a}a + {}^{0}0 = {}^{a}a$.

Rewrite rules enable natural language representation

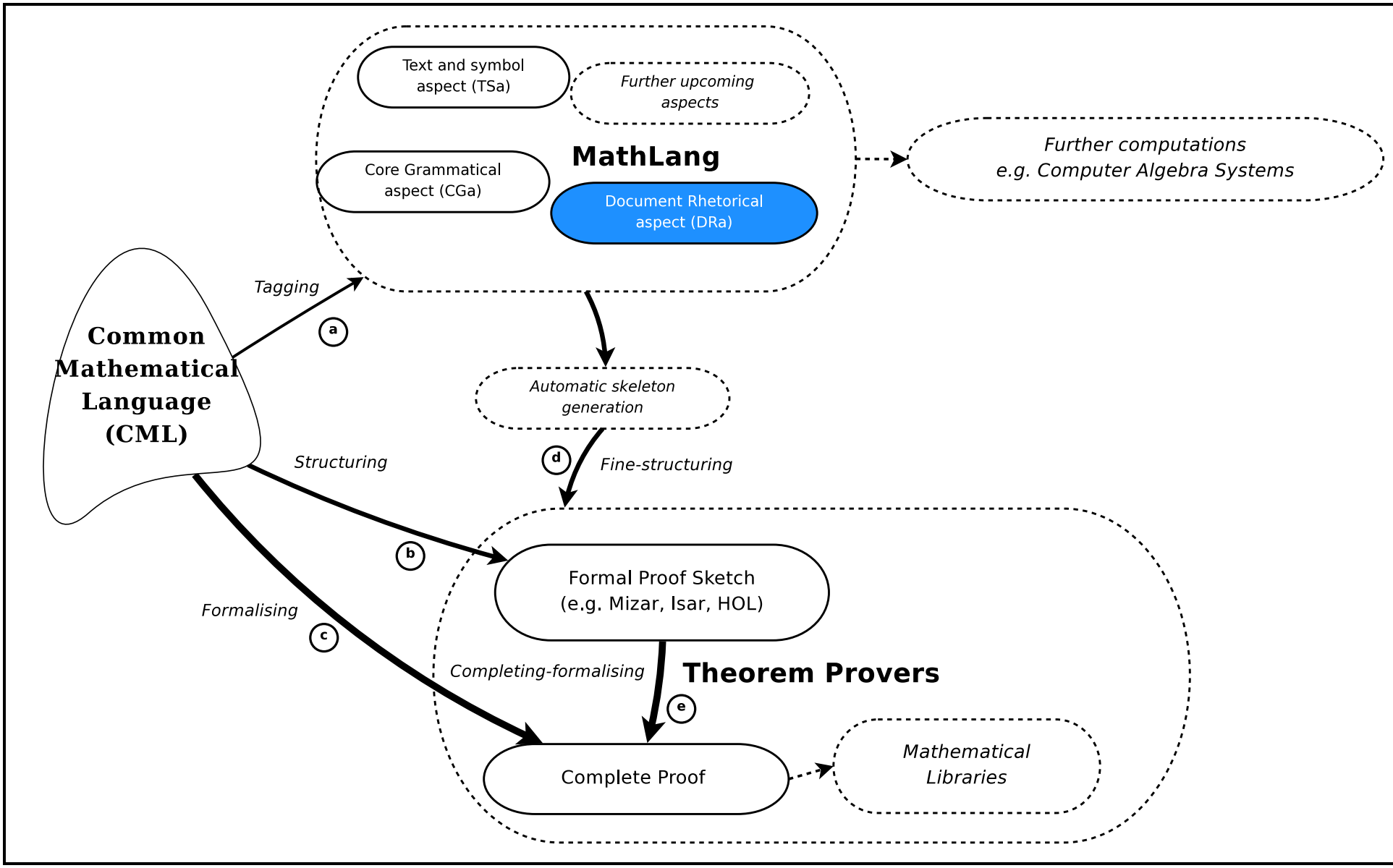


How complete is TSa?

- TSa provides useful interface facilities but it is still under development.

- So far, only simple rewrite (souring) rules are used and they are not comprehensive. E.g., unable to cope with things like $\overbrace{x = \ldots = x}^{n \text{ times}}$.

- The TSa theory and metatheory need development.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What is DRa? Retel's PhD thesis

- DRa Document Rhetorical structure aspect.
- **Structural components of a document** like *chapter, section, subsection, etc.*
- **Mathematical components of a document** like *theorem, corollary, definition, proof, etc.*
- **Relations** between above components.
- These enhance readability, and ease the navigation of a document.
- Also, these help to go into more formal versions of the document.

Relations

Description
Instances of the StructuralRhetoricalRole *class:* preamble, part, chapter, section, paragraph, *etc.*
Instances of the MathematicalRhetoricalRole *class:* lemma, corollary, theorem, conjecture, definition, axiom, claim, proposition, assertion, proof, exercise, example, problem, solution, *etc.*

Relation
Types of relations: relatesTo, uses, justifies, subpartOf, inconsistentWith, exemplifies

What does the mathematician do?

- The mathematician wraps into boxes and uniquely names chunks of text
- The mathematician assigns to each box the structural and/or mathematical rhetorical roles
- The mathematician indicates the relations between wrapped chunks of texts

Lemma 1. For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$.
Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n.m^2 = 2n^2 \ \& \ m > 0.$$

Claim. $P(m) \implies \exists m' < m.P(m')$. Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have

$$2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$$

Since $m > 0$, if follows that $m^2 > 0, n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}.\neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$ with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction. Therefore $m = 0$. But then also $n = 0$.

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$.
Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$.

Lemma 1.

For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$

Proof.

Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \ \& \ m > 0.$$

Claim. $P(m) \implies \exists m' < m. P(m')$.

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

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Now suppose $m^2 = 2n^2$

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Therefore $m = 0$. But then also $n = 0$.

□

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

$(A$, hasMathematicalRhetoricalRole, *lemma*$)$
$(E$, hasMathematicalRhetoricalRole, *definition*$)$
$(F$, hasMathematicalRhetoricalRole, *claim*$)$
$(G$, hasMathematicalRhetoricalRole, *proof*$)$
$(B$, hasMathematicalRhetoricalRole, *proof*$)$
$(H$, hasOtherMathematicalRhetoricalRole, *case*$)$
$(I$, hasOtherMathematicalRhetoricalRole, *case*$)$
$(C$, hasMathematicalRhetoricalRole, *corollary*$)$
$(D$, hasMathematicalRhetoricalRole, *proof*$)$

$(B$, justifies, $A)$
$(D$, justifies, $C)$
$(D$, uses, $A)$
$(G$, uses, $E)$
$(F$, uses, $E)$
$(H$, uses, $E)$
$(H$, subpartOf, $B)$
$(H$, subpartOf, $I)$

Lemma 1.

For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \Longrightarrow m = n = 0$ **A**

Proof.

Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n.m^2 = 2n^2 \ \& \ m > 0.$$ **E**

justifies

uses

Claim. $P(m) \Longrightarrow \exists m' < m.P(m')$. **F**

uses

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \Longrightarrow n^2 = 2k^2$ Since $m > 0$, it follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$. **G**

justifies

uses

uses

B

By the claim $\forall m \in \mathbb{N}.\neg P(m)$, since there are no infinite descending sequences of natural numbers.

subpartOf

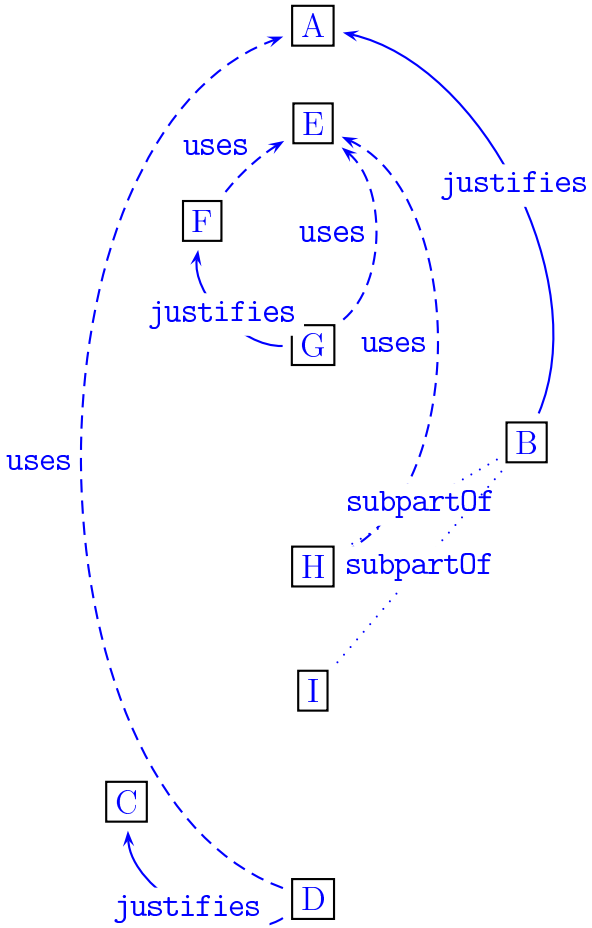
subpartOf

Now suppose $m^2 = 2n^2$

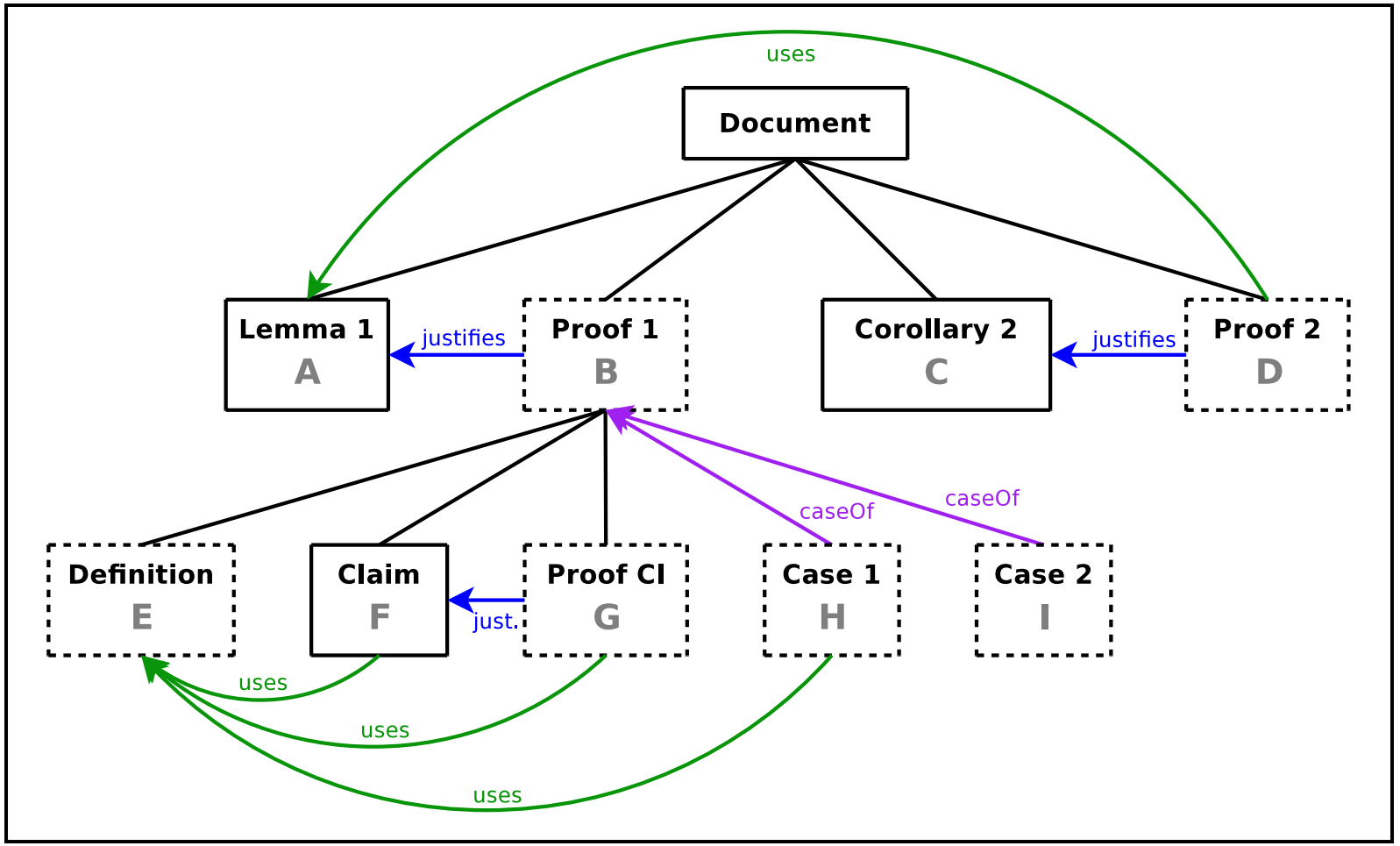
with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction. **H**

The automatically generated dependency Graph

Dependency Graph (DG)



An alternative view of the DRa

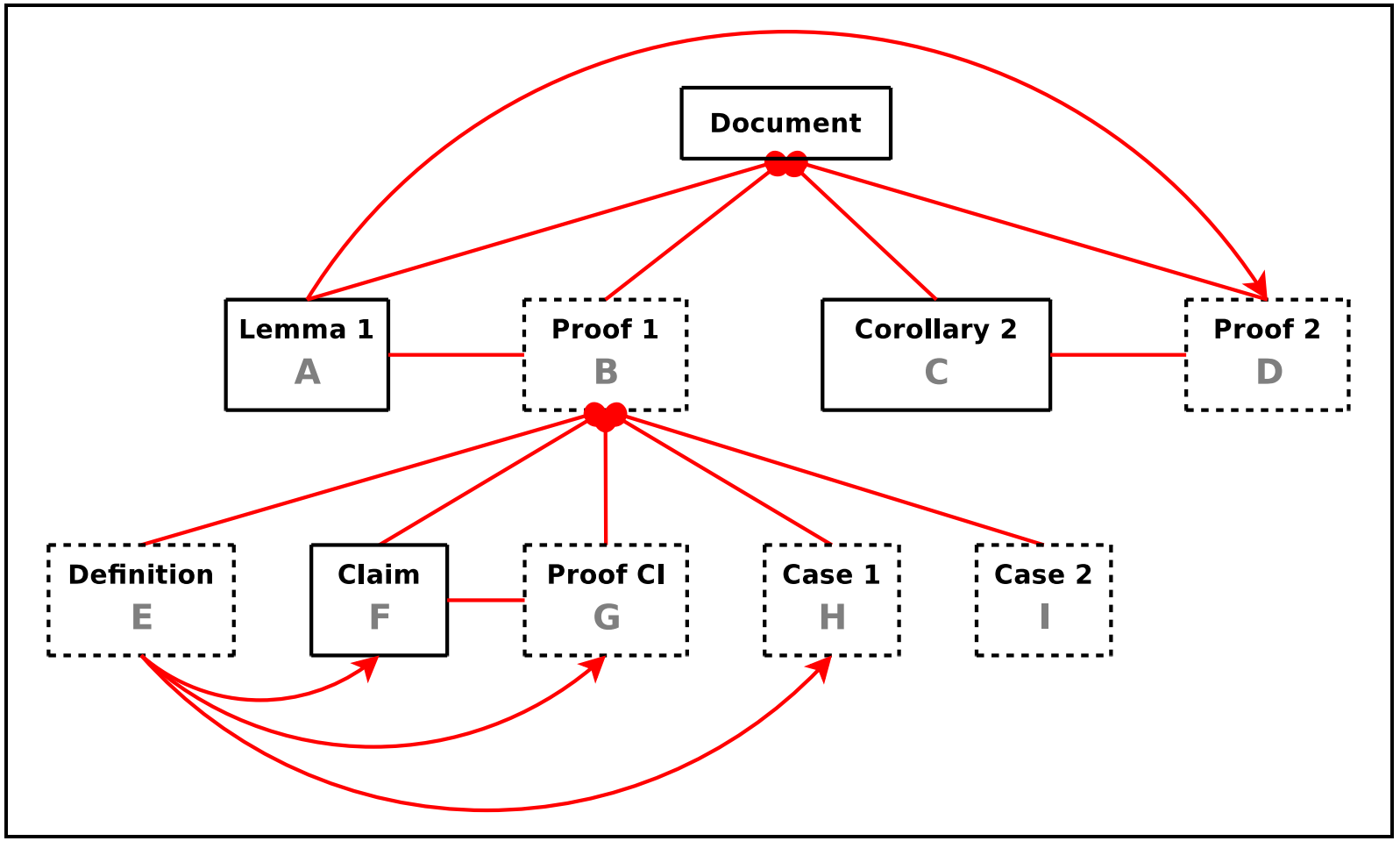


The Graph of Textual Order: GoTO
Zengler's thesis

- To be able to examine the proper structure of a DRa tree we introduce the concept of textual order between two nodes in the tree.
- Using textual orders, we can transform the dependency graph into a GoTO by transforming each edge of the DG.
- So far there are two reasons why the GoTO is produced:
  1. Automatic Checking of the GoTO can reveal errors in the document (e.g. loops in the structure of the document).
  2. The GoTO is used to automatically produce a proof skeleton for a certain prover.
- We automatically transform a DG into GoTO and automatically check the GoTO for errors in the document:

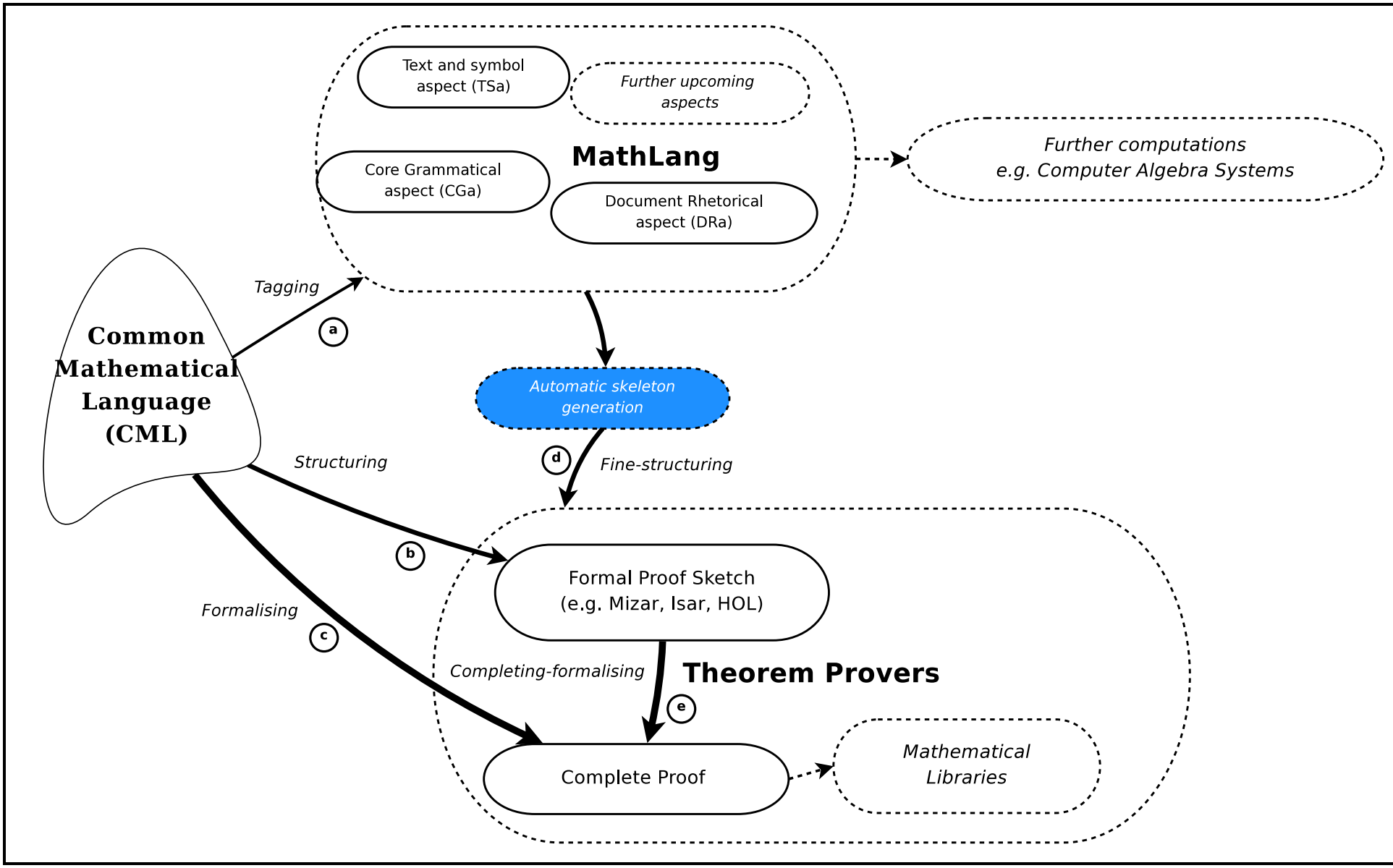
1. Loops in the GoTO (error)
2. Proof of an unproved node (error)
3. More than one proof for a proved node (warning)
4. Missing proof for a proved node (warning)

Graph of Textual Order for the DRa tree example



How complete is DRa?

- The dependency graph can be used to check whether the logical reasoning of the text is coherent and consistent (e.g., no loops in the reasoning).
- However, both the DRa language and its implementation need more experience driven tests on natural language texts.
- Also, the DRa aspect still needs a number of implementation improvements (the automation of the analysis of the text based on its DRa features).
- Extend TSa to also cover DRa (in addition to CGa).
- Extend DRa depending on further experience driven translations.
- Establish the soundness and completeness of DRa for mathematical texts.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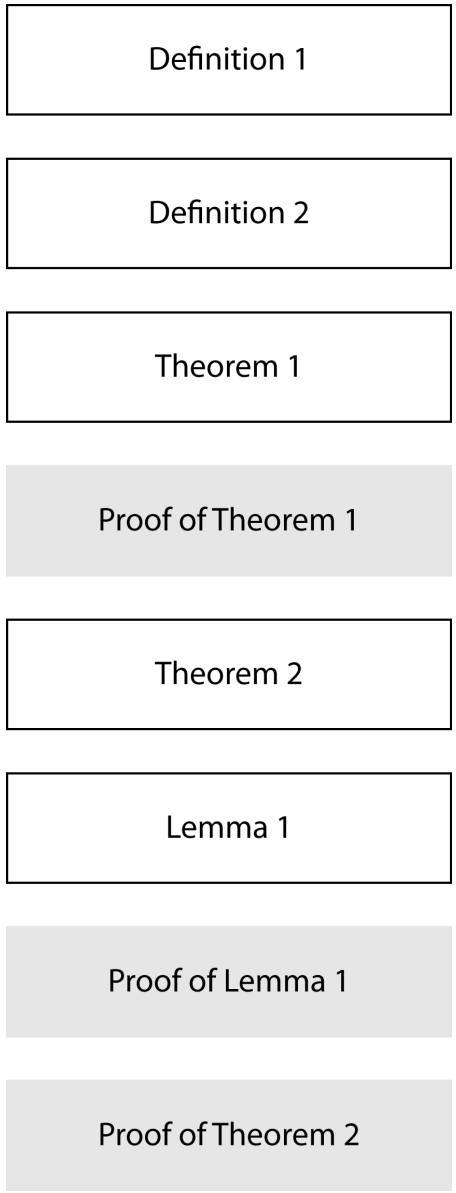
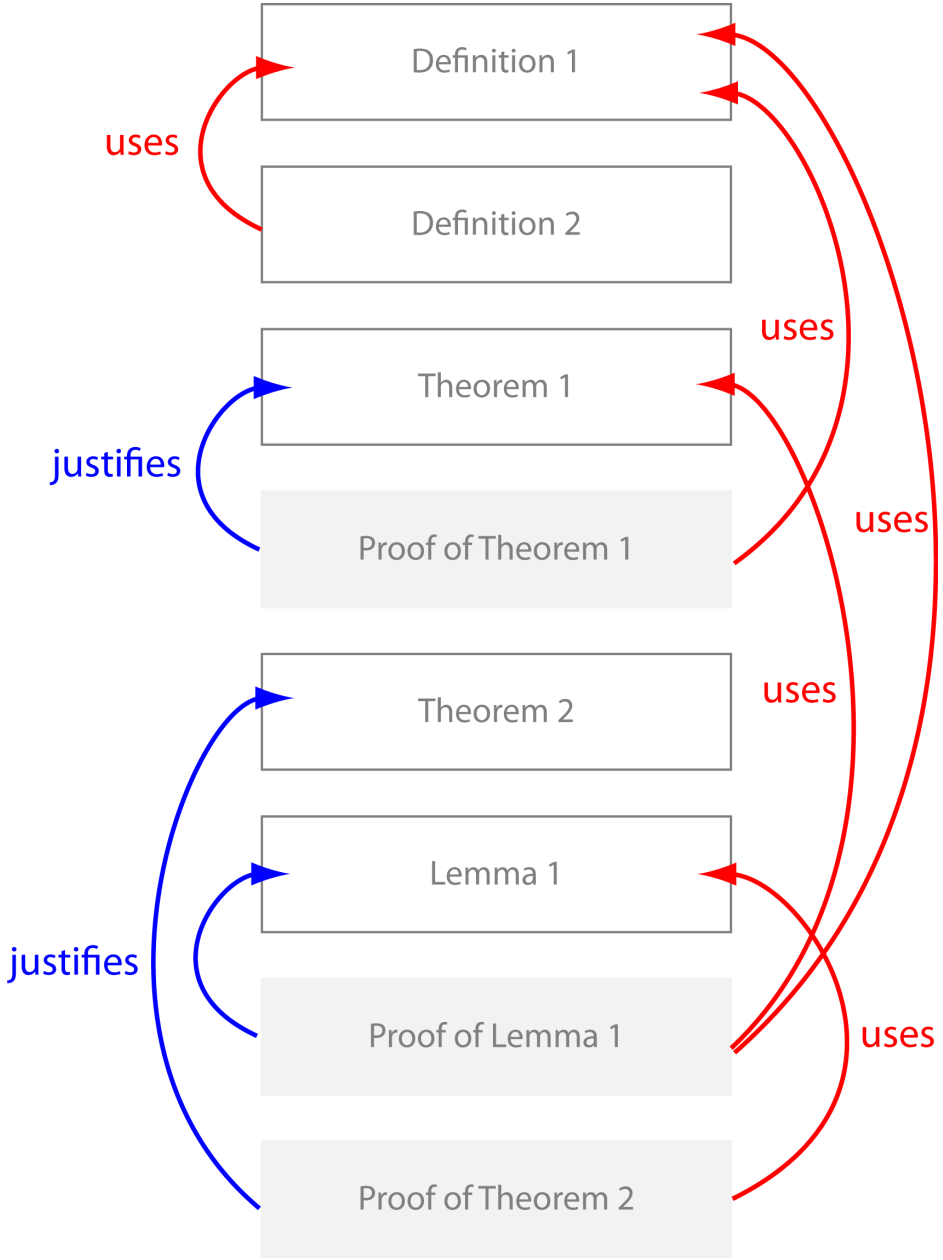
Different provers have

- different syntax
- different requirements to the structure of the text
  e.g.
  - no nested theorems/lemmas
  - only backward references
  - ...
- Aim: Skeleton should be as close as possible to the mathematician's text but with re-arrangements when necessary

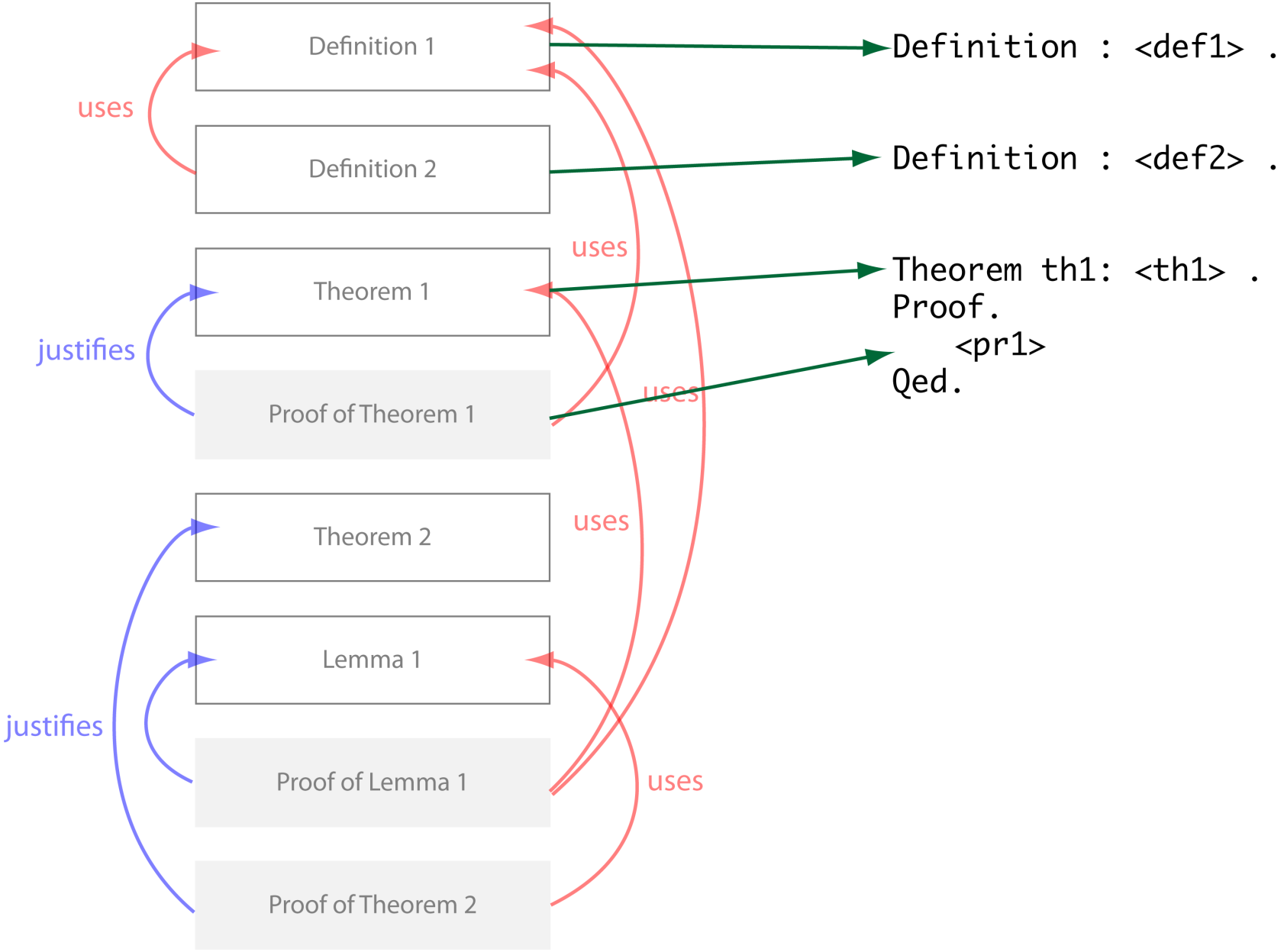
Example of nested theorems/lemmas (Moller, 03, Chapter III,2)



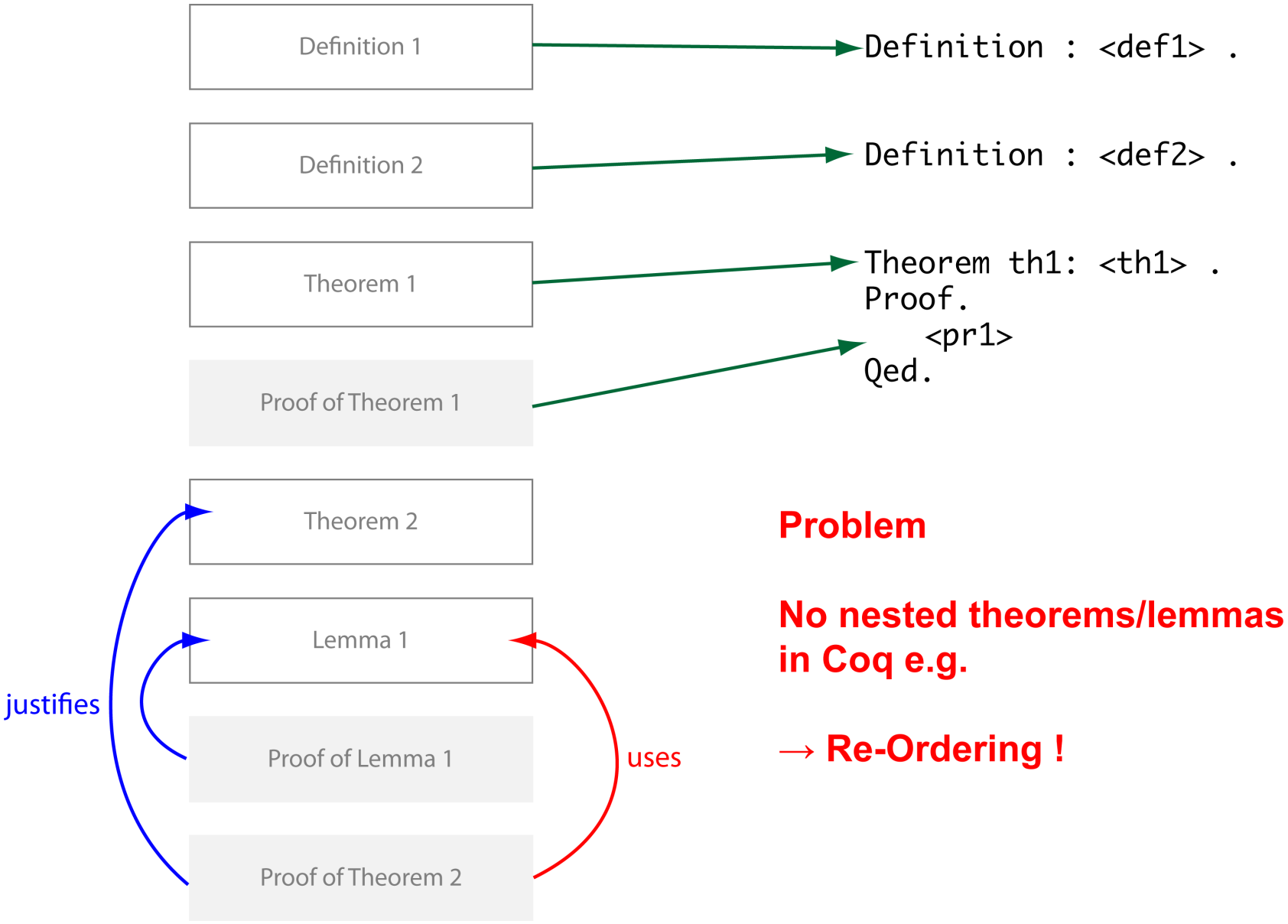
The automatic generation of a proof skeleton



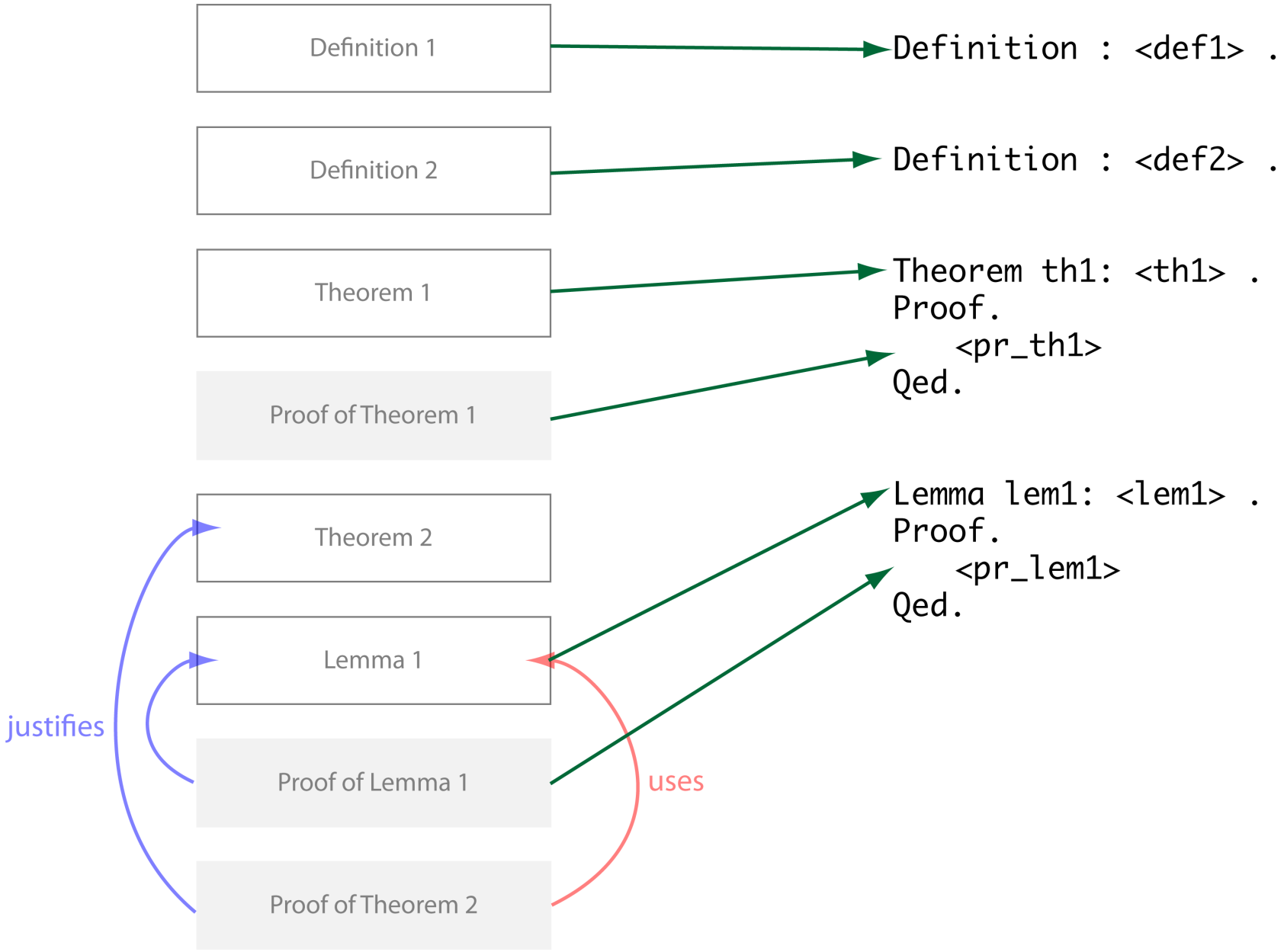
The DG for the example



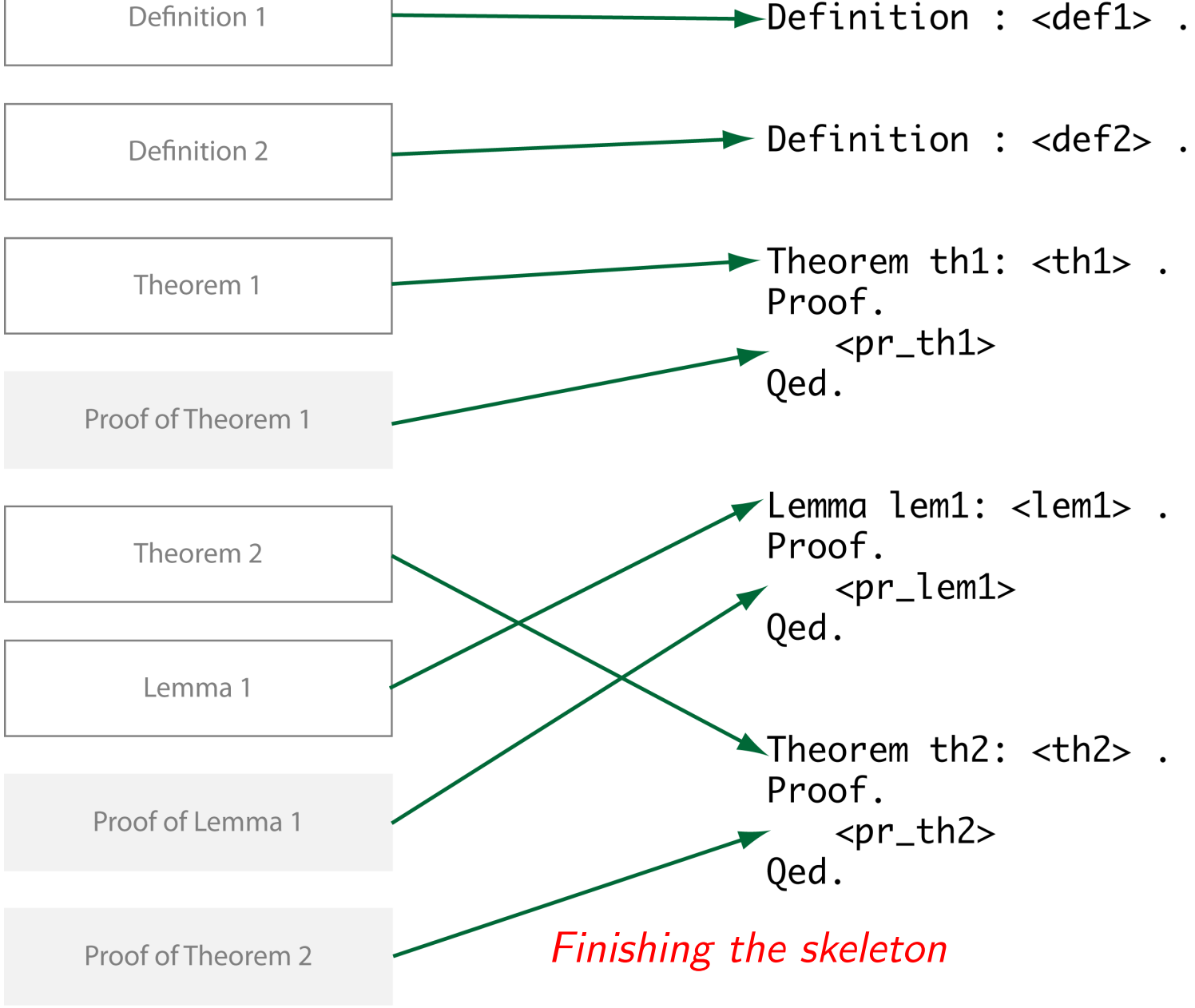
Straight-forward translation of the first part

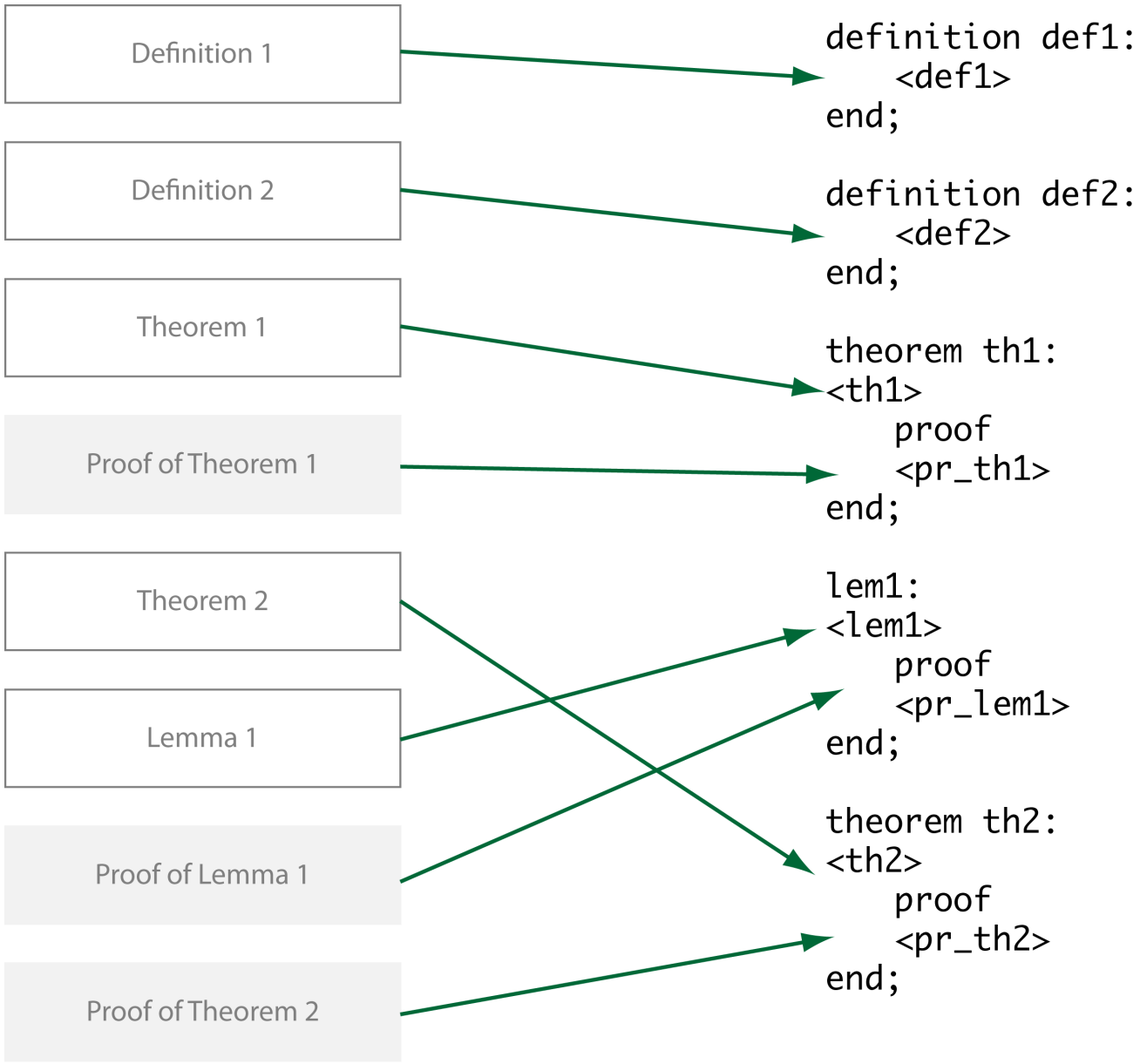


Problem: nested theorems



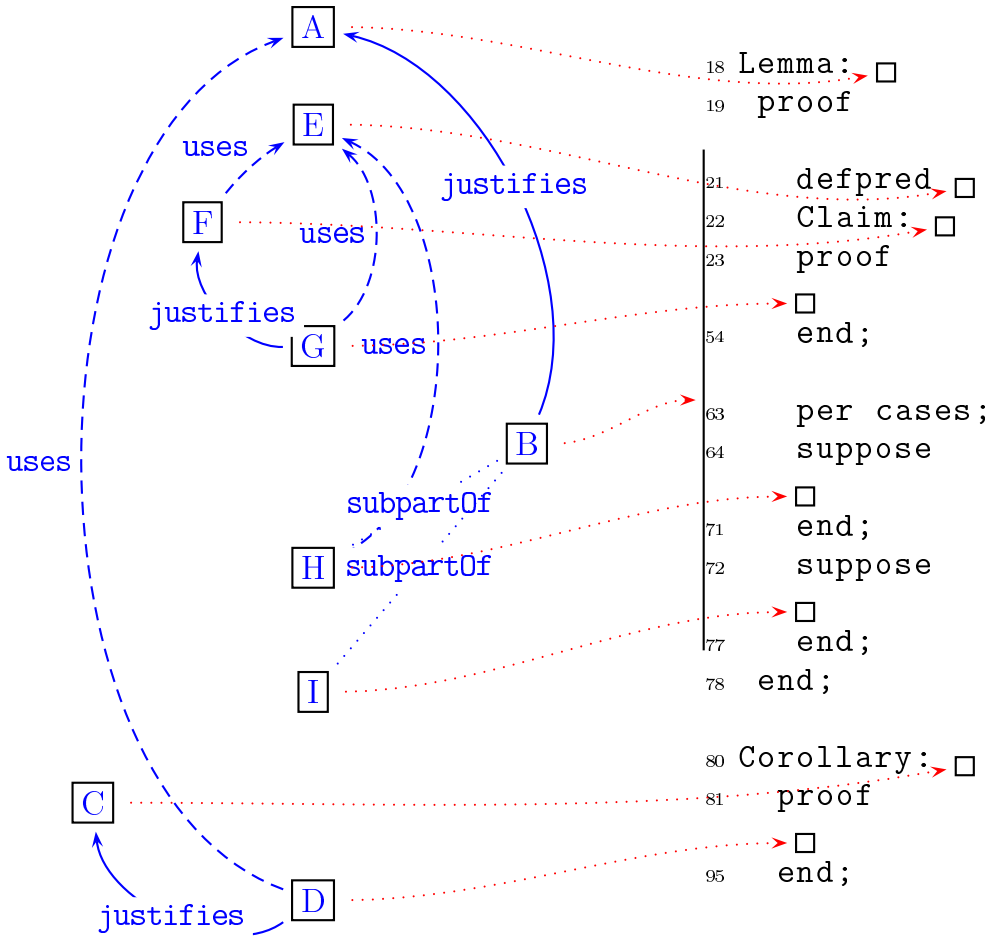
Solution: Re-ordering


Definition 1
Definition 2
Theorem 1
Proof of Theorem 1
Theorem 2
Lemma 1
Proof of Lemma 1
Proof of Theorem 2
Definition : <def1> .
Definition : <def2> .
Theorem th1: <th1> .
Proof.
<pr_th1>
Qed.
Lemma lem1: <lem1> .
Proof.
<pr_lem1>
Qed.
Theorem th2: <th2> .
Proof.
<pr_th2>
Qed.
Finishing the skeleton


Definition 1
Definition 2
Theorem 1
Proof of Theorem 1
Theorem 2
Lemma 1
Proof of Lemma 1
Proof of Theorem 2
definition def1:
<def1>
end;
definition def2:
<def2>
end;
theorem th1:
<th1>
proof
<pr_th1>
end;
lem1:
<lem1>
proof
<pr_lem1>
end;
theorem th2:
<th2>
proof
<pr_th2>
end;

Skeleton for Mizar

DRa annotation into Mizar skeleton for Barendregt's example (Retel's PhD thesis)



The remaining very rough path into Mizar

- We have not built the remaining aspects all the way into Mizar, but we have a rough path.
- Recall that GoTO gives a Mizar skeleton of the text.
- Next, the CGa encoding of the text is used to build relevant parts of the Mizar FPS (Wiedijk 2003) of the text (e.g., the CGa **preamble** could be used to find counterparts in Mizar MML and to build parts of the *Environment* in Mizar).
- At this stage, a Mizar expert would be able to complete the Mizar FPS version of the text.
- Now, the Mizar experts can complete the formalisation by filling all the gaps in the reasoning (i.e., filling the holes in sentences labelled with the error *4 by the Mizar system.)

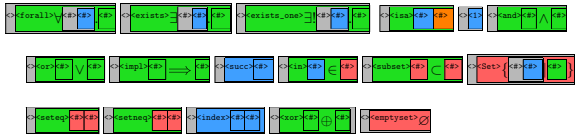
The Mizar FPS version of Barendregt's example

```
20 Lemma:  for m,n being Nat holds
21                m^2 = 2*n^2 implies m = 0 & n = 0
22  proof
23    let m,n being Nat;
24    defpred P[Nat] means
25             ex n being Nat st $1^2 = 2*n^2 & $1 > 0;
26    Claim: for m being Nat holds
27           P[m] implies ex m' being Nat st m' < m & P[m']
28    proof
29      let m being Nat;
30      assume P[m];
31      then consider n being Nat such that
32      m^2 = 2*n^2 & m > 0;
33      m^2 is even ;
34 ::>             *4
35      m is even;
36 ::>           *4
37      consider k being Nat such that m = 2*k;
38 ::>                                      *4
39      2*n^2 = m^2
40 ::>             *4
41           .= 4*k^2;
42 ::>               *4
43      then n^2 = 2*k^2;
44      m > 0 implies m^2 > 0 & n^2 > 0 & n > 0;
45 ::>                                     *4,4,4
46      then P[n];
47 ::>           *4,4
48      m^2 = n^2 + n^2;
49 ::>                *4
50      n^2 + n^2 >  n^2;
51 ::>                  *4
52      then m^2 > n^2;
53 ::>                *4
54      then m > n;
55 ::>             *4
56      take m' = n;
57      thus thesis;
58 ::>             *4,4
59    end;
```

```
67    A2: for k being Nat holds not P[k]
68    proof
69      not ex q being Seq_of_Nat
70          st q is infinite decreasing by Claim;
71 ::>                                          *4
72      hence thesis;
73 ::>              *4
74    end;
75    assume A0: m^2 = 2*n^2;
76    per cases by A0;
77    suppose B1: m <> 0;
78      then m > 0;
79 ::>            *4
80      then P[m] by B1;
81 ::>              *4
82      then contradiction by A2;
83      hence thesis;
84    end;
85    suppose S1: m = 0;
86      then n = 0;
87 ::>             *4
88      thus thesis by S1;
89 ::>                *4
90    end;
91  end;
92
93 Corollary:  sqrt 2 is irrational
94   proof
95     assume sqrt 2 is rational;
96     then ex p,q being Integer st
97     q <> 0 & sqrt 2 = p/q;
98 ::>                       *4
99     then consider m,n being Integer such that
100    A0: sqrt 2 = m/n & m = abs m & n = abs n & n <> 0;
101 ::>                                                *4
102    m^2 = 2*n^2;
103 ::>           *4
104    n = 0 by Lemma;
105 ::>         *4
106    hence contradiction;
107 ::>                    *4
108  end;
109
110 ::> 4: This inference is not accepted
```

Chapter 1

Natural Numbers



1.1 Axioms

We assume the following to be given:

A set (i.e. totality) of objects called natural numbers, possessing the properties - called axioms- to be listed below.

Before formulating the axioms we make some remarks about the symbols $=$ and $\neq$ which be used.

Unless otherwise specified, small italic letters will stand for natural numbers throughout this book.

If x is given and y is given, then either x and y are the same number; this may be written

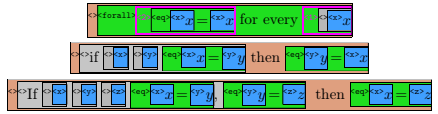
$$x = y$$

($=$ to be read “equals"); or x and y are not the same number; this may be written

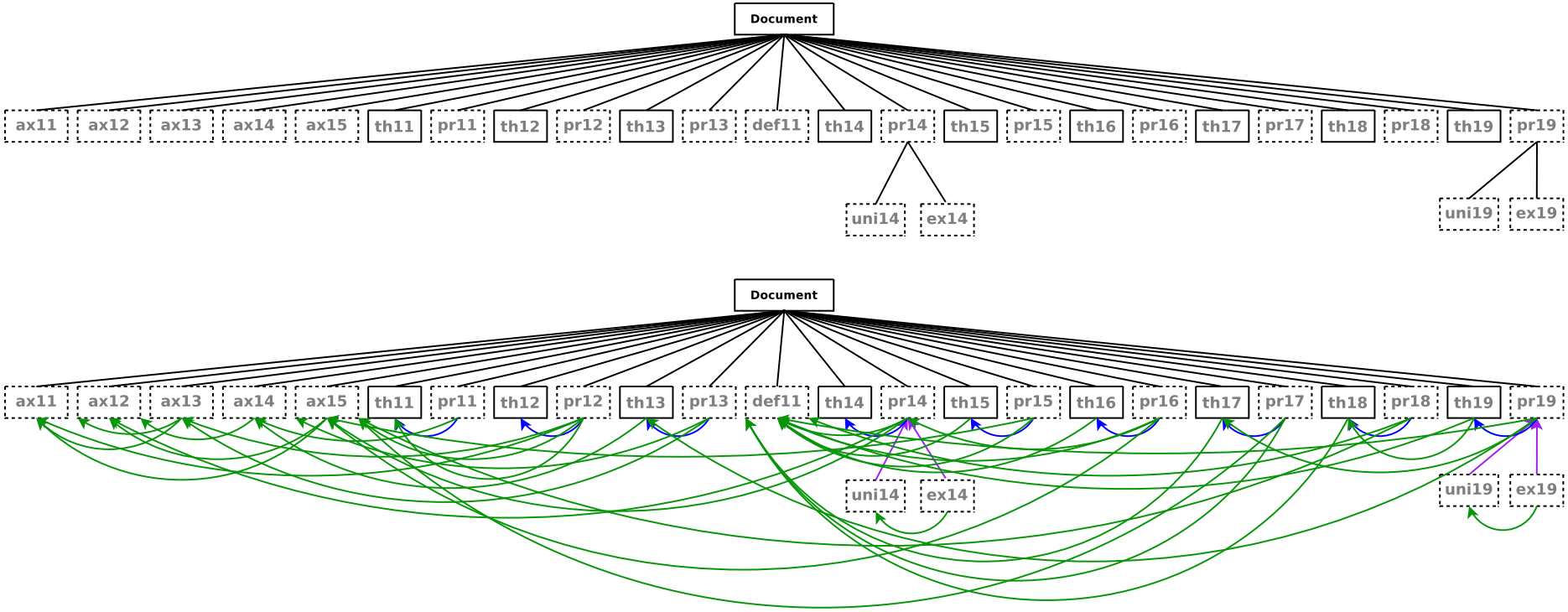
$$1x \neq y$$

($\neq$ to be read “is not equal to").

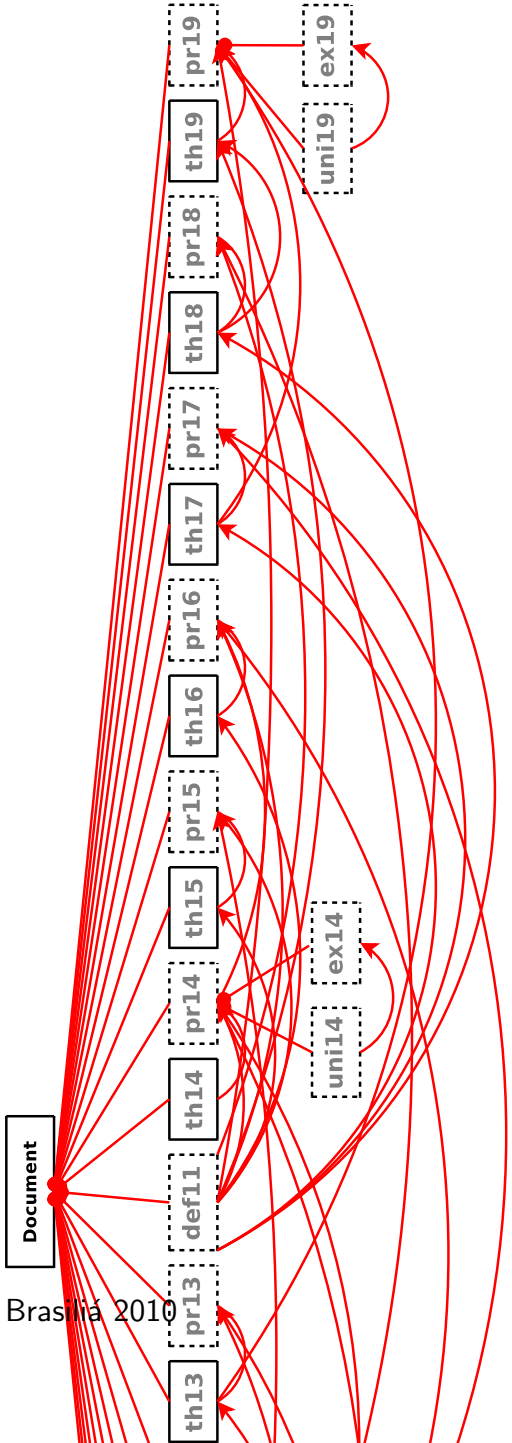
Accordingly, the following are true on purely logical grounds:



The DRa tree and the DG of sections 1 and 2 of chapter 1 of Landau's book



The GoTO of sections 1 and 2 of chapter 1 of Landau's book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Extending proof skeletons with CGa hints

- Translations into Coq for large parts of the chapter can be automated. E.g.,by just viewing the interpretation of the annotations of axiom 3 we get:

  forall x (neq (succ(x), 1)) (a)

  The automatically generated Coq proof skeleton for this axiom is:

  Axiom ax13 : <ax13> . (b)

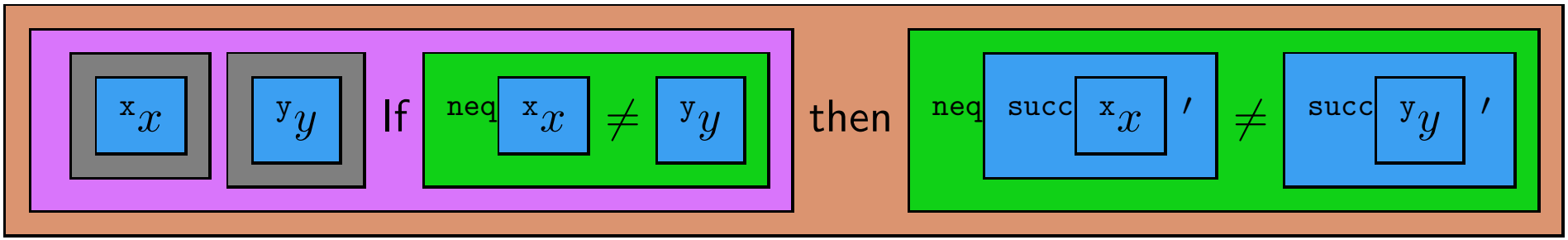
  Now, we simply replace the <ax13> placeholder of (b) with the literal translation of the interpretations in (a) to get the valid Coq axiom:

  Axiom ax13 : forall x:nats, neq (succ x) I .

Similarly for the theorems of chapter 1 of Landau's book, the work needed to get the full formalisation is straightforward: E.g. Theorem 1 is written by Landau as:

$$\text{If } x \neq y \text{ then } x' \neq y'$$

Its annotation in MathLang CGa is:



The CGa annotation of the context can also be seen as the premise of an implication. So the upper statement can be translated to:

$$\text{decl(x), decl(y) : neq x y} \rightarrow \text{neq (succ x) (succ y)}$$

And when we compare this line with its Coq translation we see again, it is just a literal transcription of the interpretation parts of CGa and therefore could be easily performed by an algorithm.

$$\text{Theorem th11 (x y:nats) : neq x y} \rightarrow \text{neq (succ x) (succ y) .}$$

From the 36 theorems of the chapter 28 could be translated literally into their corresponding Coq theorems.

Some points to consider

- We do not at all assume/prefer one type/logical theory instead of another.
- The formalisation of a language of mathematics should separate the questions:
  - *which type/logical theory is necessary for which part of mathematics*
  - *which language should mathematics be written in*.
- Mathematicians don't usually know or work with type/logical theories.
- Mathematicians usually *do* mathematics (manipulations, calculations, etc), but are not interested in general in reasoning *about* mathematics.
- The steps used for computerising books of mathematics written in English, as we are doing, can also be followed for books written in Arabic, French, German, or any other natural language.

- MathLang aims to support non-fully-formalized mathematics practiced by the ordinary mathematician as well as work toward full formalization.
- MathLang aims to handle mathematics as expressed in natural language as well as symbolic formulas.
- MathLang aims to do some amount of type checking even for non-fully-formalized mathematics. This corresponds roughly to grammatical conditions.
- MathLang aims for a formal representation of CML texts that closely corresponds to the CML conceived by the ordinary mathematician.
- MathLang aims to support automated processing of mathematical knowledge.
- MathLang aims to be independent of any foundation of mathematics.
- MathLang allows anyone to be involved, whether a mathematician, a computer engineer, a computer scientist, a linguist, a logician, etc.

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