Formal Specification F28FS2, Lecture 3
(The rest of Chapter 3 of Currie's book)

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Set membership / set elementhood

Suppose $n : T$ and $S : \mathbb{P}(T)$; so S is a set of things from T.

$n \in S$ is a predicate, read 'n is (an element) in S' or 'n is a member of S'.

$n \in S$ is true when when n is an element of S.

$n \in S$ is false when n is not an element of S.

What is the truth-value of $JOHN \in \{JOHN,\ DICK,\ HARRY\}$?

What is the truth-value of $7 \in 1..8 = \{1, 2, 3, 4, 5, 6, 7, 8\}$?

What is the truth-value of $1 \in \{n : \mathbb{N} \bullet 2 * n\}$?

What is the truth-value of $\text{-}1 \in \{n : \mathbb{Z} \mid \neg(n = \text{-}1)\}$?

What is the truth-value of $\text{-}1 \in \{n : \mathbb{N} \mid \neg(n = \text{-}1)\}$?

Type errors

If $n : T$ and $S : \mathbb{P}(T')$ and T and T' are distinct types, then $n \in S$ is not a valid proposition.

Recall the example types from Lecture 2:

$$\begin{aligned} \text{COLOUR} &::= \text{red}|\text{green}|\text{blue} \\ \text{FUEL} &::= \text{petrol}|\text{diesel}|\text{electricity}. \end{aligned}$$

Then if $n : \mathbb{Z}$ and $S == \{\text{red}, \text{green}\}$ then $n \in S$ is not a valid proposition.

Some propositions with truth-value T

$n \notin S$ is shorthand for $\neg(n \in S)$.

green $\notin \{\text{red}, \text{blue}\}$ is true.

-4 $\notin \{x : \mathbb{Z} \mid x > \text{-}1 \bullet x\}$ is also true, because -4 is not a member of $\{0, 1, 2, 3, 4, \ldots\}$.

Predicates

Which of the following are (valid) predicates, and to which would you assign truth-value T?

$$\text{diesel} \in \{\text{diesel}, \text{petrol}\} \qquad \{\text{diesel}\} \in \{\text{diesel}, \text{petrol}\}$$

$$\{\text{diesel}\} \in \{\{\text{diesel}\}, \{\text{diesel}, \text{petrol}\}\}$$

$$1 \in 1..4 \qquad 1 \in 2..4$$

$$\{\} \in \{\{\}\} \qquad \{\} \in \{\}$$

$\{\}$ is the empty set, with no elements.

$\{\} : \mathbb{P}(T)$ for any type T.

More predicates

$$\text{diesel} \notin \{\text{diesel}, \text{petrol}\} \qquad \{\text{diesel}\} \notin \{\text{diesel}, \text{petrol}\}$$

$$\{\text{diesel}\} \notin \{\{\text{diesel}\}, \{\text{diesel}, \text{petrol}\}\}$$

$$1 \notin 1..4 \qquad 1 \notin 2..4$$

$$\{\} \notin \{\{\}\} \qquad \{\} \notin \{\}$$

Cardinality and equality

Suppose $S : \mathbb{P}(T)$. The cardinality of S is the number of elements in S. The cardinality of S has type $\mathbb{N}$, of course.

Write $\#S$ for the cardinality of S.

What is $\#\{2, 3\}$?

What is $\#\{2\}$?

What is $\#\{\}$?

What is $\#\{\{2, 3\}\}$?

Equality

Suppose $S : \mathbb{P}(T)$ and $U : \mathbb{P}(T)$.

$S = U$ is a predicate; it is true when S is equal to U.

$\{petrol\} = \{petrol\}$ and $\{petrol\} = \{petrol, petrol\}$ are true.

$\{n : \mathbb{Z} \mid n > 1 \wedge n < 3\} = \{2\}$ is true.

$4..9 = \{n : \mathbb{Z} \mid n \geq 4 \wedge n \leq 9\}$ is true.

We write $S \neq U$ when $\neg(S = U)$.

Watch yourself: Sets are hard

Sets are hard. First-order predicates about sets are also hard.

Students get confused. What is the difference between ... a set and a type? a set and a subset? what is the exact notation for predicates? for sets comprehension (next slide)?

These are not quick'n'easy boil-in-the-bag microwavable just-add-water-and-stir concepts.

You have to do plenty of exercises. Discuss them on the forum.

I'm only trying to help you when I give you the following warning: if you don't run over this material repeatedly, by doing exercises until they are second nature, then I promise you'll regret it.

Important: sets comprehension

A set is a collection of objects. The general way to build sets is this:

$$\{somevars \mid condition \bullet function\ on\ vars\}$$

A term of the form above is called a sets comprehension.

For example:

$$\begin{array}{ll} \{n : \mathbb{Z} \mid n \geq 4 \bullet 2 * n\} & : \mathbb{P}(\mathbb{Z}) \\ \{8, 10, 12, 14, 16, \ldots\} & : \mathbb{P}(\mathbb{Z}) \\ & \\ \{n : \mathbb{Z} \mid \neg(n = 0) \bullet n\} & : \mathbb{P}(\mathbb{Z}) \\ \{1, \text{-}1, 2, \text{-}2, 3, \text{-}3, \ldots\} & : \mathbb{P}(\mathbb{Z}) \\ & \\ \{n : \mathbb{Z} \mid \neg(n = 0)\} & : \mathbb{P}(\mathbb{Z}) \\ \{1, \text{-}1, 2, \text{-}2, 3, \text{-}3, \ldots\} & : \mathbb{P}(\mathbb{Z}) \\ & \\ \{n : \mathbb{Z} \mid T \bullet 3 * n\} & : \mathbb{P}(\mathbb{Z}) \\ \{0, 3, \text{-}3, 6, \text{-}6, \ldots\} & : \mathbb{P}(\mathbb{Z}) \end{array}$$

Make sure you understand this slide, too

$\{n : \mathbb{Z} \bullet n\}$	$: \mathbb{P}(\mathbb{Z})$
$\{0, 1, \text{-}1, 2, \text{-}2, 3, \text{-}3, \ldots\}$	$: \mathbb{P}(\mathbb{Z})$
$\{n, m : \mathbb{Z} \mid n = m \bullet \{n, m\}\}$	$: \mathbb{P}(\mathbb{P}(\mathbb{Z}))$
$\{\{0, 0\},\ \{1, 1\},\ \{\text{-}1, \text{-}1\},\ \{2, 2\},\ \ldots\}$	$: \mathbb{P}(\mathbb{P}(\mathbb{Z}))$
$\{n, m : \mathbb{Z} \mid n + m = 7 \bullet \{n, m\}\}$	$: \mathbb{P}(\mathbb{P}(\mathbb{Z}))$
$\{\{0, 7\},\ \{1, 6\},\ \{\text{-}1, 8\},\ \{2, 5\},\ \ldots\}$	$: \mathbb{P}(\mathbb{P}(\mathbb{Z}))$
$\{n : \mathbb{Z} \bullet \{n\}\}$	$: \mathbb{P}(\mathbb{P}(\mathbb{Z}))$
$\{\{0\}, \{1\}, \{\text{-}1\}, \{2\}, \{\text{-}2\}, \{3\}, \{\text{-}3\}, \ldots\}$	$: \mathbb{P}(\mathbb{P}(\mathbb{Z}))$

Subset inclusion

Sometimes we may wish to say that one set is included in another. Suppose $S, T : \mathbb{P}(\mathbb{Z})$ (so S and T are sets of integers). Then

$$S \subseteq T \quad \text{means} \quad \forall x : \mathbb{Z} \bullet x{\in}S \Rightarrow x{\in}T.$$

In words, “if x is an element of S then it is an element of T”.

Note that $\{\} \subseteq S$ and $S \subseteq S$.

Is $\{\} \subseteq \{\}$? How about $\{1\} \subseteq \{1\}$? How about $\{\text{-}1, 1\} \subseteq \{0\}$?

Write $S \not\subseteq T$ for $\neg(S \subseteq T)$.

Write $S \subset T$ for $S \subseteq T \land S \neq T$ (strict subset inclusion).

Note: we call S a proper subset of T when $\varnothing \subset S \subset T$.

Powerset

Suppose $S : \mathbb{P}(\mathbb{Z})$ is a set of integers.

Then $\mathbb{P}(S) : \mathbb{P}(\mathbb{P}(\mathbb{Z}))$ is the set of subsets of S.

$$\mathbb{P}\{\} = \{\{\}\} \qquad \mathbb{P}\{1\} = \{\{\}, \{1\}\}$$
$$\mathbb{P}\{1, 2\} = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$$

Lemma: $\#(\mathbb{P}S)$ equals $2^{\#S}$ (big!).

Proof: To build a subset x of S, it suffices to make $\#S$ binary choices, one for each $z \in S$, as to whether $z \in x$.

E.g. $\#\mathbb{P}\{1, 2\} = 4 \quad \#\mathbb{P}\{1, 2, 3, 4, 5, 6, 7, 8\} = 256$
$\#\mathbb{P}(1..32) = 4294967296$

Powerset type

If $\mathbb{T}$ is a type then so is $\mathbb{P}\mathbb{T}$.

If numset $==$ 4..9 (recall this is $\{4, 5, 6, 7, 8, 9\}$) then numset : $\mathbb{P}\mathbb{Z}$.

And finally — as shorthand for '$x : \mathbb{Z}$ and $x \in$ numset' we may just write x : numset.

I lied — $\mathbb{N} = \{x : \mathbb{Z} \mid x \geq 0 \bullet x\}$ is a set, not a type.

$x : \mathbb{N}$ declares '$x : \mathbb{Z}$' and '$x \in \mathbb{N}$ has truth-value T'.

Note that the type of $\{\}$ may be inferred from context:

$$\{\} \subseteq \text{numset}$$

means that $\{\} : \mathbb{P}\mathbb{Z}$.

Unions and intersections

Suppose $\mathbb{T}$ is a type and $S, T : \mathbb{P}\mathbb{T}$. Then define

$$S \cap T == \{x : \mathbb{T} \mid x \in S \land x \in T \bullet x\}$$
$$S \cup T == \{x : \mathbb{T} \mid x \in S \lor x \in T \bullet x\}$$

$S \cap T$ is the elements x of type $\mathbb{T}$ that are in S and in T.

Unions and intersections

As always, watch out for type errors: $\{1, \{\}\}$ is not Z notation.

For example $\{1, 2, 3\} \cap \{2, 4\} = \{2\}$.

For example $\{1, 2, 3\} \cup \{2, 4\} = \{1, 2, 3, 4\}$.

What is $\{1, 2, 3\} \cap \{\}$?

What is $\{1, 2, 3\} \cup \{\}$?

What is the type of $S \cap T$?

Set subtraction (set difference)

Suppose $S, T : \mathbb{PT}$.

$$S \setminus T == \{x : | x \in S \wedge x \notin T\}.$$

What is $\{1, 2\} \setminus \{1, 3\}$?

What is $\{1, 2\} \setminus \{3\}$?

Unions and intersections

Suppose that $\mathcal{S} : \mathbb{P}\mathbb{P}\mathbb{T}$ ($\mathcal{S}$ is a set of $S \subseteq \mathbb{T}$) Define:

$$\bigcup \mathcal{S} == \{x : \mathbb{T} \mid (\exists S : \mathbb{P}\mathbb{T} \bullet S \in \mathcal{S} \wedge x \in S)\}$$

$$\bigcap \mathcal{S} == \{x : \mathbb{T} \mid (\forall S : \mathbb{P}\mathbb{T} \bullet S \in \mathcal{S} \wedge x \in S)\}$$

For example $\bigcup\{\{1\}, \{1, 2, 3\}\} = \{1, 2, 3\}$.

For example $\bigcap\{\{1\}, \{1, 2, 3\}\} = \{1\}$.

That's it for section 3

All done.

Warning: do exercises, in the next three or four days.

I'm going to use the stuff from the lectures so far, in forthcoming lectures.

This means that you have limited time from right now to understand this material, before I start using it.

The rest of this course is not terribly difficult if you have a really good handle on the basics.

If you do not make sure you've got the basics—NOW—then you're just making trouble for yourself (and for me; and for the rest of the class).

There's no point. Sort it out now; relax later.