Types and Functions since Principia and the Computerisation of Language and Mathematics

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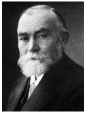
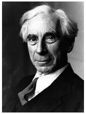
USEFUL LOGICS, TYPES, REWRITING and AUTOMATION

Calgary, January 2015

Summary

- *General definition of function 1879* [22] is key to Frege's *formalisation of logic*.
- *Self-application of functions* was at the heart of *Russell's paradox 1902* [50].
- To *avoid paradox* Russell controled function application via *type theory*.
- Russell [51] *1903* gives the first type theory: the *Ramified Type Theory* (RTT).
- RTT is used in Russell and Whitehead's Principia Mathematica [54] 1910–1912.

Summary

- *Simple theory of types* (STT): Ramsey [47] *1926*, Hilbert and Ackermann [31] *1928*.
- Church's *simply typed λ-calculus* $\lambda\!\rightarrow$ [17] 1940 = λ-calculus + STT.
- The hierarchies of types (and orders) as found in RTT and STT are *unsatisfactory*.
- The *notion of function adopted in the λ-calculus* is *unsatisfactory* [34].
- Hence, birth of *different systems of functions and types*, each with *different functional power*.
- We discuss the evolution of functions and types and their use in logic, language and computation.

Summary

- Frege's functions $\neq$ Principia's functions $\neq$ *λ-calculus* functions.
- Not all functions need to be *fully abstracted* as in the λ-calculus. For some functions, their values are enough.
- *Non-first-class functions* allow us to stay at a lower order (keeping decidability, typability, computability, etc.) without losing the flexibility of the higher-order aspects.
- Furthermore, non-first-class functions allow placing the type systems of modern theorem provers/programming languages like ML, LF and Automath more accurately in the modern formal hierarchy of types.
- Another issue that we touch on is the lessons learned from formalising mathematics in logic (à la Principia) and in proof checkers (à la Automath, or any modern proof checker).

Prehistory of Types (formal systems in 19th century) In the 19th century, the need for a more *precise* style in mathematics arose, because controversial results had appeared in analysis.

- 1821: Many of these controversies were solved by the work of Cauchy. E.g., he introduced *a precise definition of convergence* in his *Cours d'Analyse* [16].
- 1872: Due to the more *exact definition of real numbers* given by Dedekind [21], the rules for reasoning with real numbers became even more precise.
- 1895-1897: Cantor began formalizing *set theory* [14, 15] and made contributions to *number theory*.
- 1889: Peano formalized *arithmetic* [46], but did not treat logic or quantification.

- 1879:
  Frege was not satisfied with the use of natural language in mathematics:

  > *"... I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required."*
  >
  > *(*Begriffsschrift, *Preface)*

  Frege therefore presented *Begriffsschrift* [22], the first formalisation of logic giving logical concepts via symbols rather than natural language.

  > *"[Begriffsschrift's] first purpose is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated."*
  >
  > *(*Begriffsschrift, *Preface)*

Prehistory of Types (Begriffsschrift's functions)

The introduction of a *very general definition of function* was the key to the formalisation of logic. Frege defined what we will call the Abstraction Principle.

Abstraction Principle

"If in an expression, [. . .] a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function."

(*Begriffsschrift*, Section 9)

Prehistory of Types (Begriffsschrift's functions)

- Frege put *no restrictions* on what could play the role of *an argument*.
- An argument could be a *number* (as was the situation in analysis), but also a *proposition*, or a *function*.
- the *result of applying* a function to an argument did not have to be a number.
- Frege was aware of some typing rule that does not allow to substitute functions for object variables or objects for function variables:

  > *" Now just as functions are fundamentally different from objects, so also functions whose arguments are and must be functions are fundamentally different from functions whose arguments are objects and cannot be anything else. I call the latter first-level, the former second-level."*
  >
  > *(*Function and Concept, *pp. 26–27)*

Prehistory of Types (Grundgesetze's functions) The *Begriffsschrift*, however, was only a prelude to Frege's writings.

- In Grundlagen der Arithmetik [23] he argued that mathematics can be seen as a branch of logic.
- In Grundgesetze der Arithmetik [24, 26] he described the elementary parts of arithmetics within an extension of the logical framework of *Begriffsschrift*.
- Frege approached the *paradox threats for a second time* at the end of Section 2 of his *Grundgesetze*.
- He did not *apply a function to itself*, but to its course-of-values.
- "the function $\Phi(x)$ has the same *course-of-values* as the function $\Psi(x)$" if:

  > "*$\Phi(x)$ and $\Psi(x)$ always have the same value for the same argument.*"
  >
  > *(*Grundgesetze*, p. 7)*

- E.g., let $\Phi(x)$ be $x \wedge \neg x$, and $\Psi(x)$ be $x \leftrightarrow \neg x$, for all propositions x.

Prehistory of Types (Grundgesetze's functions)

- All essential information of a function is contained in its graph.
- So a system in which a function can be applied to its own graph should have similar possibilities as a system in which a function can be applied to itself.
- Frege *excluded the paradox threats* by *forbidding self-application*, but due to his *treatment of courses-of-values* these threats were able to *enter his system through a back door*.
- In 1902, Russell wrote to Frege [50] that he had *discovered a paradox* in his *Begriffsschrift* (*Begriffsschrift does not suffer from a paradox*).
- *Only six days later*, Frege answered that *Russell's derivation of the paradox was incorrect* [25]. That *self-application $f(f)$ is not possible in the* Begriffsschrift. And that *Russell's argument could be amended to a paradox* in the system of his Grundgesetze, using the *course-of-values* of functions.

Prehistory of Types (paradox in Peano and Cantor's systems)

- Frege's system was *not the only paradoxical* one.
- The Russell Paradox can be derived in *Peano's system* as well, as well as on *Cantor's Set Theory* by defining the class $K =_{\mathtt{def}} \{x \mid x \notin x\}$ and deriving $K \in K \longleftrightarrow K \notin K$.
- Paradoxes were already widely known in *antiquity*.
- The oldest logical paradox: the *Liar's Paradox* "This sentence is not true", also known as the Paradox of Epimenides. It is referred to in the Bible (Titus 1:12) and is based on the confusion between language and meta-language.
- The *Burali-Forti paradox* ([13], 1897) is the first of the modern paradoxes. It is a paradox within Cantor's theory on ordinal numbers.
- *Cantor's paradox* on the largest cardinal number occurs in the same field. It discovered by Cantor around 1895, but was not published before 1932.

Prehistory of Types (paradoxes)

- Logicians considered these paradoxes to be *out of the scope of logic*: The *Liar's Paradox* can be regarded as a problem of *linguistics*. The *paradoxes of Cantor and Burali-Forti* occurred in what was considered in those days a *highly questionable* part of mathematics: Cantor's Set Theory.
- The Russell Paradox, however, was *a paradox that could be formulated in all* the systems that were presented at the end of the 19th century (except for Frege's *Begriffsschrift*). It was at the very basics of logic. It could not be disregarded, and a solution to it had to be found.
- In 1903-1908, Russell suggested the use of *types* to solve the problem [52].

Prehistory of Types (vicious circle principle)

- *"In all the above contradictions there is a common characteristic, which we may describe as self-reference or reflexiveness.* [...] *In each contradiction something is said about* all *cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which* all *were concerned in what was said."*

  (Mathematical logic as based on the theory of types)

- Russell's plan was, *to avoid the paradoxes* by *avoiding all possible self-references*. He postulated the *"vicious circle principle"*:
- "Whatever involves all of a collection *must not be one* of the collection."

  (Mathematical logic as based on the theory of types)

- Russell implements this principle *very strictly* using *types*.

Problems of Ramified Type Theory

- The main part of the *Principia* is devoted to the development of logic and mathematics using the legal pfs of the ramified type theory.
- *ramification*/division of simple types into orders make RTT not easy to use.
- **(Equality)** $\mathtt{x} =_{\mathtt{L}} \mathtt{y} \stackrel{\mathrm{def}}{\leftrightarrow} \forall \mathtt{z}[\mathtt{z}(\mathtt{x}) \leftrightarrow \mathtt{z}(\mathtt{y})]$.
  In order to express this general notion in RTT, we have to incorporate *all* pfs $\forall \mathtt{z} : (0^0)^n[\mathtt{z}(\mathtt{x}) \leftrightarrow \mathtt{z}(\mathtt{y})]$ for $n > 1$, and this cannot be expressed in one pf.
- Not possible to give a constructive proof of the theorem of the least upper bound within a ramified type theory.
- It is not possible in RTT to give a definition of an object that refers to the class to which this object belongs (because of the Vicious Circle Principle). Such a definition is called an *impredicative definition*.

Axiom of Reducibility

- Russell and Whitehead tried to solve problems with the axiom of reducibility:
  For each formula f, there is a formula g with a predicative *type such that f and g are (logically) equivalent.*
- The validity of the Axiom of Reducibility has been questioned from the moment it was introduced.
- Though Weyl [53] made an effort to develop analysis within the Ramified Theory of Types (without the Axiom of Reducibility),
- and various parts of mathematics can be developed within RTT and without the Axiom,
- the general attitude towards RTT (without the axiom) was that the system was too restrictive, and that a *better solution* had to be found.

Deramification

- Ramsey considers it essential to divide the paradoxes into two parts:
- logical or syntactical paradoxes (like the Russell paradox, and the Burali-Forti paradox) are removed

> *"by pointing out that a propositional function cannot significantly take itself as argument, and by dividing functions and classes into a hierarchy of types according to their possible arguments."*
>
> *(The Foundations of Mathematics, p. 356)*

- Semantical paradoxes are excluded by the hierarchy of orders. These paradoxes (like the Liar's paradox, and the Richard Paradox) are based on the confusion of language and meta-language. These paradoxes are, therefore, not of a purely mathematical or logical nature. When a proper distinction between object language and meta-language is made, these so-called semantical paradoxes disappear immediately.

The Simple Theory of Types

- Ramsey [47], and Hilbert and Ackermann [31], *simplified* the Ramified Theory of Types RTT by removing the orders. The result is known as the Simple Theory of Types *(STT)*.
- Nowadays, STT is known via Church's formalisation in λ-calculus. However, STT *already existed (1926) before λ-calculus did (1932)*, and is therefore not inextricably bound up with λ-calculus.
- How to obtain STT from RTT? Just *leave out all the orders* and the references to orders (including the notions of predicative and impredicative types).

Limitation of the simply typed λ-calculus

- $\lambda\!\rightarrow$ is very restrictive.
- Numbers, booleans, the identity function have to be defined at every level.
- We can represent (and type) terms like $\lambda x : o.x$ and $\lambda x : \iota.x$.
- We cannot type $\lambda x : \alpha.x$, where α can be instantiated to any type.
- This led to new (modern) type theories that allow more general notions of functions (e.g, *polymorphic*).

The evolution of functions with Frege, Russell and Church

- Historically, *functions* have long been treated as a kind of *meta-objects*.
- Function *values* were the important part, not *abstract functions*.
- In the *low level/operational approach* there are only function values.
- The *sine-function*, is always expressed with a value: $\sin(\pi)$, $\sin(x)$ and properties like: $\sin(2x) = 2\sin(x)\cos(x)$.
- In many mathematics courses, one calls $f(x)$—and not f—the *function*.
- *Frege*, *Russell* and *Church* wrote $x \mapsto x + 3$ resp. as $x + 3$, $\hat{x} + 3$ and $\lambda x.x + 3$.
- Principia's *functions are based on Frege's Abstraction Principles* but can be first-class citizens. Frege used courses-of-values to speak about functions.
- Church made every function a first-class citizen. This is *rigid* and does not represent the development of logic in 20th century.

Functionalisation and Instantiation [40] assessed evolution of the function concept from two points of vue:

- *Functionalisation*: the construction of a function out of an expression, as in constructing the function $\lambda_x.x \times 3 + x$ from the expression $2 \times 3 + 2$.
- Functionalisation is
  - *Abstraction from a subexpression* e.g., moving from $2 \times 3 + 2$ to $x \times 3 + x$
  - *Function construction* e.g., turning $x \times 3 + x$ into $\lambda_x.x \times 3 + x$.
- *Instantiation*: the calculation of a function value when a suitable argument is assigned to the function,
  as in the construction of $2 \times 3 + 2$ by applying the function $\lambda_x.x \times 3 + x$ to 2.
- Instantiation is:
  - *Application construction* e.g., $(\lambda_x.x \times 3 + x)2$ the application of $\lambda_x.x \times 3 + x$ to 2
  - *Concretisation to a subexpression* e.g., calculating $(\lambda_x.x \times 3 + x)2$ to $2 \times 3 + 2$.

λ-calculus does not fully represent functionalisation

1. *Abstraction from a subexpression* $2 + 3 \mapsto x + 3$
2. *Function construction* $x + 3 \mapsto \lambda x.x + 3$
3. *Application construction* $(\lambda x.x + 3)2$
4. *Concretisation to a subexpression* $(\lambda x.(x + 3))2 \rightarrow 2 + 3$

- cannot abstract only half way: $x + 3$ is not a function, $\lambda x.x + 3$ is.
- cannot apply $x + 3$ to an argument: $(x + 3)2$ does not evaluate to 2+3.

Common features of modern types and functions

- We can *construct* a type by abstraction. (Write $A : *$ for *A is a type*)
  - $\lambda_{y:A}.y$, the identity over A *has type* $A \to A$
  - $\lambda_{A:*}.\lambda_{y:A}.y$, the polymorphic identity *has type* $\Pi_{A:*}.A \to A$
- We can *instantiate* types. E.g., if $A = \mathbb{N}$, then the identity over $\mathbb{N}$
  - $(\lambda_{y:A}.y)[A := \mathbb{N}]$ *has type* $(A \to A)[A := \mathbb{N}]$ or $\mathbb{N} \to \mathbb{N}$.
  - $(\lambda_{A:*}.\lambda_{y:A}.y)\mathbb{N}$ *has type* $(\Pi_{A:*}.A \to A)\mathbb{N} = (A \to A)[A := \mathbb{N}]$ or $\mathbb{N} \to \mathbb{N}$.
- $(\lambda x{:}\alpha.A)B \to_\beta A[x := B]$ $\qquad$ $(\Pi x{:}\alpha.A)B \to_\Pi A[x := B]$
- Write $A \to A$ as $\Pi_{y:A}.A$ when y not free in A.

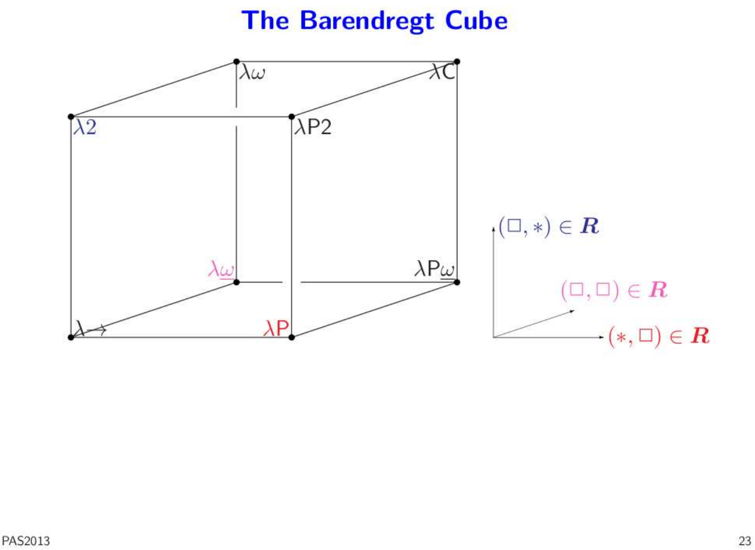
The Barendregt Cube

- Syntax: $A ::= x \mid * \mid \Box \mid AB \mid \lambda x{:}A.B \mid \Pi x{:}A.B$
- Formation rule:

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2} \qquad \textit{if } (s_1, s_2) \in \mathbf{R}$$

	Simple	Poly-morphic	Depend-ent	Constr-uctors	Related system	Refs.
$\lambda\!\to$	$(*,*)$				λ^τ	[17, 7, 32]
$\lambda 2$	$(*,*)$	$(\Box,*)$			F	[**?** **?**]
λP	$(*,*)$		$(*,\Box)$		AUT-QE, LF	[11, 28]
$\lambda\underline{\omega}$	$(*,*)$			$(\Box,\Box)$	POLYREC	[49]
λP2	$(*,*)$	$(\Box,*)$	$(*,\Box)$			[44]
$\lambda\omega$	$(*,*)$	$(\Box,*)$		$(\Box,\Box)$	Fω	[**?**]
λP$\underline{\omega}$	$(*,*)$		$(*,\Box)$	$(\Box,\Box)$		
λC	$(*,*)$	$(\Box,*)$	$(*,\Box)$	$(\Box,\Box)$	CC	[18]

The 8 Systems



Typing Polymorphic identity needs $(\Box, *)$

- $\dfrac{y:*\vdash y:* \qquad y:*, x{:}y \vdash y:*}{y:*\vdash \Pi x{:}y.y:*}$ by (Π) $(*,*)$
- $\dfrac{y:*, x:y\vdash x:y \qquad y:*\vdash \Pi x{:}y.y:*}{y:*\vdash \lambda x:y.x:\Pi x{:}y.y}$ by (λ)
- $\dfrac{\vdash *:\Box \qquad y:*\vdash \Pi x{:}y.y:*}{\vdash \Pi y:*.\Pi x{:}y.y:*}$ by (Π) by $(\Box,*)$
- $\dfrac{y:*\vdash \lambda x:y.x:\Pi x{:}y.y \qquad \vdash \Pi y:*.\Pi x{:}y.y:*}{\vdash \lambda y:*.\lambda x:y.x:\Pi y:*.\Pi x{:}y.y}$ by (λ)

The story so far of the evolution of functions and types

- Functions have gone through a long process of evolution involving various degrees of abstraction/construction/instantiation/concretisation/evaluation.
- Types too have gone through a long process of evolution involving various degrees of abstraction/construction/instantiation/concretisation/evaluation.
- During their progress, some aspects have been added or removed.
- The development of types and functions have been interlinked and their abstraction/construction/instantiation/concretisation/evaluation have much in common.
- We also argue that some of the aspects that have been dismissed during their evolution need to be re-incorporated.

From the point of vue of ML

- The language ML is not based on all of system F (2nd order polymorphic λ-calculus).
- This was not possible since it was not known then whether type checking and type finding are decidable.
- ML is based on a fragment of system F for which it was known that type checking and type finding are decidable.
- 23 years later after the design of ML, Wells showed that type checking and type finding in system F are undecidable.
- ML has polymorphism but not all the polymorphic power of system F.
- The question is, what system of functions and types does ML use?
- *A clean answer can be given when we re-incorporate the low-level function notion used by Frege and Russell (and de Bruijn) and dismissed by Church.*
- ML treats *let val id = (fn x ⇒ x) in (id id) end* as this Cube term $(\lambda id{:}(\Pi\alpha{:}*.\,\alpha \to \alpha).\, id(\beta \to \beta)(id\,\beta))(\lambda\alpha{:}*.\,\lambda x{:}\alpha.\,x)$

- To type this in the Cube, the $(\Box, *)$ rule is needed (i.e., $\lambda 2$).
- ML's typing rules forbid this expression:
  let val id = (fn x ⇒ x) in (fn y ⇒ y y)(id id) end
  Its equivalent Cube term is this well-formed typable term of $\lambda 2$:
  $(\lambda id : (\Pi\alpha{:}*.\, \alpha \to \alpha).$
  $\quad(\lambda y{:}(\Pi\alpha{:}*.\, \alpha \to \alpha).\, y(\beta \to \beta)(y\,\beta))$
  $\quad(\lambda\alpha{:}*.\, id(\alpha \to \alpha)(id\,\alpha)))$
  $(\lambda\alpha{:}*.\, \lambda x{:}\alpha.\, x)$
- Therefore, *ML should not have the full Π-formation rule* $(\Box, *)$.
- *ML has limited access to the rule* $(\Box, *)$.
- *ML's type system is none of those of the eight systems of the Cube*.
- [33] places the *type system of ML* (*between* $\lambda 2 + \lambda\underline{\omega}$).

LF

- *LF* [28] is often described as *λP* of the Barendregt Cube. However, *Use of Π-formation rule $(*, \Box)$ is restricted in LF* [27].
- We only need a type $\Pi x{:}A.B : \Box$ when PAT is applied during construction of the type $\Pi\alpha{:}\texttt{prop}.*$ of the operator Prf where for a proposition Σ, Prf(Σ) is the type of proofs of Σ.

$$\frac{\texttt{prop}{:}* \vdash \texttt{prop}: * \qquad \texttt{prop}{:}*, \alpha{:}\texttt{prop} \vdash *{:}\Box}{\texttt{prop}{:}* \vdash \Pi\alpha{:}\texttt{prop}.* : \Box}.$$

- In LF, this is the only point where the Π-formation rule $(*, \Box)$ is used. But, Prf is only used when applied to $\Sigma{:}\texttt{prop}$. We never use Prf on its own.
- This use is in fact based on a *parametric constant rather than on Π-formation*.
- Hence, the practical use of LF would not be restricted if we present Prf in a parametric form, and use $(*, \Box)$ as a parameter instead of a Π-formation rule.
- [33] precisely locate *LF* (*between $\lambda\!\rightarrow$ and λP*).

Parameters: What and Why

- We speak about functions with parameters when referring to functions with variable values in the *low-level* approach. The *x in $f(x)$ is a parameter*.
- Parameters enable the same expressive power as the high-level case, while allowing us to stay at a lower order. E.g. *first-order with parameters* versus *second-order without* [43].
- Desirable properties of the lower order theory (*decidability, easiness of calculations, typability*) can be maintained, without losing the flexibility of the higher-order aspects.
- This *low-level approach is still worthwhile for many exact disciplines*. In fact, both in logic and in computer science it has certainly not been wiped out, and for good reasons.

Automath

- The first tool for mechanical representation and verification of mathematical proofs, AUTOMATH, has a parameter mechanism.
- *Mathematical text* in AUTOMATH written as a *finite list of* lines of the form:
  $x_1 : A_1, \ldots, x_n : A_n \vdash g(x_1, \ldots, x_n) = t : T.$
  Here g is a new name, an abbreviation for the expression t of type T and $x_1, \ldots, x_n$ are the parameters of g, with respective types $A_1, \ldots, A_n$.
- Each line introduces a new definition which is inherently parametrised by the variables occurring in the context needed for it.
- Developments of ordinary mathematical theory in AUTOMATH [9] revealed that this combined definition and *parameter mechanism is vital for keeping proofs manageable and sufficiently readable for humans*.

Extending the Cube with parametric constants, see [33]

- We add *parametric constants* of the form $c(b_1, \ldots, b_n)$ with $b_1, \ldots, b_n$ terms of certain types and $c \in \mathcal{C}$.
- $b_1, \ldots, b_n$ are called the *parameters* of $c(b_1, \ldots, b_n)$.
- **R** *allows* several kinds of Π*-constructs*. We also use a set **P** of (s_1, s_2) where $s_1, s_2 \in \{*, \Box\}$ to *allow* several kinds of *parametric constants*.
- $(s_1, s_2) \in \mathbf{P}$ *means* that we *allow* parametric constants $c(b_1, \ldots, b_n) : A$ where $b_1, \ldots, b_n$ *have types* $B_1, \ldots, B_n$ *of sort* s_1, and A *is of type* s_2.
- If both $(*, s_2) \in \mathbf{P}$ and $(\Box, s_2) \in \mathbf{P}$ then *combinations of parameters allowed*.

  For example, it is allowed that B_1 *has type* $*$, whilst B_2 *has type* $\Box$.

The Cube with parametric constants

- Let $(*,*) \subseteq \mathbf{R}, \mathbf{P} \subseteq \{(*,*),(*,\Box),(\Box,*),(\Box,\Box)\}$.
- $\lambda\mathbf{RP} = \lambda\mathbf{R}$ and the two rules $(\vec{\mathbf{C}}\textbf{-weak})$ and $(\vec{\mathbf{C}}\textbf{-app})$:

$$\frac{\Gamma \vdash b : B \qquad \Gamma, \Delta_i \vdash B_i : s_i \qquad \Gamma, \Delta \vdash A : s}{\Gamma, c(\Delta) : A \vdash b : B} \quad (s_i, s) \in \mathbf{P}, c \text{ is } \Gamma\text{-fresh}$$

$$\frac{\begin{array}{lcll}\Gamma_1, c(\Delta){:}A, \Gamma_2 & \vdash & b_i{:}B_i[x_j{:=}b_j]_{j=1}^{i-1} & (i = 1, \ldots, n) \\ \Gamma_1, c(\Delta){:}A, \Gamma_2 & \vdash & A : s & (\text{if } n = 0)\end{array}}{\Gamma_1, c(\Delta){:}A, \Gamma_2 \vdash c(b_1, \ldots, b_n) : A[x_j{:=}b_j]_{j=1}^{n}}$$

$\Delta \equiv x_1{:}B_1, \ldots, x_n{:}B_n$.
$\Delta_i \equiv x_1{:}B_1, \ldots, x_{i-1}{:}B_{i-1}$

Properties of the Refined Cube

- *(Correctness of types)* If $\Gamma \vdash A : B$ then ($B \equiv \Box$ or $\Gamma \vdash B : S$ for some sort S).
- *(Subject Reduction SR)* If $\Gamma \vdash A : B$ and $A \twoheadrightarrow_\beta A'$ then $\Gamma \vdash A' : B$
- *(Strong Normalisation)* For all $\vdash$-legal terms M, we have $\mathrm{SN}_{\twoheadrightarrow_\beta}(M)$.
- Other properties such as *Uniqueness of types* and *typability of subterms* hold.
- $\lambda\mathbf{RP}$ is the system which has Π-formation rules $\mathbf{R}$ and parameter rules $\mathbf{P}$.
- Let $\lambda\mathbf{RP}$ parametrically conservative (i.e., $(s_1, s_2) \in \mathbf{P}$ implies $(s_1, s_2) \in \mathbf{R}$).
  - The parameter-free system $\lambda\mathbf{R}$ is at least as powerful as $\lambda\mathbf{RP}$.
  - If $\Gamma \vdash_{\mathbf{RP}} a : A$ then $\{\Gamma\} \vdash_{\mathbf{R}} \{a\} : \{A\}$.

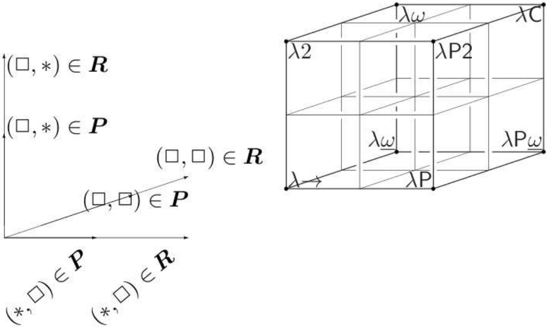
Example

- $\mathbf{R} = \{(*,*),(*,\Box)\}$
  $\mathbf{P}_1 = \emptyset$ $\mathbf{P}_2 = \{(*,*)\}$ $\mathbf{P}_3 = \{(*,\Box)\}$ $\mathbf{P}_4 = \{(*,*),(*,\Box)\}$
  All $\lambda\mathbf{RP}_i$ for $1 \leq i \leq 4$ with the above specifications are all equal in power.
- $\mathbf{R}_5 = \{(*,*)\}$ $\mathbf{P}_5 = \{(*,*),(*,\Box)\}$.
  $\lambda\!\rightarrow \; < \; \lambda\mathbf{R}_5\mathbf{P}_5 \; < \; \lambda\mathrm{P}$: we can to talk about predicates:

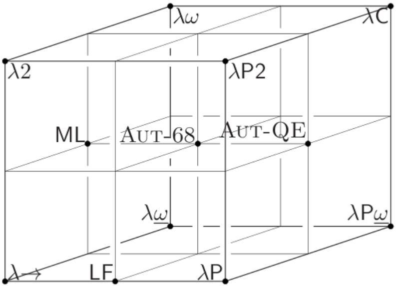
$$\begin{array}{rcl}
\alpha & : & *, \\
\mathtt{eq}(\mathtt{x}{:}\alpha, \mathtt{y}{:}\alpha) & : & *, \\
\mathtt{refl}(\mathtt{x}{:}\alpha) & : & \mathtt{eq}(\mathtt{x},\mathtt{x}), \; . \\
\mathtt{symm}(\mathtt{x}{:}\alpha, \mathtt{y}{:}\alpha, \mathtt{p}{:}\mathtt{eq}(\mathtt{x},\mathtt{y})) & : & \mathtt{eq}(\mathtt{y},\mathtt{x}), \\
\mathtt{trans}(\mathtt{x}{:}\alpha, \mathtt{y}{:}\alpha, \mathtt{z}{:}\alpha, \mathtt{p}{:}\mathtt{eq}(\mathtt{x},\mathtt{y}), \mathtt{q}{:}\mathtt{eq}(\mathtt{y},\mathtt{z})) & : & \mathtt{eq}(\mathtt{x},\mathtt{z})
\end{array}$$

eq not possible in $\lambda\!\rightarrow$.

The refined Barendregt Cube



LF, ML, AUT-68, and AUT-QE in the refined Cube

- 

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Logicians versus mathematicians: induction over numbers

- *Logician* uses `ind`: `Ind` as proof term for an application of the induction axiom.
  The type `Ind` can only be described in $\lambda\mathbf{R}$ where $\mathbf{R} = \{(*,*),(*,\Box),(\Box,*)\}$:

$$\mathtt{Ind} = \Pi p{:}(\mathbb{N}{\to}*).p0{\to}(\Pi n{:}\mathbb{N}.\Pi m{:}\mathbb{N}.pn{\to}Snm{\to}pm){\to}\Pi n{:}\mathbb{N}.pn \quad (1)$$

- Mathematician uses `ind` only with $P : \mathbb{N}{\to}*$, $Q : P0$ and $R : (\Pi n{:}\mathbb{N}.\Pi m{:}\mathbb{N}.Pn{\to}Snm{\to}Pm)$ to form a term $(\mathtt{ind}PQR){:}(\Pi n{:}\mathbb{N}.Pn)$.
- The use of the induction axiom by the mathematician is better described by the parametric scheme (p, q and r are the *parameters* of the scheme):

$$\mathtt{ind}(p{:}\mathbb{N}{\to}*, q{:}p0, r{:}(\Pi n{:}\mathbb{N}.\Pi m{:}\mathbb{N}.pn{\to}Snm{\to}pm)) : \Pi n{:}\mathbb{N}.pn \quad (2)$$

- The logician's type `Ind` is not needed by the mathematician and the types that occur in 2 can all be constructed in $\lambda\mathbf{R}$ with $\mathbf{R} = \{(*,*)(*,\Box)\}$.

Logicians versus mathematicians: induction over numbers

- *Mathematician applies* the induction axiom and doesn't need to know the proof-theoretical backgrounds.
- A *logician develops* the induction axiom (or studies its properties).
- $(\Box, *)$ is not needed by the mathematician. It is needed in logician's approach in order to form the Π-abstraction $\Pi p{:}(\mathbb{N} \to *). \cdots)$.
- Consequently, the type system that is used to describe the mathematician's use of the induction axiom can be weaker than the one for the logician.
- Nevertheless, the parameter mechanism gives the mathematician limited (but for his purposes sufficient) access to the induction scheme.

- Parameters enable the same expressive power as the high-level case, while allowing us to stay at a lower order. E.g. *first-order with parameters* versus *second-order without* [43].
- Desirable properties of the lower order theory (*decidability, easiness of calculations, typability*) can be maintained, without losing the flexibility of the higher-order aspects.
- Parameters enable us to find an exact position of type systems in the generalised framework of type systems.
- Parameters describe the difference between *developers* and *users* of systems.

Identifying λ and Π (see [37])

- In the cube, the syntax for terms (functions) and types was intermixed with the only distinction being λ- versus Π-abstraction.
- We unify the two abstractions into one.
  $\mathcal{T}_\flat ::= \mathcal{V} \mid \mathbf{S} \mid \mathcal{T}_\flat \mathcal{T}_\flat \mid \flat \mathcal{V}{:}\mathcal{T}_\flat . \mathcal{T}_\flat$
- $\mathcal{V}$ is a set of variables and $\mathbf{S} = \{*, \Box\}$.
- The β-reduction rule becomes
  $(\flat)$ $\qquad (\flat_{x:A}.B)C \to_\flat B[x := C].$
- Now we also have the old Π-reduction $(\Pi_{x:A}.B)C \to_\Pi B[x := C]$ which treats type instantiation like function instantiation.
- The type formation rule becomes

$$(\flat_1) \quad \frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\flat x{:}A.B) : s_2} \quad (s_1, s_2) \in \mathbf{R}$$

$$\text{(axiom)} \qquad \langle\rangle \vdash * : \Box$$

$$\text{(start)} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A} \quad x \notin \mathrm{DOM}(\Gamma)$$

$$\text{(weak)} \qquad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x{:}C \vdash A : B} \quad x \notin \mathrm{DOM}(\Gamma)$$

$$(\flat_2) \qquad \frac{\Gamma, x{:}A \vdash b : B \qquad \Gamma \vdash (\flat x{:}A.B) : s}{\Gamma \vdash (\flat x{:}A.b) : (\flat x{:}A.B)}$$

$$(\text{app}\flat) \qquad \frac{\Gamma \vdash F : (\flat x{:}A.B) \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x{:=}a]}$$

$$\text{(conv)} \qquad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s \qquad B =_\beta B'}{\Gamma \vdash A : B'}$$

Translations between the systems with 2 binders and those with one binder

- For $A \in \mathcal{T}$, we define $\overline{A} \in \mathcal{T}_\flat$ as follows:
  - $\overline{s} \equiv s \qquad \overline{x} \equiv x \qquad \overline{AB} \equiv \overline{A}\,\overline{B}$
  - $\overline{\lambda_{x:A}.B} \equiv \overline{\Pi_{x:A}.B} \equiv \flat_{x:\overline{A}}.\overline{B}$.
- For contexts we define: $\overline{\langle\rangle} \equiv \langle\rangle \quad \overline{\Gamma, x : A} \equiv \overline{\Gamma}, x : \overline{A}$.
- For $A \in \mathcal{T}_\flat$, we define $[A]$ to be $\{A' \in \mathcal{T}$ such that $\overline{A'} \equiv A\}$.
- For context, obviously: $[\Gamma] \equiv \{\Gamma'$ such that $\overline{\Gamma'} \equiv \Gamma\}$.

Isomorphism of the cube and the $\flat$-cube

- If $\Gamma \vdash A : B$ then $\overline{\Gamma} \vdash_\flat \overline{A} : \overline{B}$.
- If $\Gamma \vdash_\flat A : B$ then there are unique $\Gamma' \in [\Gamma]$, $A' \in [A]$ and $B' \in [B]$ such that $\Gamma' \vdash_\pi A' : B'$.
- The $\flat$-cube enjoys all the properties of the cube except the unicity of types.

Organised multiplicity of Types for $\vdash_\flat$ and $\rightarrow_\flat$ [37] For many type systems, unicity of types is not necessary (e.g. Nuprl).

We have however an organised multiplicity of types.

1. If $\Gamma \vdash_\flat A : B_1$ and $\Gamma \vdash_\flat A : B_2$, then $B_1 \overset{\diamond}{=}_\flat B_2$.
2. If $\Gamma \vdash_\flat A_1 : B_1$ and $\Gamma \vdash_\flat A_2 : B_2$ and $A_1 =_\flat A_2$, then $B_1 \overset{\diamond}{=}_\flat B_2$.
3. If $\Gamma \vdash_\flat B_1 : s_1$, $B_1 =_\flat B_2$ and $\Gamma \vdash_\flat A : B_2$ then $\Gamma \vdash_\flat B_2 : s_1$.
4. Assume $\Gamma \vdash_\flat A : B_1$ and $(\Gamma \vdash_\flat A : B_1)^{-1} = (\Gamma', A', B_1')$. Then $B_1 =_\flat B_2$ if:
    1. either $\Gamma \vdash_\flat A : B_2$, $(\Gamma \vdash_\flat A : B_2)^{-1} = (\Gamma', A'', B_2')$ and $B_1' =_\beta B_2'$,
    2. or $\Gamma \vdash_\flat C : B_2$, $(\Gamma \vdash_\flat C : B_2)^{-1} = (\Gamma', C', B_2')$ and $A' =_\beta C'$.

Extending the cube with Π-reduction loses subject reduction [38] If we change (appl) by (new appl) in the cube we lose subject reduction.

$$\text{(appl)} \quad \frac{\Gamma \vdash F : (\Pi_{x:A}.B) \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x := a]}$$

$$\text{(new appl)} \quad \frac{\Gamma \vdash F : (\Pi_{x:A}.B) \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : (\Pi_{x:A}.B)a}$$

[38] solved the problem by re-incorporating Frege and Russell's notions of low level functions (which was lost in Church's notion of function).

The same problem and solution can be repeated in our ♭-cube.

Adding type instantiation to the typing rules of the $\flat$-cube If we change (app$\flat$) by (new app$\flat$) in the $\flat$-cube we lose subject reduction.

$$(\text{app}\flat) \quad \frac{\Gamma \vdash_\flat F : (\Pi_{x:A}.B) \qquad \Gamma \vdash_\flat a : A}{\Gamma \vdash_\flat Fa : B[x := a]}$$

$$(\text{app}\flat\flat) \quad \frac{\Gamma \vdash_\flat F : (\flat_{x:A}.B) \qquad \Gamma \vdash_\flat a : A}{\Gamma \vdash_\flat Fa : (\flat_{x:A}.B)a}$$

Failure of correctness of types and subject reduction

- **Correctness of types no longer holds.** With (appl$\flat\flat$) one can have $\Gamma \vdash A : B$ without $B \equiv \Box$ or $\exists S \,.\, \Gamma \vdash B : S$.
- For example, $z : *, x : z \vdash (\flat_{y:z}.y)x : (\flat_{y:z}.z)x$ yet $(\flat_{y:z}.z)x \not\equiv \Box$ and $\forall s \,.\, z : *, x : z \not\vdash (\flat_{y:z}.z)x : s$.
- **Subject Reduction no longer holds.** That is, with (appl$\flat$): $\Gamma \vdash A : B$ and $A \twoheadrightarrow A'$ may not imply $\Gamma \vdash A' : B$.
- For example, $z : *, x : z \vdash (\flat_{y:z}.y)x : (\flat_{y:z}.z)x$ and $(\flat_{y:z}.y)x \to_\flat x$, but one can't show $z : *, x : z \vdash x : (\flat_{y:z}.z)x$.

Solving the problem Keep all the typing rules of the $\flat$-cube the same except: replace (conv) by (new-conv), (appl$\flat$) by (appl$\flat\flat$) and add three new rules as follows:

(start-def) $$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : A}{\Gamma, x = B{:}A \vdash x : A} \qquad x \notin \mathrm{DOM}\,(\Gamma)$$

(weak-def) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s \quad \Gamma \vdash D : C}{\Gamma, x = D{:}C \vdash A : B} \qquad x \notin \mathrm{DOM}\,(\Gamma)$$

(def) $$\frac{\Gamma, x = B{:}A \vdash C : D}{\Gamma \vdash (\flat x{:}A.C)B : D[x := B]}$$

(new-conv) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s \qquad \Gamma \vdash B =_{def} B'}{\Gamma \vdash A : B'}$$

(appl$\flat\flat$) $$\frac{\Gamma \vdash F : \flat_{x:A}.B \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : (\flat_{x:A}.B)a}$$

In the conversion rule, $\Gamma \vdash B =_{def} B'$ is defined as:

- If $B =_\flat B'$ then $\Gamma \vdash B =_{def} B'$
- If $x = D : C \in \Gamma$ and B' arises from B by substituting one particular free occurrence of x in B by D then $\Gamma \vdash B =_{def} B'$.
- Our 3 new rules and the definition of $\Gamma \vdash B =_{def} B'$ are trying to re-incorporate low-level aspects of functions that are not present in Church's λ-calculus.
- *In fact, our new framework is closer to Frege's abstraction principle and the principles ∗9·14 and ∗9·15 of [54].*

Correctness of types holds.

- We demonstrate this with the earlier example.
- Recall that we have $z:*, x:z \vdash (\flat_{y:z}.y)x : (\flat_{y:z}.z)x$ and want that for some s, $z:*, x:z \vdash (\flat_{y:z}.z)x : s$.
- Here is how the latter formula now holds:

$$\begin{array}{ll} z:*, x:z \vdash z:* & \text{(start and weakening)} \\ z:*, x:z.y:z\rangle x \vdash z:* & \text{(weakening)} \\ z:*, x:z \vdash (\flat_{y:z}.z)x : *[y:=x] \equiv * & \text{(def rule)} \end{array}$$

Subject Reduction holds.

- We demonstrate this with the earlier example.
- Recall that we have $z : *, x : z \vdash (\flat_{y:z}.y)x : (\flat_{y:z}.z)x$ and $(\lambda_{y:z}.y)x \to_\beta x$ and we need to show that $z : *, x : z \vdash x : (\flat_{y:z}.z)x$.
- Here is how the latter formula now holds:

a.	$z : *, x : z \vdash x : z$	(start and weakening)
b.	$z : *, x : z \vdash (\flat_{y:z}.z)x : *$	(from 1 above)
	$z : *, x : z \vdash x : (\flat_{y:z}.z)x$	(conversion, a, b, and $z =_\beta (\flat_{y:z}.z)x$)

De Bruijn's typed λ-calculi started with his Automath

- In 1967, an internationally renowned mathematician called N.G. de Bruijn wanted to do something never done before: use the computer to formally check the correctness of mathematical books.
- Such a task needs a good formalisation of mathematics, a good competence in implementation, and extreme attention to all the details so that nothing is left informal.
- Implementing extensive formal systems on the computer was never done before.
- De Bruijn, an extremely original mathematician, did every step his own way.
- He proudly announced at the ceremony of the publications of the collected Automath work: *I did it my way*.
- Dirk van Dalen said at the ceremony: *The Germans have their 3 B's, but we Dutch too have our 3 B's: Beth, Brouwer and de Bruijn*.

There is a fourth B:

They look good together

Theme 1: De Bruijn Indices and Explicit Substitutions [19]

- Classical λ-calculus: $A ::= x \mid (\lambda x.B) \mid (BC)$
  $(\lambda x.A)B \to_\beta A[x := B]$
- $(\lambda x.\lambda y.xy)y \to_\beta (\lambda y.xy)[x := y] \neq \lambda y.yy$
- $(\lambda x.\lambda y.xy)y \to_\beta (\lambda y.xy)[x := y] =_\alpha (\lambda z.xz)[x := y] = \lambda z.yz$
- $\lambda x.x$ and $\lambda y.y$ are the same function. Write this function as $\lambda 1$.
- Assume a free variable list (say $x, y, z, \ldots$).
- $(\lambda\lambda 2\,1)2 \to_\beta (\lambda 2\,1)[1 := 2] = \lambda(2[2 := 3])(1[2 := 3]) = \lambda 3\,1$

Classical λ-calculus with de Bruijn indices

- Let $i, n \geq 1$ and $k \geq 0$
- $A ::= n \,|\, (\lambda B) \,|\, (BC)$
  $(\lambda A)B \rightarrow_\beta A\{\!\{1 \leftarrow B\}\!\}$
- $$U^i_k(AB) = U^i_k(A)\, U^i_k(B)$$
  $$U^i_k(\lambda A) = \lambda(U^i_{k+1}(A))$$
  $$U^i_k(\mathtt{n}) = \begin{cases} \mathtt{n+i-1} & \text{if } n > k \\ \mathtt{n} & \text{if } n \leq k\,. \end{cases}$$
- $$\begin{array}{lcl} (A_1 A_2)\{\!\{\mathtt{i} \leftarrow B\}\!\} & = & (A_1\{\!\{\mathtt{i} \leftarrow B\}\!\})\,(A_2\{\!\{\mathtt{i} \leftarrow B\}\!\}) \\ (\lambda A)\{\!\{\mathtt{i} \leftarrow B\}\!\} & = & \lambda(A\{\!\{\mathtt{i+1} \leftarrow B\}\!\}) \end{array}$$
  $$\mathtt{n}\{\!\{\mathtt{i} \leftarrow B\}\!\} = \begin{cases} \mathtt{n-1} & \text{if } n > i \\ U^i_0(B) & \text{if } n = i \\ \mathtt{n} & \text{if } n < i\,. \end{cases}$$
- Numerous implementations of proof checkers and programming languages have been based on de Bruijn indices.

Substitution calculus λs [36]

- Write $A\{\{n \leftarrow B\}\}$ as $A\sigma^n B$ and $U^i_k(A)$ as $\varphi^i_k A$.
- $A ::= n \mid (\lambda B) \mid (BC) \mid (A\sigma^i B) \mid (\varphi^i_k B)$ *where* $i, n \geq 1, \quad k \geq 0$.

σ-generation	$(\lambda A)\, B$	$\longrightarrow$	$A\,\sigma^1\, B$
σ-λ-transition	$(\lambda A)\,\sigma^i B$	$\longrightarrow$	$\lambda(A\,\sigma^{i+1}\, B)$
σ-app-transition	$(A_1\, A_2)\,\sigma^i B$	$\longrightarrow$	$(A_1\,\sigma^i B)\,(A_2\,\sigma^i B)$
σ-destruction	$\mathtt{n}\,\sigma^i B$	$\longrightarrow$	$\begin{cases} \mathtt{n}-1 & \text{if} \quad n > i \\ \varphi^i_0\, B & \text{if} \quad n = i \\ \mathtt{n} & \text{if} \quad n < i \end{cases}$
φ-λ-transition	$\varphi^i_k(\lambda A)$	$\longrightarrow$	$\lambda(\varphi^i_{k+1}\, A)$
φ-app-transition	$\varphi^i_k(A_1\, A_2)$	$\longrightarrow$	$(\varphi^i_k\, A_1)\,(\varphi^i_k\, A_2)$
φ-destruction	$\varphi^i_k\,\mathtt{n}$	$\longrightarrow$	$\begin{cases} \mathtt{n}+\mathtt{i}-1 & \text{if} \quad n > k \\ \mathtt{n} & \text{if} \quad n \leq k \end{cases}$

1. The s-calculus (i.e., λs minus σ-generation) is strongly normalising,
2. The λs-calculus is confluent and simulates (in small steps) β-reduction
3. The λs-calculus preserves strong normalisation PSN.
4. The λs-calculus has a confluent extension with open terms λse.

- The λs-calculus was the first calculus of substitutions which satisfies all the above properties 1., 2., 3. and 4.

$\lambda\upsilon$ [8] Terms: $\Lambda\upsilon^t ::= \mathbb{N} \mid \Lambda\upsilon^t\Lambda\upsilon^t \mid \lambda\Lambda\upsilon^t \mid \Lambda\upsilon^t[\Lambda\upsilon^s]$
Substitutions: $\Lambda\upsilon^s ::= \uparrow \mid \Uparrow(\Lambda\upsilon^s) \mid \Lambda\upsilon^t$.

$$
\boxed{
\begin{array}{llcl}
\textit{(Beta)} & (\lambda a)\, b & \longrightarrow & a\,[b/] \\
\textit{(App)} & (a\, b)[s] & \longrightarrow & (a\,[s])\,(b\,[s]) \\
\textit{(Abs)} & (\lambda a)[s] & \longrightarrow & \lambda(a\,[\Uparrow(s)]) \\
\textit{(FVar)} & 1\,[a/] & \longrightarrow & a \\
\textit{(RVar)} & \mathtt{n}+1\,[a/] & \longrightarrow & \mathtt{n} \\
\textit{(FVarLift)} & 1\,[\Uparrow(s)] & \longrightarrow & 1 \\
\textit{(RVarLift)} & \mathtt{n}+1\,[\Uparrow(s)] & \longrightarrow & \mathtt{n}\,[s]\,[\uparrow] \\
\textit{(VarShift)} & \mathtt{n}\,[\uparrow] & \longrightarrow & \mathtt{n}+1
\end{array}
}
$$

$\lambda\upsilon$ satisfies 1., 2., and 3., but does not have a confluent extension on open terms.

$\lambda\sigma_{\Uparrow}$ Terms: $\Lambda\sigma^t_{\Uparrow} ::= \mathbb{N} \mid \Lambda\sigma^t_{\Uparrow}\Lambda\sigma^t_{\Uparrow} \mid \lambda\Lambda\sigma^t_{\Uparrow} \mid \Lambda\sigma^t_{\Uparrow}[\Lambda\sigma^s_{\Uparrow}]$

Substitutions: $\Lambda\sigma^s_{\Uparrow} ::= id \mid \uparrow \mid \Uparrow(\Lambda\sigma^s_{\Uparrow}) \mid \Lambda\sigma^t_{\Uparrow} \cdot \Lambda\sigma^s_{\Uparrow} \mid \Lambda\sigma^s_{\Uparrow} \circ \Lambda\sigma^s_{\Uparrow}$.

(Beta)	$(\lambda a)\, b$	$\longrightarrow$	$a\,[b \cdot id]$
(App)	$(a\, b)[s]$	$\longrightarrow$	$(a\,[s])\,(b\,[s])$
(Abs)	$(\lambda a)[s]$	$\longrightarrow$	$\lambda(a\,[\Uparrow(s)])$
(Clos)	$(a\,[s])[t]$	$\longrightarrow$	$a\,[s \circ t]$
(Varshift1)	$\mathtt{n}\,[\uparrow]$	$\longrightarrow$	$\mathtt{n}+1$
(Varshift2)	$\mathtt{n}\,[\uparrow \circ\, s]$	$\longrightarrow$	$\mathtt{n}+1\,[s]$
(FVarCons)	$1\,[a \cdot s]$	$\longrightarrow$	a
(RVarCons)	$\mathtt{n}+1\,[a \cdot s]$	$\longrightarrow$	$\mathtt{n}\,[s]$
(FVarLift1)	$1\,[\Uparrow(s)]$	$\longrightarrow$	1
(FVarLift2)	$1\,[\Uparrow(s) \circ t]$	$\longrightarrow$	$1\,[t]$
(RVarLift1)	$\mathtt{n}+1\,[\Uparrow(s)]$	$\longrightarrow$	$\mathtt{n}[s \circ \uparrow]$
(RVarLift2)	$\mathtt{n}+1\,[\Uparrow(s) \circ t]$	$\longrightarrow$	$\mathtt{n}[s \circ (\uparrow \circ\, t)]$

$\lambda\sigma_{\Uparrow}$ rules continued

(Map)	$(a \cdot s) \circ t$	$\longrightarrow$	$a[t] \cdot (s \circ t)$
(Ass)	$(s \circ t) \circ u$	$\longrightarrow$	$s \circ (t \circ u)$
(ShiftCons)	$\uparrow \circ (a \cdot s)$	$\longrightarrow$	s
(ShiftLift1)	$\uparrow \circ \Uparrow(s)$	$\longrightarrow$	$s \circ \uparrow$
(ShiftLift2)	$\uparrow \circ (\Uparrow(s) \circ t)$	$\longrightarrow$	$s \circ (\uparrow \circ t)$
(Lift1)	$\Uparrow(s) \circ \Uparrow(t)$	$\longrightarrow$	$\Uparrow(s \circ t)$
(Lift2)	$\Uparrow(s) \circ (\Uparrow(t) \circ u)$	$\longrightarrow$	$\Uparrow(s \circ t) \circ u$
(LiftEnv)	$\Uparrow(s) \circ (a \cdot t)$	$\longrightarrow$	$a \cdot (s \circ t)$
(IdL)	$id \circ s$	$\longrightarrow$	s
(IdR)	$s \circ id$	$\longrightarrow$	s
(LiftId)	$\Uparrow(id)$	$\longrightarrow$	id
(Id)	$a[id]$	$\longrightarrow$	a

$\lambda\sigma_{\Uparrow}$ satisfies 1., 2., and 4., but does not have PSN.

How is λse obtained from λs?

- They said, we can have *open terms (holes in proofs)* in $\lambda\sigma$, can you do so in λs?
- $A ::= X \,|\, n \,|\, (\lambda B) \,|\, (BC) \,|\, (A\sigma^i B) \,|\, (\varphi^i_k B) \quad where \quad i, n \geq 1\,, \quad k \geq 0\,.$
- Extending the syntax of λs with open terms without extending the λs-rules loses the confluence (even local confluence):
  $((\lambda X)Y)\sigma^1 1 \to (X\sigma^1 Y)\sigma^1 1$
  $((\lambda X)Y)\sigma^1 1 \to ((\lambda X)\sigma^1 1)(Y\sigma^1 1)$
- $(X\sigma^1 Y)\sigma^1 1$ and $((\lambda X)\sigma^1 1)(Y\sigma^1 1)$ have no common reduct.
- But, $((\lambda X)\sigma^1 1)(Y\sigma^1 1) \twoheadrightarrow (X\sigma^2 1)\sigma^1(Y\sigma^1 1)$

Simple: add de Bruijn's metasubstitution and distribution lemmas to the rules of λs

- Add the well-known meta-substitution $(\sigma - \sigma)$ and distribution $(\varphi - \sigma)$ lemmas (and the 4 extra lemmas needed to prove them).

σ-σ	$(A\sigma^i B)\,\sigma^j\, C$	$\longrightarrow$	$(A\,\sigma^{j+1}\,C)\,\sigma^i\,(B\,\sigma^{j-i+1}\,C)$	if	$i \le j$
σ-φ 1	$(\varphi^i_k\,A)\,\sigma^j\,B$	$\longrightarrow$	$\varphi^{i-1}_k\,A$	if	$k < j < k+i$
σ-φ 2	$(\varphi^i_k\,A)\,\sigma^j\,B$	$\longrightarrow$	$\varphi^i_k(A\,\sigma^{j-i+1}\,B)$	if	$k+i \le j$
φ-σ	$\varphi^i_k(A\,\sigma^j\,B)$	$\longrightarrow$	$(\varphi^i_{k+1}\,A)\,\sigma^j\,(\varphi^i_{k+1-j}\,B)$	if	$j \le k+1$
φ-φ 1	$\varphi^i_k\,(\varphi^j_l\,A)$	$\longrightarrow$	$\varphi^j_l\,(\varphi^i_{k+1-j}\,A)$	if	$l+j \le k$
φ-φ 2	$\varphi^i_k\,(\varphi^j_l\,A)$	$\longrightarrow$	$\varphi^{j+i-1}_l\,A$	if	$l \le k < l+j$

- $(\sigma - \sigma)$:
  $A[x := B][y := C] = A[y := C][x := B[y := C]]$ if $x \neq y$ and $x \notin FV(C)$.
- $(\varphi - \sigma)$:
  $updatedA[x := B] = updatedA[x := updatedB]$.

Where did the extra rules come from? In de Bruijn's classical λ-calculus we have the lemmas:

1. *($\sigma - \varphi$ 1)* For $k < j < k + i$ we have: $U_k^{i-1}(A) = U_k^i(A)\{\!\{\mathtt{j} \leftarrow B\}\!\}$.
2. *($\varphi - \varphi$ 2)* For $l \leq k < l + j$ we have: $U_k^i(U_l^j(A)) = U_l^{j+i-1}(A)$.
3. *($\sigma - \varphi$ 2)* For $k + i \leq j$ we have: $U_k^i(A)\{\!\{\mathtt{j} \leftarrow B\}\!\} = U_k^i(A\{\!\{\mathtt{j} - \mathtt{i} + 1 \leftarrow B\}\!\})$.
4. *($\sigma - \sigma$) [Meta-substitution lemma]* For $i \leq j$ we have: $A\{\!\{\mathtt{i} \leftarrow B\}\!\}\{\!\{\mathtt{j} \leftarrow C\}\!\} = A\{\!\{\mathtt{j} + 1 \leftarrow C\}\!\}\{\!\{\mathtt{i} \leftarrow B\{\!\{\mathtt{j} - \mathtt{i} + 1 \leftarrow C\}\!\}\}\!\}$.

- The proof of $(\sigma - \sigma)$ uses $(\sigma - \varphi\ 1)$ and $(\sigma - \varphi\ 2)$ both with $k = 0$.
- The proof of $(\sigma - \varphi\ 2)$ requires $(\varphi - \varphi\ 2)$ with $l = 0$.

Where did the extra rules come from (continued)? In de Bruijn's classical λ-calculus we also have the lemmas:

1. *($\varphi - \varphi$ 1)* For $j \leq k+1$ we have:
   $U^i_{k+p}(U^j_p(A)) = U^j_p(U^i_{k+p+1-j}(A))$.
2. *($\varphi - \sigma$) [Distribution lemma]*
   For $j \leq k+1$ we have:
   $U^i_k(A\{\!\{\mathrm{j} \leftarrow B\}\!\}) = U^i_{k+1}(A)\{\!\{\mathrm{j} \leftarrow U^i_{k+1-j}(B)\}\!\}$.

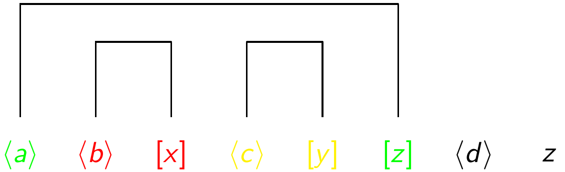
- $(\varphi - \varphi\ 1)$ with $p = 0$ is needed to prove $(\varphi - \sigma)$.

Theme 2: Lambda Calculus à la de Bruijn

- $\mathcal{I}(x) = x, \qquad \mathcal{I}(\lambda x.B) = [x]\mathcal{I}(B), \qquad \mathcal{I}(AB) = \langle \mathcal{I}(B) \rangle \mathcal{I}(A)$
- $(\lambda x.\lambda y.xy)z$ translates to $\langle z \rangle [x][y] \langle y \rangle x$.
- The *applicator wagon* $\langle z \rangle$ and *abstractor wagon* $[x]$ occur NEXT to each other.
- $(\lambda x.A)B \to_\beta A[x := B]$ becomes $\langle B \rangle [x]\ A \to_\beta [x := B]A$
- The "bracketing structure" of $((\ \lambda_x.(\lambda_y.\lambda_z.-)c)ba)$ is $[_1\ \ [_2\ \ [_3\ \ ]_2\ \ ]_1\ \ ]_3$', where '$[_i$' and '$]_i$' match.
- The bracketing structure of $\langle a \rangle \langle b \rangle [x] \langle c \rangle [y][z]\ \langle d \rangle$ is simpler: $[\ [\ ][\ ]]$.
- $\langle b \rangle [x]$ and $\langle c \rangle [y]$ are AT-pairs whereas $\langle a \rangle [z]$ is an AT-couple.

Redexes in de Bruijn's notation

Classical Notation	de Bruijn's Notation
$(\underline{(\lambda_x.(\lambda_y.\lambda_z.zd)c)b})a$	$\langle a\rangle\underline{\langle b\rangle[x]}\langle c\rangle[y][z]\langle d\rangle z$
$\downarrow_\beta$	$\downarrow_\beta$
$(\underline{(\lambda_y.\lambda_z.zd)c})a$	$\langle a\rangle\underline{\langle c\rangle[y]}[z]\langle d\rangle z$
$\downarrow_\beta$	$\downarrow_\beta$
$\underline{(\lambda_z.zd)a}$	$\underline{\langle a\rangle[z]}\langle d\rangle z$
$\downarrow_\beta$	$\downarrow_\beta$
ad	$\langle d\rangle a$



- This maks it easy to study local/global/mini reductions into the λ-calculus [12, 3, 35]

Some notions of reduction studied in the literature

Name	*In Classical Notation*	*In de Bruijn's notation*
(θ)	$((\lambda_x.N)P)Q$ $\downarrow$ $(\lambda_x.NQ)P$	$\langle Q\rangle\langle P\rangle[x]N$ $\downarrow$ $\langle P\rangle[x]\langle Q\rangle N$
(γ)	$(\lambda_x.\lambda_y.N)P$ $\downarrow$ $\lambda_y.(\lambda_x.N)P$	$\langle P\rangle[x][y]N$ $\downarrow$ $[y]\langle P\rangle[x]N$
(γ_C)	$((\lambda_x.\lambda_y.N)P)Q$ $\downarrow$ $(\lambda_y.(\lambda_x.N)P)Q$	$\langle Q\rangle\langle P\rangle[x][y]N$ $\downarrow$ $\langle Q\rangle[y]\langle P\rangle[x]N$
(g)	$((\lambda_x.\lambda_y.N)P)Q$ $\downarrow$ $(\lambda_x.N[y:=Q])P$	$\langle Q\rangle\langle P\rangle[x][y]N$ $\downarrow$ $\langle P\rangle[x][y:=Q]N$
(β_e)	? $\downarrow$?	$\langle Q\rangle\overline{s}[y]N$ $\downarrow$ $\overline{s}[y:=Q]N$

A Few Uses of these reductions/term reshuffling

- [48] uses θ and γ in analyzing perpetual reduction strategies.
- Term reshuffling is used in [2, 41] in analyzing typability problems.
- [1, 20, 42, 5] use generalised reduction and/or term reshuffling in relating SN to WN.
- [6] uses a form of term-reshuffling in obtaining a calculus that corresponds to lazy functional evaluation.
- [3, 39, 4, 10] shows that they could reduce space/time needs.
- All these works have been heavily influenced by de Bruijn's Automath whose λ-notation facilitated the manipulation of redexes.
- All can be represented clearer in de Bruijn's notation.

Even more: de Bruijn's generalised reduction has better properties

$$
\begin{array}{llll}
(\beta) & (\lambda_x.M)N \to M[x := N] & \\
(\beta_I) & (\lambda_x.M)N \to M[x := N] & \text{if } x \in FV(M) \\
(\beta_K) & (\lambda_x.M)N \to M & \text{if } x \notin FV(M) \\
(\theta) & (\lambda_x.N)PQ \to (\lambda_x.NQ)P & \\
(\beta_e) & (M)\overline{s}[x]N \to \overline{s}\{N[x := M] & \text{for } \overline{s} \text{ well-balanced.}
\end{array}
$$

- [5] shows that β_e satisfies PSN, postponment of K-contraction and conservation (latter 2 properties fail for β-reduction).
- Conservation of β_e: If A is $\beta_e I$-normalisable then A is β_e-strongly normalisable.
- Postponment of K-contraction : Hence, discard arguments of K-redexes after I-reduction. This gives flexibility in implementation: unnecessary work can be delayed, or even completely avoided.

- Attempts have been made at establishing some reduction relations for which postponement of K-contractions and conservation hold.
- The picture is as follows (-N stands for normalising and $r \in \{\beta_I, \theta_K\}$).

(β_K-postponement for r)	If $M \to_{\beta_K} N \to_r O$ then $\exists P$ such that $M \twoheadrightarrow^+_{\beta_I \theta_K}$	
(Conservation for β_I)	If M is β_I-N then M is β_I-SN	Barendre
(Conservation for $\beta + \theta$)	If M is $\beta_I \theta_K$-N then M is β-SN	[20]

- De Groote does not produce these results for a single reduction relation, but for $\beta + \theta$ (this is more restrictive than β_e).
- β_e is the first single relation to satisfy β_K-postponement and conservation.
- [5] shows that:

(β_{eK}-postponement for β_e)	If $M \to_{\beta_{eK}} N \to_{\beta_{eI}} O$ then $\exists P$ such that $M \to_{\beta_{eI}}$
(Conservation for β_e)	If M is β_{eI}-N then M is β_e-SN

Common Mathematical Language of mathematicians: CML

+ CML is *expressive*: it has linguistic categories like *proofs* and *theorems*.
+ CML has been refined by intensive use and is rooted in *long traditions*.
+ CML is *approved* by most mathematicians as a communication medium.
+ CML *accommodates many branches* of mathematics, and is adaptable to new ones.
– Since CML is based on natural language, it is *informal* and *ambiguous*.
– CML is *incomplete:* Much is left implicit, appealing to the reader's intuition.
– CML is *poorly organised:* In a CML text, many structural aspects are omitted.
– CML is *automation-unfriendly:* A CML text is a plain text and cannot be easily automated.

A CML-text

From chapter 1, § 2 of E. Landau's *Foundations of Analysis* (Landau 1930, 1951).

Theorem 6. [Commutative Law of Addition]

$$x + y = y + x.$$

Proof Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.
I) We have

$$y + 1 = y',$$

and furthermore, by the construction in the proof of Theorem 4,

$$1 + y = y',$$

so that

$$1 + y = y + 1$$

and 1 belongs to $\mathfrak{M}$.
II) If x belongs to $\mathfrak{M}$, then

$$x + y = y + x,$$

Therefore

$$(x + y)' = (y + x)' = y + x'.$$

By the construction in the proof of Theorem 4, we have

$$x' + y = (x + y)',$$

hence

$$x' + y = y + x',$$

so that x' belongs to $\mathfrak{M}$. The assertion therefore holds for all x. □

The problem with formal logic

- No logical language is an alternative to CML
    - A logical language does not have *mathematico-linguistic* categories, is *not universal* to all mathematicians, and is *not a good communication medium*.
    - Logical languages make fixed choices (*first versus higher order, predicative versus impredicative, constructive versus classical, types or sets*, etc.). But different parts of mathematics need different choices and there is no universal agreement as to which is the best formalism.
    - A logician reformulates in logic their *formalization* of a mathematical-text as a formal, complete text which is structured considerably *unlike* the original, and is of little use to the *ordinary* mathematician.
    - Mathematicians do not want to use formal logic and have *for centuries* done mathematics without it.
- *So, mathematicians kept to CML.*
- We would like to find an alternative to CML which avoids some of the features of the logical languages which made them unattractive to mathematicians.

What are the options for computerization?

Computers can handle mathematical text at various levels:

- Images of pages may be stored. While useful, this is not a good representation of *language* or *knowledge*.
- Typesetting systems like LaTeX, TeXmacs, can be used.
- Document representations like OpenMath, OMDoc, MathML, can be used.
- Formal logics used by theorem provers (Coq, Isabelle, HOL, Mizar, Isar, etc.) can be used.

We are gradually developing a system named Mathlang which we hope will eventually allow building a bridge between the latter 3 levels.

This talk aims at discussing the motivations rather than the details.

The issues with typesetting systems

+ A system like LaTeX, TeXmacs, provides good defaults for visual appearance, while allowing fine control when needed.
+ LaTeX and TeXmacs support commonly needed document structures, while allowing custom structures to be created.
– Unless the mathematician is amazingly disciplined, the *logical structure of symbolic formulas is not represented* at all.
– The *logical structure of mathematics as embedded in natural language text is not represented.* Automated discovery of the semantics of natural language text is still too primitive and requires human oversight.

LaTeX example

draft documents	✓
public documents	✓
computations and proofs	✗

```
\begin{theorem}[Commutative Law of Addition]\label{theorem:6}
$$x+y=y+x.$$
\end {theorem}
\begin{proof}
Fix $y$, and $\mathfrak{M}$ be the set of all $x$ for which
the assertion holds.
\begin{enumerate}
\item We have $$y+1=y',$$
and furthermore, by the construction in
the proof of Theorem \ref{theorem:4}, $$1+y=y',$$
so that $$1+y=y+1$$
and $1$ belongs to $\mathfrak{M}$.
```

```latex
\item If $x$ belongs to $\mathfrak{M}$, then $$x+y=y+x,$$
Therefore
$$(x+y)'=(y+x)'=y+x'.$$
By the construction in the proof of
Theorem \ref{theorem:4}, we have $$x'+y=(x+y)',$$
hence
$$x'+y=y+x',$$
so that $x'$ belongs to $\mathfrak{M}$.
\end{enumerate}
The assertion therefore holds for all $x$.
\end{proof}
```

Full formalization difficulties: choices

A CML-text is structured differently from a fully formalized text proving the same facts. *Making the latter involves extensive knowledge and many choices:*

- The choice of the *underlying logical system.*
- The choice of *how concepts are implemented* (equational reasoning, equivalences and classes, partial functions, induction, etc.).
- The choice of the *formal foundation*: a type theory (dependent?), a set theory (ZF? FM?), a category theory? etc.
- The choice of the *proof checker*: Automath, Isabelle, Coq, PVS, Mizar, HOL, ...

An issue is that one must in general commit to one set of choices.

Full formalization difficulties: informality

Any informal reasoning in a CML-text will cause various problems when fully formalizing it:

- A single (big) step may need to expand into a (series of) syntactic proof expressions. *Very long expressions can replace a clear CML-text.*
- The entire CML-text may need *reformulation* in a fully *complete* syntactic formalism where every detail is spelled out. New details may need to be woven throughout the entire text. The text may need to be *turned inside out.*
- Reasoning may be obscured by *proof tactics*, whose meaning is often *ad hoc* and implementation-dependent.

Regardless, ordinary mathematicians do not find the new text useful.

Coq example	draft documents	✗
	public documents	✗
	computations and proofs	✓

From Module `Arith.Plus` of Coq standard library (`http://coq.inria.fr/`).

```
Lemma plus_sym: (n,m:nat)(n+m)=(m+n).
Proof.
Intros n m ; Elim n ; Simpl_rew ; Auto with arith.
Intros y H ; Elim (plus_n_Sm m y) ; Simpl_rew ; Auto with arith.
Qed.
```

Mathlang's Goal: Open borders between mathematics, logic and computation

- Ordinary mathematicians *avoid* formal mathematical logic.
- Ordinary mathematicians *avoid* proof checking (via a computer).
- Ordinary mathematicians *may use* a computer for computation: there are over 1 million people who use Mathematica (including linguists, engineers, etc.).
- Mathematicians may also use other computer forms like Maple, LaTeX, etc.
- But we are not interested in only *libraries* or *computation* or *text editing*.
- We want *freedeom of movement* between mathematics, logic and computation.
- At every stage, we must have *the choice* of the level of formalilty and the depth of computation.

Aim for Mathlang? (Kamareddine and Wells 2001, 2002)

Can we formalise a mathematical text, avoiding as much as possible the ambiguities of natural language, while still guaranteeing the following four goals?

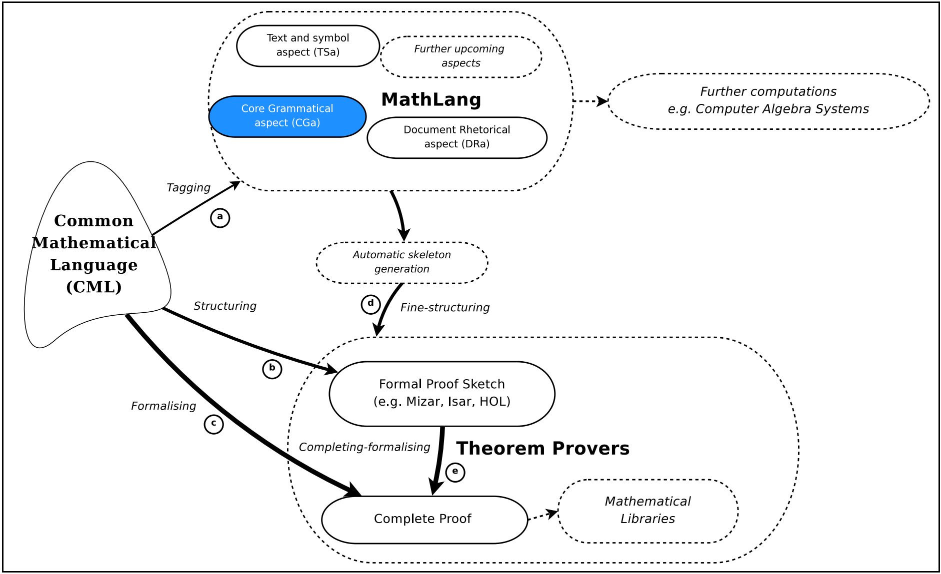
1. The formalised text looks very much like the original mathematical text (and hence the content of the original mathematical text is respected).
2. The formalised text can be fully manipulated and searched in ways that respect its mathematical structure and meaning.
3. Steps can be made to do computation (via computer algebra systems) and proof checking (via proof checkers) on the formalised text.
4. This formalisation of text is not much harder for the ordinary mathematician than LaTeX. *Full formalization down to a foundation of mathematics is not required*, although allowing and supporting this is one goal.

(No theorem prover's language satisfies these goals.)

Mathlang	draft documents	✓
	public documents	✓
	computations and proofs	✓

- A Mathlang text captures the grammatical and reasoning aspects of mathematical structure for further computer manipulation.
- A *weak type system* checks Mathlang documents at a grammatical level.
- A Mathlang text remains *close* to its CML original, allowing confidence that the CML has been captured correctly.
- We have been developing ways to weave natural language text into Mathlang.
- Mathlang aims to eventually support *all encoding uses*.
- The CML view of a Mathlang text should match the mathematician's intentions.
- The formal structure should be suitable for various automated uses.

Example of a MathLang Path



What is CGa? (Maarek's PhD thesis)

- CGa is a formal language derived from MV (N.G. de Bruijn 1987) and WTT (Kamareddine and Nederpelt 2004) which aims at expliciting the grammatical role played by the elements of a CML text.
- The structures and common concepts used in CML are captured by CGa with a finite set of grammatical/linguistic/syntactic categories: *Term* "$\sqrt{2}$", *set* "$\mathbb{Q}$", *noun* "number", *adjective* "even", *statement* "$a = b$", *declaration* "Let a be a number", *definition* "An even number is..", *step* "a is odd, hence $a \neq 0$", *context* "Assume a is even".
  term | set | noun | adjective | statement | declaration | definition | step | context .
- Generally, each syntactic category has a corresponding *weak type*.
- CGa's type system derives typing judgments to check whether the reasoning parts of a document are coherently built.

Weak Type Theory In Weak Type Theory (or $\textsc{Wtt}$) we have the following linguistic categories:

- On the *atomic* level: *variables*, *constants* and *binders*,
- On the *phrase* level: *terms* $\mathcal{T}$, *sets* $\mathbb{S}$, *nouns* $\mathcal{N}$ and *adjectives* $\mathcal{A}$,
- On the *sentence* level: *statements* P and *definitions* $\mathcal{D}$,
- On the *discourse* level: *contexts* $\mathbf{\Gamma}$, *lines* $\mathbf{l}$ and *books* $\mathbf{B}$.

Categories of syntax of WTT

Other category	abstract syntax	symbol
expressions	$\mathcal{E} = T\vert\mathbb{S}\vert\mathcal{N}\vert P$	E
parameters	$\mathcal{P} = T\vert\mathbb{S}\vert P$ (note: $\overrightarrow{\mathcal{P}}$ is a list of $\mathcal{P}$s)	P
typings	$\mathbf{T} = \mathbb{S} :\ \mathsf{SET}\ \vert\mathcal{S} :\ \mathsf{STAT}\ \vert T : \mathbb{S}\vert T : \mathcal{N}\vert T : \mathcal{A}$	T
declarations	$\mathcal{Z} =\ \mathtt{V}^S :\ \mathsf{SET}\ \vert\mathtt{V}^P :\ \mathsf{STAT}\ \vert\mathtt{V}^T : \mathbb{S}\vert\mathtt{V}^T : \mathcal{N}$	Z

level	category	abstract syntax	symbol
atomic	*variables*	$\mathtt{V} = \mathtt{V}^T \vert \mathtt{V}^S \vert \mathtt{V}^P$	x
	constants	$\mathtt{C} = \mathtt{C}^T \vert \mathtt{C}^S \vert \mathtt{C}^N \vert \mathtt{C}^A \vert \mathtt{C}^P$	c
	binders	$\mathtt{B} = \mathtt{B}^T \vert \mathtt{B}^S \vert \mathtt{B}^N \vert \mathtt{B}^A \vert \mathtt{B}^P$	b
phrase	*terms*	$\mathcal{T} = \mathtt{C}^T(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^T_{\mathcal{Z}}(\mathcal{E}) \vert \mathtt{V}^T$	t
	sets	$\mathbb{S} = \mathtt{C}^S(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^S_{\mathcal{Z}}(\mathcal{E}) \vert \mathtt{V}^S$	s
	nouns	$\mathcal{N} = \mathtt{C}^N(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^N_{\mathcal{Z}}(\mathcal{E}) \vert \mathcal{A}\mathcal{N}$	n
	adjectives	$\mathcal{A} = \mathtt{C}^A(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^A_{\mathcal{Z}}(\mathcal{E})$	a
sentence	*statements*	$\mathcal{P} = \mathtt{C}^P(\overrightarrow{\mathcal{P}}) \vert \mathtt{B}^P_{\mathcal{Z}}(\mathcal{E}) \vert \mathtt{V}^P$	S
	definitions	$\mathcal{D} = \mathcal{D}^\varphi \vert \mathcal{D}^P$ $\mathcal{D}^\varphi = \mathtt{C}^T(\overrightarrow{V}) := \mathcal{T} \vert \mathtt{C}^S(\overrightarrow{V}) := \mathbb{S} \vert$ $\mathtt{C}^N(\overrightarrow{V}) := \mathcal{N} \vert \mathtt{C}^A(\overrightarrow{V}) := \mathcal{A}$ $\mathcal{D}^P = \mathtt{C}^P(\overrightarrow{V}) := \mathcal{P}$	D
discourse	*contexts*	$\mathbf{\Gamma} = \emptyset \mid \mathbf{\Gamma}, \mathcal{Z} \mid \mathbf{\Gamma}, \mathcal{P}$	Γ
	lines	$\mathbf{l} = \mathbf{\Gamma} \triangleright \mathcal{P} \mid \mathbf{\Gamma} \triangleright \mathcal{D}$	l
	books	$\mathbf{B} = \emptyset \mid \mathbf{B} \circ \mathbf{l}$	B

Derivation rules

(1) B is a weakly well-typed book: $\vdash\ B :: \texttt{book}.$

(2) Γ is a weakly well-typed context relative to book B: $B\ \vdash\ \Gamma :: \texttt{cont}.$

(3) t is a weakly well-typed term, etc., relative to book B and context Γ:

$$\begin{array}{lll} B;\Gamma\ \vdash\ t :: T, & B;\Gamma\ \vdash\ s :: S, & B;\Gamma\ \vdash\ n :: N, \\ B;\Gamma\ \vdash\ a :: A, & B;\Gamma\ \vdash\ p :: P, & B;\Gamma\ \vdash\ d :: D \end{array}$$

$OK(B;\Gamma).$ stands for: $\vdash\ B :: \texttt{book}$, *and* $B\ \vdash\ \Gamma :: \texttt{cont}$

Examples of derivation rules

- $\mathtt{dvar}(\emptyset) = \emptyset \qquad \mathtt{dvar}(\Gamma', x : W) = \mathtt{dvar}(\Gamma'), x$
  $\mathtt{dvar}(\Gamma', P) = \mathtt{dvar}(\Gamma')$

$$\frac{OK(B;\Gamma), \quad x \in \mathtt{V}^{\mathtt{T/S/P}}, \quad x \in \mathtt{dvar}(\Gamma)}{B;\Gamma \;\vdash\; x :: T/S/P} \quad (var)$$

$$\frac{B;\Gamma \;\vdash\; n :: N, \quad B;\Gamma \;\vdash\; a :: A}{B;\Gamma \;\vdash\; an :: N} \quad (adj{-}noun)$$

$$\frac{}{\vdash \;\; \emptyset :: \mathtt{book}} \quad (emp{-}book)$$

$$\frac{B;\Gamma \;\vdash\; p :: P}{\vdash \;\; B \circ \Gamma \triangleright p :: \mathtt{book}} \qquad \frac{B;\Gamma \;\vdash\; d :: D}{\vdash \;\; B \circ \Gamma \triangleright d :: \mathtt{book}}$$

$$(book{-}ext)$$

Properties of WTT

- *Every variable is declared* If $B;\Gamma \;\vdash\; \Phi :: \mathbf{W}$ then $FV(\Phi) \subseteq \mathtt{dvar}(\Gamma)$.
- *Correct subcontexts* If $B \;\vdash\; \Gamma :: \mathtt{cont}$ and $\Gamma' \subseteq \Gamma$ then $B \;\vdash\; \Gamma' :: \mathtt{cont}$.
- *Correct subbooks* If $\;\vdash\; B :: \mathtt{book}$ and $B' \subseteq B$ then $\;\vdash\; B' :: \mathtt{book}$.
- *Free constants are either declared in book or in contexts* If $B;\Gamma \;\vdash\; \Phi :: \mathbf{W}$, then $FC(\Phi) \subseteq \mathtt{prefcons}(B) \cup \mathtt{defcons}(B)$.
- *Types are unique* If $B;\Gamma \;\vdash\; A :: \mathbf{W_1}$ and $B;\Gamma \;\vdash\; A :: \mathbf{W_2}$, then $\mathbf{W_1} \equiv \mathbf{W_2}$.
- *Weak type checking is decidable* there is a decision procedure for the question $B;\Gamma \;\vdash\; \Phi :: \mathbf{W}$?.
- *Weak typability is computable* there is a procedure deciding whether an answer exists for $B;\Gamma \;\vdash\; \Phi ::$? and if so, delivering the answer.

Definition unfolding

- Let $\vdash B :: \texttt{book}$ and $\Gamma \rhd c(x_1, \ldots, x_n) := \Phi$ a line in B.
- We write $B \vdash c(P_1, \ldots, P_n) \xrightarrow{\delta} \Phi[x_i := P_i]$.
- *Church-Rosser* If $B \vdash \Phi \overset{\delta}{\twoheadrightarrow} \Phi_1$ and $B \vdash \Phi \overset{\delta}{\twoheadrightarrow} \Phi_2$ then there exists Φ_3 such that $B \vdash \Phi_1 \overset{\delta}{\twoheadrightarrow} \Phi_3$ andf $B \vdash \Phi_2 \overset{\delta}{\twoheadrightarrow} \Phi_3$.
- *Strong Normalisation* Let $\vdash B :: \texttt{book}$. For all subformulas Ψ occurring in B, relation $\xrightarrow{\delta}$ is strongly normalizing (i.e., definition unfolding inside a well-typed book is a well-founded procedure).

CGa Weak Type Checking

Let $\mathfrak{M}$ be a set,

y and x are natural numbers,

if x belongs to $\mathfrak{M}$

then $x + y = y + x$

CGa Weak Type checking detects grammatical errors

Let $\mathfrak{M}$ be a set,

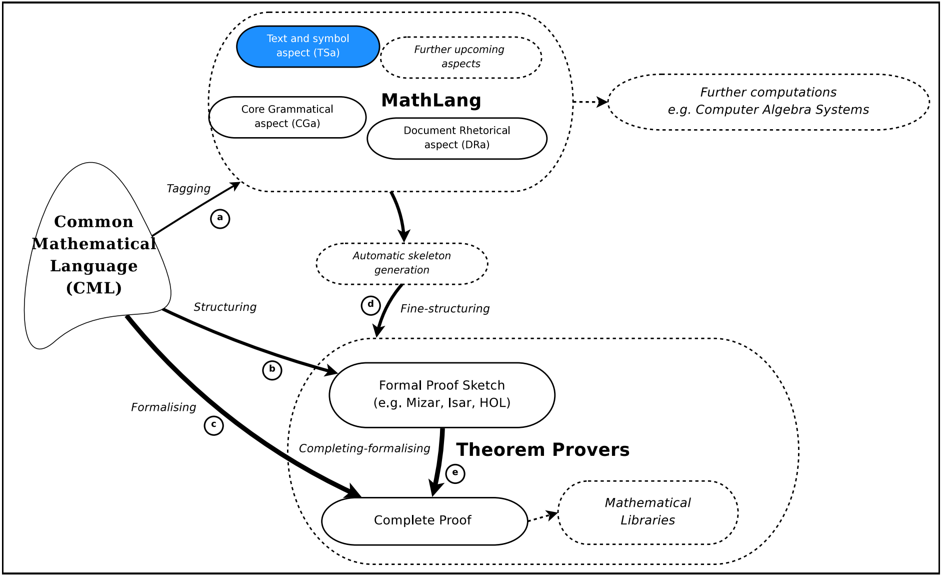
y and x are natural numbers,

if x belongs to $\mathfrak{M}$

then $x + y$ $\Leftarrow$ **error**

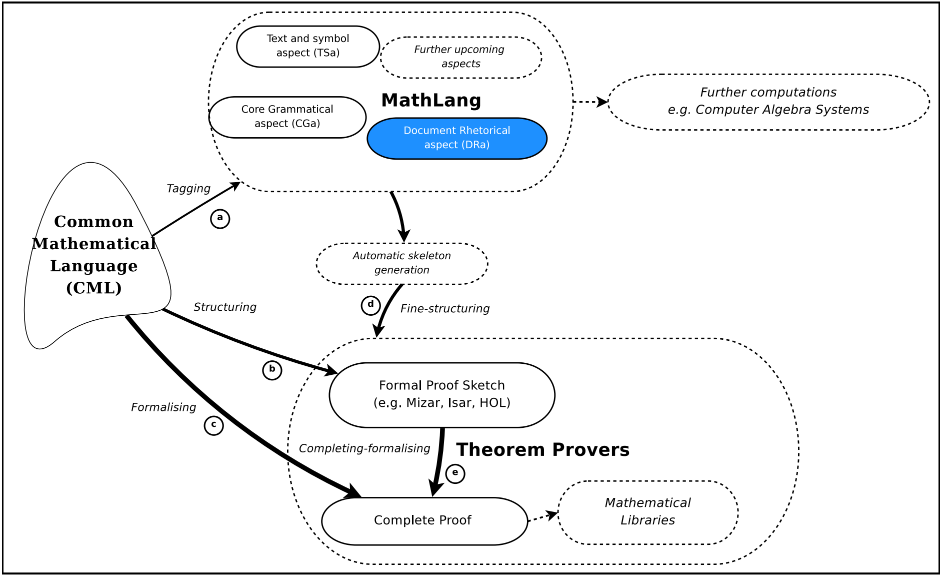
How complete is the CGa?

- CGa is quite advanced but remains under development according to new translations of mathematical texts. Are the current CGa categories sufficient?
- The metatheory of WTT has been established in (Kamareddine and Nederepelt 2004). That of CGa remains to be established. However, since CGa is quite similar to WTT, its metatheory might be similar to that of WTT.
- The type checker for CGa works well and gives some useful error messages. Error messages should be improved.


Text and symbol aspect (TSa)
Further upcoming aspects
Core Grammatical aspect (CGa)
MathLang
Document Rhetorical aspect (DRa)
Further computations e.g. Computer Algebra Systems
Tagging
a
Common Mathematical Language (CML)
Automatic skeleton generation
Structuring
d
Fine-structuring
b
Formal Proof Sketch (e.g. Mizar, Isar, HOL)
Formalising
c
Completing-formalising
Theorem Provers
e
Complete Proof
Mathematical Libraries

What is TSa? Lamar's PhD thesis

- TSa builds the bridge between a CML text and its grammatical interpretation and adjoins to each CGa expression a string of words and/or symbols which aims to act as its CML representation.
- TSa plays the role of a user interface
- TSa can flexibly represent natural language mathematics.
- The author wraps the natural language text with boxes representing the grammatical categories (as we saw before).
- The author can also give interpretations to the parts of the text.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Completing-formalising
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Mathematical Libraries

What is DRa? Retel's PhD thesis

- DRa Document Rhetorical structure aspect.
- **Structural components of a document** like *chapter, section, subsection, etc.*
- **Mathematical components of a document** like *theorem, corollary, definition, proof, etc.*
- **Relations** between above components.
- These enhance readability, and ease the navigation of a document.
- Also, these help to go into more formal versions of the document.

Relations

Description
Instances of the *StructuralRhetoricalRole* *class:* preamble, part, chapter, section, paragraph, *etc.*
Instances of the *MathematicalRhetoricalRole* *class:* lemma, corollary, theorem, conjecture, definition, axiom, claim, proposition, assertion, proof, exercise, example, problem, solution, *etc.*

Relation
Types of relations: relatesTo, uses, justifies, subpartOf, inconsistentWith, exemplifies

What does the mathematician do?

- The mathematician wraps into boxes and uniquely names chunks of text
- The mathematician assigns to each box the structural and/or mathematical rhetorical roles
- The mathematician indicates the relations between wrapped chunks of texts

Lemma 1. For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$.
Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \;\&\; m > 0.$$

Claim. $P(m) \implies \exists m' < m. P(m')$. Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have

$$2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$$

Since $m > 0$, if follows that $m^2 > 0, n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}. \neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$ with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction. Therefore $m = 0$. But then also $n = 0$.

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$.
Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$.

Barendregt

Lemma 1.

For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$

Proof.

Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \ \&\ m > 0.$$

Claim. $P(m) \implies \exists m' < m. P(m')$.

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}. \neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$

with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction.

Therefore $m = 0$. But then also $n = 0$. □

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

Lemma 1.

For $m, n \in \mathbb{N}$ *one has:* $m^2 = 2n^2 \implies m = n = 0$

Proof.

Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \; \& \; m > 0.$$

Claim. $P(m) \implies \exists m' < m. P(m')$.

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds squared are odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

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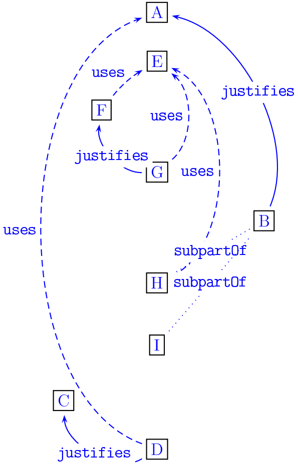
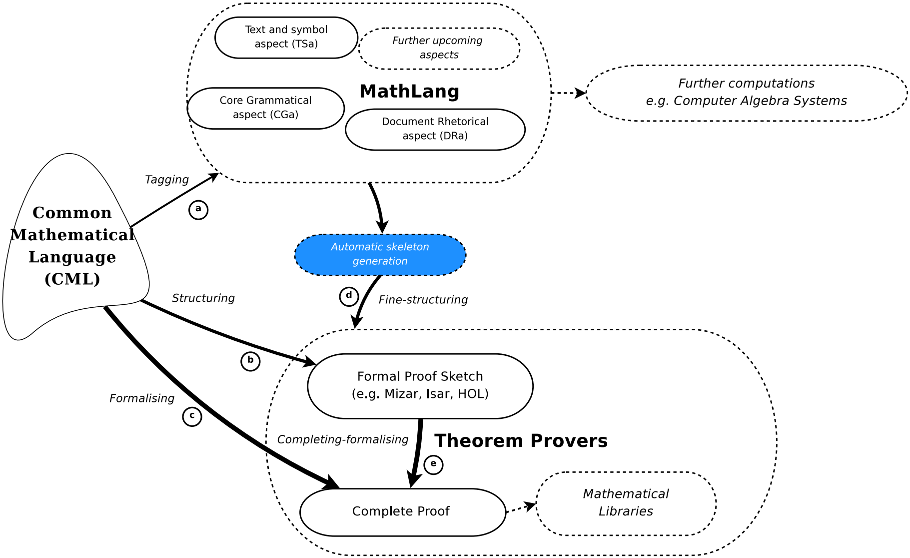
The automatically generated dependency Graph

Dependency Graph (DG)




Text and symbol aspect (TSa)
Further upcoming aspects
MathLang
Further computations e.g. Computer Algebra Systems
Core Grammatical aspect (CGa)
Document Rhetorical aspect (DRa)
Common Mathematical Language (CML)
Tagging
a
Automatic skeleton generation
Structuring
d
Fine-structuring
b
Formal Proof Sketch (e.g. Mizar, Isar, HOL)
Formalising
c
Completing-formalising
Theorem Provers
e
Complete Proof
Mathematical Libraries

An example of a full formalisation in Coq via MathLang

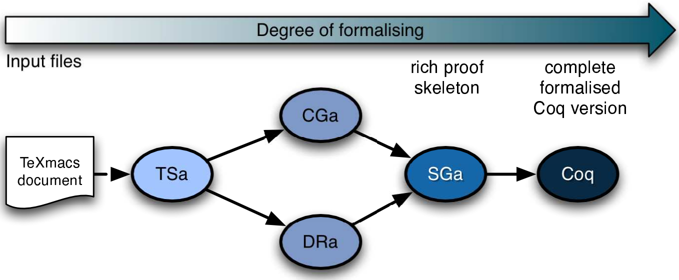


Figure 1: The path for processing the Landau chapter

Chapter 1

Natural Numbers

<><forall> $\forall$ <#><#>, <#> <><exists> $\exists$ <#><#>, <#> <><exists_one> $\exists!$ <#><#>, <#> <><isa><#> <#> <><1> <><and><#> $\wedge$ <#>

<><or><#> $\vee$ <#> <><impl><#> $\longrightarrow$ <#> <><succ><#> <><in><#> $\in$ <#> <><subset><#> $\subset$ <#> <><Set> $\{$ <#><#> <#> $\}$

<><seteq><#><#> <><setneq><#><#> <><index><#><#> <><xor><#> $\oplus$ <#> <><emptyset> $\emptyset$

1.1 Axioms

We assume the following to be given:

A set (i.e. totality) of objects called natural numbers, possessing the properties - called axioms- to be listed below.

Before formulating the axioms we make some remarks about the symbols $=$ and $\neq$ which be used.

Unless otherwise specified, small italic letters will stand for natural numbers throughout this book.

If x is given and y is given, then either x and y are the same number; this may be written

$$x = y$$

($=$ to be read "equals"); or x and y are not the same number; this may be written

$$1x \neq y$$

($\neq$ to be read "is not equal to").

Accordingly, the following are true on purely logical grounds:

$x = x$ for every x

if x y $x = y$ then $y = x$

If x y z $x = y$, $y = z$ then $x = z$

1

Chapter 1 of Landau

- 5 axioms which we annotate with the mathematical role "axiom", and give them the names"ax11" - "ax15".
- 6 definitions which we annotate with the mathematical role "definition", and give them names "def11" - "def16".
- 36 nodes with the mathematical role "theorem", named "th11" - "th136" and with proofs "pr11" - "pr136".
- Some proofs are partitioned into an existential part and a uniqueness part.
- Other proofs consist of different cases which we annotate as unproved nodes with the mathematical role "case".

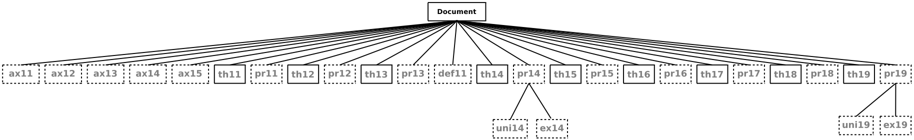


Figure 2: The DRa tree of sections 1 and 2 of chapter 1 of Landau's book

- The relations are annotated in a straightforward manner.
- Each proof *justifies* its corresponding theorem.
- Axiom 5 ("ax15") is the axiom of induction. So every proof which uses induction, *uses* also this axiom.
- Definition 1 ("def11") is the definition of addition. Hence every node which uses addition also *uses* this definition.
- Some theorems *use* other theorems via texts like: "By Theorem ...".
- In total we have 36 *justifies* relations, 154 uses relations, 6 caseOf, 3 existencePartOf and 3 uniquenessPartOf relations.
- The DG and GoTO are automatically generated.
- The GoTO is automatically checked and no errors result. So, we proceed to the next stage: automatically generating the SGa.

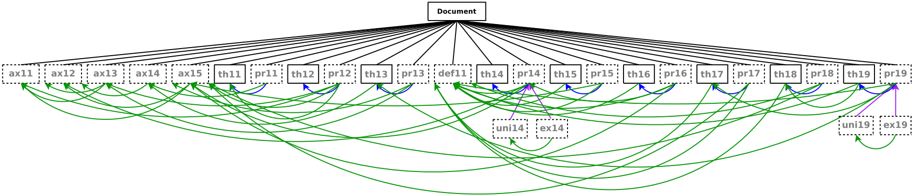
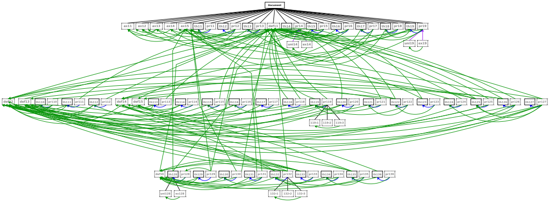


Figure 3: The DG of sections 1 and 2 of chapter 1 of Landau's book



DG of sections 1..4

With the help of the CGa annotations and the automatically generated rich proof skeleton, Zengler (who was not familiar with Coq) completed the Coq proofs of the whole of chapter one in a couple of hours.

Some points to consider

- We do not at all assume/prefer one type/logical theory instead of another.
- MathLang aims to do some amount of type checking even for non-fully-formalized mathematics. This corresponds roughly to grammatical conditions.
- MathLang aims to support automated processing of mathematical knowledge.
- MathLang aims to be independent of any foundation of mathematics.
- MathLang allows anyone to be involved, whether a mathematician, a computer engineer, a computer scientist, a linguist, a logician, etc.

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