Probability approximations using Stein's method

Tutorial 1: Solutions

1. For any set A, we may write

$$\begin{aligned}\mathbb{P}(X \in A) - \mathbb{P}(Y \in A) &= \mathbb{P}(X \in A, X = Y) + \mathbb{P}(X \in A, X \neq Y) \\ &\quad - \mathbb{P}(Y \in A, X = Y) - \mathbb{P}(Y \in A, X \neq Y) \\ &= \mathbb{P}(X \in A, X \neq Y) - \mathbb{P}(Y \in A, X \neq Y) \\ &\leq \mathbb{P}(X \neq Y)\,.\end{aligned}$$

A similar argument also gives $\mathbb{P}(Y \in A) - \mathbb{P}(X \in A) \leq \mathbb{P}(X \neq Y)$, so that

$$|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \leq \mathbb{P}(X \neq Y)\,,$$

as required.

2. We construct our coupling $(\widehat{X}, \widehat{Y})$ by letting $\widehat{X} = X$ and $\widehat{Y} = \widehat{X} + Z$, where $Z \sim \text{Po}(\mu - \lambda)$. The inequality from the previous question then gives

$$d_{TV}(X, Y) \leq \mathbb{P}(\widehat{X} \neq \widehat{Y}) = \mathbb{P}(Z > 0) = 1 - \mathbb{P}(Z = 0) = 1 - e^{\lambda - \mu}\,.$$

3. In each case we have $np = 10$. In the setting (as we have here) where the p_i are all equal to p, the Poisson approximation bound of Theorem 2.9 that we obtained through Stein's method becomes

$$d_{TV}(W, Z) \leq (1 - e^{-np})p \leq p\,.$$

In the two cases given this is

(a)
$$d_{TV}(W, Z) \leq 0.1\,.$$

(b)
$$d_{TV}(W, Z) \leq 0.01\,.$$

Of course, we can get slightly better bounds by including the factor of $1 - e^{-np}$, but this is very close to 1 here, so doesn't make much of a difference. Note also that the bounds we obtain here are significantly better than we would get from Theorem 2.6.

4. (a) We have

$$\mathbb{E}f(Z) = p\sum_{j=0}^{\infty}(1-p)^j f(j) = p(1-p)\sum_{j=1}^{\infty}(1-p)^{j-1}f(j)$$
$$= p(1-p)\sum_{j=0}^{\infty}(1-p)^j f(j+1) = (1-p)\mathbb{E}f(Z+1)\,,$$

as required.

(b) Substituting the given expression for f into the Stein equation confirms that it is a solution to that equation. We have that

$$f(j) - f(k) = \sum_{i\in A, i\geq k}(1-p)^{i-k} - \sum_{i\in A, i\geq j}(1-p)^{i-j}\,,$$

and since neither term on the RHS can be larger than $\sum_{i=0}^{\infty}(1-p)^i = \frac{1}{p}$, the stated bound on f follows.

(c) From part (a) we have that $\mathbb{E}f(Z') = (1-p')\mathbb{E}f(Z'+1)$.

Following Stein's method, we use the Stein equation from part (b) to write

$$\begin{aligned}\mathbb{P}(Z'\in A) - \mathbb{P}(Z\in A) &= (1-p)\mathbb{E}f(Z'+1) - \mathbb{E}f(Z')\\ &= (1-p)\mathbb{E}f(Z'+1) - (1-p')\mathbb{E}f(Z'+1)\\ &= (p'-p)\mathbb{E}f(Z'+1)\,.\end{aligned}$$

Taking the supremum over sets $A \subseteq \mathbb{Z}^+$ and using the bound on f from part (b) gives us the required result.