Explaining Concepts in Compositional Type-Based Program Analysis: Principality, Intersection Types, Expansion, etc.

Joe Wells

Heriot-Watt University

Sébastien Carlier and Christian Haack helped with these overheads.

Overview.

- **Basic concepts of types.**
- Type polymorphism.
- Compositionality and principality.
- Case study: Type error slicing made possible by compositionality.
- Case study: Getting principal typings in the λ-calculus with polymorphism.
- Conclusion.

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- For programming flexibility, it is best to automatically calculate optimal types, because programmers might write type information that is not “most general”, preventing typable programs from being accepted and/or making modules reusable in fewer combinations.
- Programming language type systems are getting more and more complex (e.g., Cyclone, a “safe C”) and it is getting harder for programmers to supply the types.

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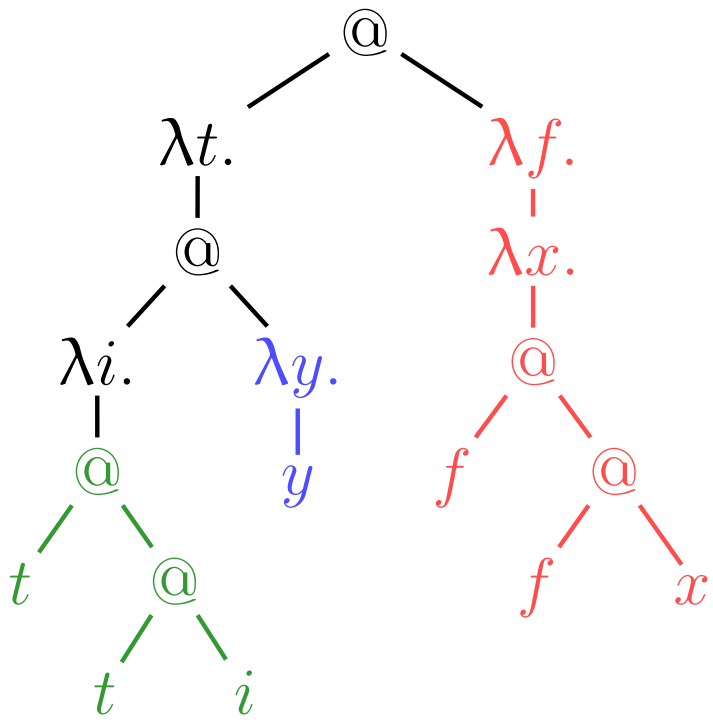
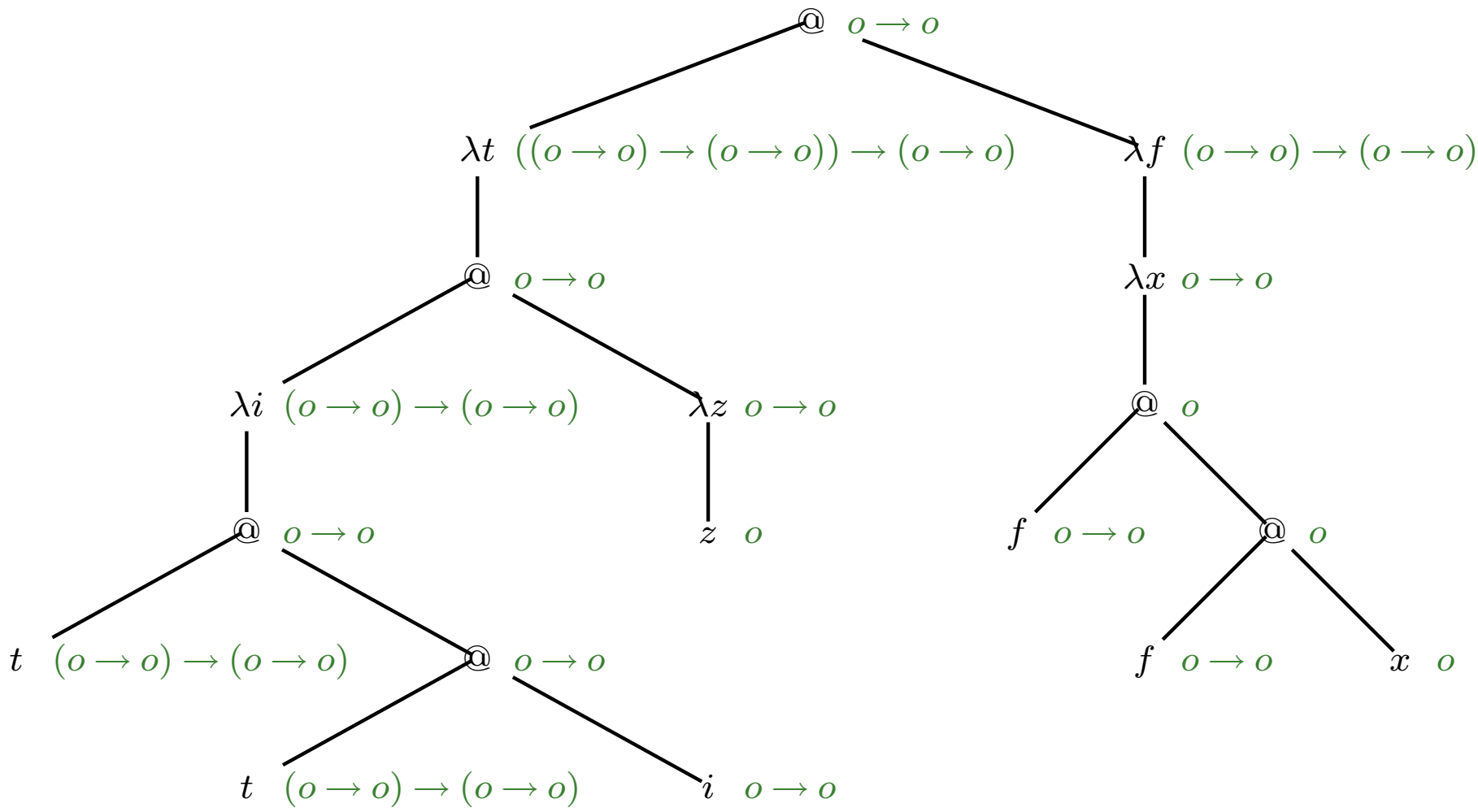
is the same as this λ-term:



$$= (\lambda t.(\lambda i.t\ (t\ i))\ (\lambda y.y))(\lambda f.\lambda x.f\ (f\ x))$$

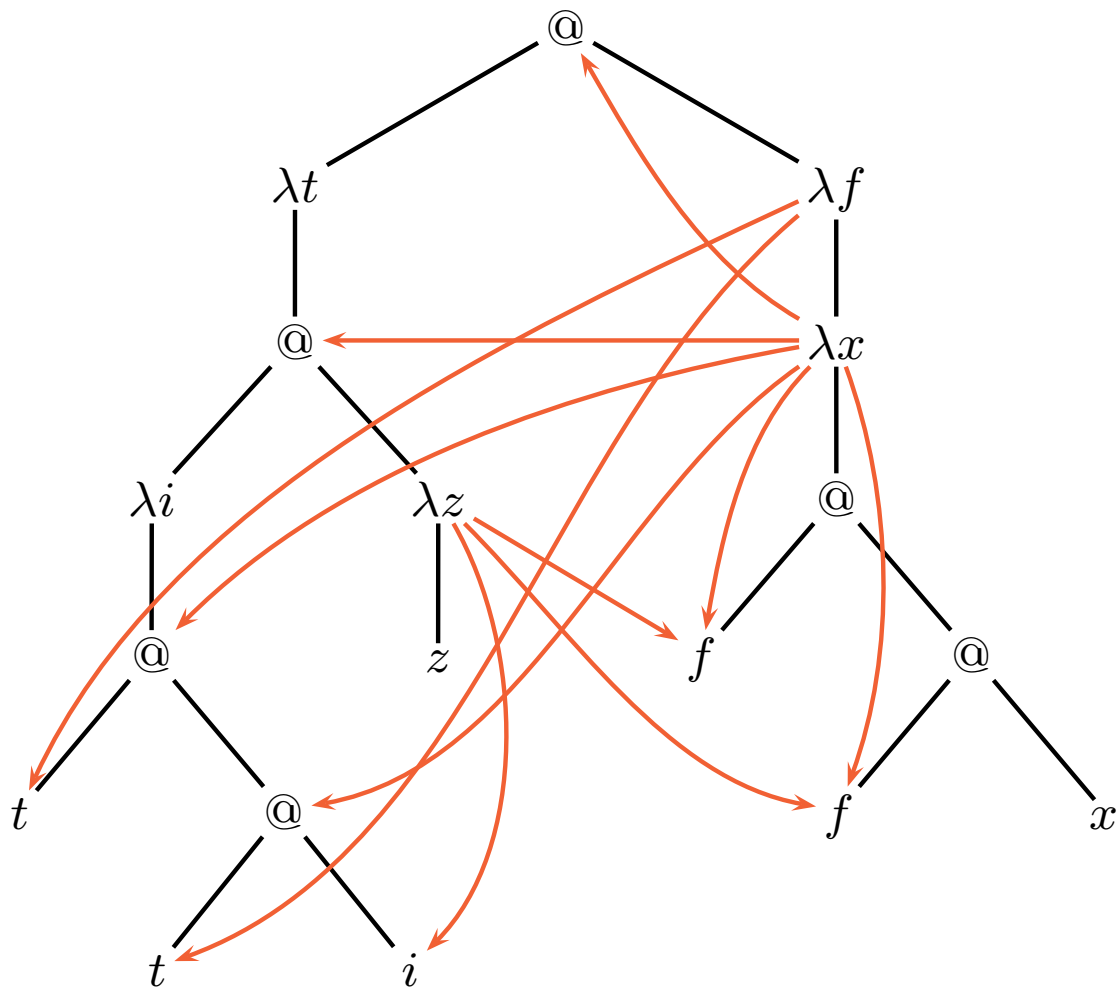
Example: Types.

Our example analyzed using the simply typed λ-calculus:



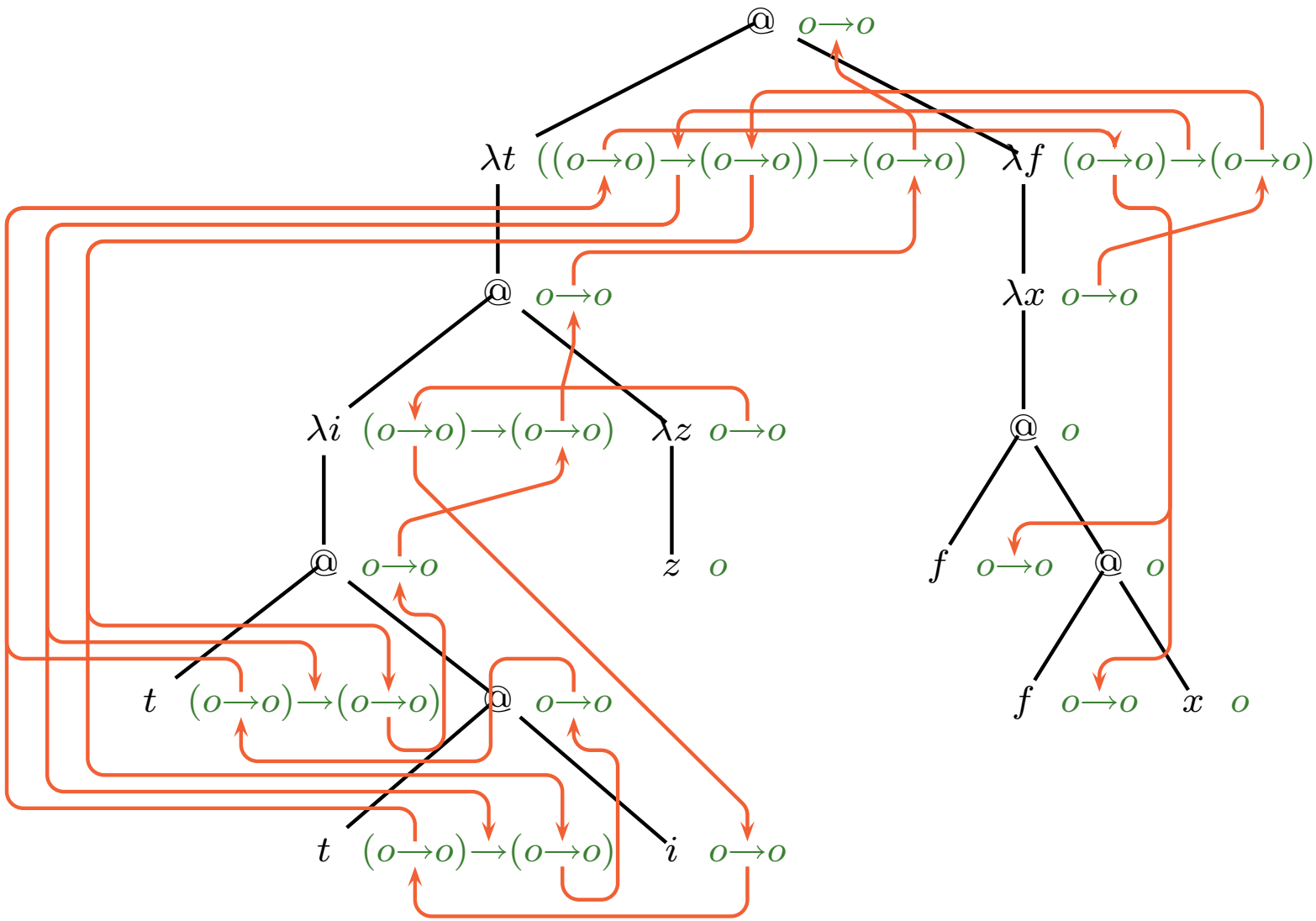
Example: Flow.

Our example analyzed using 0CFA [Shivers, 1991]:



Type analysis *is* flow analysis.

Illustrating how the type and flow analyses are intertwined:



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- Allows a program fragment to be viewed in different ways, depending on where its output is used or where its inputs come from.
- Is *essential* for code reuse [Reynolds, 1974] and abstract data types [Mitchell and Plotkin, 1988].
- Is traditionally treated formally using "for all" ($\forall$) quantifiers [Girard, 1972] or "there exists" ($\exists$) quantifiers and/or by a notion of subtyping ($T_1 \leq T_2$).

Example: “for all” quantifiers.

$$\texttt{val swap}^{\forall \texttt{a,b}.(\texttt{a}\times\texttt{b})\rightarrow(\texttt{b}\times\texttt{a})} = (\texttt{fn}\ (\texttt{x}^{\texttt{a}}, \texttt{y}^{\texttt{b}}) \Rightarrow (\texttt{y}^{\texttt{b}},\ \texttt{x}^{\texttt{a}}));$$

$$\texttt{val pair1}^{\texttt{int}\times\texttt{bool}} = (1, \texttt{true});$$

$$\texttt{val pair2}^{\texttt{real}\times\texttt{real}} = (2.7, 9.9);$$

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- In body of polymorphic function, the usage types are hidden behind type variables.

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$$\texttt{val closure1}^{\texttt{int}\times(\texttt{int}\rightarrow\texttt{bool})} = (5,\ (\texttt{fn x} \Rightarrow \texttt{x} > 1));$$

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  Example is semantically like $\forall \texttt{a} \in \{\texttt{int}, \texttt{real}\}.\texttt{a} \to \texttt{a}$, but the typing rules have significant practical differences.
- Named "intersection types" because in traditional model theory, semantic denotations $[\![T_1]\!]$ and $[\![T_2]\!]$ are program fragment sets and $[\![T_1 \cap T_2]\!] = [\![T_1]\!] \cap [\![T_2]\!]$ (usually).

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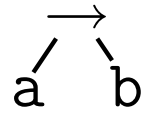
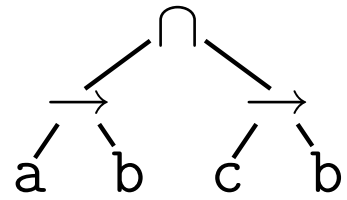
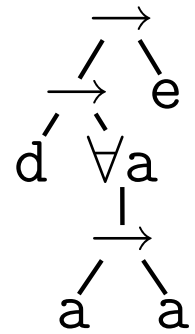
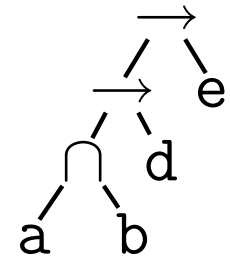
Type				
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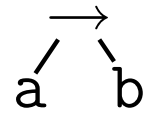
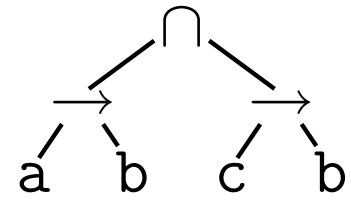
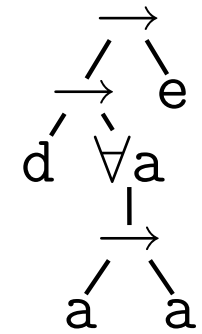
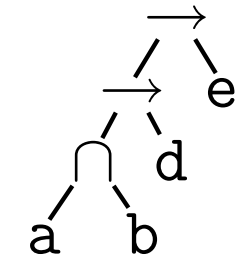
Type				
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- Rank-k bounds how far into the future evaluation a type system can look in making distinctions when predicting behavior. The rank-k restrictions of intersection types are decidable.

Terms typed by type systems.

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Λ_k: rank-k System F.

$\bigcap$: intersection types.

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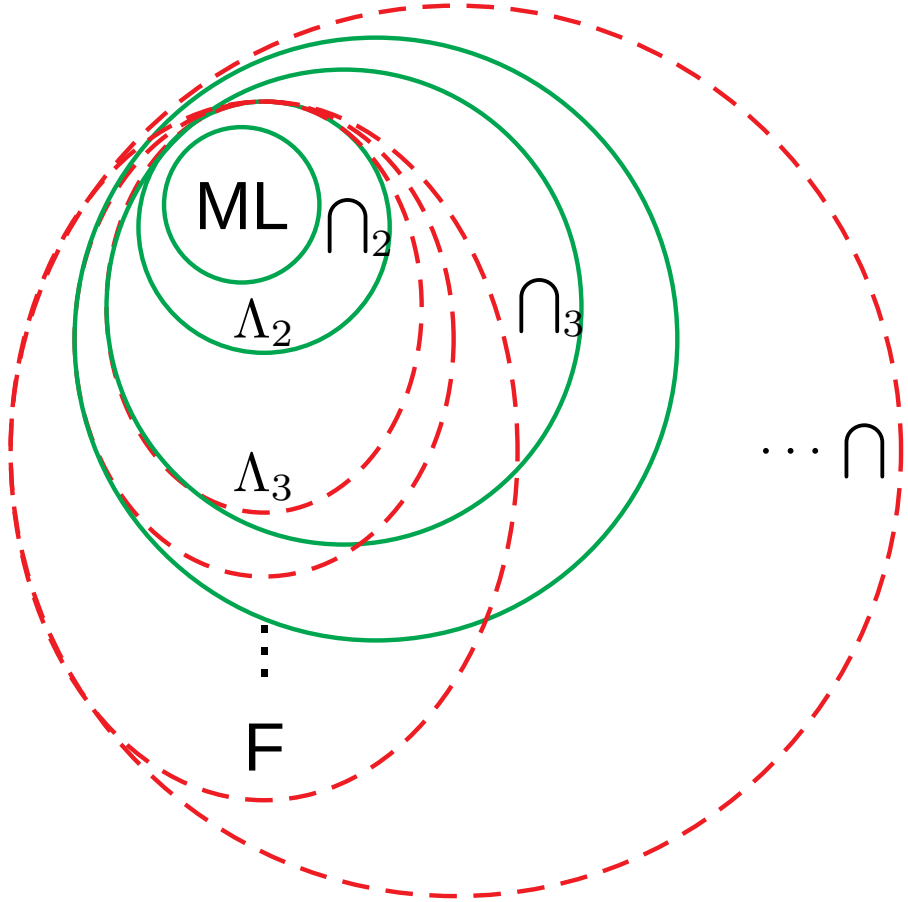

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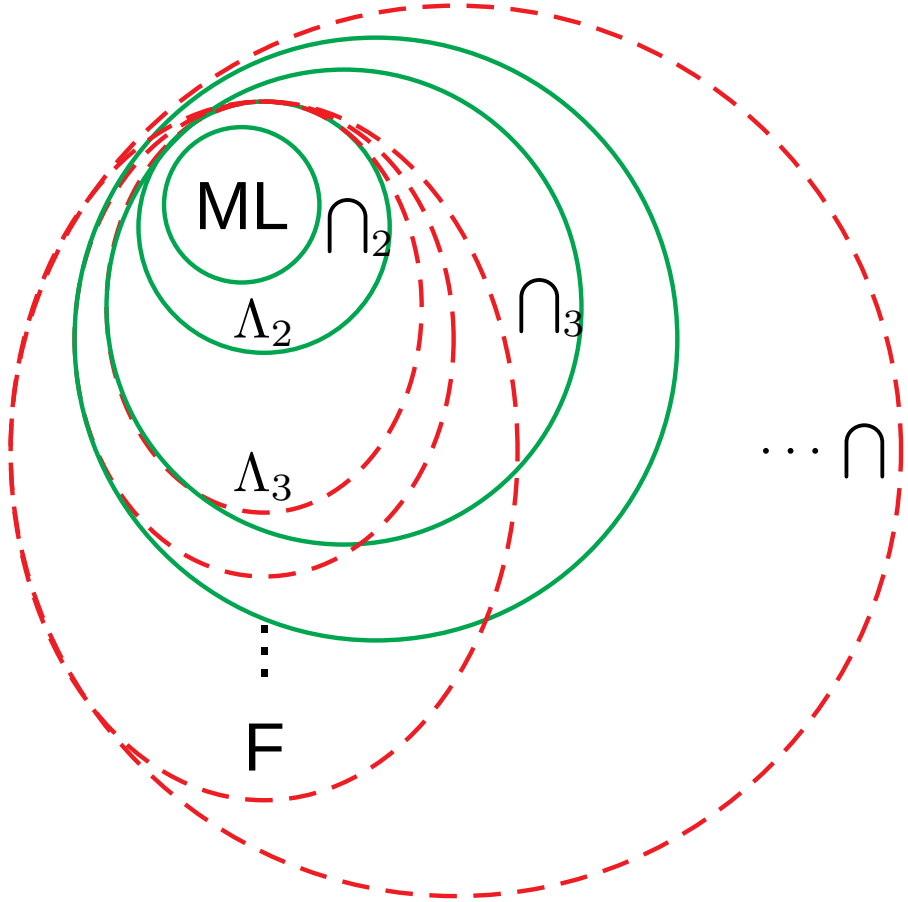

(Decision procedure complexity now known [Kfoury, Mairson, Turbak, and Wells, 1999].)

Example untypable with $\forall$-quantifiers.

$$M = \left(\begin{array}{l} \texttt{fun self_apply2 z} \Rightarrow (\texttt{z z})\ \texttt{z}; \\ \texttt{fun apply f x} \Rightarrow \texttt{f x}; \\ \texttt{fun reverse_apply y g} \Rightarrow \texttt{g y}; \\ \texttt{fun id w} \Rightarrow \texttt{w}; \\ (\texttt{self_apply2 apply not true}, \\ \ \texttt{self_apply2 reverse_apply id false not}); \end{array} \right)$$

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- Program fragment M *safely* computes $(\texttt{false}, \texttt{true})$.

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$$M = \left(\begin{array}{l} \texttt{fun self_apply2 z} \Rightarrow (\texttt{z z})\ \texttt{z}; \\ \texttt{fun apply f x} \Rightarrow \texttt{f x}; \\ \texttt{fun reverse_apply y g} \Rightarrow \texttt{g y}; \\ \texttt{fun id w} \Rightarrow \texttt{w}; \\ (\texttt{self_apply2 apply not true}, \\ \ \texttt{self_apply2 reverse_apply id false not}); \end{array} \right)$$

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- M needs only rank-3 intersection types.

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Intersection types seem to have the potential to be a general, flexible framework for many program analyses.

Overview.

- Basic concepts of types.
- Type polymorphism.
- **Compositionality and principality.**
- Case study: Type error slicing made possible by compositionality.
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- Many type systems do not support compositional analysis, because they lack *principal typings*. (More later.)

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- Modern systems like Java and C♯ have broken the link needed by separate compilation between the compile-time and link-time environments, so it is better not to use *any* compile-time environment.
- A network node without global knowledge can gradually learn more about other entities and predict possible failures as soon as sufficient information is available.

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A type system can thus be seen as a set of pairs of the form $(M : \Theta)$ where Θ is usually of the form $\langle A \vdash T \rangle$.

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- Principal typings (PTs) allow *compositional* analysis.
- Until Wells [2002], each system with PTs had its own definition via syntactic operations like *substitution*, *subtyping*, *weakening*, etc.

Which systems have principal typings?

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- The popular W algorithm [Damas and Milner, 1982] for HM is not compositional and compositional analysis for HM can not use HM typings for intermediate results.
- Fortunately, a restricted rank-2 intersection type system [Damas, 1985] types the same terms and has PTs. My first case study uses this to analyze HM compositionally.
- Getting PTs usually needs types or type constraints that closely follow the language semantics. For the λ-calculus, adding intersection types can generally gain PTs (e.g., [Margaria and Zacchi, 1995]).

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For example, HM's lack of principal typings means an HM analysis algorithm must do one of these:

- Be incomplete (failing on some typable terms).
- Be noncompositional (not strictly bottom-up). For example, the W algorithm [Damas and Milner, 1982] is noncompositional because for $(\texttt{let}\ x = M\ \texttt{in}\ N)$ it first analyzes M and then uses the result in analyzing N.
- Not use HM typings for intermediate results. E.g., the typing of $(\texttt{xx})$ in the Chap. 1 system of Damas [1985]:

  $$\langle(\texttt{x} : \texttt{a}, \texttt{x} : \texttt{a} \rightarrow \texttt{b}) \vdash \texttt{b}\rangle$$

  This is essentially intersection types, i.e.:

  $$\langle(\texttt{x} : \texttt{a} \cap (\texttt{a} \rightarrow \texttt{b})) \vdash \texttt{b}\rangle$$

  Essentially the same was done by Shao and Appel [1993] and Bernstein and Stark [1995].

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The system I will describe uses a type system that types the same terms as HM, but uses intersection types instead of “for all” quantifiers internally, so it is compositional. This made it much easier to generate and solve constraints.

Type error example.

```
val average = fn weight => fn list =>
  let val iterator = fn (x,(sum,length)) =>
                        (sum + weight x, length + 1)
      val (sum,length) = foldl iterator (0,0) list
  in sum div length end

val find_best = fn weight => fn lists =>
  let val average = average weight
      val iterator = fn (list,(best,max)) =>
                        let val avg_list = average list
                        in if avg_list > max then
                             (list,avg_list)
                           else
                             (best,max)
                        end
  val (best,_) = foldl iterator (nil,0) lists
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val find_best_simple = find_best 1
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Wrong type error location.

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Type error slice.

type constructor clash,
endpoints: function vs. int

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(.. val average = fn weight =>
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A possible fix.

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(.. val average = fn weight =>
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The key mechanism to understand is expansion, which is presented here via a well chosen example.

A problematic type inference example.

Consider typing this example λ-term:

$$M = \underbrace{(\lambda x.x\ (\lambda y.y\ z))}_{N}\ \underbrace{(\lambda f.\lambda x.f\ (f\ x))}_{P}$$

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$N : \langle (z : a) \vdash T_1 \rightarrow c \rangle$ where $T_1 = ((a \rightarrow b) \rightarrow b) \rightarrow c$

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To type M, we must find derivable judgements such that:

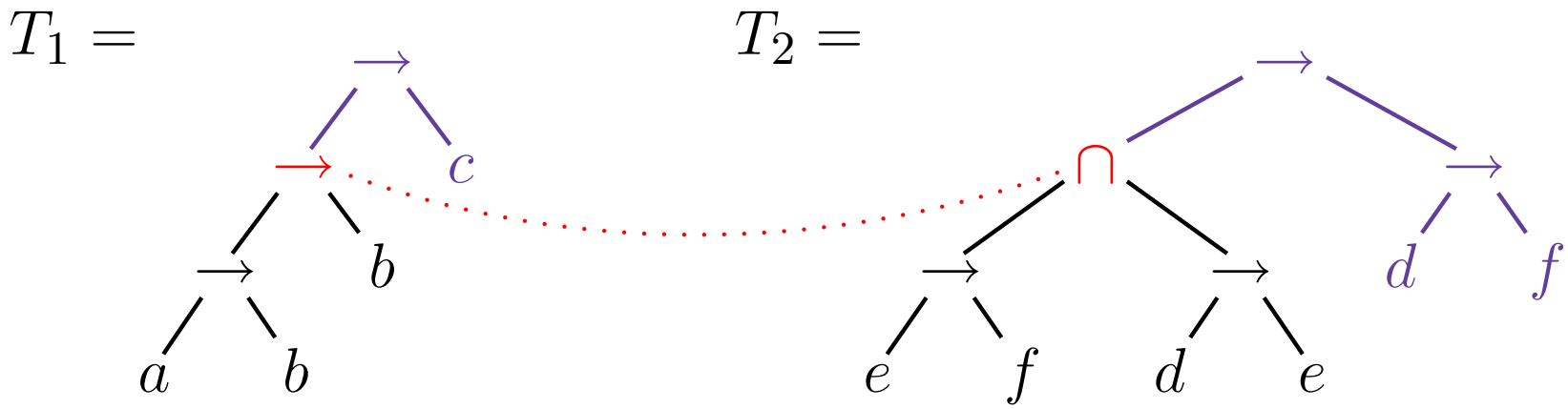
$$N : \langle (z : T'') \vdash T \rightarrow T' \rangle \qquad P : \langle () \vdash T \rangle$$

They ought to be obtainable from the principal typings.

Can we unify the example types? (1)

Can we unify T_1 and T_2 merely by substitution?

$$T_1 = ((a \to b) \to b) \to c$$
$$T_2 = ((e \to f) \cap (d \to e)) \to (d \to f)$$



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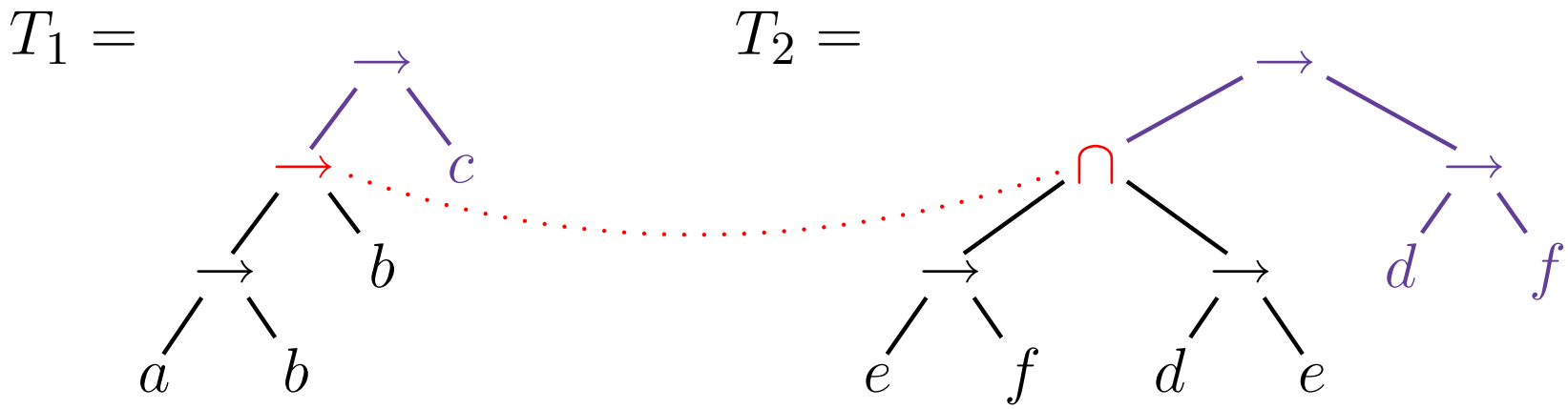

Problem: clash between $\to$ and $\cap$.

Could we use $T \cap T = T$ to make the intersection go away?

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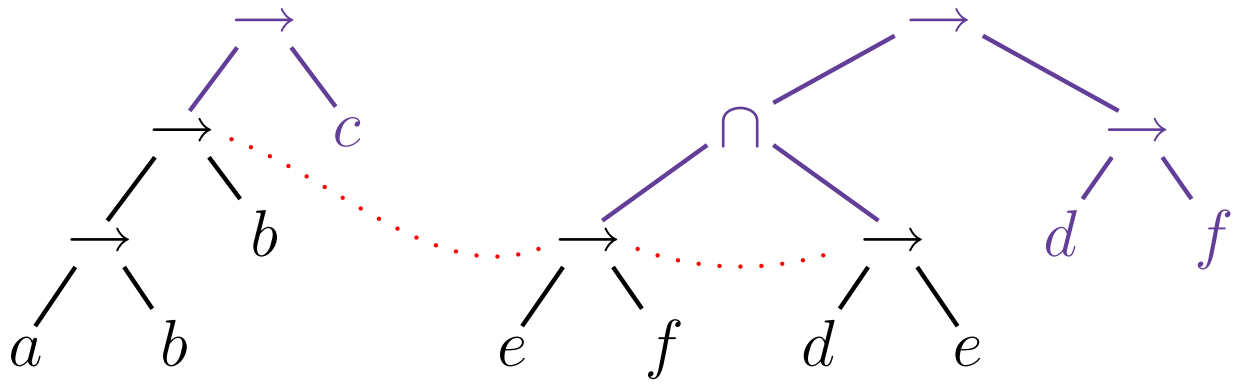
If using $T \cap T = T$, we now have 3 types to unify together:



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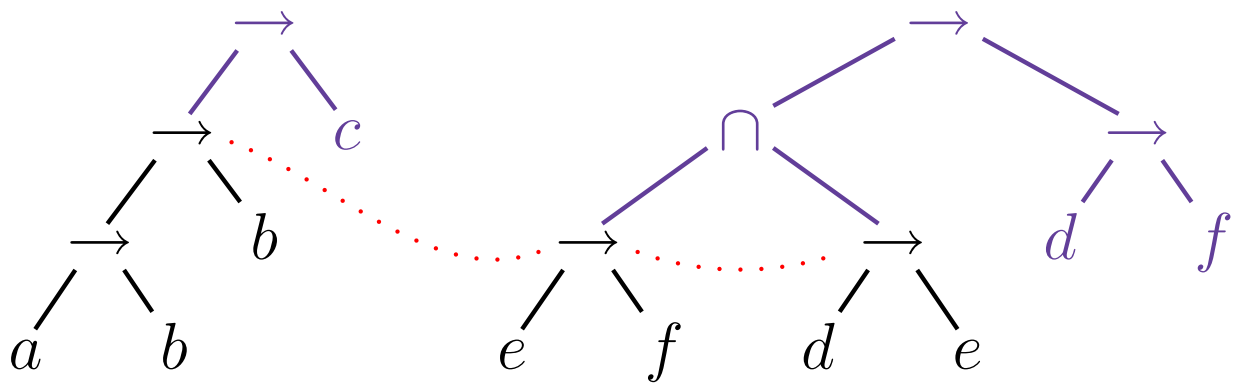
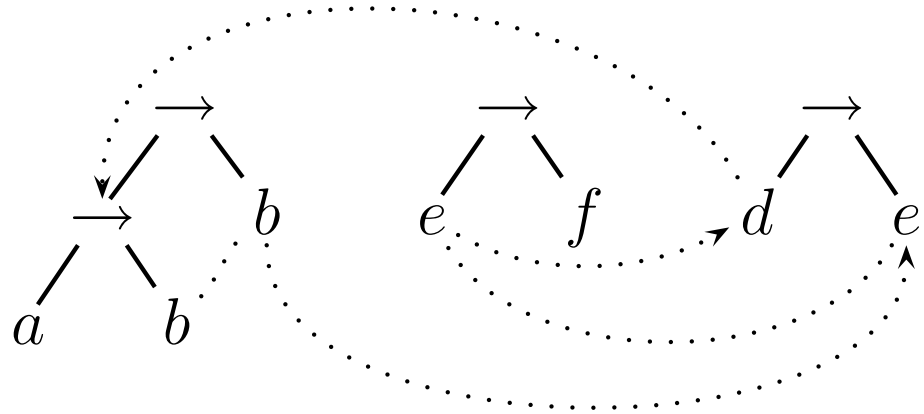


Oh, no! We cannot solve $a \rightarrow b = b$ (without recursive types).



Solving the example with expansion.

Instead, we do *expansion* [Coppo, Dezani-Ciancaglini, and Venneri, 1980] on the typing of N to solve the problem:

$$N : \langle \quad (z : a) \quad \vdash (\qquad ((a \to b) \to b) \qquad \to \quad c \quad) \to c \rangle$$

$$\downarrow$$

$$N : \langle (z : a_1 \cap a_2) \vdash (((a_1 \to b_1) \to b_1) \cap ((a_2 \to b_2) \to b_2) \to \quad c \quad) \to c \rangle$$

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Solving the example with expansion.

Instead, we do *expansion* [Coppo, Dezani-Ciancaglini, and Venneri, 1980] on the typing of N to solve the problem:

$$N : \langle \quad (z : a) \quad \vdash (\quad ((a \to b) \to b) \quad \to \quad c \quad) \to c \rangle$$

$$\downarrow$$

$$N : \langle (z : a_1 \cap a_2) \vdash (((a_1 \to b_1) \to b_1) \cap ((a_2 \to b_2) \to b_2) \to \quad c \quad) \to c \rangle$$

$$P : \langle \quad () \quad \vdash (\quad e \quad \to f) \cap (\quad d \quad \to e) \to (d \to f) \rangle$$

Then we apply this substitution (dotted lines above):

$$S_{\mathsf{f}} = (e := a_1 \to b_1,\ f := b_1,\ d := a_2 \to a_1 \to b_1,$$
$$b_2 := a_1 \to b_1,\ c := (a_2 \to a_1 \to b_1) \to b_1 \quad)$$

Huh? What did you just do?

But how precisely did expansion go from the 1st to the 2nd typing for N?

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Recently this has become much easier to understand due to a new definition using *expansion variables* (E-variables) [Kfoury and Wells, 1999; Carlier et al., 2004], which I will now show you.

How to do expansion with E-variables.

We apply $S = (e := (((a := a_1), b := b_1) \cap ((a := a_2), b := b_2)))$:

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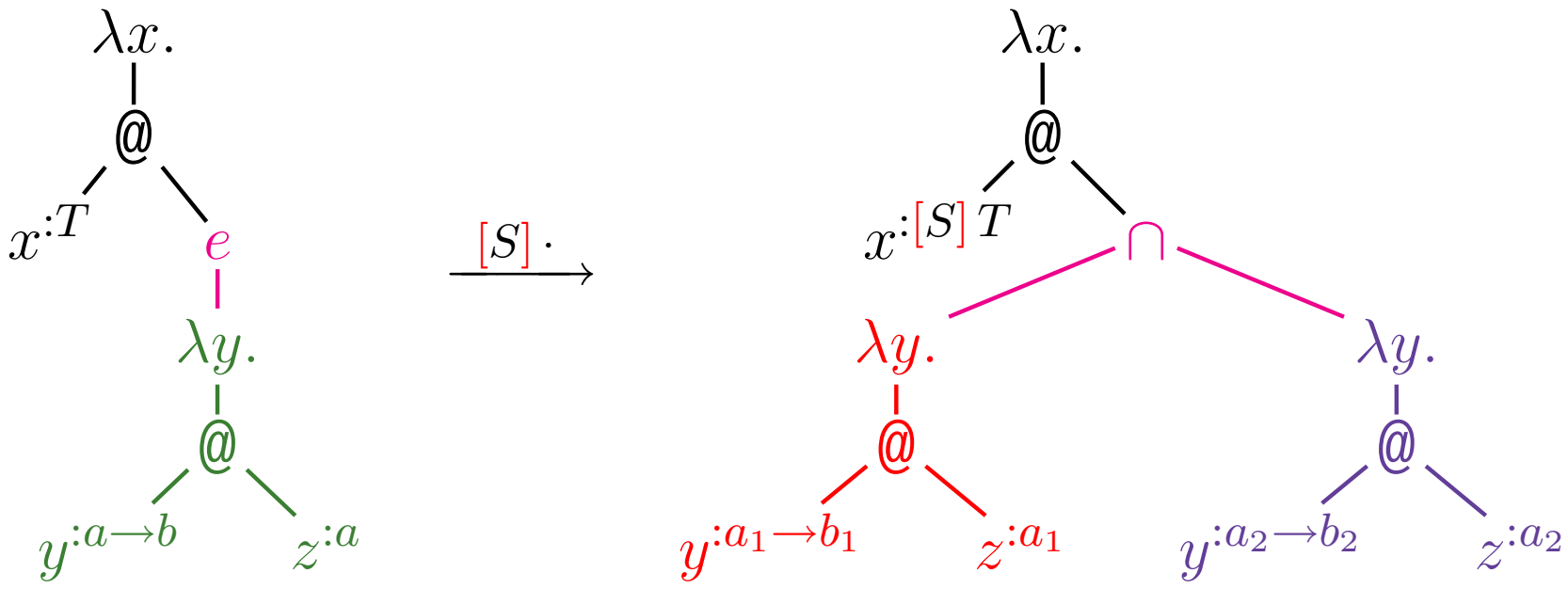

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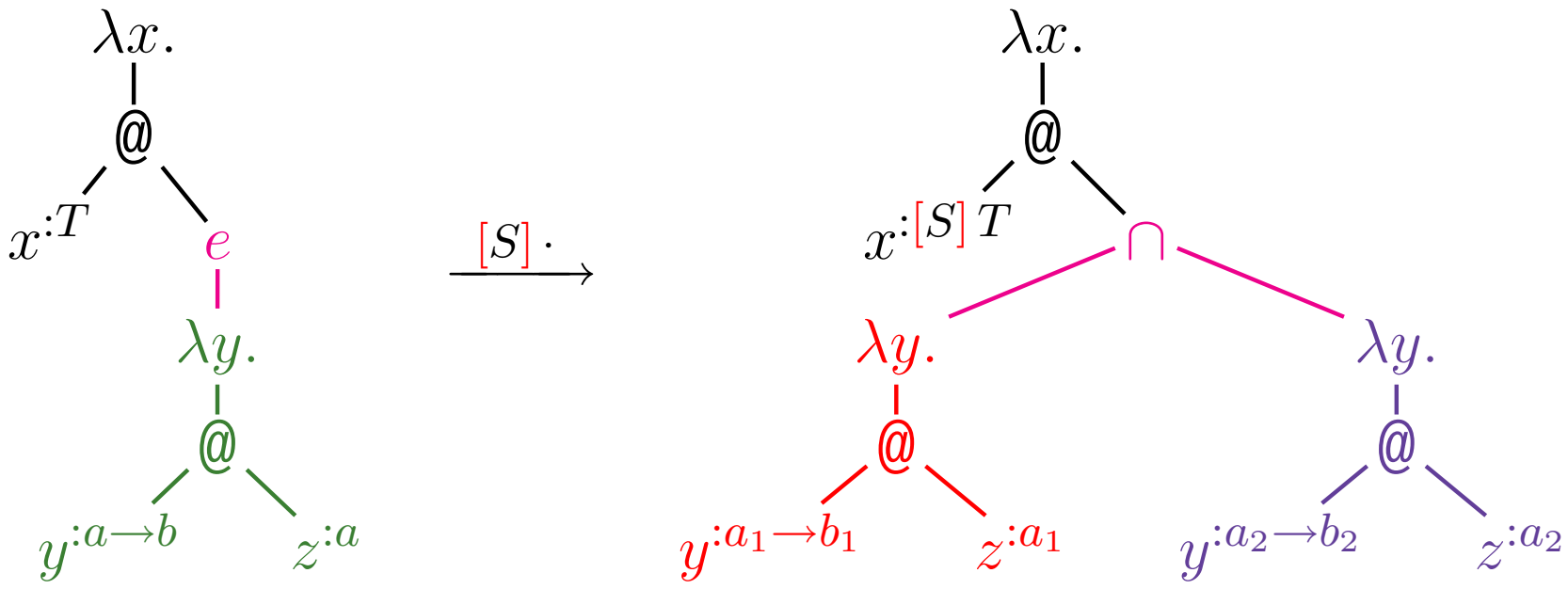

Effect on typings:

$$\langle (z : e\, a) \vdash (e\, ((a \to b) \to b) \to c) \to c \rangle$$

$$\xrightarrow{[S]\,\cdot} \langle (z : a_1 \cap a_2) \vdash (((a_1 \to b_1) \to b_1) \cap ((a_2 \to b_2) \to b_2) \to c) \to c \rangle$$

Overview.

- Basic concepts of types.
- Type polymorphism.
- Compositionality and principality.
- Case study: Type error slicing made possible by compositionality.
- Case study: Getting principal typings in the λ-calculus with polymorphism.
- **Conclusion.**

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- Getting compositionality is hard with “for all” quantifiers, so there may be motivation to learn *intersection types* and similar technologies.
- Doing compositional analysis with intersection types requires *expansion*. This is now much better understood and can be done with *E-variables*.

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