Rewriting in the Design of Type Systems

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Overview

- **Goals for this talk**
- Type inference as rewriting via an example with simple types
- Intersection types and why you might want them
- Type inference as rewriting via an example with intersection types
- Various concluding remarks

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- Show examples where it is reasonable to use similar syntax for types and terms.
- Show how the definition of a type system might be based on rewriting on the terms of the system.
- Give what may be a clearer explanation of type inference in System I [Kfoury and Wells, 1999].

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  - Types may be *easy* or *hard* to determine.
- It is not immoral/wrong if types are not formulas of well known independently interesting logics.
- Reasoning about a software system is *compositional* if the pieces are reasoned about independently and the results are composed without reinspecting the pieces.

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An Example Program

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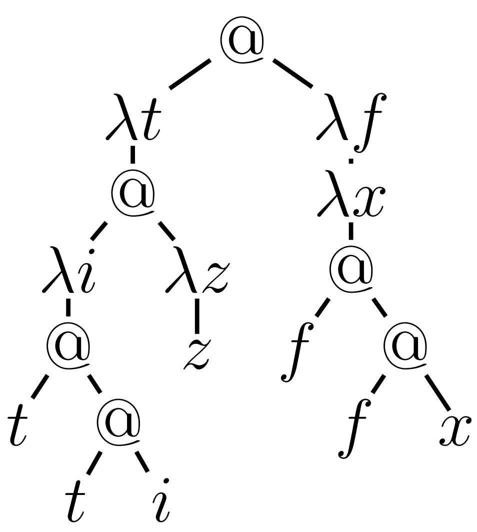
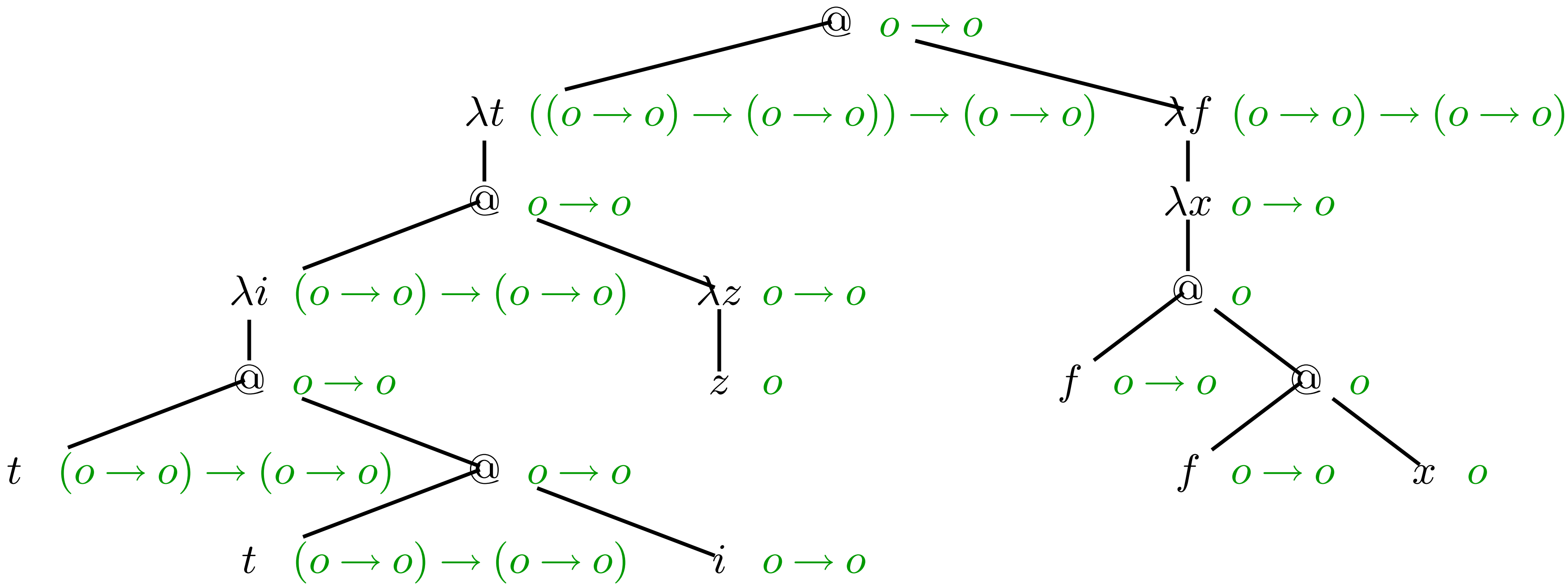
which can be drawn as this tree:



Simple Types for the Example

Our example analyzed using the simply typed λ-calculus:



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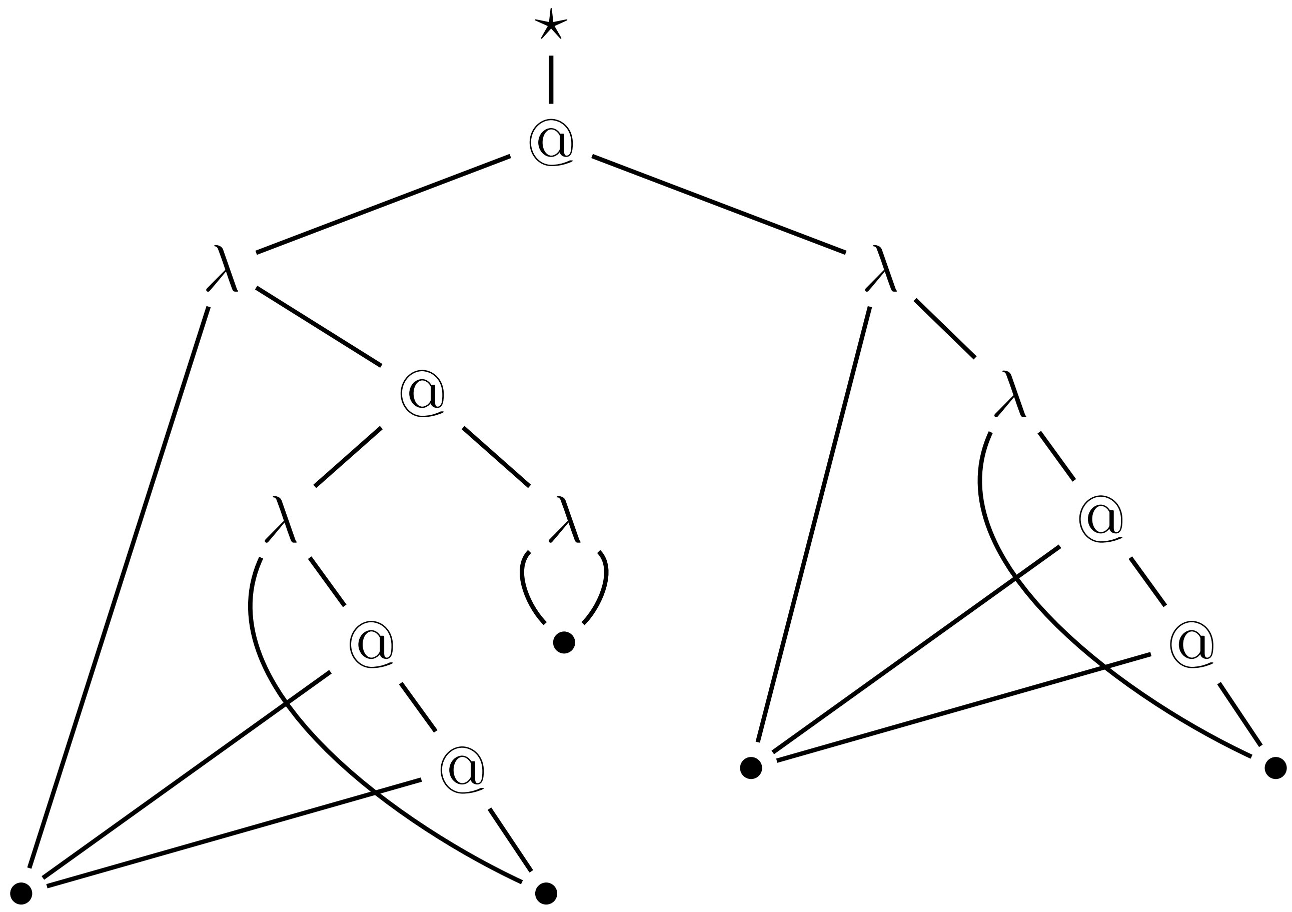
- However, I think it will be helpful to view the process from a different angle. Probably someone else has done something like this before, perhaps not quite the way I will do it.
- It is not essential that the following diagrams are DAGs, but any implementation would do so and it makes the examples fit.

Another Look at the Lambda Term



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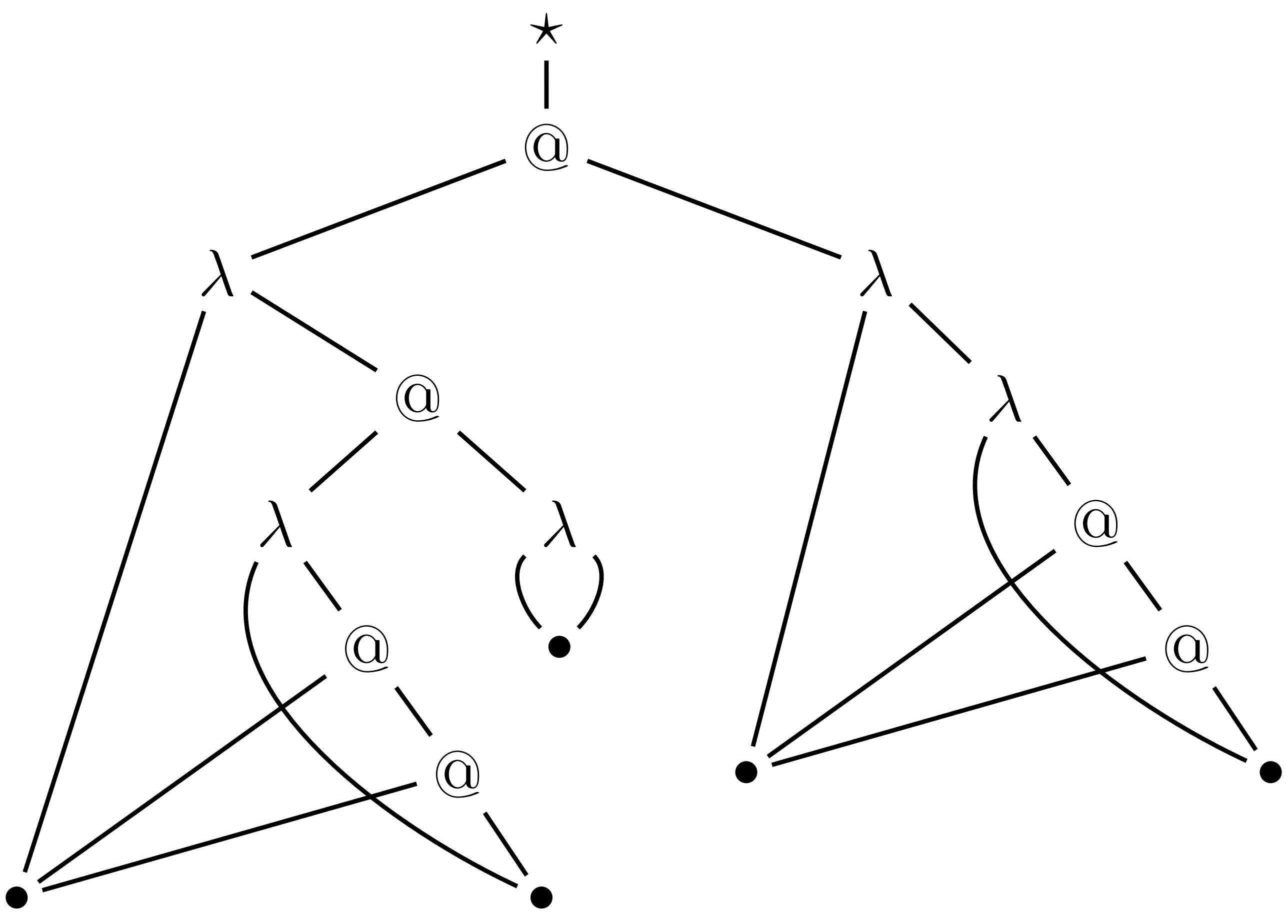
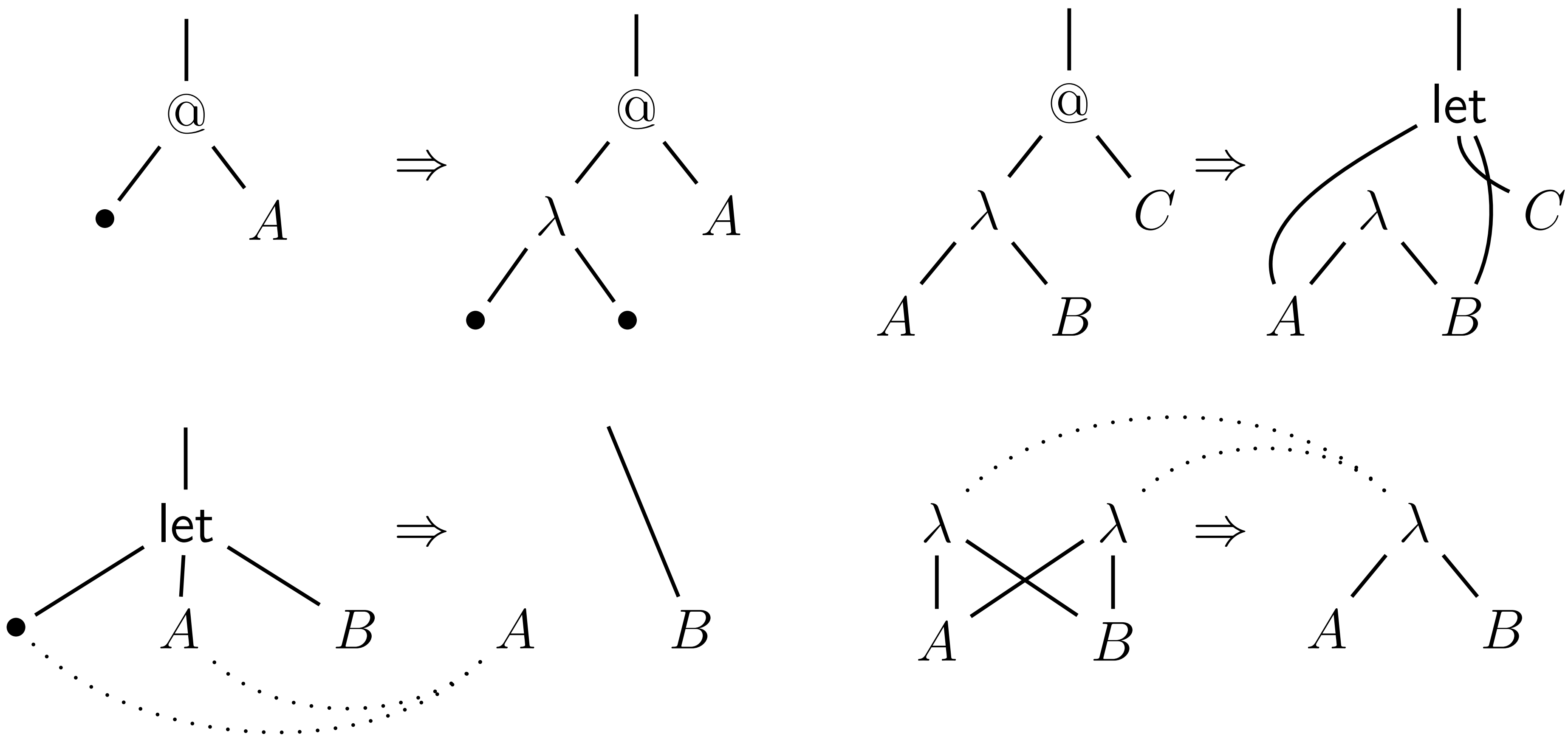


This is just another view of the same term. However, I will also say it is the term's *type*, just not yet normalized.

Some of the Type Rewriting Rules



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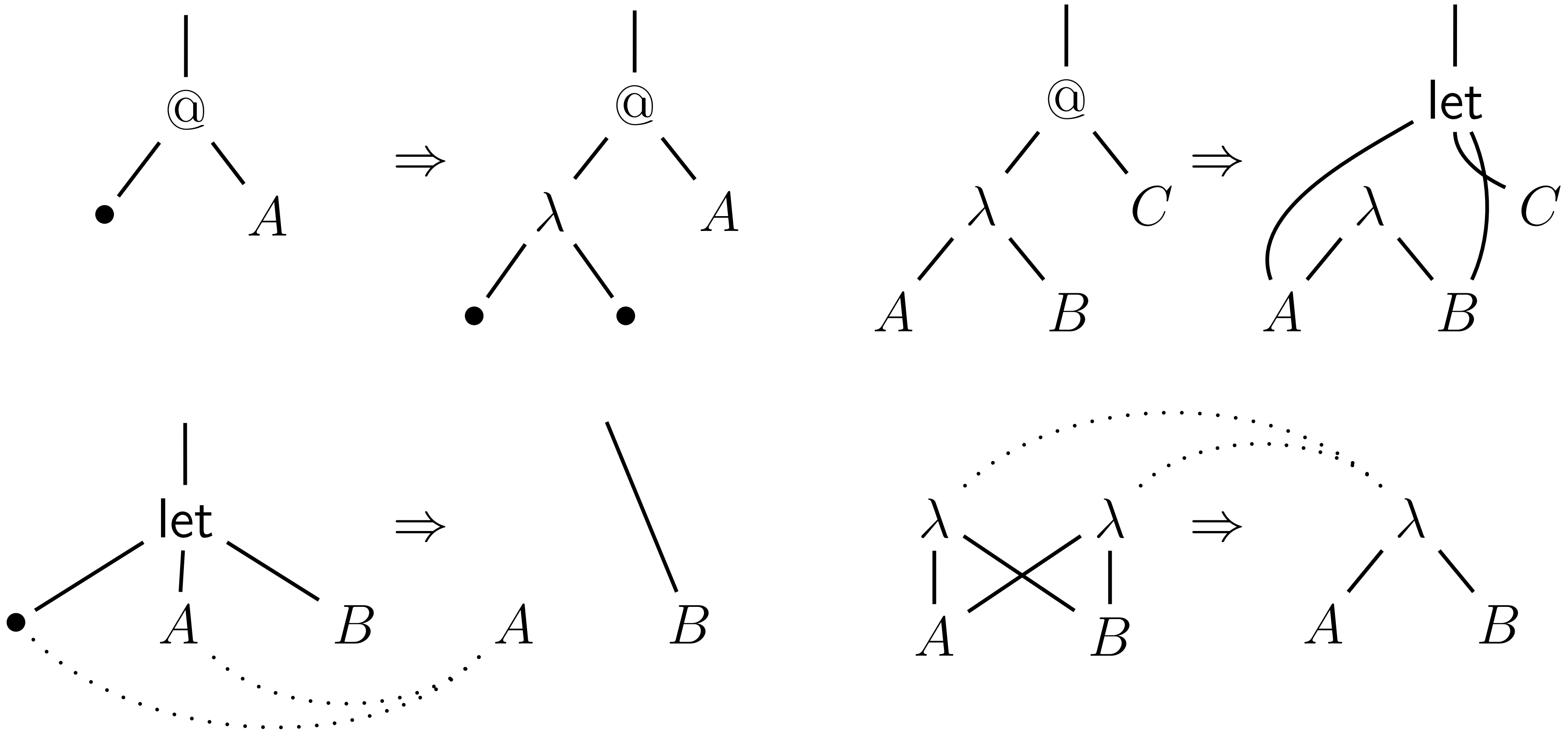
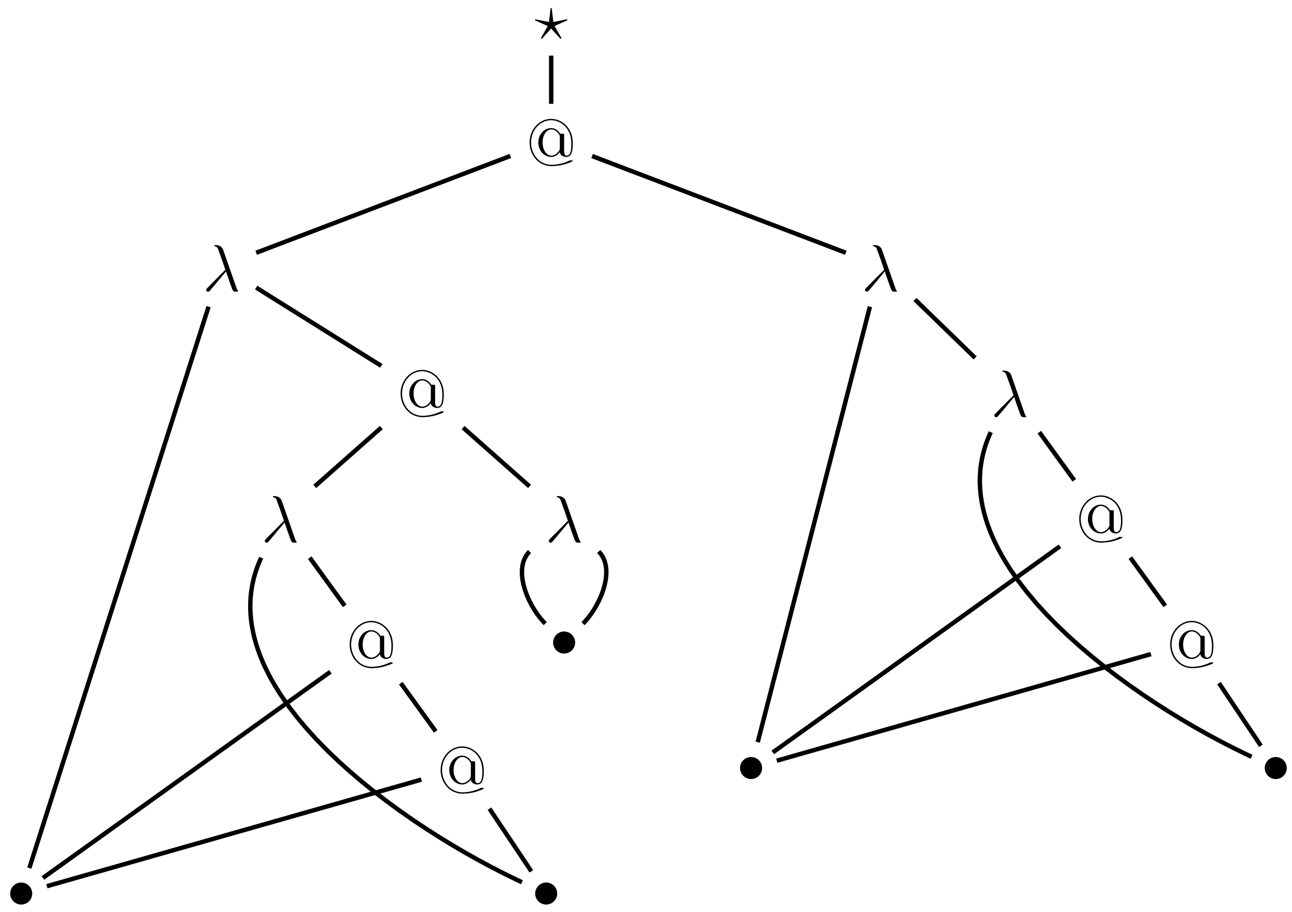


Some rules omitted. There are rules for garbage collection.
Formalism unverified, for discussion only.

Normalizing the Term’s Type (1)



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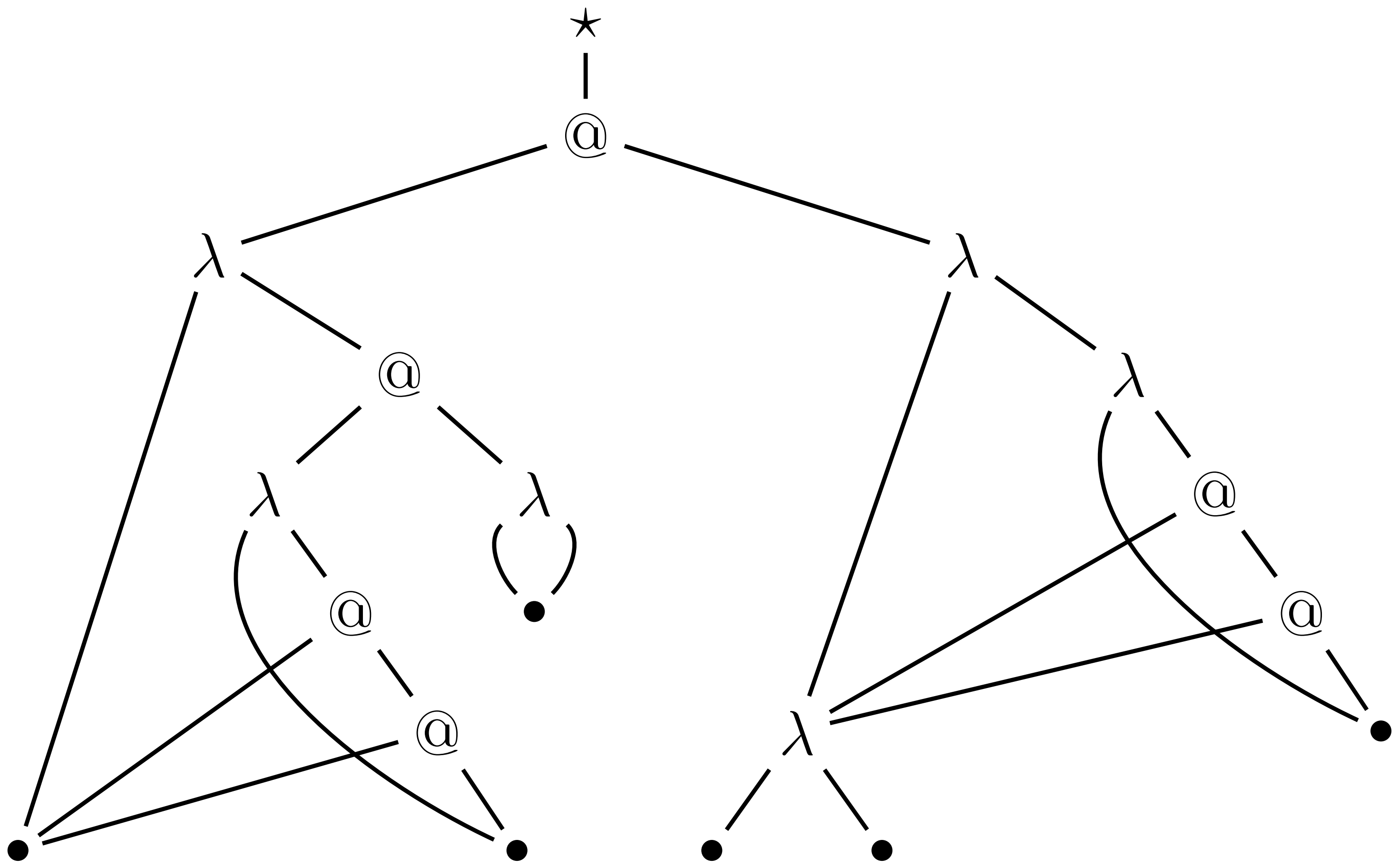

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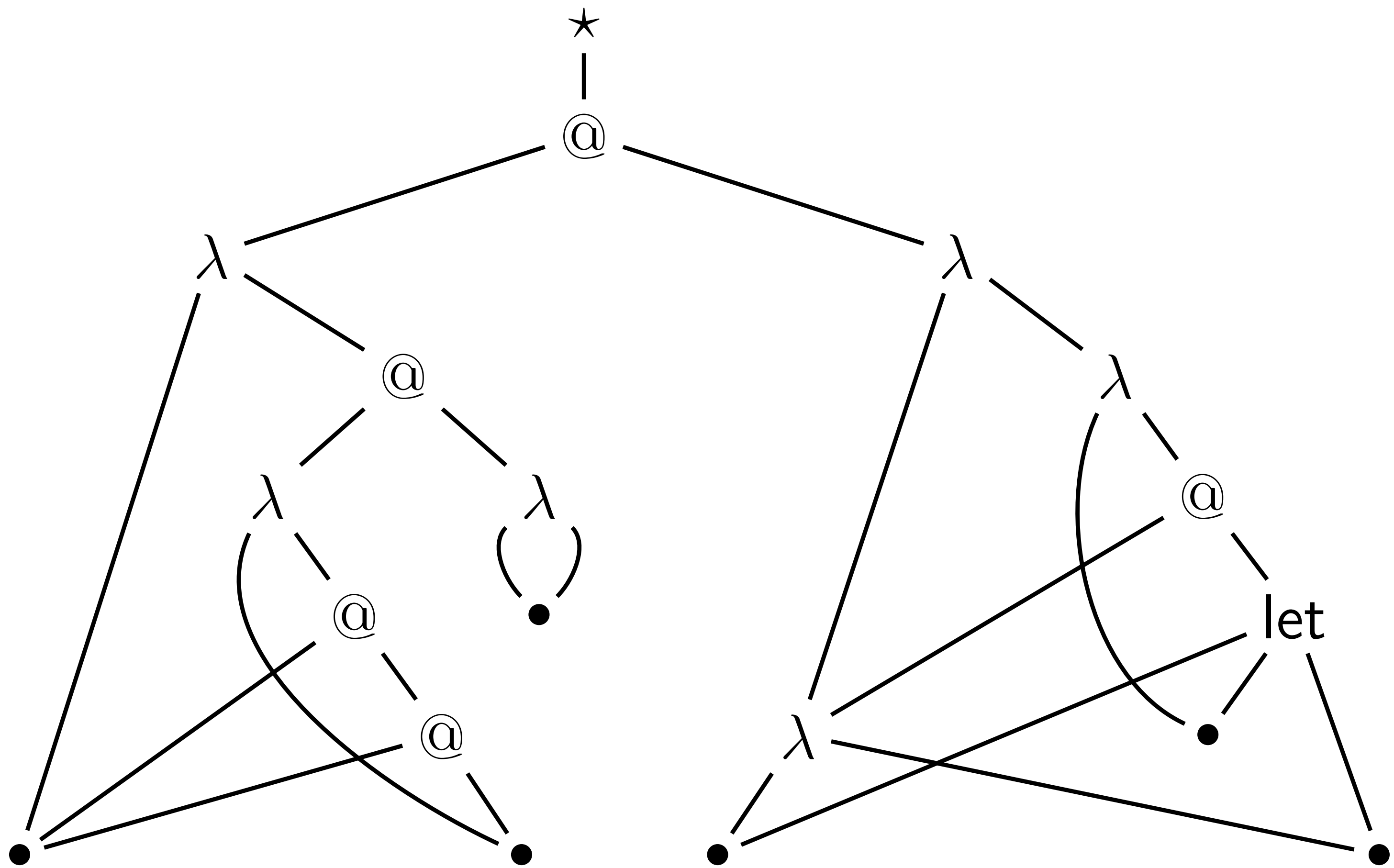

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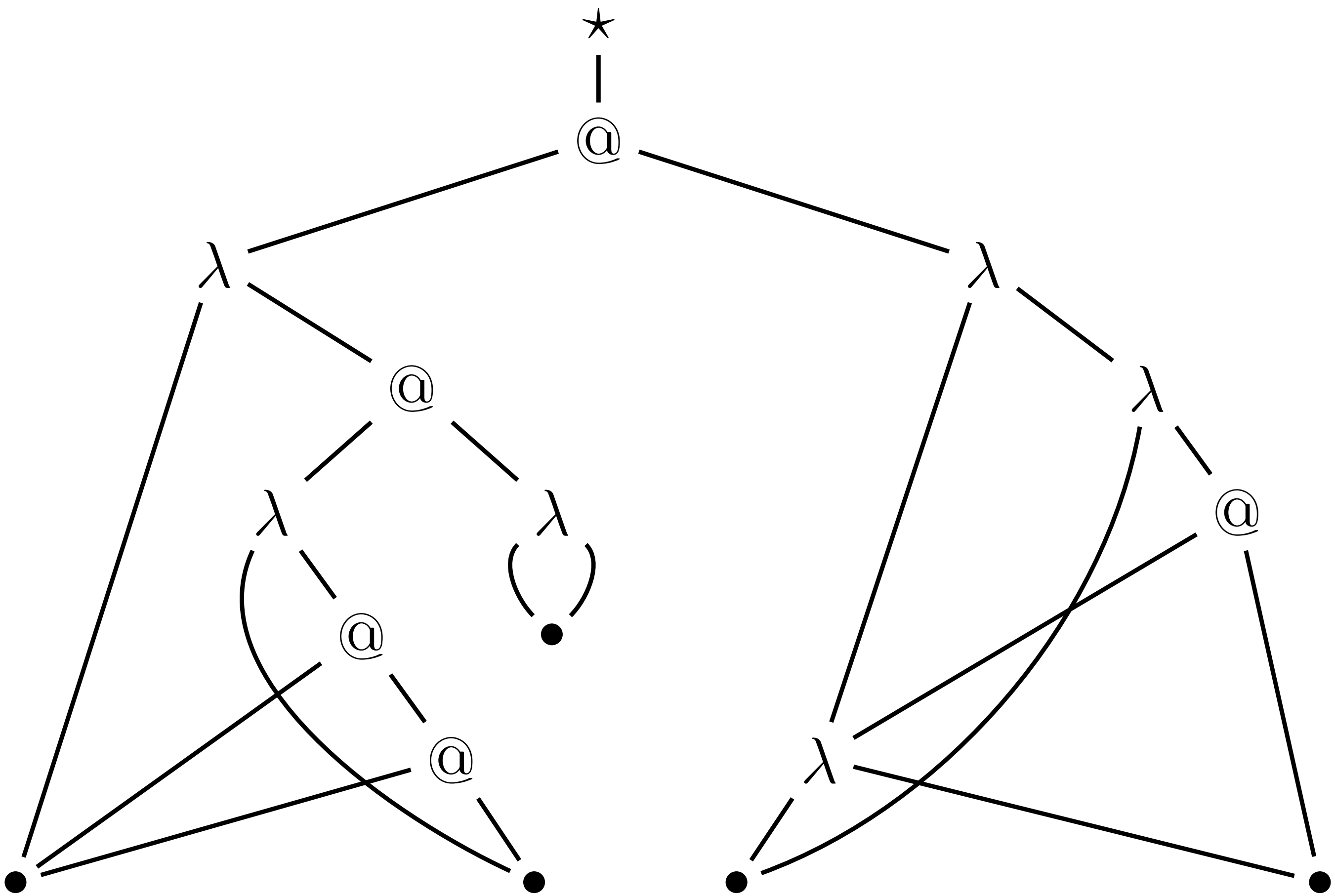
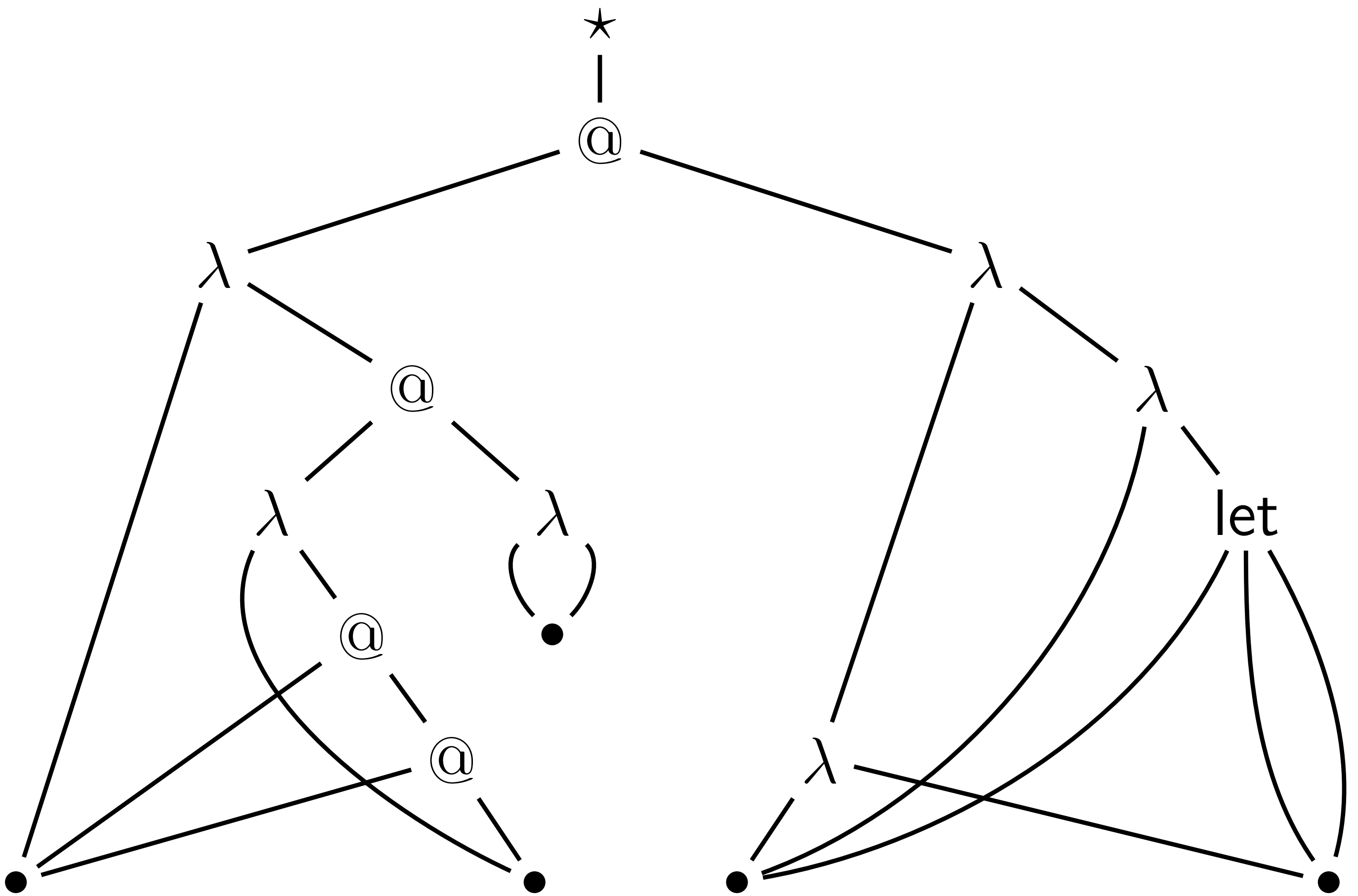


Normalizing the Term's Type (5)



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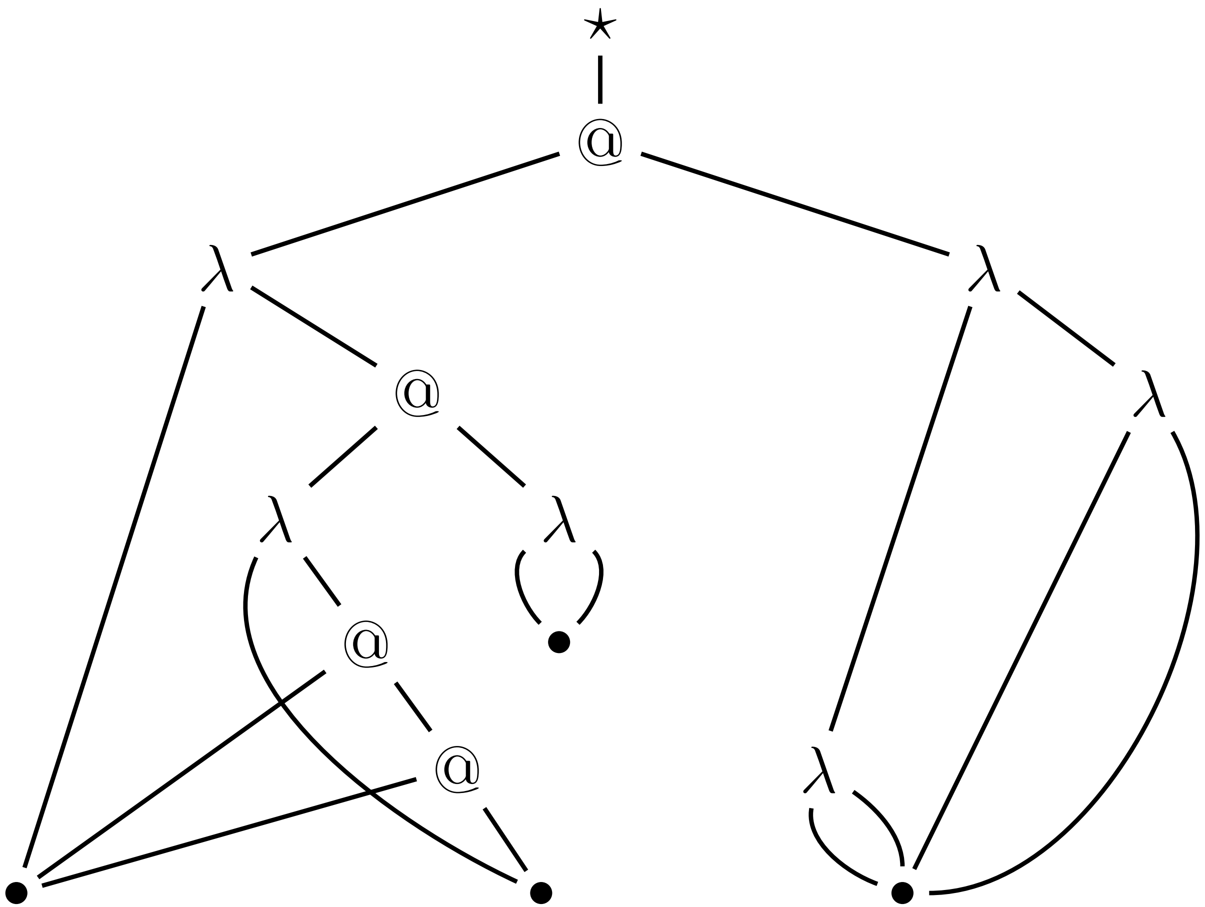

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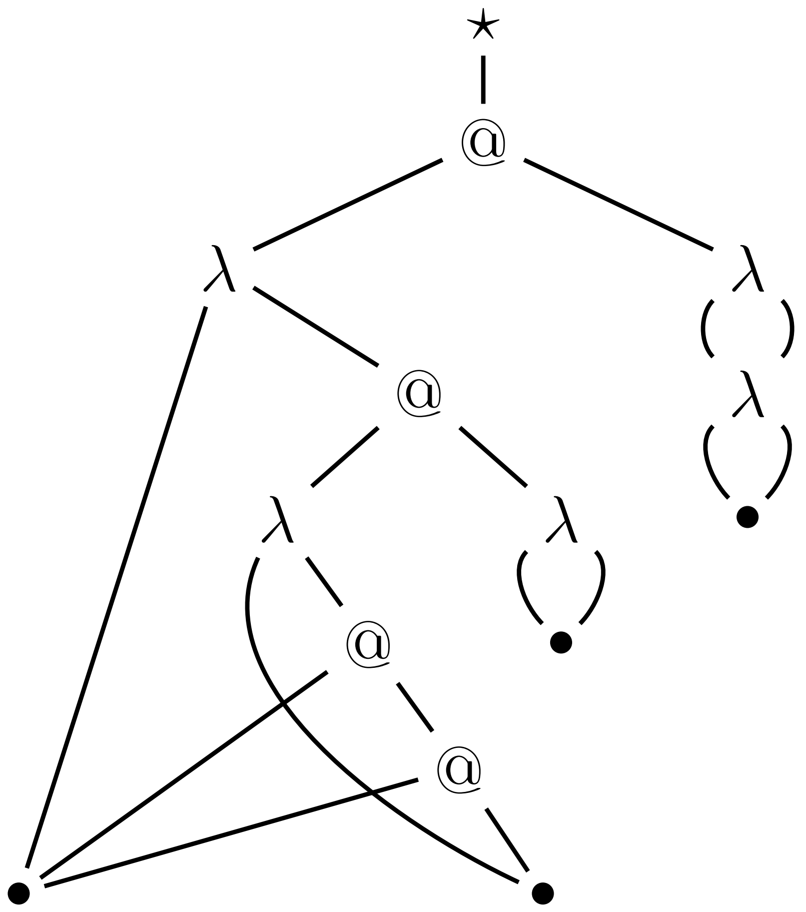

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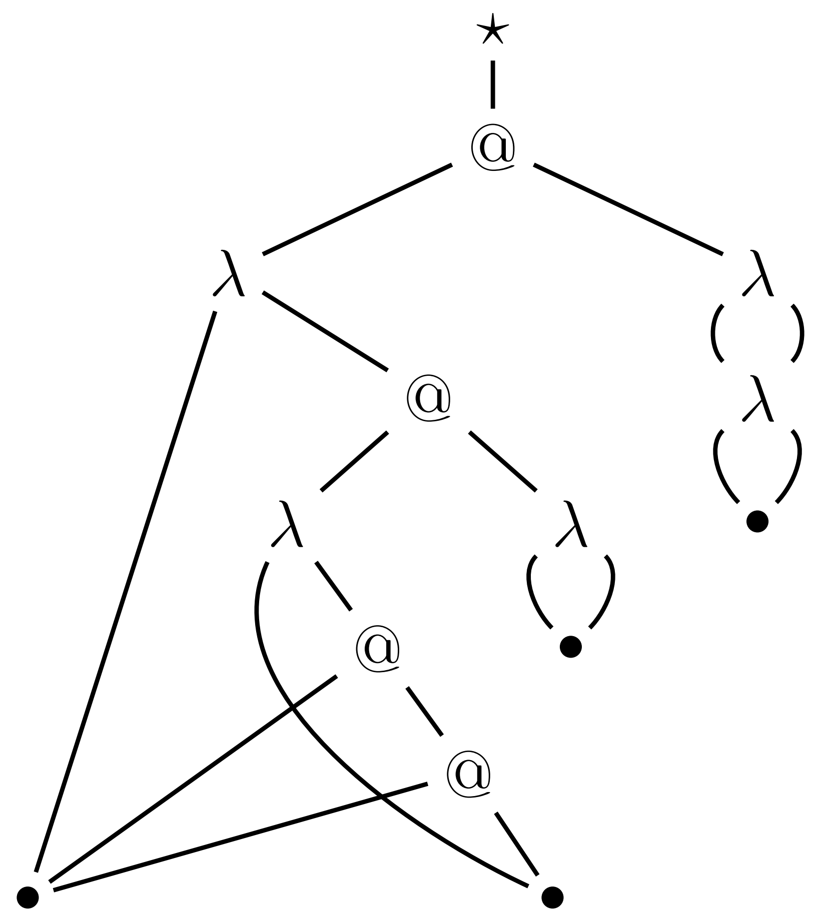
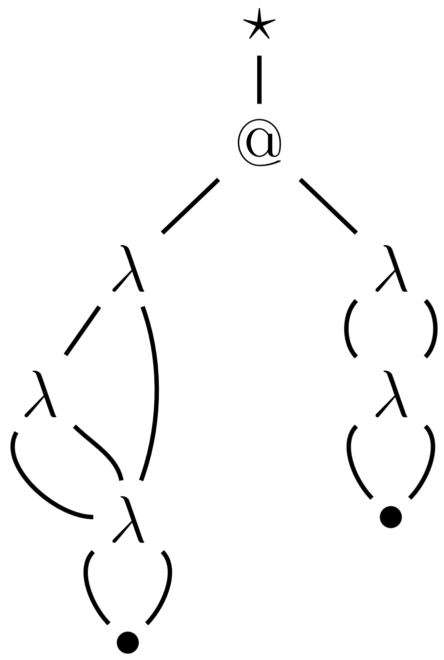


Read the "λ" as "$\rightarrow$" to see a traditional principal typing for the right subterm.

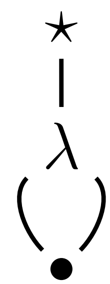
Normalizing the Term's Type (8)



In a number of additional steps, the left subterm's principal typing is found.

Normalizing the Term's Type (9)

- In a number of additional steps, the principal typing of the entire term is reached:



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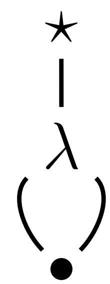

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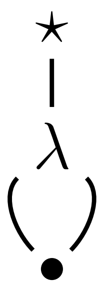

- So type inference for simple types can be viewed as simply applying an unusual set of rewrite rules to the λ term.
- What about for more complex type systems? Now I will consider intersection types.

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Intersection Types

- Type polymorphism by *listing* usage types [Coppo, Dezani-Ciancaglini, and Venneri, 1980].

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- Why "intersection"? If semantic denotations $[\![\sigma]\!]$ and $[\![\tau]\!]$ are program fragment sets, then $[\![\sigma \cap \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!]$.

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- Why "intersection"? If semantic denotations $[\![\sigma]\!]$ and $[\![\tau]\!]$ are program fragment sets, then $[\![\sigma \cap \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!]$.
- Example comparing intersection and $\forall$-quantified types:

  intersection types: $(\mathtt{fn\ x} \Rightarrow \mathtt{x})^{(\mathtt{int} \to \mathtt{int}) \cap (\mathtt{bool} \to \mathtt{bool})}$

  $\forall$-quantified types: $(\mathtt{fn\ x} \Rightarrow \mathtt{x})^{\forall \alpha . (\alpha \to \alpha)}$

  Example is semantically like $\forall \alpha \in \{\mathtt{int}, \mathtt{bool}\} . \alpha \to \alpha$, but has significant practical differences.

Typability for Various Systems

F: System F.

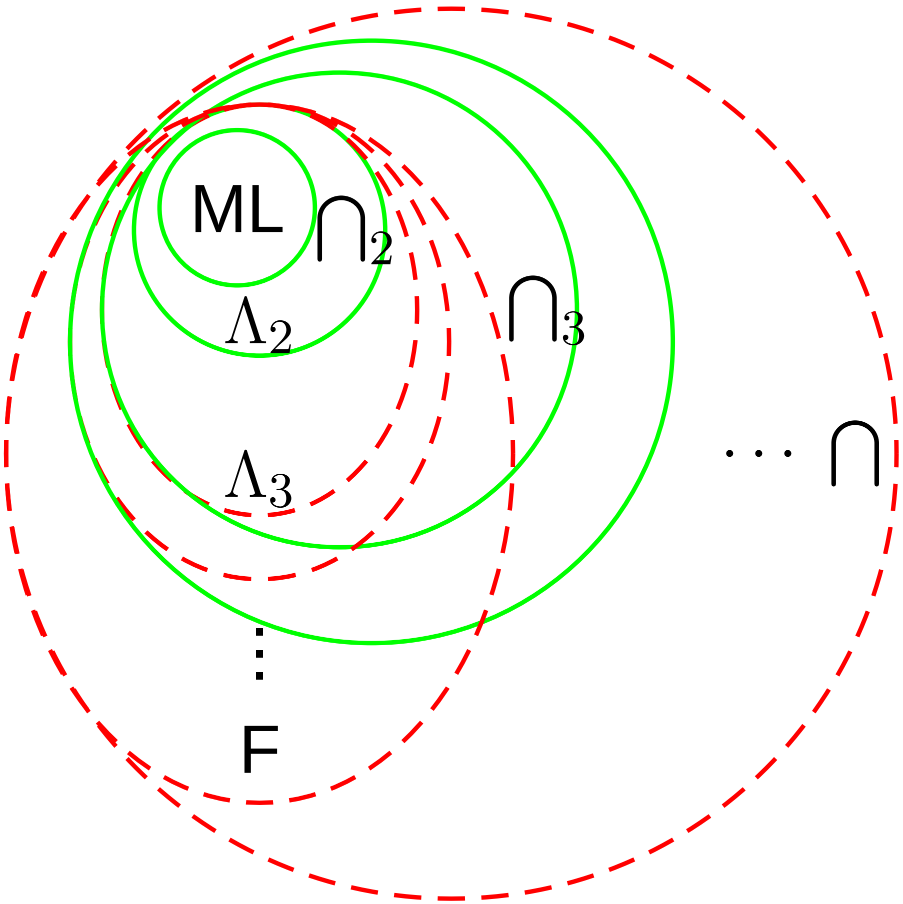
Λ_k: rank-k System F.

$\bigcap$: intersection types.

$\bigcap_k$: rank-k of $\bigcap$.

Decidable.

Undecidable.



(Asymptotic complexity now known [Kfoury, Mairson, Turbak, and Wells, 1999].)

Flexibility of Intersection Types

```
fun self_apply2 z  ⇒ (z z) z;
fun apply f x ⇒ f x;
fun reverse_apply y g ⇒ g y;
fun id w ⇒ w;
(self_apply2 apply not true,
 self_apply2 reverse_apply id false not);
```

- The example *safely* computes (false, true).
- Urzyczyn [1997] proved this example is not typable in F_ω, considered the most powerful type system with universal quantifiers.
- The example is typable in the rank-3 restriction of intersection types.

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Another Example

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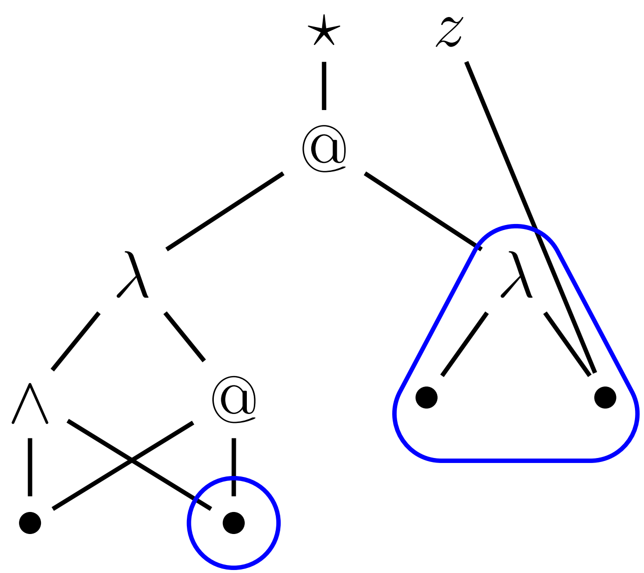
The DAG is formed a bit differently from before because intersection types are more flexible:



The colored boundaries correspond to *expansion variables* in System I.

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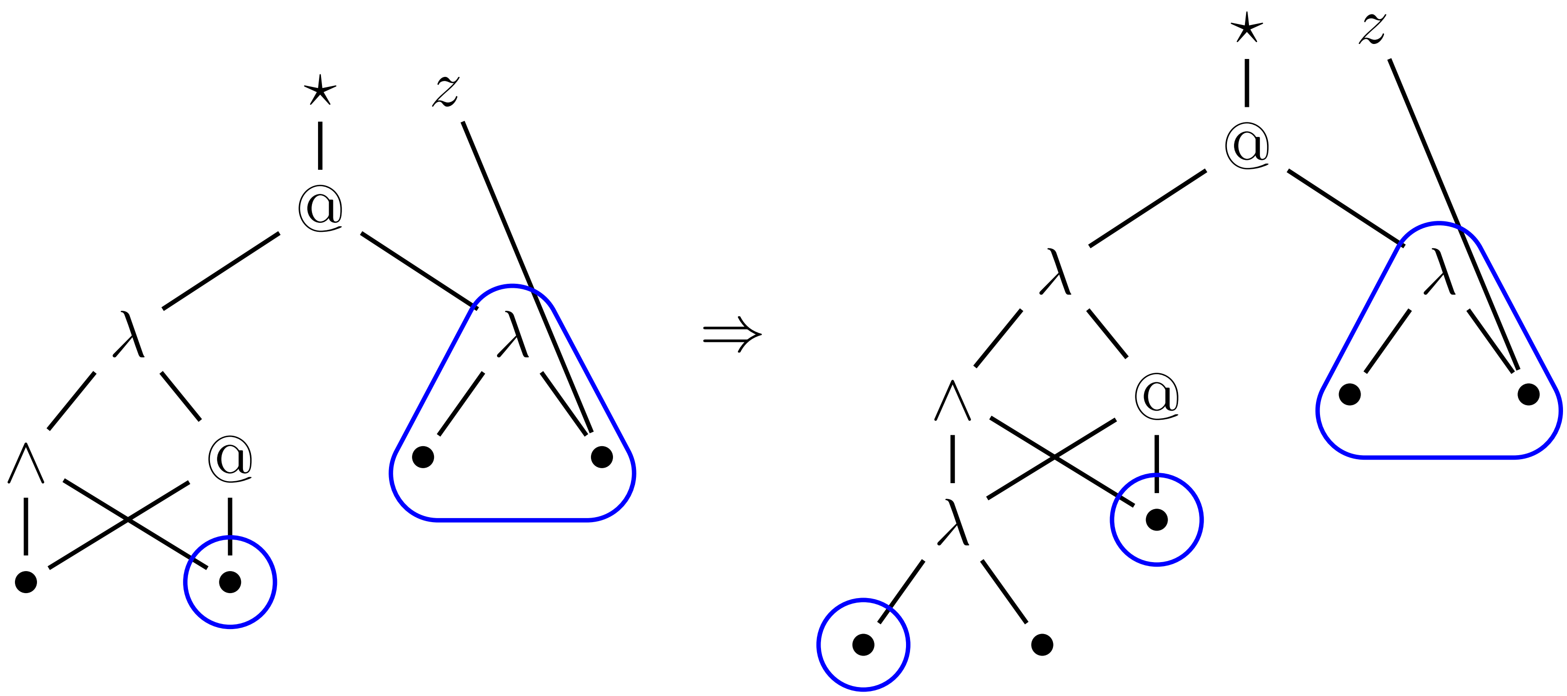

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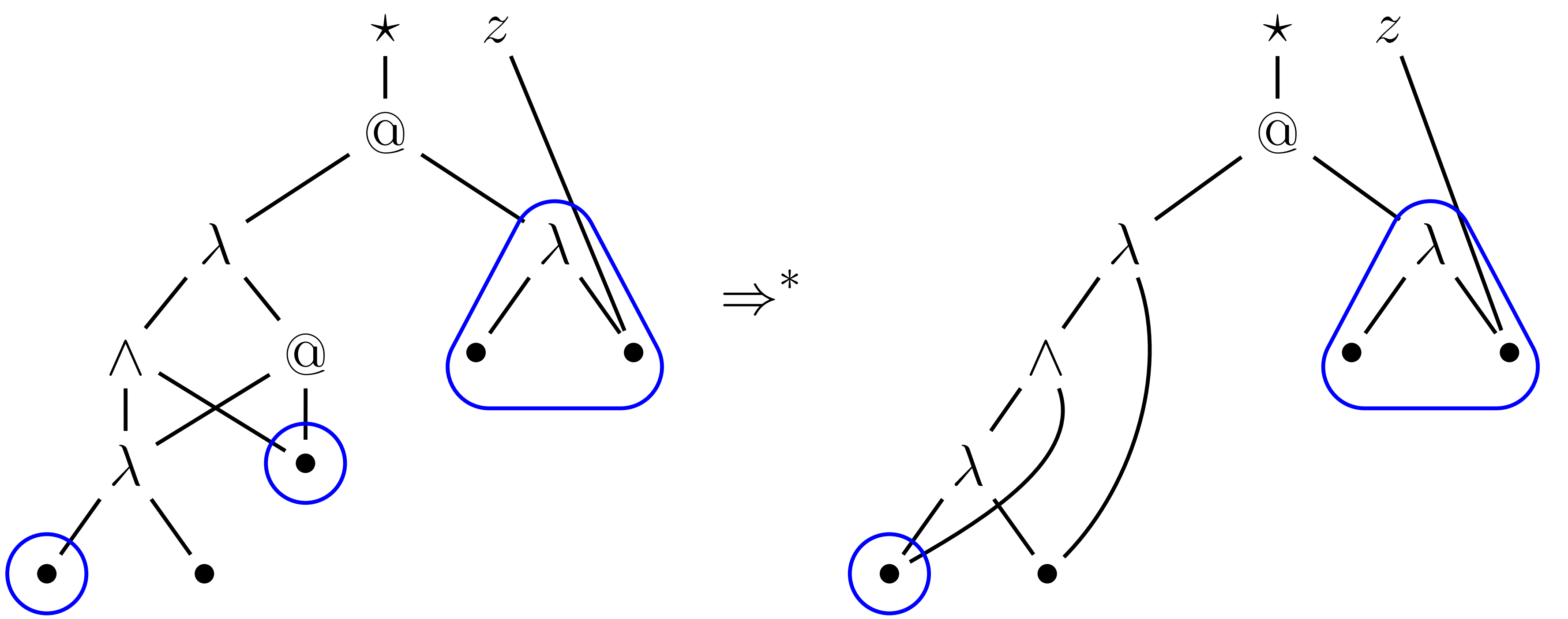

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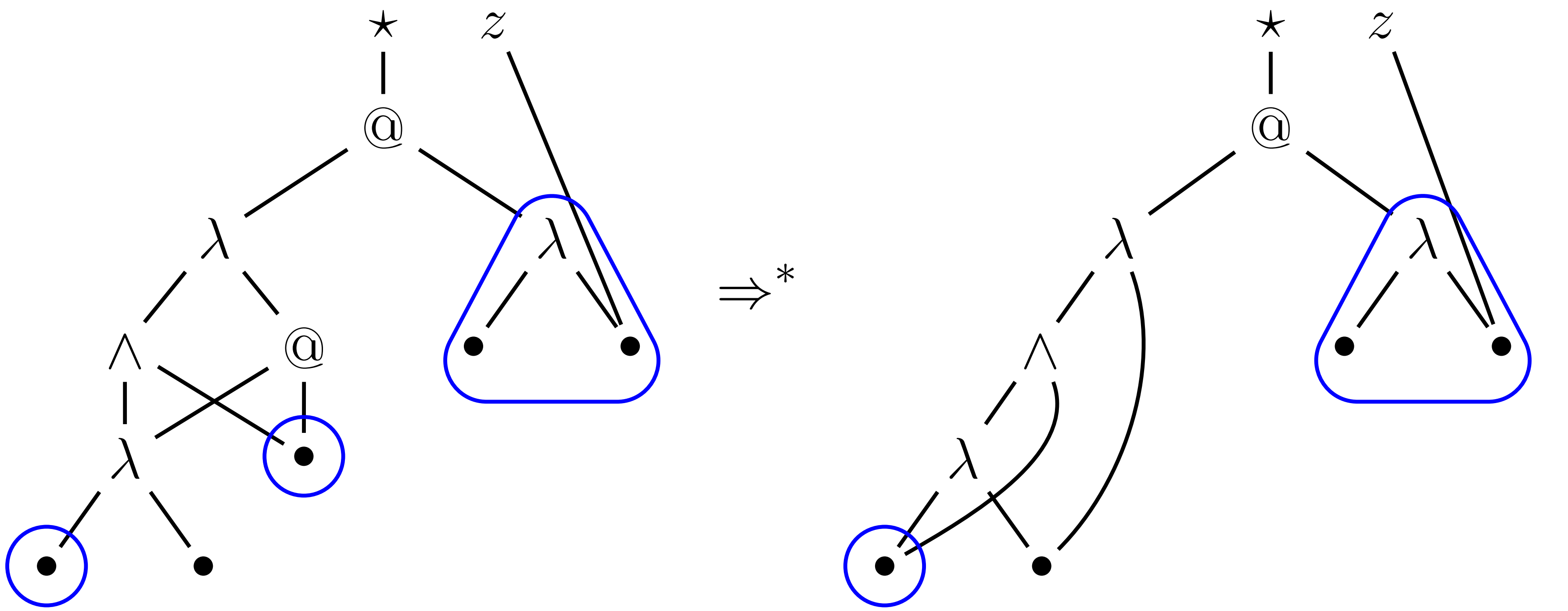
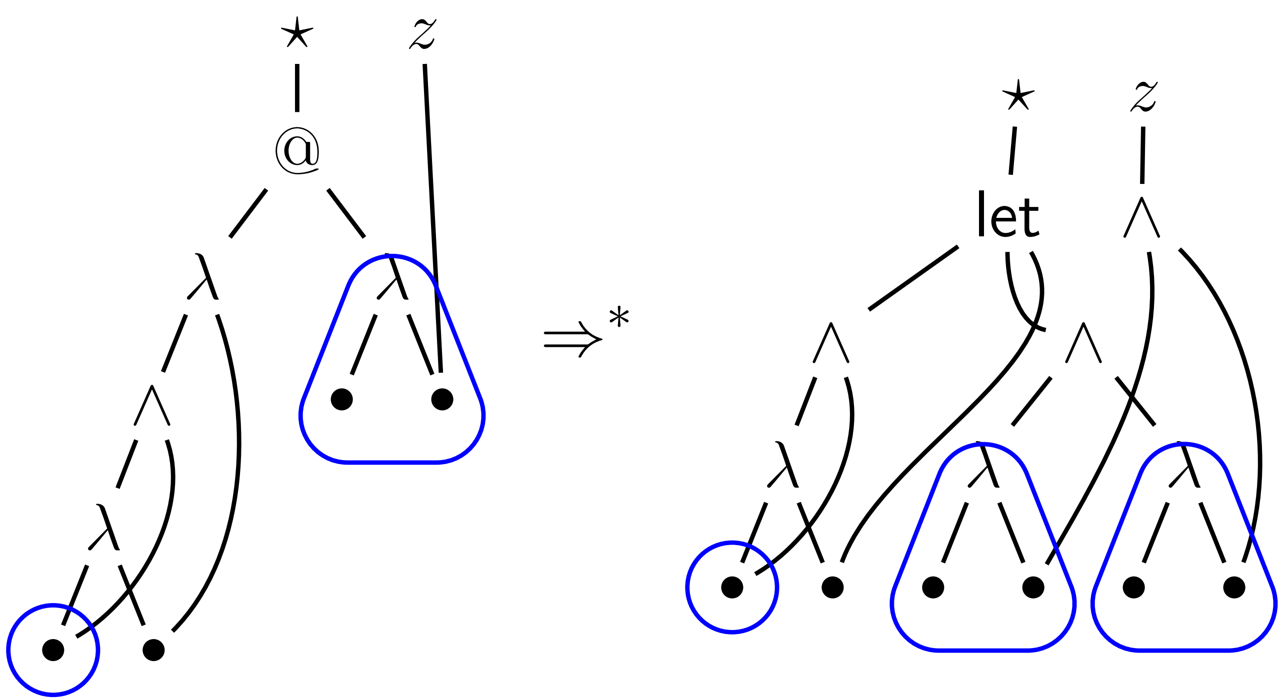


The principal typing of $(\lambda x.xx)$ now appears on the left. The typing on the right is already the principal typing of $(\lambda y.z)$.

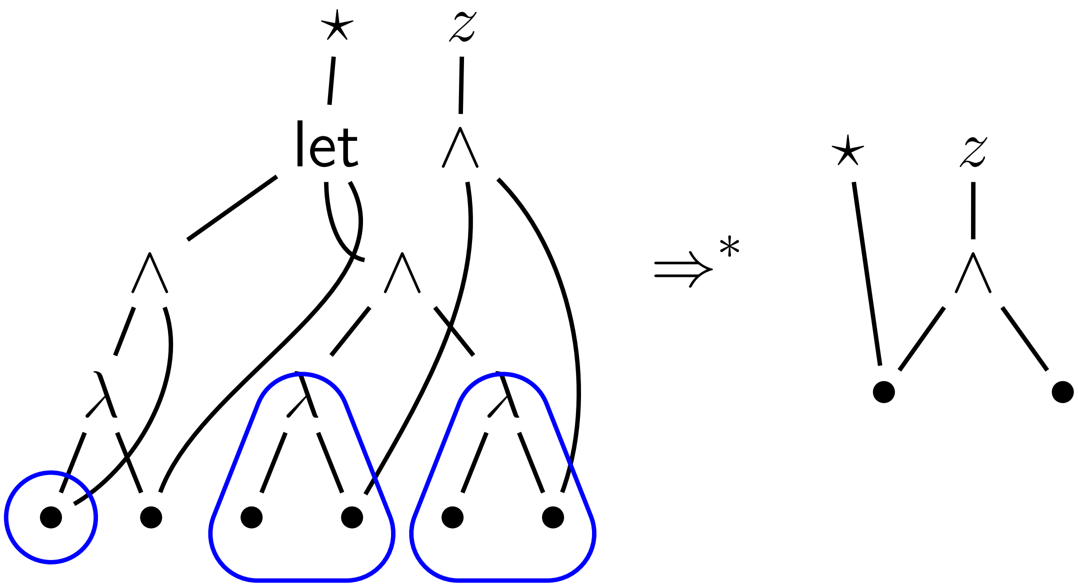
Normalizing the Typing (3)

In 2 more steps, the boundaries (expansion variables) play a role:



Only the contents of a boundary may be duplicated and then all incoming edges must be split with a $\wedge$ node connected to the corresponding nodes in the split copies.

Normalizing the Typing (4)



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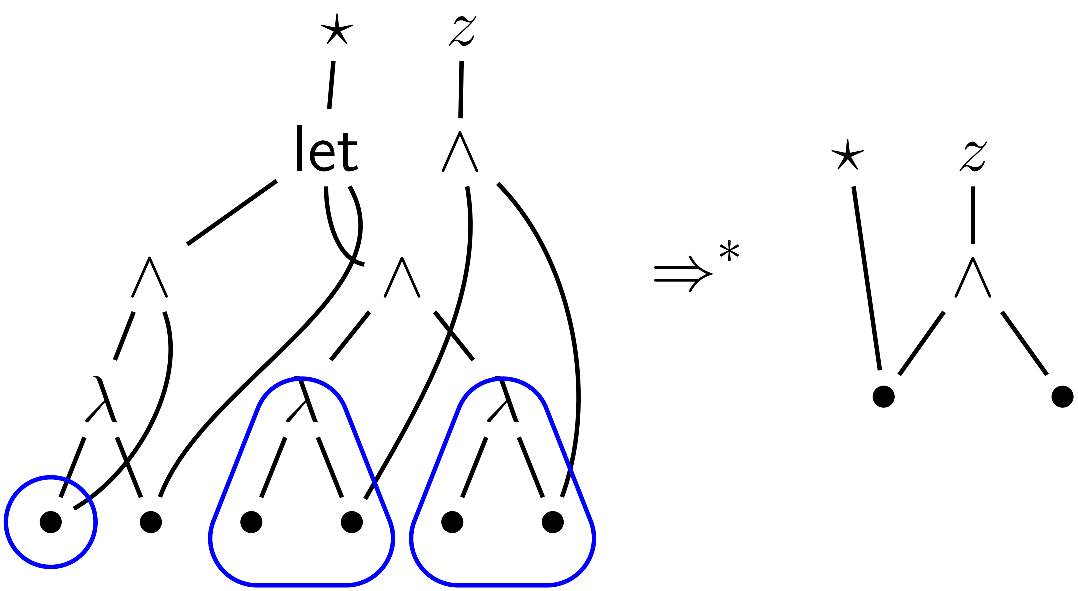

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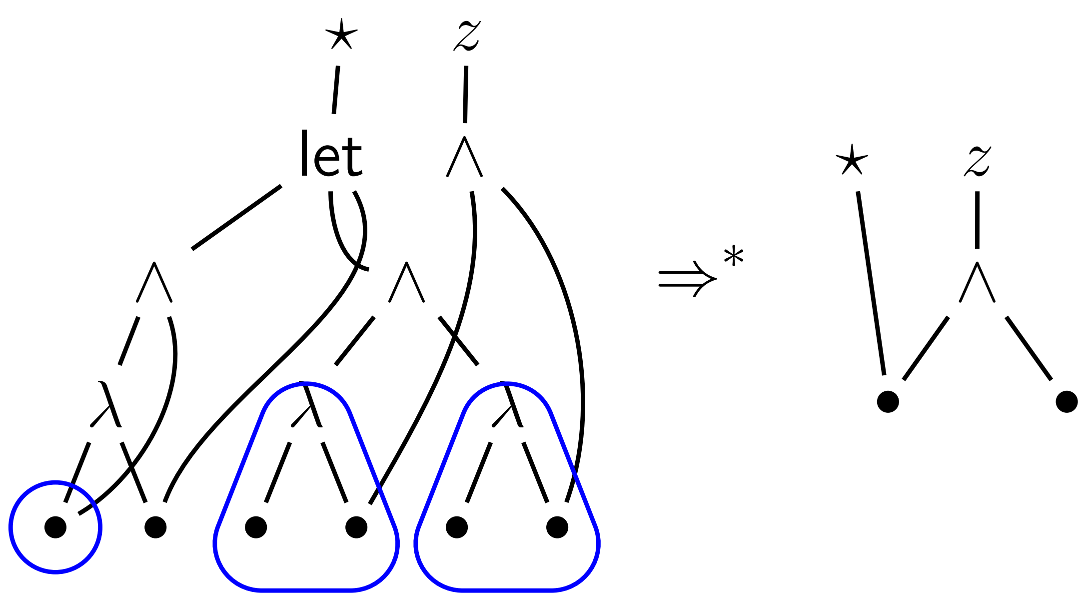

- The typing's normal form indicates that the result is obtained from the variable z and that a copy of z is discarded in the process.
- Thus, a complex type inference problem is just applying a set of rewrite rules to the λ term.

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- Extend with conditional types.
- Implement in a compiler.

Related Work in the Church Project

- Kfoury [1996, 2000]: “A linearization of the lambda calculus”.
- Kfoury [1999]: “Beta-reduction as unification”.
- Kfoury and Wells [1999]: “Principality and Decidable Type Inference for Finite-Rank Intersection Types”, the paper which introduced System I and its principal typing algorithm.
- Updated work on System I: Carlier [2002] and Kfoury, Washburn, and Wells [2002].
- Ongoing work extending expansion variables to work for more programming language features: tagged variants (usually handled with sum types), mutually recursive definitions, etc.

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- These rules may yield results more abstract than those yielded by the usual evaluation rules.
- This may be able to give a clearer explanation of how some type systems work.
- The close connection between rewriting and types is made more apparent.

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