Non-commutative Stone duality

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1. Classical/Commutative Stone duality

A topological space X is called a *Boolean space* if it is compact, Hausdorff and 0-dimensional (that is, it has a base of clopen sets).

Theorem: Classical/Commutative Stone duality. Stone.

1. *With each Boolean algebra B, we can associate a Boolean space* $\mathsf{X}(B)$, *called the* Stone space *of B.*

2. *With each Boolean space X, we can associate a Boolean algebra,* $\mathsf{B}(X)$, *of clopen subsets.*

3. $B \cong \mathsf{B}(\mathsf{X}(B))$ *for each Boolean algebra B.*

4. $X \cong \mathsf{X}(\mathsf{B}(X))$ *for each Boolean space X.*

2. Ideas behind non-commutative Stone duality

1. Replace the Boolean algebra by some kind of semigroup which has a Boolean character equipped with an order.

2. Replace the topological space by a (1-sorted) topological (small) category. We assume all maps are continuous, the space of identities is open, and the multiplication map is open.

3. Non-commutative Stone duality: Boolean inverse monoids

This is the version of non-commutative Stone duality of most interest to those working in operator algebras.

An *inverse monoid* is a monoid in which for each element s there is a unique element t such that $s = sts$ and $t = tst$. We usually denote t by s^{-1} and refer to the *inverse* of s.

Inverse monoids are ordered when we define $s \leq t$ iff $s = ts^{-1}s$. This is called the *natural partial order*.

An inverse monoid is said to be *Boolean* if it satisfies three conditions:

1. The idempotents form a Boolean algebra wrt the natural partial order.
2. If If $st^{-1}t = ts^{-1}s$ and $ss^{-1}t = tt^{-1}s$ then $s \vee t$ exists.
3. If $s \vee t$ exists then $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$ for any $u \in S$.

A topological groupoid is said to be *étale* if domain and range maps are local homeomorphisms and the space of identities is open.

A *Boolean groupoid* is an étale groupoid whose space of identities is a Boolean space.

Theorem: Non-commutative Stone duality I. Lawson and Lenz.

1. *With each Boolean inverse monoid S, we can associate a Boolean groupoid* $\mathsf{G}(S)$, *called the* Stone groupoid *of S.*

2. *With each Boolean groupoid G, we can associate a Boolean inverse monoid,* $\mathsf{KB}(G)$, *of compact-open local bisections.*

3. $S \cong \mathsf{KB}(\mathsf{G}(S))$ *for each Boolean inverse monoid S.*

4. $G \cong \mathsf{G}(\mathsf{KB}(G))$ *for each Boolean groupoid G.*

4. Non-commutative Stone duality: Boolean bi-restriction monoids

We now replace groupoids by categories. We say that a category is *étale* if its domain and range maps are both local homeomorphisms and the space of identities is open. A category is said to be *Boolean* if it is étale and the space of identities is a Boolean space.

We replace inverse monoids by bi-restriction monoids (see next slide). One approach to understanding these semigroups is that they are defined by axiomatizing the behaviour of the idempotents $s^{-1}s$ and ss^{-1} in an inverse semigroup.

We define a monoid S to be a *right restriction monoid* if it is equipped with a unary operation $a \mapsto a^*$ satisfying the following axioms:

(RR1) $(s^*)^* = s^*$.

(RR2) $(s^*t^*)^* = s^*t^*$.

(RR3) $s^*t^* = t^*s^*$.

(RR4) $ss^* = s$.

(RR5) $(st)^* = (s^*t)^*$.

(RR6) $t^*s = s(ts)^*$.

Those elements a such that $a^* = a$ are called *projections*. *The element a^* in fact axiomatizes the domain of definition of a partial function.*

We define a *left restriction monoid*, dually, and use $a \mapsto a^+$ for the unary operation.

A *bi-restriction monoid* is a monoid which is both a left and right restriction monoid and the sets of projections are the same.

Let S be a bi-restriction monoid. Define

$$y \leq x \text{ iff } y = xy^* \text{ equivalently } y = y^+x.$$

This is a partial order with respect to which the monoid is partially ordered. This is called the *natural partial order*.

A bi-restriction monoid is said to be *Boolean* if it satisfies three conditions:

1. The idempotents form a Boolean algebra wrt the natural partial order.
2. If $st^* = ts^*$ and $s^+t = t^+s$ then $s \vee t$ exists.
3. If $s \vee t$ exists then $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$ for any $u \in S$.

Theorem: Non-commutative Stone duality II. Kudryavtseva and Lawson.

1. *With each Boolean bi-restriction monoid S, we can associate a Boolean category* $\mathsf{C}(S)$, *called the* Stone category *of S.*

2. *With each Boolean category C, we can associate a Boolean bi-restriction monoid,* $\mathsf{KB}(C)$, *of compact-open local bisections.*

3. $S \cong \mathsf{KB}(\mathsf{C}(S))$ *for each Boolean bi-restriction monoid S.*

4. $C \cong \mathsf{C}(\mathsf{KB}(G))$ *for each Boolean category C.*

5. Non-commutative Stone duality: Boolean right restriction monoids

We now replace étale categories by *domain-étale* catgeories where we only require the domain map to be a local homeomorphism.

A *Boolean domain-étale category* is a domain-étale category whose space of identities is a Boolean space.

Let S be a right restriction monoid. Define

$$y \leq x \text{ iff } y = xy^{*}.$$

This is a partial order with respect to which the monoid is partially ordered. This is called the *natural partial order*.

A right restriction is said to be *Boolean* if it satisfies three conditions:

1. The idempotents form a Boolean algebra wrt the natural partial order.

2. If $st^{*} = ts^{*}$ then $s \vee t$ exists.

3. If $s \vee t$ exists then $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$ for all $u \in S$.

Theorem: Non-commutative Stone duality III. Cockett and Garner.

1. *With each Boolean right restriction monoid* S*, we can associate a Boolean domian-etale category* $\mathsf{C}(S)$*, called the* Stone category *of* S*.*

2. *With each Boolean domain-etale category* C*, we can associate a Boolean right restriction monoid,* $\mathsf{KS}(C)$*, of compact-open local* sections.

3. $S \cong \mathsf{KS}(\mathsf{C}(S))$ *for each Boolean right restriction monoid* S*.*

4. $C \cong \mathsf{C}(\mathsf{KS}(C))$ *for each Boolean domain-etale category* C*.*

6. In conclusion . . .

- Garner showed that the Boolean right restriction monoids are intimately connected with those varieties (in the sense of universal algebra) which are Cartesian closed.

- The work of Cockett and Garner suggests that we may generalize non-commutative Stone duality further, perhaps by using some ideas of Resende.

References

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