

Is Economic Planning Hypercomputational? The Argument from Cantor Diagonalisation

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[26] argues that the diagonal argument of the number theorist Cantor can be used to elucidate issues that arose in the socialist calculation debate of the 1930s. In particular he contends that the diagonal argument buttresses the claims of the Austrian economists regarding the impossibility of rational planning. We challenge Murphy's argument, both at the number theoretic level and from the standpoint of economic realism.

Key words: hypercomputation, prices, planning, economic calculation, Lange

1 INTRODUCTION

Since the 1920s there has been an ongoing debate on the feasibility of “socialist economic calculation”, which is generally taken to mean calculation of costs and benefits in the absence of markets, on the capitalist pattern, for consumer goods, labour and means of production. This debate was most active in the 1930s. Proponents of socialism such as [11] and [22] argued that rational economic calculation was feasible under socialism while, on the other side, economists of the Austrian school such as [36] and [15, 16] argued that socialist calculation was impossible, either *a priori* or on the basis of computational intractability. The debate was renewed in the 1980s with a notable

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contribution from [27], who argued that the calculations required for socialist plan formation would take millions of years even with the best computers. Against this [7] and [9, 8] have argued that improved algorithmic techniques make the calculations tractable. More recently, [4] and [5] have drawn on the critique of AI made by [28] to defend Hayek's original arguments. [26] argues that the problem of economic planning is actually hypercomputational, and thus impossible in principle. Similar arguments have been made by [23].

2 IS PLANNING TRACTABLE?

Murphy summarises his argument as follows:

[I]f the socialist planners really are to mimic the market outcome, they would need to publish a list containing, not merely a huge number of prices, and not merely an infinite number of prices, but rather a list containing an uncountably infinite number of prices. But as we have seen above, it is literally impossible, even in principle, for socialist planners to publish such a list. That is, even if we granted them a sheet of paper infinitely long and gave them an infinite amount of time, they still could not, even in theory, write down the entire set of “accounting prices” at which their managers would be required to exchange factors of production. Therefore the purported mathematical solution to Mises's challenge is truly impossible to implement, in every sense of the word.

Why do we need an infinite list of accounting prices? Because, Murphy says, “all *conceivable* goods and services that might be offered, must have corresponding prices included in the planners official lists” (emphasis added). This, he contends, includes goods which have not yet been produced—such as weekend trips to Mars—but which may become possible with some future technology. The set of goods whose accounting prices are required, he says, includes every possible book that might be written in the future. On this basis he claims that [17] grossly underestimated how many equations would actually be required to implement the mathematical solution to the planning problem.

Since computation over infinite domains is impossible in principle, he concludes that the preparation of a socialist plan is not merely intractable, but entirely out of the question.

Arguments about computability in economics are tricky. They may reveal more about the axiomatic foundations of economic theories than they do about the operation of real world economies. [1], for example, supposedly established the existence of equilibrium for competitive economies. We will term such an equilibrium ‘classical mechanical’ following [25], who showed that the conceptual apparatus used to define it is equivalent to that used in energy minimisation problems in classical mechanics. As [35] showed, the Arrow–Debreu proof rested on theorems that are valid only in non-constructive mathematics.

Arrow’s use of non-constructive mathematics is critical because only constructive mathematics has an algorithmic implementation and is guaranteed to be effectively computable. But even if

1. a ‘classical mechanical’ economic equilibrium can be proven to exist, and
2. it can be shown that there is an effective procedure by which this can be determined (that is, the equilibrium is *in principle* computable)

there is still the question of computation tractability; that is, of determining the complexity order governing the computation process that arrives at the solution.

An equilibrium might exist, yet all algorithms to search for it might be NP-hard. [10] have shown that subject to Leontief utility functions (according to which each consumer demands inputs in fixed proportions) the problem of finding a market equilibrium with maximum social welfare is NP hard. This result might at first seem to support the contention of the Austrian school of economics, that the problem of rational economic planning is computationally intractable. The notion of NP-hardness had not been invented in Hayek’s day, but he would seem to have been retrospectively vindicated.

Such results would seem to be particularly telling against neoclassical socialist economists (Lange, Dickinson) whose arguments were based around the ease with which a planning system could achieve neoclassical welfare maximising equilibrium.

While [10] might be taken to show that the neoclassical problem of economic equilibrium was intractable for economic planners, even with large scale computers, this conclusion does not necessarily follow. NP complete problems are not always intractable in practice. [14] show that NP problems have phase transition regions, within which they are hard to solve, as well as other less constrained regions, where solutions are easy to find. It might be

the case that, in practice, the problem of finding a social welfare-maximising equilibrium falls into a non-critical region of the constraint space. An analogous situation exists with linear programming proposed as a technique of rational economic planning by [18]. The simplex algorithm for linear programming is in worst cases exponential but for most practical cases it is only polynomial, [32].

If, on the other hand, we assume that real economies fall into the phase transition region of the problem space, then neither central planners nor a collection of millions of individuals interacting via the market could solve the social welfare maximisation problem. In neoclassical economics the number of constraints on the equilibrium will be proportional, among other things, to the number of economic actors, n . The computational resource constituted by the actors will be proportional to n but, in the phase change region, the cost of the computation will grow as e^n . Computational resources grow linearly, because they are proportional to the number of people available to make decisions, while computational costs grow exponentially. This implies that a market economy could never have sufficient computational resources to find its equilibrium.

Clearly, we cannot conclude from this that market economies are impossible: we have plentiful empirical evidence that they exist. It would seem to follow that the problem of finding the neoclassical equilibrium is a mirage: no planning system could discover it, but nor could the market. The “problem of neoclassical equilibrium” misrepresents what capitalist economies actually do, and by the same token sets an impossible goal for socialist planning.

If we dispense with the notion of ‘classical mechanical’ equilibrium, in relation to the economy, and replace it with the concept of *statistical* mechanical equilibrium ([13]), we arrive at a problem that is much more tractable. The simulations of [37, 38] and [12] show that a market economy can rapidly converge on this sort of equilibrium.

It should be noted that the notion of a statistical mechanical equilibrium, while quite alien to neoclassical economics, has something in common with the presumptions of the Austrian school, who tend emphasize the chaotic, non-equilibrium nature of capitalist economies. According to Hayek, under the price system individual producers merely have to watch the movement of a “few pointers” in order to orient themselves in the chaos of the market.

How can a single vector of prices act as a regulator for a complex matrix of inter-sectoral flows in a market economy? We can identify two possible reasons.

First, [30][1776] argued that human labour was the universal resource by means of which all other goods were purchased from nature. The universality of human labour means that it is possible to associate with each commodity a single scalar number—price—which indirectly and on average represents the amount of labour that was used to make that commodity. Let us define the “value” (or labour-value) of a commodity as the amount of labour required for its production. Deviations of relative prices from relative values can then allow labour to move from where it is less socially necessary to where it is more necessary. But this is only possible because all economic activity comes down, in the end, to human activity. If that were not the case, a single indicator would not be sufficient to regulate the consumption of inputs that were fundamentally of different dimensions. It is only because the dimension of all inputs is ultimately labour (direct or indirect) that scalar prices can effectively regulate activity.

A second answer lies in the computational tractability of systems of linear equations. [33] showed that the determination of the labour values of goods is equivalent to solving a set of linear equations. Solutions to such equations can be tractably approached by iterative techniques.

Firms add up wage costs and costs of other commodity inputs, add a mark-up, and set their prices accordingly. This distributed algorithm, which is nowadays carried out by a combination of people and company computers, is structurally similar to the solution of linear equations by an iterative method. This models one phase of the iterative solution of Sraffa’s equations. Empirical evidence indicates that the price vector upon which the process converges lies somewhere near the vector of labour values see [29, 24].

The exact attractor is not relevant at this point; what is relevant is that the iterative functional system has a stable attractor. It has such an attractor because the process of economic production can be well approximated by a piecewise contractive linear transform on price or value space. If production processes were strongly nonlinear, such that the output of (say) corn were a polynomial along the lines of

$$C_{\text{out}} = aC_{\text{in}} + bC_{\text{in}}^2 + dC_{\text{in}}^3 + eL + fL^2 + gL^3 + hI + kI^2$$

(with C representing corn, L labour and I iron), then the iterative functional system would be highly unstable, and the evolution of the entire price system would be completely chaotic and unpredictable. Prices would then be useless as a guide to economic activity. For the instability of such systems see [3] or [2].

Neither of the two factors above are specific to a market economy. Labour is the key universal resource in any society (prior to full robotisation). By the full version of the Church–Turing thesis, if a problem can be solved by a distributed collection of human computers, then it can be solved by a Universal Computer. If it is tractable for humans interacting via a market it is also algorithmically tractable when calculated by the computers of a socialist planning agency. The very factors which make the price system relatively stable and useful are the factors which make socialist economic calculation tractable. We contend that economic planning does not have to solve the impossible problem of neoclassical equilibrium, but merely has to apply the classical “law of value” more efficiently.

3 MUST PLANNING CONSIDER AN INFINITE NUMBER OF PRICES?

Against this background, consider Murphy’s thesis that the problem domain of economic calculation is not merely NP hard, but actually transfinite. If the problem domain is infinite, it is not at all clear how a market economy is supposed to provide an effective solution. No finite computational resource—whether it be state planners with computers or capitalist supermarkets and wholesalers with their computers and databases—can scan an infinite search space.* Either the market economy must also be deficient, by Murphy’s criteria, or his criteria are misplaced. Murphy is demanding that an economic system today take into account information which can only exist in the future, information about products that will one day be invented in the future. He is demanding the impossible: the backward transmission of information through time.

If he is, alternatively, demanding computation over all possible futures, it is hard to see how he thinks a market economy is able to solve the problem. No system, whether capitalistic or socialistic, planned or unplanned, could do this. Economic systems can only allocate resources between products that have already been thought of or invented.

This seems blatantly obvious, so why might Murphy be claiming otherwise? His idea seems to be that only on condition that all *possible* prices are considered, could a Lange/Dickinson-type system (patterned after the neoclassical fiction of the Walrasian auctioneer) be a “perfect substitute” for the market mechanism, with regard to the issue of *innovation* (the production of new products or use of new production processes).

* We here disregard the highly contentious recent claims of [19] for the reasons given in [34, 31].

There is a small kernel of sense in this, though it is expressed perversely. A persistent theme in Austrian economics is that the neoclassical representation of the market system, with its stress on static allocative efficiency, is misleading and in a sense sells short the virtues of capitalism. The principal virtue of capitalism, according to the Austrians, is not that it produces an optimally efficient, perfectly competitive, equilibrium with prices everywhere equal to marginal cost (as in the standard economics textbooks), but that it spawns an effective process of discovery and innovation—the notion encapsulated by the word “entrepreneurship”.

If an economic system were to entrust its process of product- and process-innovation purely and simply to a mechanism in which managers make decisions on what to produce, and how to produce it, based on accounting prices handled out by a planning authority, then in a sense Murphy is right: the accounting prices would have to include the prices of all the things they *might* produce as well as things they’re currently producing.

Our response is twofold. First, historically, the Lange/Dickinson scheme was not supposed to be a solution to the problem of innovation: that was not the problem [36] originally posed in the opening salvo of the socialist calculation debate. Second, the market does not handle innovation purely via passive responses to price signals, and by the same token a socialist economy will not handle innovation via passive responses to computed prices (or, for that matter, labour values) of currently non-existent goods.

In any system, what is needed is some mechanism for exploring options “in the neighbourhood of” the current input–output matrix that are rendered feasible by scientific advances (or, in some cases, just by leaps of the imagination). This inevitably involves experimentation, trial and error, and so on. This task is beyond the scope of the Lange/Dickinson mechanism, just as it is beyond the scope of the textbook process of market equilibration (migration of capital from low-profit fields to high-profit fields). Creating an effective mechanism for this job is non-trivial. [8] discuss this, suggesting that one would need some kind of agreed annual innovation budget, and that it might be a good idea to have more than one agency in the business of disbursing resources for innovation experiments. The parlaying of scientific advances into new products that people want, or new processes that are more efficient than the old ones, is not an issue that invites a simple “capitalism vs socialism” split. Capitalist economies have differed quite widely in their effectiveness in this regard (for example, Britain versus the USA), and socialist economies might be expected to differ too.

4 IS THERE AN UNCOUNTABLY INFINITE NUMBER OF PRICES?

Murphy claims to use Cantor’s diagonal argument to demonstrate that there is an uncountable infinity of prices. In fact, he does no such thing. Rather, he explains diagonalisation and then asserts that it is applicable to the alleged infinity of prices without actually applying it. Nonetheless, let us, for the sake of argument, assume that there is an infinite number of prices and explore its cardinality.

Cantor’s argument may be summarised briefly as follows. We may *enumerate* (i.e. list or write down) all the integers starting from one by repeatedly adding one:

$$1 \quad 2 \quad 3 \quad \dots$$

We may also enumerate all the rational numbers—that is, numbers made from ratios of integers—by systematically listing all possible successive ratios of integers:

$$1/1 \quad 1/2 \quad 2/1 \quad 2/2 \quad 1/3 \quad 2/3 \quad 3/3 \quad 3/2 \quad 3/1 \quad \dots$$

Note that many rationals recur. For example, $1 = 1/1 = 2/2 = 3/3$ and so on. Note also that the *cardinality* of the rationals, that is the “type of infinity” that characterises how many there are, is the same as that of the integers, because we can put the rationals into one to one correspondence with the integers:

$$\begin{array}{l} 1 \leftrightarrow 1/1 \\ 2 \leftrightarrow 1/2 \\ 3 \leftrightarrow 2/1 \\ 4 \leftrightarrow 2/2 \\ 5 \leftrightarrow 1/3 \end{array}$$

In other words, there are as many integers as rationals. We say that the rationals are *countable*.

It is important to note that every integer and rational has a *finite representation*, even though some rationals have *infinite decimal expansions*. For example, if we try to evaluate $1/3$, we get $0.33333\dots$ (decimal) with 3 repeating forever. Nonetheless, $1/3$ is a perfectly good finite representation of that value.

Cantor introduced diagonalisation to show that the number of real numbers—that is, numbers consisting of an integer followed by an arbitrary number of decimal places—has a higher cardinality than the integers and rationals. In

other words, there are more reals than integers or rationals. Following [20], we consider all the real numbers between 0 and 1, where each is represented uniquely by a decimal fraction that doesn't terminate. If a number has a last decimal digit of 0 we replace this with an infinite number of 9s. Now, suppose there is an enumeration of reals x_1, x_2, x_3, \dots between 0 and 1. Suppose x_i has decimal digits x_{i1}, x_{i2}, x_{i3} and so on. Then we can write down the sequence of decimal fractions as:

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \dots \\ x_{21} & x_{22} & x_{23} & \dots \\ x_{31} & x_{32} & x_{33} & \dots \end{array}$$

and so on.

We now construct a new decimal fraction x' such that x'_1 differs from x_{11} , x'_2 differs from x_{22} , x'_3 differs from x_{33} , and so on, so that in general x'_i differs from x_{ii} [†]. Thus, x' is different from all of the reals that we have listed between 0 and 1. We conclude that the cardinality of the reals is higher than that of the integers and rationals; in other words, the reals are not a countable set.

It is now straightforward to demonstrate that this argument does *not* apply to prices.

First of all, we are not interested in prices *per se* but in prices of *commodities*, and the number of different commodities is necessarily countable. We note that, as in [33], every commodity is produced from a discrete and finite amount of other commodities, and base commodities (typically raw materials) are composed of finite numbers of atoms. Thus, we could represent every commodity by some archetype and give it a unique integer identifier based on, say, a Gödel number composed from the number of atoms of each element it contains. As there is only a finite number of elements we can again enumerate all possible finite combinations and hence all possible commodities.

Of course, if the number of commodities is countable then so is the number of corresponding prices. Nonetheless, let us further explore the enumerability of prices. Unit prices are only representable to a finite number of places, as monetary systems are based on integer quantities of their smallest denominations: pence or cents, say. We might argue that we wish to deal in arbitrary fractions of prices, for example in selling different proportions of a pound of

[†] Note that we may not replace a 9 with a 0

haggis or a litre of whisky. Noting the physical limitations on measurement which ensure that we can only distinguish discrete quantities of things at the microscopic level (see [6]), every fractional price is a ratio of integers and so must be rational and therefore countable.

5 COMPUTING WITH “INFINITE” DATA

As discussed above, Murphy claims that economic planning must take all possible future commodities into account. While we have already disputed this, let us explore its implications further.

A classic input–output system ([33]) requires the solution of a system of equations of the form:

$$\begin{aligned}
 A_a p_a + B_a p_b + \cdots + K_a p_k &= A p_a \\
 A_b p_b + B_b p_b + \cdots + K_b p_k &= B p_b \\
 &\vdots \\
 A_k p_k + B_k p_k + \cdots + K_k p_k &= K p_k
 \end{aligned}$$

where X is the quantity produced annually of commodity x , X_y is the quantity of y used to produce X and p_x is the unknown unit value of commodity x . Given known X s and X_y s, we wish to solve the system for the p_x s.

Granting, for the sake of argument, that there is an infinite number of commodities, we have already shown that it is countable. However, at any given moment in time only a finite number of *actual* commodities may be in production, even if there is a countable infinity of *potential*, but not currently produced, commodities.

Let us assume that at any given point in time there are finite N actual commodities in production. Their production can only involve a finite number of other actual commodities. Thus, we can set the quantity produced, quantity required and price of all the potential commodities to zero.

Clearly, while a finite number of commodities is used in production, there is an infinite number of potential commodities. Thus, most elements of the matrices and vectors for the X s, X_y s and p_x s are zeros. Even if we ignore the potential commodities, most actual commodities require only a small subset of all actual commodities as inputs, and so the *I/O* matrix has mostly zero elements. Structures whose elements are primarily zeros are said to be *sparse*.

There are numerous long-established techniques for representing and manipulating sparse structures: for a classic account see [21]. Typically, rather

than allocating all potentially necessary space as arrays (which is plainly impossible for an infinite number of elements) only the non-zero elements are represented, as composite structures of explicit indices and values.

For example, for an input–output matrix of N commodities, if N is finite and most elements are non-zero, then we might define the C-like array:

```
int commodities [N,N];
```

so the number of commodity Y needed to produce commodity X would be `commodities[X,Y]`. If the elements are primarily zeros, then instead we could represent non-zero elements with the structure:

```
struct commodity {int i; int j; int value};
```

where `i` is the index corresponding to X , `j` is the index corresponding to Y , and `value` holds the former `commodities[X,Y]`. We then construct a linked structure to chain together all the non-zero elements represented in this form. When we need `commodities[X,Y]` for a calculation, rather than directly accessing the array, we search the linked structure for an entry whose `i` index is X and whose `j` index is Y . If we find one then we return the corresponding `value`. If we can't find one then the element must be zero. When a new commodity is first produced, we add new entries to the end of the chain. If a commodity goes out of production then we remove all entries that refer to it.

While this increases the cost of finding a matrix element from constant time to linear time, it enables us to represent and process an apparently infinite number of elements in finite space and time. And for finite N , this technique saves space compared with the original array representation if no more than 25% of the elements are non-zero.

6 A SHORT EXPERIMENT

If we assume that the socialist economy retains some form of market for consumer goods, as proposed by Lange to provide information on final requirements, then the process of deriving a balanced plan is tractable.

Let us take a very simple example, an economy with four types of goods which we will call bread, corn, coal and iron. In order to mine coal, both iron and coal are used as inputs. To make bread we need corn for the flour and coal to bake it. To grow the corn, iron tools and seed corn are required. The making of iron itself demands coal and more iron implements. We can describe this as a set of four processes:

TABLE 1
Convergence of gross production on that required for the final net product

iron	coal	corn	bread	labour	
0	20000	0	1000	0	Net output
2000	24500	1500	1000	61000	1st estimate gross usage
2580	29400	1650	1000	129500	
3102	31540	1665	1000	157300	
3342	33012	1666	1000	174310	
..	hidden steps
3708	34895	1667	1000	196510	
3708	34895	1667	1000	196515	
3708	34896	1667	1000	196517	20th estimate gross usage

1 ton iron	←	0.05 ton iron	+ 2 ton coal	+ 20 days labour
1 ton coal	←	0.2 ton coal	+ 0.1 ton iron	+ 3 days labour
1 ton corn	←	0.1 ton corn	+ 0.02 ton iron	+ 10 days labour
1 ton bread	←	1.5 ton corn	+ 0.5 ton coal	+ 1 days labour

Assume, following [22], that the planning authorities have a current estimate of consumer demand for final outputs. The planners start with the required net output. This is shown on the first line of Table 1. We assume that 20000 tons of coal and 1000 tons of bread are the consumer goods required.

The planners estimate how much iron, corn, coal, and labour would be directly consumed in producing the final output: namely, 2000 tons of iron, 1500 tons of corn and 4500 additional tons of coal.

They add the intermediate inputs to the net output to get a first estimate of the gross usage of goods. Since this estimate involves producing more iron, coal and corn than they had at first allowed for, they repeat the calculation to get a second estimate of the gross usage of goods.

The answers differ each time round, but the differences between successive answers get smaller and smaller. Eventually, (assuming integer quantities are used) after 20 attempts in this example, the planners get a consistent result: if the population is to consume 20000 tons of coal and 1000 tons of bread, then the gross outputs must be 3708 tons of iron, 34896 tons of coal and 1667 tons of corn.

Is it feasible to scale this up to the number of goods produced in a real

economy? While the calculations would have been impossibly tedious to do by hand in the 1930s, they are readily automated. Table 1 was produced by running a computer algorithm. If detailed planning is to be feasible, we need to know:

1. How many types of goods an economy produces.
2. How many types of inputs are used to produce each output.
3. How fast a computer program running the algorithm would be for the scale of data provided in (1) and (2).

Table 2 illustrates the effect of running the planning algorithm on a cheap personal computer of 2004 vintage. We determined the calculation time for economies whose number of industries ranged from one thousand to one million.

Two different assumptions were made regarding the functional relationship between the mean number of inputs used to make a good and the complexity of the economy. Clearly, the number of direct inputs used to manufacture any given product is only a tiny fraction of the total number of goods produced. It is plausible that as industrial complexity develops, the mean number of inputs used to produce each product will also grow, but more slowly. In the first part of Table 2 we assume that the mean number of inputs (M) grows as the square root of the number of final outputs (N). In the second part of the table we assume that the growth of M follows a logarithmic law.

It can be seen that calculation times are modest even for very big economies. The daunting “million-equation” foe of the 1930s yields gracefully to the modest home computer of the early 21st century! The limiting factor in the experiments is computer memory. The largest model tested required 1.5 Gigabytes of memory; larger models would have required a 64-bit computer.

The experiment went up to 1 million products. The number of products in the Soviet economy was, according to an estimate cited in [27], around 10 million. Nove believed this number was so huge as to rule out any possibility of constructing a balanced disaggregated plan. This may well have been true with the computer technology available in the 1970s, but the situation is now quite different.

7 CONCLUSION

We have

TABLE 2

Timings for applying the planning algorithm to model economies of different sizes. Timings were performed on a 3 Ghz Intel Zeon running Linux, with 2 GB of memory.

	Industries N	Mean Inputs M	CPU Time seconds	Memory bytes
Law $M = \sqrt{N}$				
	1,000	30	0.1	150KB
	10,000	100	3.8	5MB
	40,000	200	33.8	64MB
	160,000	400	77.1	512MB
	320,000	600	166.0	1.5G
Law $M \approx \log N$				
	1,000	30	0.1	150KB
	10,000	40	1.6	2.4MB
	100,000	50	5.8	40MB
	1,000,000	60	68.2	480MB

- questioned Murphy's requirement that planning requires pre-knowledge of all possible prices,
- argued that the domain of prices to which planning is applied is in principle finite rather than infinite and that thus Cantor's arguments are inapplicable, or at worst prices are countable, and Cantor's arguments are applicable but irrelevant because there is no conceivable requirement that this domain be closed under diagonalisation,
- argued that planning over finite prices is tractable,
- shown that diagonalisation is not applicable to prices or commodities, and
- discussed how infinite structures of predominantly zero values may be given finite representations.

In conclusion we have shown that Murphy's arguments are ill founded. The computational feasibility of economic planning at a detailed level is an issue that must be investigated in its own right, and cannot be settled by appeal to Cantor. We have presented specific arguments that suggest that detailed planning is indeed feasible.

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