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Global Solutions to a Reactive Boussinesq System with Front Data on an Infinite Domain

Simon Malham[1], **Jack X. Xin**[2]

[1] Department of Theoretical Mechanics, University of Nottingham, Nottingham, UK.
E-mail: etzsjam@thmech.nottingham.ac.uk
[2] Department of Mathematics, University of Arizona, Tucson AZ 85721, USA.
E-mail: xin@math.arizona.edu

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Abstract: We prove the existence of global solutions to a coupled system of Navier–Stokes, and reaction-diffusion equations (for temperature and mass fraction) with prescribed front data on an infinite vertical strip or tube. This system models a one-step exothermic chemical reaction. The heat release induced volume expansion is accounted for via the Boussinesq approximation. The solutions are time dependent moving fronts in the presence of fluid convection. In the general setting, the fronts are subject to intensive Rayleigh-Taylor and thermal-diffusive instabilities. Various physical quantities, such as fluid velocity, temperature, and front speed, can grow in time. We show that the growth is at most $e^{\mathcal{O}(t)}$ for large time t by constructing a nonlinear functional on the temperature and mass fraction components. These results hold for arbitrary order reactions in two space dimensions and for quadratic and cubic reactions in three space dimensions. In the absence of any thermal-diffusive instability (unit Lewis number), and with weak fluid coupling, we construct a class of fronts moving through shear flows. Although the front speeds may oscillate in time, we show that they are uniformly bounded for large t. The front equation shows the generic time-dependent nature of the front speeds and the straining effect of the flow field.

1. Introduction

We study the existence of global solutions to the following Boussinesq combustion system on the infinite tube $\Omega := \{(\boldsymbol{x}, y) \in \Sigma \times \mathbb{R}\}$, where $\Sigma \subset \mathbb{R}^{d-1}$ is an open, bounded, simply connected domain with smooth boundary $\partial\Sigma = \overline{\Sigma}/\Sigma$, outward normal $\hat{\boldsymbol{n}}$, and $d = 2, 3$ is our spatial dimension:

$$\partial_t \psi + \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi = \Delta \psi - \psi f(\theta), \tag{1.1a}$$

$$\partial_t \theta + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \ell \Delta \theta + \psi f(\theta), \tag{1.1b}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \nu \Delta \boldsymbol{u} - \boldsymbol{\nabla} p + \sigma \theta \boldsymbol{e}, \tag{1.1c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0. \tag{1.1d}$$

Physically we interpret: $\boldsymbol{u}(\boldsymbol{x}, y, t)\colon \Omega \times \mathbb{R} \to \mathbb{R}^d$ as the fluid velocity; $p(\boldsymbol{x}, y, t)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ the pressure; $\psi(\boldsymbol{x}, y, t)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ the concentration of the reactant in a one-step irreversible exothermic reaction; $\theta(\boldsymbol{x}, y, t)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ the temperature of the reactant-product mixture; ν the normalised fluid viscosity or Prandtl number; ℓ the Lewis number; σ the Rayleigh number; $\boldsymbol{e}$ denotes the unit vertical direction opposite to the propagation direction of flame (aligned with the y direction). For convenience we shall write $\boldsymbol{\theta} := (\psi, \theta)\colon \Omega \times \mathbb{R} \to \mathbb{R}^2$, i.e. the 2-tuple of reactant concentration and temperature. We assume non-homogeneous boundary conditions which allow front type initial data to be prescribed (for our main results $\boldsymbol{b}^* \equiv 0$):

$$\begin{array}{lll} \partial_{\hat{\boldsymbol{n}}} \psi = 0,\ \partial_{\hat{\boldsymbol{n}}} \theta = 0, & \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Sigma \times \mathbb{R} \times \mathbb{R}_+, \\ \psi \to 0,\ \ \theta \to 1, & \boldsymbol{u} \to \boldsymbol{b}^*, & \text{as } y \to +\infty, \\ \psi \to 1,\ \ \theta \to 0, & \boldsymbol{u} \to \boldsymbol{b}^*, & \text{as } y \to -\infty. \end{array} \tag{1.2}$$

We will suppose for some $m \in \mathbb{N}$,

$$f(\theta) = \begin{cases} \theta^m, & \theta > 0, \\ 0, & \theta \le 0. \end{cases} \tag{1.3}$$

System (1.1) models the vertical movement of flame fronts. Thermal volume expansion of the fluid due to the irreversible exothermic combustion reaction is accounted for by the Overbeck-Boussinesq approximation [10, 28]. The nonlinear chemical reaction term $\psi f(\theta)$ usually takes the Arrhenius form $\psi f(\theta) \exp\{-\mathcal{E}/R\theta\}$, or its normalised version $\psi f(\theta) \exp\{(\theta - 1)/(1 + \chi(\theta - 1))\epsilon\}$, where $f(\theta)$ is usually of the form (1.3) though m is not always a positive integer (in general). The constant $\mathcal{E}$ is the activation energy, R is the universal gas constant, $\chi \in (0, 1)$ is the thermal expansion coefficient and $\epsilon > 0$ is the reciprocal of the Zel'dovich number (see Buckmaster and Ludford [8] or Berestycki and Larrouturou [2]). Since the supremum of $\exp\{-\mathcal{E}/R\theta\}$ over $\Omega \times \mathbb{R}$ is always bounded, the inclusion of this factor in the chemical nonlinearity would not affect any of our proofs. Consequently we neglect this exponential factor and for simplicity, choose f to be a power nonlinearity.

This simple chemical nonlinearity also arises in isothermal autocatalytic chemical reactions of the form $A + mB \to (m + 1)B$, where m is the order of reaction, and ψ, θ are the concentrations of reactant A and autocatalyst B respectively. The rate of reaction is thus given by $k\psi\theta^m$, where the constant enthalpy k can be scaled out of the system. In this case the temperature remains fixed, yet the density of the A and B mixture increases with the reaction resulting in a change of fluid velocity. Thus chemical feedback plays the role of thermal feedback and system (1.1) then governs the dynamics of the moving concentration fronts in the presence of fluid convection.

Billingham, King, Merkin, Metcalf, Needham, and Scott [6, 39, 40] studied the autocatalytic reaction-diffusion system (1.1a), (1.1b), neglecting hydrodynamical effects. They proved existence and uniqueness results for an associated boundary value problem and studied the development of travelling fronts of chemical reaction. See also Focant and Gallay [18] for a recent study of existence of traveling fronts in the quadratic-cubic case and their stability when ℓ is near one. The passively convected version of the non-isothermal system was considered by Berestycki, Larrouturou and Roquejoffre [3, 48] in an infinite tube, and they proved the linear and nonlinear stability of travelling front solutions. Manley, Marion and Temam [34, 36] examined system (1.1) on a finite tube in the case of a multi-component reaction and with slightly different boundary conditions. They proved the global existence ($d = 2, 3$) of suitable weak solutions uniformly bounded ($d = 2$) in time. Further, their estimates for the Hausdorff dimension of the universal attractor indicated that for long tubes, hydrodynamical effects make a significant contribution to the complexity of the flow. Crucial to their proofs was the assumption of a bounded nonlinear reaction term.

We are interested in studying the full system (1.1) on unbounded domains while allowing for unbounded chemical nonlinearities. The attractor dimension results of Manley, Marion and Temam [34, 36] indicate the importance of studying this system when the vertical domain size is much larger than the typical length scale associated with the front width. The infinite cylindrical domain is the natural setting for examining the long time behaviour of travelling front solutions and especially for the irreversible reactions. Numerical simulations (Patnaik and Kailasanath [45], Zhu and Xin [62]) have shown that moving fronts of system (1.1) are subject to both Rayleigh-Taylor (upward fronts) and thermal diffusive instabilities. The Rayleigh-Taylor instability from the $\sigma\theta\boldsymbol{e}$ term is due to heavier (cold) fluid lying above the lighter (hot) fluid. It leads to bubble formation on the front and growth of fluid energy and vorticity. The thermal-diffusive instability due to $\ell \neq 1$ can cause chaotic front oscillation ($\ell > 1$) or formations of cellular front structures ($\ell < 1$) [51, 39]. As a result, the maximum temperature can grow in time. Last, but not least, for high Reynolds numbers (small ν) the fluid flow can become highly irregular, which in turn wrinkles the front and may induce front acceleration. In [62], power growth in time of maximum vorticity and temperature is numerically observed ($d = 2$, $\nu = 0.005$, $\ell = 0.1$ or 10). Majda and Souganidis [33] studied front acceleration (front speed of $\mathcal{O}(t^p)$, $p > 0$), in a prescribed (passive) random shear flow of Hölder regularity. All this evidence suggests that in general, one should not expect the front speed to be uniformly bounded in time, instead a power growth may well happen.

Our first result implies an exponential bound $e^{\mathcal{O}(t)}$ on the front speeds for fronts in a two dimensional infinite vertical strip (for all orders of reaction m) or in a three dimensional infinite vertical tube (for $m \in \{1, 2, 3\}$). We treat only the one-step reaction case, as the analysis of the multi-component case is practically identical. We prove the existence of global weak solutions ($d = 2, 3$) to (1.1). In the two dimensional case we prove uniqueness for a class of slightly more regular weak solutions as well as the existence of strong, smooth solutions for smooth initial data. Our sharpest norm upper bounds of solutions grow with time, so we are unable to discuss attractors. The growing bounds may be interpreted as the enhancement of instabilities in the system due to the unbounded chemical nonlinearities and unbounded domains. If $\Omega = \Sigma \times \Lambda$, $|\Lambda| < \infty$, i.e. a bounded strip or tube, we can considerably improve our growth estimates. The details however will be presented elsewhere.

Our second result concerns fronts in a reduced system when $\ell = 1$, σ is small, $\nu > 2\pi$. The fluid flows are laminar, the Rayleigh-Taylor effect is minimal, and the thermal-diffusive effect is absent. We construct a class of front solutions near the known passive fronts in smooth shear flows. The time dependent front speeds are proved to be uniformly bounded in time. Passive fronts in shear flows on infinite cylindrical domains have been studied at length by Berestycki, Larrouturou, Lions, Nirenberg and Roquejoffre [2, 3, 48, 4, 5], regarding the existence and stability of travelling front solutions. Similar issues on passive fronts in periodic flow fields have also been well studied by Papanicolaou, Xin, and Zhu, [43, 56–59]. The passive fronts in these cases all propagate with constant speeds. However, with fluid coupling turned on, front speed is no longer constant as we will see from the front equation arising in the course of the proof.

2. Global Weak Solutions and Growth Estimates

2.1. Notation and Statement of Main Result. We denote $\langle\varphi\rangle := \int_\Omega \varphi\,\mathrm{d}\boldsymbol{x}\mathrm{d}y$ for Lebesgue measurable functions $\varphi\colon \Omega \to \mathbb{R}$. For $q \in [1,\infty)$, $n \in \mathbb{N}$, $L^q(\Omega;\mathbb{R}^d)$ and $H^n(\Omega;\mathbb{R}^d)$ are the usual Lebesgue and Sobolev spaces of $\mathbb{R}^d$-valued functions, equipped with the norm and inner product

$$\|\boldsymbol{\varphi}\|^q_{L^q(\Omega;\mathbb{R}^d)} := \sum_{i=1}^{d}\langle|\varphi_i|^q\rangle,$$
$$(\boldsymbol{\varphi},\boldsymbol{\phi})_{H^n(\Omega;\mathbb{R}^d)} := \sum_{i=1}^{d}\sum_{|\alpha|\le n}\langle D^\alpha\varphi_i D^\alpha\phi_i\rangle.$$

Since Ω has a smooth boundary, an equivalent norm on $H^n(\Omega;\mathbb{R}^d)$ is

$$\|\boldsymbol{\varphi}\|^2_{H^n(\Omega;\mathbb{R}^d)} = \|\boldsymbol{\varphi}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \sum_{|\alpha|=n}\|D^\alpha\boldsymbol{\varphi}\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

The non-reflexive space $L^\infty(\Omega;\mathbb{R}^d)$ is equipped with the usual sup-norm. We adopt the notation $\mathbb{L}^q := L^q(\Omega;\mathbb{R}^2)$ for $\mathbb{R}^2$-valued functions defined on $\Omega \subset \mathbb{R}^d$. We will often use the Gagliardo–Nirenberg inequality: for all $\varphi \in H^1(\Omega)$, $q \in [2,\infty)$ when $d = 2$ and $q \in [2,6]$ when $d = 3$

$$\|\varphi\|_{L^q} \le c\,\|\boldsymbol{\nabla}\varphi\|^{\frac{d}{2}-\frac{d}{q}}_{L^2(\Omega;\mathbb{R}^d)}\|\varphi\|^{1-\frac{d}{2}+\frac{d}{q}}_{L^2} + c(\Omega)\|\varphi\|_{L^2}. \tag{2.1}$$

The last term on the right-hand side of (2.1) is zero when $\varphi \in H^1_0(\Omega)$. Also, we shall use the interpolation inequality [19, 31]: for $\varphi \in L^r(\Omega)\cap L^p(\Omega)$, $1 \le p \le q \le r \le \infty$, $\mu \in [0,1]$ and $1/q = \mu/p + (1-\mu)/r$

$$\|\varphi\|_{L^q} \le \|\varphi\|^\mu_{L^p}\|\varphi\|^{1-\mu}_{L^r}. \tag{2.2}$$

The Poincaré inequality establishes the equivalence of the norm $\|\varphi\|_{H^1(\Omega)}$ and semi-norm $\|\boldsymbol{\nabla}\varphi\|_{L^2(\Omega;\mathbb{R}^d)}$ on $H^1_0(\Omega)$: for $\varphi \in H^1_0(\Omega)$

$$\|\varphi\|_{L^2} \le c(\Sigma)\|\boldsymbol{\nabla}\varphi\|_{L^2(\Omega;\mathbb{R}^d)}. \tag{2.3}$$

For a given Hilbert space $\mathbb{X}$ with inner product $(\cdot,\cdot)_{\mathbb{X}}$, we shall use $\langle\cdot,\cdot\rangle_{\mathbb{X}\times\mathbb{X}'}$ to denote the bilinear form establishing the duality between $\mathbb{X}$ and its dual $\mathbb{X}'$. For two topological vector spaces $\mathbb{X}$ and $\mathbb{Y}$, the notation $\mathbb{X} \hookrightarrow \mathbb{Y}$ shall indicate that a continuous embedding exists from $\mathbb{X}$ into $\mathbb{Y}$ and we shall use $\mathbb{X} \hookrightarrow\hookrightarrow \mathbb{Y}$ when the embedding is compact. Given a Banach space $\mathbb{Y}$, we shall use $L^p_{\mathrm{loc}}([0,\infty);\mathbb{Y})$ to denote the space of measurable functions from $[0,\infty)$ to $\mathbb{Y}$ such that $\|\cdot\|_{\mathbb{Y}} \in L^p_{\mathrm{loc}}([0,\infty))$. The notations $\mathrm{w}\text{-}\mathbb{Y}$ and $\mathrm{w}^*\text{-}\mathbb{Y}$ are used to denote $\mathbb{Y}$ endowed with its weak and weak-star topologies respectively. By $C([0,\infty);\mathrm{w}\text{-}\mathbb{X})$ we indicate the space of functions continuous from $[0,\infty)$ into $\mathrm{w}\text{-}\mathbb{X}$.

With $\mathbb{D} := \{\boldsymbol{v} \in C_0^\infty(\Omega;\mathbb{R}^d) : \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\}$, we specify $\mathbb{H}$ to be the closure of $\mathbb{D}$ in $L^2(\Omega;\mathbb{R}^d)$ and $\mathbb{V}$ to be the closure of $\mathbb{D}$ in $H^1(\Omega;\mathbb{R}^d)$. Since there exists $\mathcal{T}_C : \{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^d) : \boldsymbol{\nabla}\cdot\boldsymbol{v} \in L^2(\Omega)\} \to \{\boldsymbol{v}|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\}$, a continuous linear trace operator such that $\mathcal{T}_C(\boldsymbol{v}) = \boldsymbol{v}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega}$ for smooth $\boldsymbol{v}$, we have the standard characterisations [14, 32]

$$
\begin{aligned}
\mathbb{H} &= \{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^d) : \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0,\ \mathcal{T}_C(\boldsymbol{v}) = 0\},\\
\mathbb{V} &= \{\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d) : \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\}.
\end{aligned}
$$

By the Riesz representation theorem we can identify $\mathbb{H} \equiv \mathbb{H}'$ which is our pivot space and then, using the inequalities above, we establish the Gelfand triple [56] $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}'$, where each space is dense in the one which follows. We use $\mathcal{P}$ to represent the Leray-Hodge orthogonal projection onto divergence free functions $\mathcal{P}\colon L^2(\Omega;\mathbb{R}^d) \to \mathbb{H}$. In the standard fashion we define the linear Stokes operator $\mathcal{A} = -\mathcal{P}\Delta\colon D(\mathcal{A}) \to \mathbb{H}$, where $D(\mathcal{A}) = H^2(\Omega;\mathbb{R}^d) \cap \mathbb{V}$.

Further, with $\mathbb{E} := \{\varphi \in C_0^\infty(\mathbb{R}^d)$ restricted to $\Omega : \partial_{\hat{\boldsymbol{n}}}\varphi = 0$ on $\partial\Sigma \times \mathbb{R}\}$ and $W := \{\varphi \in H^1(\Omega) : \|\varphi\|^2_{H^1(\Omega)} + \|\Delta\varphi\|^2_{L^2(\Omega)}$, we specify $\mathbb{W}$ to be the closure of $\mathbb{E}$ in W. Since there exist $\mathcal{T}_D : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$ and $\mathcal{T}_N : W \to H^{-\frac{1}{2}}(\partial\Omega)$, continuous linear trace operators such that $\mathcal{T}_D(\varphi) = \varphi|_{\partial\Omega}$ and $\mathcal{T}_N(\varphi) = \partial_{\hat{\boldsymbol{n}}}\varphi|_{\partial\Omega}$ for smooth φ, we can characterise [15, 56]

$$
\mathbb{W} = \{\varphi \in H^1(\Omega); \Delta\varphi \in L^2(\Omega) : \partial_{\hat{\boldsymbol{n}}}\varphi = 0 \text{ on } \partial\Sigma \times \mathbb{R}\},
$$

We will also need Green's theorem, which by density arguments, holds for $\varphi \in W$ and $v \in H^1(\Omega)$:

$$
\langle \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla}\varphi\rangle + \langle v\Delta\varphi\rangle = \langle \mathcal{T}_D(v), \mathcal{T}_N(\varphi)\rangle_{H^{\frac{1}{2}}(\partial\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)}. \tag{2.4}
$$

Suppose $\phi \in C_0^\infty(\mathbb{R};[0,\infty))$ satisfies $\int_{\mathbb{R}} \phi(y)\,\mathrm{d}y = 1$. Set $\tilde{\psi}(y) = \int_{y-y_s}^\infty \phi(s)\mathrm{d}s$ and $\tilde{\theta}(y) = \int_{-\infty}^y \phi(s)\mathrm{d}s$, where y_s is a finite constant which we can choose to be zero. Thus $\tilde{\boldsymbol{\theta}} = (\tilde{\psi}, \tilde{\theta}) \in [0,1]^2$ is smooth, satisfies the boundary conditions (1.2) and $\tilde{\psi}\cdot\tilde{\theta}$ has compact support in $\mathbb{R}$. As in Heywood [22], we assume that $\boldsymbol{b}^*$ can be continued as a function into Ω, $\boldsymbol{b} \in H^2_{\text{loc}}(\overline{\Omega};\mathbb{R}^d)$, for which there exists a scalar distribution $q(\boldsymbol{x}, y)$ such that $\boldsymbol{f} = \nu\Delta\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b} - \boldsymbol{\nabla} q \in L^2(\Omega;\mathbb{R}^d)$. This is trivial when $\boldsymbol{b}^* \equiv \boldsymbol{0}$, which we *assume* throughout the rest of this section. (However, with slight modifications to our proofs equivalent to those in Heywood [22], our results can include the case when $\boldsymbol{f}$ has finite Dirichlet integral – for example, when Σ is a disk of radius r_0, a natural choice for $\boldsymbol{b}^*$ would be a Hagan-Poiseuille flow, $\boldsymbol{b}^*(\boldsymbol{x}) = \partial_y\tilde{p}\cdot(|\boldsymbol{x}|^2 - r_0^2)\boldsymbol{e}/4\nu$ for some prescribed pressure gradient $\partial_y\tilde{p}$.)

We linearly decompose our solutions into $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}$. For initial data $(\boldsymbol{\theta}^{\text{in}}, \boldsymbol{u}^{\text{in}})$ satisfying the boundary conditions (1.2), our initial boundary value problem now takes the form:

$$\partial_t\hat{\psi} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\psi = \Delta\psi - \psi f(\theta), \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.5a}$$
$$\partial_t\hat{\theta} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\theta = \ell\Delta\theta + \psi f(\theta), \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.5b}$$
$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \boldsymbol{\nabla}p + \sigma\hat{\theta}\boldsymbol{e}, \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.5c}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.5d}$$
$$(\partial_{\hat{\boldsymbol{n}}}\hat{\boldsymbol{\theta}}, \boldsymbol{u}) = \boldsymbol{0}, \qquad \text{on } \partial\Sigma\times\mathbb{R}\times\mathbb{R}_+, \tag{2.5e}$$
$$(\hat{\boldsymbol{\theta}}, \boldsymbol{u}) \to \boldsymbol{0}, \qquad \text{as } |y|\to\infty, \tag{2.5f}$$
$$\hat{\boldsymbol{\theta}}(\boldsymbol{x}, y, 0) = \hat{\boldsymbol{\theta}}^{\text{in}}(\boldsymbol{x}, y) = \boldsymbol{\theta}^{\text{in}} - \tilde{\boldsymbol{\theta}}, \qquad \text{in } \Omega, \tag{2.5g}$$
$$\boldsymbol{u}(\boldsymbol{x}, y, 0) = \boldsymbol{u}^{\text{in}}(\boldsymbol{x}, y). \tag{2.5h}$$

The term $\tilde{\theta}\boldsymbol{e}$ is the gradient of a scalar function so p is now the modified pressure.

Definition 2.1. *For given initial data* $(\hat{\boldsymbol{\theta}}^{\text{in}}, \boldsymbol{u}^{\text{in}}) \in \mathbb{L}^q(\Omega)\times\mathbb{H}$, *for all* $q\in[1,\infty)$, *which satisfy* $\hat{\psi}^{\text{in}} \in [-1,1]$ *and* $\hat{\theta}^{\text{in}} \in [-1,\infty)$ *for a.e.* $(\boldsymbol{x}, y)\in\Omega$, *a global weak solution of* (2.5) *indicates measurable functions*

$$\hat{\boldsymbol{\theta}} \in L^\infty_{\text{loc}}([0,\infty);\mathbb{L}^q) \cap L^2_{\text{loc}}([0,\infty);H^1(\Omega;\mathbb{R}^2)) \cap C([0,\infty);\mathbb{L}^2), \tag{2.6a}$$
$$\boldsymbol{u} \in L^\infty_{\text{loc}}([0,\infty);\mathbb{H}) \cap L^2_{\text{loc}}([0,\infty);\mathbb{V}) \cap C([0,\infty);\text{w-}\mathbb{H}), \tag{2.6b}$$

for every $q\in[1,\infty)$ *when* $d = 2$ *and every* $q\in[1,6]$ *when* $d = 3$, *such that* $\hat{\psi}\in[-1,1]$ *and* $\hat{\theta}\geq -1$, *a.e. in* Ω, *for every* $t\in[0,\infty)$, *and which for all* $(\varphi, \boldsymbol{v}) \in C^\infty([0,\infty);\mathbb{E}\times\mathbb{D})$ *and* $[t_0, t_1]\subset[0,\infty)$ *satisfy*

$$\int_{t_0}^{t_1}\langle\hat{\psi}\partial_t\varphi + \psi\Delta\varphi + \psi\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi - \psi f(\theta)\varphi\rangle\,\mathrm{d}t = \langle\hat{\psi}(t_1)\varphi(t_1)\rangle - \langle\hat{\psi}(t_0)\varphi(t_0)\rangle, \tag{2.7a}$$
$$\int_{t_0}^{t_1}\langle\hat{\theta}\partial_t\varphi + \ell\theta\Delta\varphi + \theta\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi + \psi f(\theta)\varphi\rangle\,\mathrm{d}t = \langle\hat{\theta}(t_1)\varphi(t_1)\rangle - \langle\hat{\theta}(t_0)\varphi(t_0)\rangle, \tag{2.7b}$$
$$\int_{t_0}^{t_1}\langle\boldsymbol{u}\cdot\partial_t\boldsymbol{v} + \nu\boldsymbol{u}\cdot\Delta\boldsymbol{v} + \boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v} + \sigma\hat{\theta}v\boldsymbol{e}\rangle\,\mathrm{d}t = \langle\boldsymbol{u}(t_1)\boldsymbol{v}(t_1)\rangle - \langle\boldsymbol{u}(t_0)\boldsymbol{v}(t_0)\rangle, \tag{2.7c}$$

and

$$(\hat{\boldsymbol{\theta}}(0), \boldsymbol{u}(0)) = (\hat{\boldsymbol{\theta}}^{\text{in}}, \boldsymbol{u}^{\text{in}}). \tag{2.8}$$

Remark 2.1. Since for these weak solutions, $(\hat{\boldsymbol{\theta}}, \boldsymbol{u}) \in C([0,\infty);\mathbb{L}^2\times\text{w-}\mathbb{H})$, the initial condition (2.8) is satisfied in this sense.

Theorem 2.1. *Suppose* $\psi^{\text{in}} \in [0,1]$, $\theta^{\text{in}}\in[0,\infty)$ *for a.e.* $(\boldsymbol{x}, y)\in\Omega$. *If* $(\hat{\boldsymbol{\theta}}^{\text{in}}, \boldsymbol{u}^{\text{in}}) \in \mathbb{L}^q\times\mathbb{H}$ *for every* $q\in[1,\infty)$, *then there exists a global weak solution to* (2.5) *for all* $m\in\mathbb{N}$ *when* $d = 2$ *and for* $m = 1, 2$ *or* 3 *when* $d = 3$. *Moreover, as* $t\to\infty$ *there is a positive constant* c *such that*

$$\|\hat{\boldsymbol{\theta}}(t)\|_{\mathbb{L}^q}, \|\boldsymbol{u}(t)\|_{\mathbb{H}} = \mathcal{O}(e^{ct}) \tag{2.9}$$

for all $q\in[1,\infty)$ *if* $d = 2$ *and* $q\in[1,6]$ *if* $d = 3$.

When $d = 2$, *if* $(\hat{\boldsymbol{\theta}}^{\text{in}}, \boldsymbol{u}^{\text{in}}) \in H^n(\Omega;\mathbb{R}^2)\times(\mathbb{V}\cap H^n(\Omega;\mathbb{R}^2))$ *for every* $n\in\mathbb{N}$, *then there exists a unique global classical solution to* (2.5).

2.2. Existence of Weak Solutions ($d = 2, 3$). We shall prove this result through a series of lemmas. We provide a-priori estimates for an associated system – the Leray regularised form of (2.5), with renormalised chemical nonlinearities ($\boldsymbol{\theta}_\delta = \tilde{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}_\delta$)

$$\partial_t \hat{\psi}_\delta + \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla} \psi_\delta = \Delta \psi_\delta - \psi_\delta f(\theta_\delta) e^{-\delta \theta_\delta}, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.10a}$$

$$\partial_t \hat{\theta}_\delta + \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla} \theta_\delta = \ell \Delta \theta_\delta + \psi_\delta f(\theta_\delta) e^{-\delta \theta_\delta}, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.10b}$$

$$\partial_t \boldsymbol{u}_\delta + \boldsymbol{w}_\delta \cdot \boldsymbol{\nabla} \boldsymbol{u}_\delta = \nu \Delta \boldsymbol{u}_\delta - \boldsymbol{\nabla} p_\delta + \sigma \hat{\theta}_\delta \boldsymbol{e}, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.10c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_\delta = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.10d}$$

$$(\partial_{\hat{\boldsymbol{n}}} \hat{\boldsymbol{\theta}}_\delta, \boldsymbol{u}_\delta) = \boldsymbol{0}, \qquad \text{on } \partial\Sigma \times \mathbb{R} \times \mathbb{R}_+, \tag{2.10e}$$

$$(\hat{\boldsymbol{\theta}}_\delta, \boldsymbol{u}_\delta) \to \boldsymbol{0}, \qquad \text{as } |y| \to \infty, \tag{2.10f}$$

$$\hat{\boldsymbol{\theta}}_\delta(0) = \hat{\boldsymbol{\theta}}_\delta^{\text{in}} = J_\delta * \hat{\boldsymbol{\theta}}^{\text{in}}, \qquad \text{in } \Omega, \tag{2.10g}$$

$$\boldsymbol{u}_\delta(0) = \boldsymbol{u}_\delta^{\text{in}} = J_\delta * \boldsymbol{u}^{\text{in}}, \qquad \text{in } \Omega. \tag{2.10h}$$

For $\delta > 0$, let $J_\delta \in C_0^\infty(\mathbb{R}^d)$ be a Friedrichs mollifier [1] with support on $B_\delta(\boldsymbol{x}, y)$, the ball of radius δ, centered at $(\boldsymbol{x}, y)$. We define $\boldsymbol{w}_\delta(\boldsymbol{\xi}) = (J_\delta * \boldsymbol{u}_\delta)(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} J_\delta(\boldsymbol{\xi} - \boldsymbol{\eta}) \bar{\boldsymbol{u}}_\delta(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}$, i.e. the mollification of $\boldsymbol{u}_\delta$, where $\bar{\boldsymbol{u}}_\delta$ is the zero extension of $\boldsymbol{u}_\delta$ outside Ω. We recall the usual mollifiers properties: if $\boldsymbol{v} \in L^q(\Omega; \mathbb{R}^d)$, $q \in [1, \infty)$, then $\|J_\delta * \boldsymbol{v}\|_{L^q} \le \|\boldsymbol{v}\|_{L^q}$ and $\lim_{\delta \to 0^+} \|J_\delta * \boldsymbol{v} - \boldsymbol{v}\|_{L^q} = 0$.

For every $\delta > 0$, we know that a unique classical solution exists to (2.10) for the given initial data, at least on some interval $[0, T_\delta]$, $T_\delta > 0$. We remark that since the polynomial function $f(\cdot)$ is Lipschitz continuous and the nonlinear reaction terms are also bounded for this approximate system, then such an existence result on a finite domain follows classically via a Faedo-Galerkin method, projecting initially onto the first N eigen-functions of the appropriate elliptic operators on a smooth bounded boundary say of vertical diameter N (see for example Heywood [22]). Norm estimates can be shown to be independent of the domain size considered and the result follows by considering the limit $N \to \infty$. Our a-priori Lebesgue and Sobolev norm estimates on $(\hat{\boldsymbol{\theta}}_\delta, \boldsymbol{u}_\delta)$ will verify that such a solution exists on $[0, \infty)$, i.e. the interval of existence is independent of δ. We will then eventually consider the limit $\delta \to 0^+$. The following preliminary lemma is an extension of the usual parabolic maximum principles (see Smoller [52] and Marion [36]).

Lemma 2.1. *If $\psi^{\text{in}} \in [0, 1]$, $\theta^{\text{in}} \in [0, \infty)$ a.e. in Ω, then $\psi_\delta \in [0, 1]$, $\theta_\delta \ge 0$, everywhere in $\Omega \times [0, T_\delta]$.*

Proof. Consider the inner product of (2.10a) with $\psi^- := \max\{0, -\psi_\delta\}$ in $L^2(\Omega)$,

$$\|\psi^-(t)\|_{L^2}^2 + 2 \int_0^t \|\boldsymbol{\nabla} \psi^-\|_{L^2(\Omega; \mathbb{R}^d)}^2 \mathrm{d}s \le \|\psi^-(0)\|_{L^2}^2.$$

Analogous estimates follow for $\theta^- := \max\{0, -\theta_\delta\}$ and $\psi^+ := \max\{0, \psi_\delta - 1\}$. □

We now establish the main estimates we require to prove the existence of weak solutions for $d = 2, 3$. The phrase "uniformly bounded" is considered to be with respect to our regularising parameter $\delta > 0$. We shall use c and $c(\cdot)$ to denote a generic finite positive constant which might depend on the argument indicated, but which does not depend on the regularisation parameter.

Lemma 2.2. *If* $(\hat{\boldsymbol{\theta}}^{\text{in}}, \boldsymbol{u}^{\text{in}}) \in \mathbb{L}^2 \times \mathbb{H}$*, then the set* $\{\hat{\boldsymbol{\theta}}_\delta, \boldsymbol{u}_\delta\}$ *is uniformly bounded in* $L^\infty_{\text{loc}}([0,\infty); \mathbb{L}^2 \times \mathbb{H}) \cap L^2_{\text{loc}}([0,\infty); H^1(\Omega;\mathbb{R}^2) \times \mathbb{V})$*, and in fact*

$$\|\hat{\boldsymbol{\theta}}_\delta(t)\|_{\mathbb{L}^2},\ \|\boldsymbol{u}_\delta(t)\|_{\mathbb{H}},\ \int_0^t \|\hat{\boldsymbol{\theta}}_\delta(s)\|_{H^1(\Omega;\mathbb{R}^2)} \mathrm{d}s = \mathcal{O}(e^{ct}).$$

Proof. Motivated by the work of Masuda [37], Haraux and Youkana [20], Collet and Xin [12] and Bricmont, Kupiainen and Xin [7] on reaction-diffusion systems, we consider a simple nonlinear functional that allows us to take advantage of the interaction of the chemical nonlinearities. As in Collet and Xin [12], for $F \in C^2(\mathbb{R}^2; [0,\infty))$, we use (2.5a)–(2.5b) to derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle F(\hat{\psi}_\delta, \hat{\theta}_\delta)\rangle + \langle Q(\boldsymbol{\nabla}\hat{\boldsymbol{\theta}}_\delta)\rangle \\
&= \langle \boldsymbol{\nabla} \cdot (F_1 \boldsymbol{\nabla}\hat{\psi}_\delta + \ell F_2 \boldsymbol{\nabla}\hat{\theta}_\delta - \boldsymbol{u}_\delta F)\rangle \\
&\quad + \langle F_1 \Delta\tilde{\psi} + \ell F_2 \Delta\tilde{\theta}\rangle - \langle F_1 \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\tilde{\psi} + F_2 \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\tilde{\theta}\rangle \\
&\quad - \langle (F_1 - F_2)\psi_\delta f(\theta_\delta) e^{-\delta\theta_\delta}\rangle,
\end{aligned} \tag{2.11}$$

where

$$Q(\boldsymbol{\nabla}\hat{\boldsymbol{\theta}}_\delta) = F_{11}|\boldsymbol{\nabla}\hat{\psi}_\delta|^2 + (1+\ell)F_{12}\boldsymbol{\nabla}\hat{\psi}_\delta \cdot \boldsymbol{\nabla}\hat{\theta}_\delta + \ell F_{22}|\boldsymbol{\nabla}\hat{\theta}_\delta|^2. \tag{2.12}$$

Here F_1 and F_2 are the partial derivatives of F with respect to its first and second arguments. We would like to choose an $F(\hat{\psi}_\delta, \hat{\theta}_\delta)$ which satisfies:

$$F_{11}F_{22} > (1+\ell)^2 F_{12}^2/4\ell, \quad \text{for all } \hat{\psi}_\delta \in [-1,1],\ \hat{\theta}_\delta \in [-1,\infty), \tag{2.13a}$$

$$F_1 - F_2 > 0, \quad \text{for all } \hat{\psi}_\delta \in [0,1],\ \hat{\theta}_\delta \in [0,\infty). \tag{2.13b}$$

We impose (2.13a) to ensure the quadratic form (2.12) is non-negative and condition (2.13b) to partially help control the nonlinear terms.

Step 1. We show that the following inequality holds for the mean-square reactant concentration and temperature:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle F(\hat{\psi}_\delta, \hat{\theta}_\delta)\rangle + c(\ell)\big(\|\boldsymbol{\nabla}\hat{\psi}_\delta\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\boldsymbol{\nabla}\hat{\theta}_\delta\|^2_{L^2(\Omega;\mathbb{R}^d)}\big)& \\
\le c\big(\|\hat{\boldsymbol{\theta}}_\delta\|^2_{\mathbb{L}^2} + \|\boldsymbol{u}_\delta\|^2_{\mathbb{H}} + \ell^2 + 1\big)&.
\end{aligned} \tag{2.14}$$

We assume F to be the quadratic form $F(\hat{\psi}_\delta, \hat{\theta}_\delta) := \alpha\hat{\psi}_\delta^2 + \beta\hat{\psi}_\delta\hat{\theta}_\delta + \gamma\hat{\theta}_\delta^2$. Note that for arbitrary $(\alpha,\beta,\gamma) \in \mathbb{R}^3_+$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+$ we can estimate

$$\begin{aligned}
F(\hat{\boldsymbol{\theta}}_\delta) &\ge \Big(\alpha - \frac{\beta\varepsilon_1}{2}\Big)\hat{\psi}_\delta^2 + \Big(\gamma - \frac{\beta}{2\varepsilon_1}\Big)\hat{\theta}_\delta^2, \\
Q(\boldsymbol{\nabla}\hat{\boldsymbol{\theta}}_\delta) &\ge \Big(2\alpha - \frac{(1+\ell)\beta\varepsilon_2}{2}\Big)|\boldsymbol{\nabla}\hat{\psi}_\delta|^2 + \Big(2\gamma\ell - \frac{(1+\ell)\beta}{2\varepsilon_2}\Big)|\boldsymbol{\nabla}\hat{\theta}_\delta|^2.
\end{aligned}$$

Condition (2.13b) is satisfied provided we choose

$$2\alpha - \beta = k_1 > 0 \quad \text{and} \quad \beta - 2\gamma = k_2 > 0. \tag{2.15}$$

For a given $\gamma > 0$, we can always choose β large enough so that $k_2 > 0$ and then ε_1 and ε_2 large enough so that $\gamma > \beta/2\varepsilon_1$ and $2\gamma\ell > (1+\ell)\beta/2\varepsilon_2$. Finally we can choose α sufficiently large (but finite) such that $k_1 > 0$, $\alpha > \beta\varepsilon_1/2$ and $2\alpha > (1+\ell)\beta\varepsilon_2/2$ and thus we guarantee (2.15) and hence (2.13b) as well as (2.13a). In (2.11) we bound the linear convection and diffusion terms using the Hölder and Young inequalities:

$$|\langle F_1\Delta\tilde{\psi} + \ell F_2\Delta\tilde{\theta}\rangle| + |\langle F_1\boldsymbol{u}_\delta\cdot\boldsymbol{\nabla}\tilde{\psi} + F_2\boldsymbol{u}_\delta\cdot\boldsymbol{\nabla}\tilde{\theta}\rangle| \\ \le c\left(\int_{\Sigma\times\Theta}|\boldsymbol{u}_\delta\cdot\boldsymbol{e}|^2\rho(y)\,\mathrm{d}\boldsymbol{x}\mathrm{d}y + \int_{\Sigma\times\Theta}|\hat{\boldsymbol{\theta}}_\delta|^2\rho(y)\,\mathrm{d}\boldsymbol{x}\mathrm{d}y + \ell^2 + 1\right),$$

where $\Theta = \operatorname{supp}\{\partial_y\tilde{\psi}\} = \operatorname{supp}\{\partial_y\tilde{\theta}\}$, $\rho(y) = \max\{|\partial_y\tilde{\psi}|, |\partial_y\tilde{\theta}|, |\partial_y^2\tilde{\psi}|, |\partial_y^2\tilde{\theta}|\}$. Using (2.13b) we estimate the nonlinear term

$$\begin{aligned}\langle\big(k_1\hat{\psi}_\delta + k_2\hat{\theta}_\delta\big)\psi_\delta f(\theta_\delta)e^{-\delta\theta_\delta}\rangle \ge & -\int_{\Omega_K}|-k_1|\hat{\psi}_\delta| + k_2\hat{\theta}_\delta|\psi_\delta f(\theta_\delta)\,\mathrm{d}\boldsymbol{x}\mathrm{d}y \\ & -\int_{\Omega_-}|k_1\hat{\psi}_\delta - k_2|\hat{\theta}_\delta||\psi_\delta f(\theta_\delta)\,\mathrm{d}\boldsymbol{x}\mathrm{d}y - c \\ \ge & -c\big(\|\hat{\boldsymbol{\theta}}_\delta\|_{\mathbb{L}^2}^2 + 1\big),\end{aligned} \tag{2.16}$$

where $\Omega_K = \{(\boldsymbol{x},y)\in\Omega\colon \hat{\psi}_\delta\in[-1,0],\ \hat{\theta}_\delta\in[0, K = 2k_1/k_2]\}$ and $\Omega_- = \{(\boldsymbol{x},y)\in\Omega\colon \hat{\psi}_\delta\in[0,1],\ \hat{\theta}_\delta\in[-1,0]\}$ and we have made use of the fact that $\tilde{\psi}\cdot\tilde{\theta}^m$ has compact support. Combining these last two inequalities yields (2.14).

Step 2. We establish the following energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_\delta\|_{\mathbb{H}}^2 + 2\nu\|\boldsymbol{u}_\delta\|_{\mathbb{V}}^2 \le \|\boldsymbol{u}_\delta\|_{\mathbb{H}}^2 + \sigma^2\|\hat{\theta}_\delta\|_{L^2}^2. \tag{2.17}$$

This inequality follows by considering the inner product of (2.10c) with $\boldsymbol{u}_\delta$ in $\mathbb{H}$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_\delta\|_{\mathbb{H}}^2 + \nu\|\boldsymbol{u}_\delta\|_{\mathbb{V}}^2 = \sigma(\boldsymbol{u}_\delta, \hat{\theta}_\delta\boldsymbol{e})_{\mathbb{H}},$$

and then using the Hölder and Young inequalities to estimate

$$\sigma(\boldsymbol{u}_\delta, \hat{\theta}_\delta\boldsymbol{e})_{\mathbb{H}} \le \sigma\|\boldsymbol{u}_\delta\|_{\mathbb{H}}\|\hat{\theta}_\delta\|_{L^2} \le \frac{1}{2}\|\boldsymbol{u}_\delta\|_{\mathbb{H}}^2 + \frac{\sigma^2}{2}\|\hat{\theta}_\delta\|_{L^2}^2.$$

Combine the inequalities in Steps 1 and 2 and apply the Grönwall inequality, noting that by the usual mollifier properties $\|\boldsymbol{u}_\delta^{\mathrm{in}}\|_{\mathbb{H}} \le \|\boldsymbol{u}^{\mathrm{in}}\|_{\mathbb{H}}$ and $\|\hat{\boldsymbol{\theta}}_\delta^{\mathrm{in}}\|_{L^2} \le \|\hat{\boldsymbol{\theta}}^{\mathrm{in}}\|_{L^2}$, to establish the norm growth estimates. □

We are now in a position to estimate the $L^q(\Omega)$-norms of the mass-fraction and temperature fields and in particular $\mathbb{L}^1$ estimates which indicate the growth rates of the reacting fronts.

Lemma 2.3. *If, in addition to the assumptions of Lemma 2.2, we assume $\hat{\boldsymbol{\theta}}_\delta^{\mathrm{in}}\in\mathbb{L}^q$, for all $q\in[1,\infty)$, then $\{\hat{\boldsymbol{\theta}}_\delta\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}([0,\infty);\mathbb{L}^q)$ for every $q\in[1,\infty)$ when $d = 2$ and $q\in[2,6]$ when $d = 3$, and: $\|\hat{\boldsymbol{\theta}}(t)\|_{\mathbb{L}^q} = \mathcal{O}(e^{ct})$. We can extend this estimate to include $q\in[1,2)$ when $d = 3$, provided we additionally assume $m\in\{1,\dots,6\}$.*

Proof. Step 1. We establish that if $\hat{\boldsymbol{\theta}}_\delta^{\text{in}} \in \mathbb{L}^q$ for some $q \in [2,\infty)$ when $d = 2$ or some $q \in [2,6]$ when $d = 3$, then $\hat{\boldsymbol{\theta}}_\delta$ lies in $L^\infty_{\text{loc}}([0,\infty);\mathbb{L}^q)$, which follows from the inequality: for any $n \in \mathbb{N}$,

$$\|\hat{\boldsymbol{\theta}}_\delta(t)\|_{\mathbb{L}^{2n}} \le \|\hat{\boldsymbol{\theta}}_\delta(0)\|_{\mathbb{L}^{2n}} + ct + c\int_0^t \|\hat{\boldsymbol{\theta}}_\delta(s)\|_{\mathbb{L}^2}^{1/n} + \|\boldsymbol{u}_\delta(s)\|_{L^{2n}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s. \quad (2.18)$$

For $n \in \mathbb{N}$, let F be the polynomial $F(\hat{\psi}_\delta,\hat{\theta}_\delta) := \sum_{k=0}^{2n}\alpha_k\hat{\psi}_\delta^{2n-k}\hat{\theta}_\delta^k$, where α_k, $k = 0,1,\ldots,2n$ are positive constants. We can choose the α_k so that F satisfies the conditions (2.13) and also that $F \ge c\hat{\psi}_\delta^{2n} + c\hat{\theta}_\delta^{2n}$, for all $\hat{\psi}_\delta \in [-1,1]$, $\hat{\theta}_\delta \in [-1,\infty)$, to ensure $\langle F\rangle$ is a positive definite functional – we provide a proof in the appendix (analogous to the quadratic case). Using (2.11), since there exists $A \in \mathbb{R}$ such that for all $\hat{\theta}_\delta > A$, $F_1 - F_2 > 0$, then with $\Omega_A = \{(\boldsymbol{x},y)\in\Omega\colon \hat{\psi}_\delta \in [-1,1],\ \hat{\theta}_\delta \in [-1,A]\}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle F\rangle \le{}& c\int_{\Omega_A}\left(|\hat{\boldsymbol{\theta}}_\delta|^{2n-1}\right)\psi_\delta f(\theta_\delta)\mathrm{d}\boldsymbol{x}\mathrm{d}y + c\left(\|\boldsymbol{u}_\delta\|_{L^{2n}(\Omega;\mathbb{R}^d)}+1\right)\sum_{i=1,2}\|F_i\|_{L^{\frac{2n}{2n-1}}(\Omega)}\\ \le{}& c\int_{\Omega_A}\left(|\hat{\boldsymbol{\theta}}_\delta|^{2n-1}\right)\tilde{\psi}\cdot\tilde{\theta}^m\,\mathrm{d}\boldsymbol{x}\mathrm{d}y + c(A)\left(\int_{\Omega_A}|\hat{\boldsymbol{\theta}}_\delta|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}y\right)^{\frac{1}{2n}}\langle F\rangle^{1-\frac{1}{2n}}\\ &+ c\left(\|\boldsymbol{u}_\delta\|_{L^{2n}(\Omega;\mathbb{R}^d)}+1\right)\langle F\rangle^{1-\frac{1}{2n}}.\end{aligned}$$

We have used that $\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}\|_{L^\infty} \le c$, $\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}\|_{L^{2n}(\Omega;\mathbb{R}^{d+2})} \le c$ and $\|\Delta\tilde{\boldsymbol{\theta}}\|_{\mathbb{L}^{2n}} \le c$. Now using that $\tilde{\psi}\cdot\tilde{\theta}^m$ has compact support the last inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle F\rangle^{\frac{1}{2n}}\right) \le c(A)\left(\int_{\Omega_A}|\hat{\boldsymbol{\theta}}_\delta|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}y\right)^{\frac{1}{2n}} + c\left(\|\boldsymbol{u}_\delta\|_{L^{2n}(\Omega;\mathbb{R}^d)}+1\right),$$

which yields (2.18) after integration in time. If we use the Gagliardo–Nirenberg inequality (2.1) to estimate the L^{2n}-norm on velocity field in (2.18) we must restrict ourselves to $n = 1, 2$ or 3 when $d = 3$. Intermediate L^q-estimates follow from (2.2).

Step 2. If $\hat{\boldsymbol{\theta}}_\delta^{\text{in}} \in \mathbb{L}^1$ and provided $\psi_\delta f(\theta_\delta) \in L^1(\Omega)$, then $\{\hat{\boldsymbol{\theta}}_\delta\}$ is uniformly bounded in $L^\infty_{\text{loc}}([0,\infty);\mathbb{L}^q(\Omega))$ for every $q \in [1,2)$. In particular, by Step 1, the latter assumption here is true for all $m \in \mathbb{N}$ when $d = 2$, but when $d = 3$, we are restricted to reactions for which $m \in \{1,\ldots,6\}$.

As in Constantin and Fefferman [13], consider $F(\xi) = \int_0^\xi(\xi-\eta)\phi(\eta)\mathrm{d}\eta \in C^2(\mathbb{R})$, where: $\phi(\xi) \ge 0$, for all $\xi \in \mathbb{R}$; $\phi \to 0$ as $\xi \to 0$; $\phi(\xi) = 0$, for all $\xi > \rho_0$ and $\int_0^{\rho_0}\phi(\xi)\,\mathrm{d}\xi = 1$. Hence $F'(\xi) = \int_0^\xi\phi(\eta)\,\mathrm{d}\eta \in [0,1]$ and $F''(\xi) = \phi(\xi) \ge 0$. We form the estimate ($\Psi := \{(\boldsymbol{x},y)\in\Omega\colon |\hat{\psi}_\delta| > \rho_0\}$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Psi F(|\hat{\psi}_\delta|)\mathrm{d}\boldsymbol{x}\mathrm{d}y + \int_\Psi F''(|\hat{\psi}_\delta|)|\boldsymbol{\nabla}|\hat{\psi}_\delta||^2\mathrm{d}\boldsymbol{x}\mathrm{d}y&\\ \le \int_\Omega |\psi_\delta f(\theta_\delta)| + |\boldsymbol{u}_\delta\cdot\boldsymbol{\nabla}\tilde{\psi}| + |\Delta\tilde{\psi}|\,\mathrm{d}\boldsymbol{x}\mathrm{d}y&.\end{aligned}$$

Taking the limit as $\rho_0 \to 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\psi}_\delta\|_{L^1(\Omega)} \le \|\psi_\delta f(\theta_\delta)\|_{L^1(\Omega)} + c\|\boldsymbol{u}_\delta\|_{\mathbb{H}} + c.$$

Now use that $\|\psi_\delta f(\theta_\delta)\|_{L^1(\Omega)} \le c\big(1 + \|\hat{\psi}_\delta\|_{L^1(\Omega)} + \|\hat{\theta}_\delta\|_{L^1(\Omega)} + \sum_{k=2}^{m} \|\hat{\theta}_\delta\|^k_{L^k(\Omega)}\big)$, where, by step 1, the last term on the right-hand side here is uniformly bounded for every $m \in \mathbb{N}$ when $d = 2$, but only for $m \in \{1, \dots, 6\}$ when $d = 3$. We can derive an analogous estimate for $\|\hat{\theta}_\delta\|_{L^1(\Omega)}$. Adding the two estimates together, using Lemma 2.2 and the interpolation inequality (2.2), the result follows.

Consider the following weak form of our regularised system (2.10), for all $(\varphi, \boldsymbol{v}) \in C^\infty([0,\infty); \mathbb{E} \times \mathbb{D})$ and $[t_0, t_1] \subset [0, \infty)$,

$$\int_{t_0}^{t_1} \langle \hat{\psi}_\delta \partial_t \varphi + \psi_\delta \Delta\varphi + \psi_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\varphi - \psi_\delta f(\theta_\delta) e^{-\delta\theta_\delta}\varphi \rangle \,\mathrm{d}t = \langle \hat{\psi}_\delta(t)\varphi(t)\rangle\big|_{t_0}^{t_1}, \tag{2.19a}$$

$$\int_{t_0}^{t_1} \langle \hat{\theta}_\delta \partial_t \varphi + \ell\theta_\delta \Delta\varphi + \theta_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\varphi + \psi_\delta f(\theta_\delta) e^{-\delta\theta_\delta}\varphi \rangle \,\mathrm{d}t = \langle \hat{\theta}_\delta(t)\varphi(t)\rangle\big|_{t_0}^{t_1}, \tag{2.19b}$$

$$\int_{t_0}^{t_1} \langle \boldsymbol{u}_\delta \cdot \partial_t \boldsymbol{v} + \nu \boldsymbol{u}_\delta \cdot \Delta \boldsymbol{v} + \boldsymbol{w}_\delta \otimes \boldsymbol{u}_\delta : \boldsymbol{\nabla}\boldsymbol{v} + \sigma\hat{\theta}_\delta v \boldsymbol{e}\rangle \,\mathrm{d}t = \langle \boldsymbol{u}_\delta(t)\boldsymbol{v}(t)\rangle\big|_{t_0}^{t_1}. \tag{2.19c}$$

From Lemmas 2.2 and 2.3, there exists a subsequence $\{\hat{\boldsymbol{\theta}}_\delta, \boldsymbol{u}_\delta\}$ (which we label by the same subscript) such that for all $q \in [1,\infty)$ when $d = 2$ and $q \in [2, 6]$ when $d = 3$, as $\delta \to 0^+$,

$$\begin{aligned} \hat{\boldsymbol{\theta}}_\delta &\to \hat{\boldsymbol{\theta}} \text{ in } \mathrm{w}^*\text{-}\mathrm{L}^\infty_{\mathrm{loc}}([0,\infty); \mathrm{w}\text{-}\mathbb{L}^{\mathrm{q}}) \cap \mathrm{w}\text{-}\mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{w}\text{-}\mathrm{H}^1(\Omega;\mathbb{R}^2)), \\ \boldsymbol{u}_\delta &\to \boldsymbol{u} \text{ in } \mathrm{w}^*\text{-}\mathrm{L}^\infty_{\mathrm{loc}}([0,\infty); \mathrm{w}\text{-}\mathbb{H}) \cap \mathrm{w}\text{-}\mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{w}\text{-}\mathbb{V}). \end{aligned} \tag{2.20}$$

Convergence of the linear terms in (2.19) to the appropriate terms in (2.7) follows from (2.20). By standard results we can deduce from (2.19c) that $\{\partial_t \boldsymbol{u}_\delta\}$ is uniformly bounded in $L^{4/d}_{\mathrm{loc}}([0,\infty); \mathbb{V}')$. The Aubin-Lions theorem [14, 54] for Bochner spaces (based on the Rellich-Kondrachov compact imbedding) implies

$$\begin{aligned} &\{\phi \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)) \colon \partial_t\phi \in L^{4/d}_{\mathrm{loc}}([0,\infty); H^1(\Omega)')\} \\ &\qquad \hookrightarrow\hookrightarrow L^2_{\mathrm{loc}}([0,\infty); L^2_{\mathrm{loc}}(\Omega)), \end{aligned} \tag{2.21}$$

where the target space is equipped with the strong topology. Hence $\{\boldsymbol{u}_\delta\}$ is relatively compact in $L^2_{\mathrm{loc}}([0,\infty); L^2(B_R))$ for any $R \in (0,\infty)$. So $\boldsymbol{u}_\delta \to \boldsymbol{u}$ strongly in both $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathrm{supp}\{\boldsymbol{v}\}))$ and $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathrm{supp}\{\varphi\}))$, which together with $\hat{\boldsymbol{\theta}}_\delta \to \hat{\boldsymbol{\theta}}$ weakly in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathrm{supp}\{\varphi\}))$, is sufficient to establish the convergence of the nonlinear convective terms in our weak formulation since

$$\begin{aligned} \int_{t_0}^{t_1} \langle \boldsymbol{\nabla}\varphi \cdot (\boldsymbol{u}_\delta \hat{\boldsymbol{\theta}}_\delta - \boldsymbol{u}\hat{\boldsymbol{\theta}})\rangle \,\mathrm{d}t &\le \int_{t_0}^{t_1} \|\boldsymbol{u}_\delta - \boldsymbol{u}\|_{\mathbb{H}} \|\hat{\boldsymbol{\theta}}_\delta\|_{\mathbb{L}^2} \|\boldsymbol{\nabla}\varphi\|_{L^\infty(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\ &\quad + \left| \int_{t_0}^{t_1} \langle \boldsymbol{u} \cdot \boldsymbol{\nabla}\varphi(\hat{\boldsymbol{\theta}}_\delta - \hat{\boldsymbol{\theta}})\rangle \,\mathrm{d}t \right|. \end{aligned}$$

By applying Green's theorem we can establish that

$$\begin{aligned} \|\ell\Delta\hat{\theta}_\delta + \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\hat{\theta}_\delta\|_{H^1(\Omega)'} &\le \sup_{\|\varphi\|_{H^1}=1} \Big\{ \ell |\langle \varphi\Delta\hat{\theta}_\delta\rangle| + |\langle \varphi \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\hat{\theta}_\delta\rangle| \Big\} \\ &= \sup_{\|\varphi\|_{H^1}=1} \Big\{ \ell |\langle \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\hat{\theta}_\delta\rangle| + |\langle \boldsymbol{\nabla}\varphi \cdot \boldsymbol{u}_\delta\hat{\theta}_\delta\rangle| \Big\} \\ &\le \ell \|\boldsymbol{\nabla}\hat{\theta}_\delta\|_{L^2(\Omega)} + \|\boldsymbol{u}_\delta\|_{L^4(\Omega;\mathbb{R}^d)} \|\hat{\theta}_\delta\|_{L^4(\Omega)}. \end{aligned}$$

Using the conclusions of Lemma 2.3 and recalling that $\tilde{\psi} \cdot \tilde{\theta}$ has compact support, we know that $\{\psi_\delta f(\theta_\delta) e^{-\delta\theta_\delta}\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$ provided we assume $m = 1, 2$ or 3 when $d = 3$ (no assumption when $d = 2$). Hence $\partial_t \hat{\theta}_\delta = \ell\Delta\theta_\delta - \boldsymbol{u}_\delta \cdot \boldsymbol{\nabla}\theta_\delta + \psi_\delta f(\theta_\delta)e^{-\delta\theta_\delta} \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)')$ for $d = 2, 3$. Analogous estimates hold for $\partial_t \hat{\psi}_\delta$. By employing the compact embedding (2.21) we can establish that $\hat{\boldsymbol{\theta}}_\delta \to \hat{\boldsymbol{\theta}}$ strongly in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathrm{supp}\{\varphi\}))$. We can now guarantee the convergence of the nonlinear reaction terms in (2.19) to the appropriate terms in (2.7) by, for example, using the identity

$$\int_{t_0}^{t_1} \Big\langle \varphi\big(\psi_\delta f(\theta_\delta)e^{-\delta\theta_\delta} - \psi f(\theta)\big) \Big\rangle \, \mathrm{d}t \equiv \int_{t_0}^{t_1} \Big\langle \varphi\Big((\psi_\delta - \psi) f(\theta_\delta) e^{-\delta\theta_\delta} + \psi\big(f(\theta_\delta)e^{-\delta\theta_\delta} - f(\theta)\big)\Big) \Big\rangle \, \mathrm{d}t,$$

and noting that strong convergence of $\hat{\psi}_\delta \to \hat{\psi}$ in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathrm{supp}\{\varphi\}))$ and weak convergence of $f(\theta_\delta) \to f(\theta)$ in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathrm{supp}\{\varphi\}))$ is sufficient to ensure the right-hand side converges to zero.

By choosing $\boldsymbol{v} \in \mathbb{D}$ independent of time, that $\boldsymbol{u} \in C([0,\infty); \mathrm{w}\text{-}\mathbb{H})$ follows from (2.7) and that $\mathbb{D}$ is dense in $\mathbb{H}$. Further, from the embedding

$$\{\phi \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)) \colon \partial_t\phi \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)')\} \hookrightarrow C([0,\infty); L^2(\Omega)),$$

we deduce that $\hat{\boldsymbol{\theta}} \in C([0,\infty); \mathbb{L}^2)$ after taking the limit $\delta \to 0^+$. This last embedding also implies that when $d = 2$, we have $\boldsymbol{u} \in C([0,\infty); \mathbb{H})$.

2.3. Stronger Solutions (d = 2). We prove uniqueness for slightly more regular solutions and then show that classical solutions exist provided we assume our initial data is smooth.

Lemma 2.4. *When $d = 2$, for initial data $\hat{\boldsymbol{\theta}}^{\mathrm{in}} \in H^1(\Omega; \mathbb{R}^2)$ and $\boldsymbol{u}^{\mathrm{in}} \in \mathbb{H}$, weak solutions also satisfy $\hat{\boldsymbol{\theta}} \in L^\infty_{\mathrm{loc}}([0,\infty); H^1(\Omega;\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty); \mathbb{W}^2)$ and this additional regularity is sufficient to establish uniqueness.*

Proof. Consider the inner product of (2.10b) with $-\Delta\hat{\theta}_\delta$ in $L^2(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\nabla}\hat{\theta}_\delta\|^2_{\mathbb{L}^2} + 2\ell\|\Delta\hat{\theta}_\delta\|^2_{L^2} \leq c\|\Delta\hat{\theta}_\delta\|_{L^2}\big(\|\boldsymbol{\nabla}\hat{\theta}_\delta\|_{\mathbb{L}^4}\|\boldsymbol{u}_\delta\|_{\mathbb{L}^4} + \|\psi_\delta f(\theta_\delta)e^{-\delta\theta_\delta}\|_{L^2} + \|\boldsymbol{u}\|_{\mathbb{H}} + \|\boldsymbol{\nabla}\hat{\theta}_\delta\|_{\mathbb{L}^2} + \ell + 1\big).$$

Using the Gagliardo–Nirenberg and Young inequalities, for arbitrary $\sigma > 0$,

$$\|\Delta\hat{\theta}_\delta\|_{L^2}\|\boldsymbol{\nabla}\hat{\theta}_\delta\|_{L^4}\|\boldsymbol{u}_\delta\|_{L^4} \leq \sigma\|\Delta\hat{\theta}_\delta\|^2_{L^2} + \frac{c}{\sigma}\|\boldsymbol{\nabla}\hat{\theta}_\delta\|^2_{L^2}\|\boldsymbol{u}_\delta\|^{\frac{8}{4-d}}_{L^4}.$$

Also note from the Gagliardo–Nirenberg inequality and Lemma 2.2 that $\|\boldsymbol{u}_\delta\|^{\frac{8}{4-d}}_{L^4}$ lies in $L^1_{\mathrm{loc}}([0,\infty))$ when $d = 2$. Hence noting regularity already established, forming an analogous estimate for $\|\boldsymbol{\nabla}\psi_\delta(t)\|_{\mathbb{L}^2}$, integrating with respect to time and considering the limit $\delta \to 0^+$, the regularity result follows.

We know [54] that for $\varphi \in C([0,\infty); L^2(\Omega))$, $\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2_{L^2} = 2(\varphi, \partial_t\varphi)_{L^2}$, in the scalar distribution sense on compact subsets of $[0,\infty)$. Let $(\boldsymbol{\theta}_1, \boldsymbol{u}_1)$ and $(\boldsymbol{\theta}_2, \boldsymbol{u}_2)$ be two weak solutions to our system (2.7) with the same initial data $(\hat{\boldsymbol{\theta}}^{\mathrm{in}}, \boldsymbol{u}^{\mathrm{in}}) \in H^1(\Omega;\mathbb{R}^2) \times \mathbb{H}$. Set

$\bar{\boldsymbol{\theta}} = \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1$ and $\bar{\boldsymbol{u}} = \boldsymbol{u}_2 - \boldsymbol{u}_1$. Since $\mathbb{E}$ is dense in $\mathbb{W}$ we can choose $\varphi = \hat{\theta}$ in (2.7b) and use Green's theorem to form

$$\frac{1}{2}\|\bar{\theta}(t_1)\|_{L^2}^2 - \frac{1}{2}\|\bar{\theta}(t_0)\|_{L^2}^2 + \ell\int_{t_0}^{t_1}\|\boldsymbol{\nabla}\bar{\theta}\|_{\mathbb{L}^2}^2 \mathrm{d}t = \int_{t_0}^{t_1}\langle\boldsymbol{\nabla}\bar{\theta}\cdot(\boldsymbol{u}_2\theta_2 - \boldsymbol{u}_1\theta_1)\rangle\,\mathrm{d}t + \int_{t_0}^{t_1}\langle\bar{\theta}(\psi_2 f(\theta_2) - \psi_1 f(\theta_1))\rangle\,\mathrm{d}t,$$

where

$$\langle\boldsymbol{\nabla}\bar{\theta}\cdot(\boldsymbol{u}_2\theta_2 - \boldsymbol{u}_1\theta_1)\rangle \le \|\boldsymbol{\nabla}\bar{\theta}\|_{\mathbb{L}^2}\big(\|\bar{\boldsymbol{u}}\|_{\mathbb{L}^4}\|\hat{\theta}_2\|_{L^4} + \|\bar{\theta}\|_{L^4}\|\boldsymbol{u}_1\|_{\mathbb{L}^4}\big) + c\|\boldsymbol{\nabla}\bar{\theta}\|_{\mathbb{L}^2}\big(\|\bar{\boldsymbol{u}}\|_{\mathbb{H}} + \|\bar{\theta}\|_{L^2}\big).$$

We can estimate $\langle\bar{\theta}(\psi_2 f(\theta_2) - \psi_1 f(\theta_1))\rangle$ in a similiar fashion. Using the Gagliardo–Nirenberg and Young inequalities, forming analogous estimates for $\bar{\psi}$ and $\bar{\boldsymbol{u}}$ and adding them together and choosing $t_0 = 0$, the result follows. □

Lemma 2.5. *If* $(\hat{\boldsymbol{\theta}}^{\text{in}}, \boldsymbol{u}^{\text{in}}) \in \mathbb{W}^2 \times (H^2(\Omega;\mathbb{R}^2)\cap\mathbb{V})$*, then* $\hat{\boldsymbol{\theta}}$ *lies in* $L^\infty_{\text{loc}}([0,\infty);\mathbb{W}^2)$.

Proof. Consider the $L^2(\Omega)$-inner product of the time derivative of (2.10b) with $\partial_t\hat{\theta}_\delta$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\hat{\theta}_\delta\|_{L^2}^2 + \ell\|\boldsymbol{\nabla}(\partial_t\hat{\theta}_\delta)\|_{\mathbb{L}^2}^2 \le \|\partial_t\hat{\theta}_\delta\|_{L^4}\|\partial_t\hat{\psi}_\delta\|_{L^2}\|\hat{\theta}_\delta\|_{L^{4m}}^m + \|\partial_t\hat{\theta}_\delta\|_{L^2}\|\partial_t\hat{\psi}_\delta\|_{L^2} + c\|\partial_t\hat{\theta}_\delta\|_{L^4}^2\|\psi_\delta f'(\theta_\delta)\|_{L^2} + \delta c\|\partial_t\hat{\theta}_\delta\|_{L^4}^2\|\psi_\delta f(\theta_\delta)\|_{L^2} + \|\partial_t\hat{\theta}_\delta\|_{L^4}\|\boldsymbol{\nabla}\hat{\theta}_\delta\|_{\mathbb{L}^4}\|\partial_t\boldsymbol{u}_\delta\|_{\mathbb{H}} + c\|\partial_t\hat{\theta}_\delta\|_{L^2}\|\partial_t\boldsymbol{u}_\delta\|_{\mathbb{H}}.$$

We can obtain analogous estimates for $\partial_t\hat{\psi}_\delta$ and $\partial_t\boldsymbol{u}_\delta$. Using the Gagliardo–Nirenberg and Young inequalities and the regularity already established, add the inequalities for $\partial_t\hat{\psi}_\delta$, $\partial_t\hat{\theta}_\delta$ and $\partial_t\boldsymbol{u}_\delta$, integrate with respect to time, and note from (2.10) that $\lim_{t_1\to 0^+}\|(\partial_t\hat{\boldsymbol{\theta}}_\delta, \partial_t\boldsymbol{u}_\delta)(t_1)\|_{\mathbb{L}^2\times\mathbb{H}} \le c\,\|(\hat{\boldsymbol{\theta}}_\delta^{\text{in}}, \boldsymbol{u}_\delta^{\text{in}})\|_{\mathbb{W}^2\times(H^2(\Omega;\mathbb{R}^2)\cap\mathbb{V})}$.

From (2.10b) and the regularity already established, it follows that $\ell\Delta\hat{\theta}_\delta = g$, where $g = \partial_t\hat{\theta}_\delta + \boldsymbol{u}_\delta\cdot\boldsymbol{\nabla}\theta_\delta - \psi_\delta f(\theta_\delta)e^{-\delta\theta_\delta} - \ell\Delta\tilde{\theta} \in L^\infty_{\text{loc}}([0,\infty);L^2(\Omega))$. An analogous estimate for $\hat{\psi}_\delta$ follows. Then take the limit $\delta\to 0^+$. □

Finally, we deduce from standard estimates that $\boldsymbol{u}$ lies in $L^\infty_{\text{loc}}([0,\infty);\mathbb{V}\cap H^2(\Omega;\mathbb{R}^2))$. That all components of $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{u}$ lie in $C^\infty(\overline{\Omega}\times\mathbb{R}_+)$, when the initial data is smooth, follows from the standard elliptic estimates in Constantin and Foias [14], p. 26, and Temam [54], pp. 302–3, as well as estimates for passively convected reaction-diffusion systems found in Smoller [52] or Gilbarg and Trudinger [19]. The proof of Theorem 2.1 is complete.

3. Front Solutions to a System with Unit Lewis Number ($d = 2$)

3.1. Orientation and Statement of Results. Let us consider the front solutions of the following simpler system:

$$u_t + u\cdot\nabla_x u = -\nabla_x p + \nu\Delta_x u + \epsilon T\hat{x}_2 + f(x),\ \nabla_x\cdot u = 0, \tag{3.1}$$

$$T_t + u\cdot\nabla_x T = \Delta_x T + g(T), \tag{3.2}$$

where we denote by T the temperature; $x = (x_1, x_2) \in \Omega \equiv (0, 2\pi) \times R^1$; $f(x) = (0, \sin x_1)$, a shearing force; $g(T) = T(1-T)(T-\mu)$, $\mu \in (0, \frac{1}{2})$, the bistable nonlinearity. The other notations are the same as before except that ϵ is now the Rayleigh number. System (3.1)-(3.2) is the unit Lewis number case of the original system studied in early sections. The identity $\psi + \theta = 1$ makes possible the reduction of the chemistry part to the temperature equation.

Our goal is to find an asymptotic regime where the structures of front solutions can be explicitly demonstrated, and the front speeds are uniformly bounded in time. Since the main issue is to deal with the fluid coupling, we choose the technically simpler bistable nonlinearity for the reaction term g in (3.2). Taking g as the combustion type nonlinearity of Arrhenius form with ignition temperature cutoff would require working in weighted function spaces, and only produce similar results. In the same spirit, we choose a special shear forcing function for $f(x)$, which was proposed by Kolmogorov for studying two dimensional turbulence, see [38, 50] and references therein.

The boundary conditions for system (3.1)-(3.2) are:

$$u|_{x_1=0,2\pi} = 0, \quad \forall \ x_2, \tag{3.3}$$

$$T_{x_1}|_{x_1=0,2\pi} = 0, \quad \forall \ x_2, \tag{3.4}$$

respectively the no slip boundary condition for velocity u and the adiabatic boundary condition for temperature T.

When $\epsilon = 0$, Eq. (3.1) decouples from (3.2), and a simple stationary solution is:

$$u_0 = (0, \nu^{-1} \sin x_1), \tag{3.5}$$

with pressure $p_0 = 0$. For $u = u_0$, Eq. (3.2) admits traveling front solutions of the form $T = T_0(x_1, x_2 - c_0 t) \equiv T_0(y, s)$, $y = x_1$, $s = x_2 - c_0 t$, and T_0 satisfies the elliptic equation:

$$\Delta_{y,s} T_0 + (\nu^{-1} \sin y + c_0) T_{0,s} + g(T_0) = 0, \tag{3.6}$$

for $(y, s) \in (0, 2\pi) \times R^1$, with the boundary conditions:

$$T_0(y, -\infty) = 0, T_0(y, +\infty) = 1, \max_{y \in [0,2\pi]} T_0(y, 0) = \frac{1}{2}, T_{0,y}(y, s)|_{y=0,2\pi} = 0. \tag{3.7}$$

Existence, uniqueness, and asymptotic stability of such traveling fronts are studied at length in a series of papers by Berestycki, Larrouturou, Nirenberg, and Roquejoffre, see [4, 5, 3], and [48]. The linearised operator around T_0(see Subsect. 3.2 for details) has an eigenfunction $T_{0,s}(x_1, x_2)$ for the simple eigenvalue zero on $L^2(\Omega)$, and its L^2 adjoint operator also has an eigenfunction corresponding to the simple eigenvalue zero, which we denote by $T^\star_{0,s}$. The L^2 orthogonal complement of $T^\star_{0,s}$ is denoted by W.

Now let us consider front solutions to system (3.1)-(3.2) ($\epsilon \neq 0$) of the form:

$$u = u_0 + \epsilon u_1(t, x, \epsilon), \tag{3.8}$$

$$p = \epsilon p_1(t, x, \epsilon), \tag{3.9}$$

$$T = T_0(x_1, x_2 - c(t, \epsilon)) + \epsilon T_1(t, x, \epsilon), \tag{3.10}$$

$$c(t, \epsilon) = c_0 t + \epsilon c_1(t, \epsilon). \tag{3.11}$$

We will show that for ϵ small enough, (3.8)-(3.11) are valid for all $t \geq 0$, with u_1, p_1, T_1 uniformly bounded in proper norms, and $|c_1| \leq O(t)$, as $t \to +\infty$. Substituting (3.8)–(3.11) into (3.1)–(3.2), and using Eq. (3.6), we have:

$$u_{1,t} + u_0 \cdot \nabla u_1 + u_1 \cdot \nabla u_0 + \epsilon u_1 \cdot \nabla u_1 = -\nabla p_1 + \nu \Delta u_1 + T_0 \hat{x_2} + \epsilon T_1 \hat{x_2}, \quad (3.12)$$

$$\nabla \cdot u_1 = 0, \quad (3.13)$$

$$T_{1,t} + u_0 \cdot \nabla T_1 + u_1 \cdot \nabla T_0 + \epsilon u_1 \cdot \nabla T_1 - c_1'(t,\epsilon) T_{0,s} = \Delta T_1 + g'(T_0) T_1 + \epsilon N(T_1, \epsilon), \quad (3.14)$$

where

$$\epsilon N(T_1, \epsilon) = \frac{g(T_0 + \epsilon T_1) - g(T_0)}{\epsilon} - g'(T_0) T_1, \quad (3.15)$$

and so $N(T_1, \epsilon)$ contains quadratic and cubic terms in T_1. The prime on c denotes the time derivative. The boundary conditions for u_1 and T_1 are:

$$u_1|_{x_1=0,2\pi} = 0, \quad T_{1,x_1}|_{x_1=0,2\pi} = 0. \quad (3.16)$$

Let us introduce some notations for this section. For any open subset $G \in R^2$ and measurable vector function $u(x) = (u_1(x), u_2(x))$, define:

$$||u||_G^2 = \sum_{i=1,2} \int_G u_i^2(x) dx, \quad L^2(G) = \{u : ||u||_G < +\infty\},$$

$$||\nabla u||_G^2 = \sum_{i,j=1,2} \int_G |u_{i,x_j}|^2 dx, \quad H^1(G) = \{u : ||u||_G^2 + ||\nabla u||_G^2 < +\infty\},$$

$$||D^2 u||_G^2 = \sum_{i,j,k=1,2} \int_G |u_{i,x_j,x_k}|^2 dx, \quad H^2(G) = \{u : \ ||D^2 u||_G < +\infty\},$$

$$D(G) = \{u \in C_0^\infty(G) : \nabla \cdot u = 0\},$$

$J(G)$ = closure of $D(G)$ in the norm $||u||_G$,
$J_0(G)$ = closure of $D(G)$ in the norm $H^1(G)$,
$H_0^1(G)$ = closure of $C_0^\infty(G)$ in the norm $H^1(G)$.

It is well known that the orthogonal complement of $J(G)$ in $L^2(G)$ is:

$$J^\perp \equiv \{u : u = \nabla p, \text{for some p} \in \mathrm{H}^1_{\mathrm{loc}}(\mathrm{G}), \ \text{with } \nabla \mathrm{p} \in \mathrm{L}^2_{\mathrm{loc}}(\mathrm{G})\}.$$

Let P be the orthogonal projection from $L^2(G)$ to $J(G)$, then the Stokes operator is:

$$-A = P\Delta, \quad (3.17)$$

with domain of definition $D(A) = H^2(G) \cap J_0$. If G is bounded, ∂G is C^3, then $A : D(A) \to J(G)$ is one to one and onto. Moreover, A^{-1} exists, is completely continuous, and symmetric. The eigenfunctions of A, denoted by $a^l(x)$, $l = 1, 2, \cdots$, with eigenvalues λ_l, i.e, $Aa^l = \lambda_l a^l$, are orthogonal and complete in $J_0(G)$.

The main result of the section is:

Theorem 3.1. *Let $u_1(0,x) \in J_0(\Omega)$, $T_1(0,x) \in H^1(\Omega) \cap W$, $c_1(0,\epsilon) = 0$. Let $< T_0 >$ be the integral average of $T_0(x_1,x_2)$ over $x_1 \in [0,2\pi]$, and so $< T_0 >$ is a bounded smooth function of x_2. Then there exists a positive number ϵ_0 depending on the H^1 norms of $u_1(0,x)$ and $T_1(0,x)$, and ν, such that if $\nu > 2\pi$, $\epsilon \in (0,\epsilon_0)$, system (3.12)-(3.14) admits unique solutions $u_1 = u_1(t,x,\epsilon)$, $T_1 = T_1(t,x,\epsilon)$, $c_1 = c_1(t,\epsilon)$, $p_1 = p_1(t,x,\epsilon)$ on $(0,+\infty) \times \Omega$, which satisfy:*

$$u_1 \in L^\infty((0,+\infty); J_0) \cap C([0,+\infty); J_0) \cap C^1((0,+\infty); J_0), \tag{3.18}$$

$$u_{1,t},\ D_x^2 u_1, \nabla p_1 - < T_0 > \hat{x}_2 \in L^2_{loc}((0,+\infty); L^2(\Omega)), \tag{3.19}$$

$$T_1 \in L^\infty((0,+\infty); H^1 \cap W) \cap C([0,+\infty); H^1 \cap W) \cap C^1((0,+\infty); H^1 \cap W), \tag{3.20}$$

$$c_1 \in C^1[0,+\infty), \tag{3.21}$$

$$\lim_{t\to 0} ||u_1 - u_1(0,x)||_{H^1} = \lim_{t\to 0} ||T_1 - T_1(0,x)||_{H^1} = 0. \tag{3.22}$$

Moreover, the following estimates hold:

$$||u_1||_{H^1} + ||T_1||_{H^1} + |c_1'| \le C, \quad \forall t \ge 0, \tag{3.23}$$

for some positive constant C, depending on the H^1 norm of initial data $u_1(0,x)$, $T_1(0,x)$, and ν. The front solutions to system (3.1)-(3.2) are then given by (3.8)-(3.11).

Remark 3.1. The solutions satisfying (3.18)–(3.22) are strong solutions. Front structures are seen from (3.8)-(3.11). The theorem does not specify the asymptotic behaviour of c_1' as $t \to +\infty$, whether it converges to a constant or it is oscillatory in time. Numerical simulations ([45, 62]) indicate that for downward propagating fronts, the front speeds tend to nearly constant values, while for upward propagating fronts, front speeds tend to oscillate in time due to the Rayleigh-Taylor instability induced by ϵ. Without the constraint on ν and ϵ, we do not expect that front solutions and their speeds will remain bounded in time. Power growth in t is observed for vorticity field, and front shape can evolve into a bubble like structure for an upward moving front, [62].

Remark 3.2. The condition that $T_1 \in W$ is not a restriction on the initial data. If it is not in W, one can always shift T_0 in x_2 by a suitable constant, or change the initial position for c, so that the new T_1 belongs to W.

The idea of proof is to seek energy estimates on u_1, and spectral-semigroup type estimates for T_1 as often used in stability analysis for traveling waves in reaction-diffusion equations. Combining the two types of estimates, we show that the H^1 norm of both u_1 and T_1 are bounded for all time. The condition $\nu > 2\pi$ comes up in the energy inequality for controlling the convective terms with the dissipative term $\nu\Delta u_1$. The basic ingredient for the energy estimate is the Poincaré inequality available when no slip boundary condition is imposed on u_1. If we impose periodic boundary condition instead, then due to unboundedness of our domain Ω, Poincaré inequality no longer holds. It seems that one has to come up with a different approach for analyzing solutions.

The proof of the theorem is organised as follows. In Subsect. 3.2, we consider solutions to Eqs. (3.12)-(3.13) with a given forcing term in $L^\infty((0,t_0); L^2(\Omega))$, for any $t_0 > 0$, and derive energy inequalities. These inequalites are along the line of those in Constantin and Foias [14]. To handle unbounded domains, we estimate the nonlinear terms differently using inequalities on the Stokes operator as given in Heywood [22]. In Subsect. 3.3, we present estimates based on the analytical semigroup generated by the linearised reaction-diffusion operator around the basic traveling front T_0. That the linearised operator is sectorial and so a generator of the analytical semigroup follows from the works of Berestycki, Larrouturou, and Roquejoffre, see [3, 48], and references therein. The nonlinear term $u_1 \cdot \nabla T_1$ is handled with an inequality in Kozono and Ogawa [27] for bounding convective terms in unbounded domains with fractional powers of differential operators. In Subsect. 3.4, we complete the proof by combining the above estimates and show the existence of global strong solutions uniformly bounded in H^1 norms for all time.

3.2. Solutions to a Forced Navier–Stokes Equation. We discuss the strong solutions to the following forced Navier–Stokes equation:

$$v_t + u_0 \cdot \nabla v + v \cdot \nabla u_0 + \epsilon v \cdot \nabla v = -\nabla p + \nu \Delta v + F(t,x), \tag{3.24}$$

$$\nabla \cdot v = 0, \quad x \in \Omega = (0,1) \times R^1, \tag{3.25}$$

$$v|_{t=0} = v_0(x) \in J_0(\Omega), \quad v|_{\partial\Omega} = 0, \tag{3.26}$$

where the forcing function $F(t,x) \in L^\infty((0,t_0); L^2(\Omega))$, for any $t_0 > 0$.

Following Heywood [22], we first consider (3.24)-(3.26) on any bounded domain with at least C^3 smooth boundary, then approximate Ω with an enlarging sequence of such domains. They can be domains enclosed by two parallel straight lines, with distance 2π apart, on the left and right, and two C^∞ curves on the top and bottom that connect to the straight lines with C^∞ smoothness. We will derive estimates on solutions that are independent of the x_2 diameter of these approximate domains, then pass to the limit. Since the approximate domains and Ω itself have width 2π in the x_1 direction, the Poincaré inequality:

$$||u||_{L^2} \le 2\pi ||\nabla u||_{L^2}, \quad \forall \ u \in H_0^1, \tag{3.27}$$

holds.

For any bounded domain, still denoted by Ω, the n^{th} Galerkin approximate solution is:

$$v^n(x) = \sum_{k=1}^{n} c_{kn} a^k(x), \tag{3.28}$$

with $c_{kn} = c_{kn}(t)$, and v^n satisfies:

$$\int_\Omega (v_t^n + \epsilon v^n \cdot \nabla v^n + u_0 \cdot \nabla v^n + v^n \cdot \nabla u_0 - \nu \Delta v^n) \cdot a^l(x) dx = \int_\Omega F \cdot a^l dx, \tag{3.29}$$

or

$$(v^n, a^l)_t + \epsilon (v^n \cdot \nabla v^n, a^l) + (u_0 \cdot \nabla v^n, a^l) + (v^n \cdot \nabla u_0, a^l) - \nu (\Delta v^n, a^l) = (F, a^l), \tag{3.30}$$

where $l = 1, 2, \cdots, n$, $(\cdot,\cdot)$ is the usual L^2 inner product. System (3.29) or (3.30) is an ODE system for $c_{kn}(t)$, $k = 1, 2, \cdots, n$ with quadratic nonlinearities. In the following, we skip the superscript n on v. Multiply (3.29) by c_{ln} and summing over l gives the identity:

$$\frac{1}{2} \frac{d}{dt} ||v||_2^2 + (v \cdot \nabla u_0, v) + \nu ||\nabla v||_2^2 = (F, v). \tag{3.31}$$

Poincaré inequality (3.27) implies that:

$$(v\cdot\nabla u_0, v) + \nu||\nabla v||_2^2 \geq -\nu^{-1}||v||_2^2 + \frac{\nu}{(2\pi)^2}||v||_2^2 = \delta||v||_2^2,$$

with $\delta \equiv (2\pi)^{-2}\nu - \nu^{-1} > 0$. It follows that

$$||v||_{2,t} \leq -\delta||v||_2 + ||F||_2, \tag{3.32}$$

or

$$\sup_{t\in[0,t_0]}||v||_2(t) \leq ||v(0)||_2 + C(t)\sup_{t\in[0,t_0]}||F||_2, \quad \forall\ t\geq 0, \tag{3.33}$$

where $C(t)$ is a bounded smooth function in $t \geq 0$, $C(0) = 0$, $C(t) \leq \frac{1}{\delta}$. Since

$$||v^n||_2^2 = \sum_{k=1}^{n} c_{kn}^2(t),$$

(3.33) implies the existence of smooth solutions of the ODE system (3.30) for all time.
Multiplying $\lambda_l c_{ln}$ to both sides of (3.30), and summing over l gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla v||_2^2 + \nu||P\Delta v||_2^2 = {} & \epsilon(v\cdot\nabla v, P\Delta v) + (u_0\cdot\nabla v, P\Delta v)\\ & + (v\cdot\nabla u_0, P\Delta v) - (F, P\Delta v).\end{aligned} \tag{3.34}$$

The terms on the right hand side of (3.34) are estimated below. By the Cauchy-Schwartz and Gagliardo–Nirenberg inequalities, we have:

$$\begin{aligned}R_1 \equiv |(v\cdot\nabla v, P\Delta v)| &\leq ||v||_4||\nabla v||_4||P\Delta v||_2\\ &\leq ||v||_2^{\frac{1}{2}}||\nabla v||_2^{\frac{1}{2}}||\nabla v||_2^{\frac{1}{2}}||D^2 v||_2^{\frac{1}{2}}||P\Delta v||_2.\end{aligned}$$

Recall that (see Lemma 1 of [22] for the three dimensional case):

$$||D^2 u||_2 \leq C(||P\Delta u||_2 + ||\nabla u||_2), \tag{3.35}$$

where C depends only on smoothness (at least C^3) of the boundary. It follows that:

$$\begin{aligned}R_1 &\leq C||v||_2^{\frac{1}{2}}||\nabla v||_2(||Av||_2 + ||\nabla v||_2)^{\frac{1}{2}}||Av||_2\\ &\leq C||v||_2^{\frac{1}{2}}||\nabla v||_2||Av||_2^{\frac{3}{2}} + C||v||_2^{\frac{1}{2}}||\nabla v||_2^{\frac{3}{2}}||Av||_2,\end{aligned}$$

and by Young's inequality:

$$\begin{aligned}R_1 \leq & \frac{1}{4}C\alpha^{-4}||v||_2^2||\nabla v||_2^4 + \frac{3}{4}C\alpha^{\frac{4}{3}}||Av||_2^2 + 2\nu^{-1}C^2||v||_2||\nabla v||_2^3 + \frac{\nu}{8}||Av||_2^2\\ \leq & \frac{1}{4}\alpha^{-4}C||v||_2^2||\nabla v||_2^4 + 4\pi\nu^{-1}C^2||\nabla v||_2^4 + (\frac{3}{4}C\alpha^{\frac{4}{3}} + \frac{\nu}{8})||Av||_2^2, \quad \text{(using (3.27))}\\ = & C(4^{-1}\alpha^{-4}||v||_2^2 + 4\pi\nu^{-1}C^2)||\nabla v||_2^4 + (\frac{3}{4}C\alpha^{\frac{4}{3}} + \frac{\nu}{8})||Av||_2^2,\end{aligned} \tag{3.36}$$

for some constant α to be chosen. The other three terms are bounded as:

$$\begin{aligned}R_2 \equiv |(u_0\cdot\nabla v, P\Delta v)| &\leq |u_0|_\infty||\nabla v||_2||Av||_2\\ &\leq 2\nu^{-1}|u_0|_\infty^2||\nabla v||_2^2 + \frac{\nu}{8}||Av||_2^2,\end{aligned} \tag{3.37}$$

$$R_3 \equiv |(v \cdot \nabla u_0, P\Delta v)| \le |\nabla u_0|_\infty ||v||_2 ||Av||_2$$
$$\le 2\nu^{-1}|\nabla u_0|_\infty^2 ||v||_2^2 + \frac{\nu}{8}||Av||_2^2, \tag{3.38}$$

$$|(F, P\Delta v)| \le ||F||_2 ||Av||_2 \le \nu^{-1}||F||_2^2 + \frac{\nu}{4}||Av||_2^2. \tag{3.39}$$

Here and in the rest of this section, $|\cdot|_\infty$ denotes the L^∞ norm.

Combining (3.34)-(3.39) and choosing $6C\alpha^{\frac{4}{3}} = \nu$, we have (with C denoting a generic constant independent of ν, and skipping the subscript 2 for the L^2 norm):

$$\frac{1}{2}\frac{d}{dt}||\nabla v||^2 + \frac{\nu}{4}||Av||^2 \le \epsilon C(\nu^{-3}||v||^2 + \nu^{-1})||\nabla v||^4$$
$$+\ 2\nu^{-1}|u_0|_\infty^2||\nabla v||^2 + 2\nu^{-1}|\nabla u_0|_\infty^2||v||^2 + \nu^{-1}||F||^2.$$

Substituting (3.33) and (3.27), and denoting $\sup_{t\in[0,t_0]} ||\cdot||_2$ by $||\cdot||_\infty$, we continue:

$$\frac{1}{2}\frac{d}{dt}||\nabla v||^2 + \frac{\nu}{4}||Av||^2 \le \epsilon C(\nu^{-3}||v(0)||^2 + \nu^{-3}\delta^{-2}||F||_\infty^2 + \nu^{-1})||\nabla v||^4$$
$$+\ C\nu^{-3}||\nabla v||^2 + \nu^{-1}||F||_\infty^2. \tag{3.40}$$

It follows from (3.31) and (3.27) that:

$$\frac{1}{2}\frac{d}{dt}||v||^2 + (2\pi)^2\delta||\nabla v||^2 \le ||F||\cdot||v||, \tag{3.41}$$

or

$$\frac{1}{2}\frac{d}{dt}||v||^2 + 2\pi^2\delta||\nabla v||^2 + \frac{1}{2}\delta||v||^2 \le \frac{1}{2\delta}||F||^2 + \frac{\delta}{2}||v||^2,$$

or

$$\frac{1}{2}\frac{d}{dt}||v||^2 + 2\pi^2\delta||\nabla v||^2 \le \frac{1}{2\delta}||F||^2. \tag{3.42}$$

Integrating (3.42) and using (3.33) to get:

$$2\pi^2\delta\int_t^{t+\tau}||\nabla v||^2(s)ds \le ||v(0)||^2 + \frac{1}{\delta^2}||F||_\infty^2 + \frac{1}{2\delta}||F||_\infty^2\tau,$$

assuming $t_0 \ge t + \tau$. Thus,

$$\int_t^{t+\tau}||\nabla v||^2(s)ds \le C\delta^{-1}(||v(0)||^2 + \delta^{-1}||F||_\infty^2(\tau + \delta^{-1})), \tag{3.43}$$

with constant C independent of ν and δ.
It follows from (3.43) that the Lebesque measure

$$|\{s \in [t, t+\tau] : ||\nabla v|| \ge \rho\}| \le C\rho^{-2}\delta^{-1}(||v(0)||^2 + \delta^{-1}||F||_\infty^2(\tau + \delta^{-1})).$$

Choose $\rho = \frac{\sqrt{2}}{\sqrt{\tau}}\sqrt{C}\delta^{-1}(\delta||v(0)||^2 + ||F||_\infty(\delta^{-1} + \tau))^{1/2}$, then

$$|\{s \in [t, t+\tau] : ||\nabla v|| \ge \rho\}| \le \frac{\tau}{2}, \tag{3.44}$$

and thus $\exists\, t_1 \in [t, t+\tau]$ such that

$$||\nabla v||^2(t_1) \leq C\tau^{-1}\delta^{-2}(\delta||v(0)||^2 + ||F||_\infty^2(\delta^{-1} + \tau)). \tag{3.45}$$

Multiplying (3.40) by:

$$I \equiv \exp\left\{-\int_{t_1}^{t} \epsilon C(\nu^{-3}||v(0)||^2 + \nu^{-3}\delta^{-2}||F||_\infty^2 + \nu^{-1})||\nabla v||^2 ds - (t - t_1)C\nu^{-3}\right\},$$

where the constant C is twice that in (3.40), to obtain:

$$\frac{d}{dt}[||\nabla v||^2 I] \leq \frac{2}{\nu}||F||_\infty^2 I. \tag{3.46}$$

Integrating (3.46) on $[t_1, t]$ to get:

$$||\nabla v||^2(t) \leq ||\nabla v(t_1)||^2 I^{-1} + \frac{2}{\nu}||F||_\infty^2(t - t_1)I^{-1}. \tag{3.47}$$

Let $t_1 \in [t - \tau, t]$, $t \geq \tau \geq 0$, such that (3.45) holds. In view of (3.43) and (3.45), we have from (3.47):

$$\begin{aligned} ||\nabla v||^2(t) \leq {} & [C\tau^{-1}\delta^{-2}(\delta||v(0)||^2 + ||F||_\infty^2(\delta^{-1} + \tau)) + 2\nu^{-1}||F||_\infty^2\tau]\times \\ & \exp\{C\epsilon(\nu^{-3}||v(0)||^2 + \nu^{-3}\delta^{-2}||F||_\infty^2 + \nu^{-1})\times \\ & \delta^{-1}(||v(0)||^2 + \delta^{-1}||F||_\infty^2(\tau + \delta^{-1})) + C\tau\nu^{-3}\}. \end{aligned} \tag{3.48}$$

Fix $\tau = \delta^{-2}$, then for $t \geq \tau$, we have:

$$\begin{aligned} ||\nabla v||^2(t) \leq {} & C(\delta||v(0)||^2 + ||F||_\infty^2(\delta^{-1} + \delta^{-2}) + \nu^{-1}\delta^{-2}||F||_\infty^2)\times \\ & \exp\{C\epsilon\delta^{-1}(\nu^{-3}||v(0)||^2 + \nu^{-3}\delta^{-2}||F||_\infty^2 + \nu^{-1})\times \\ & (||v(0)||^2 + \delta^{-2}(1 + \delta^{-1})||F||_\infty^2) + C\delta^{-2}\nu^{-3}\}, \end{aligned} \tag{3.49}$$

while for $t \in [0, \tau]$, we set $t_1 = 0$ in (3.47) to have:

$$\begin{aligned} ||\nabla v||^2(t) \leq {} & (||\nabla v(0)||^2 + 2\delta^{-2}\nu^{-1}||F||_\infty^2)\times \\ & \exp\{C\epsilon\delta^{-1}(\nu^{-3}||v(0)||^2 + \nu^{-3}\delta^{-2}||F||_\infty^2 + \nu^{-1})\times \\ & (||v(0)||^2 + \delta^{-1}||F||_\infty^2(\delta^{-1} + \delta^{-2})) + C\delta^{-2}\nu^{-3}\}. \end{aligned} \tag{3.50}$$

Combining (3.49) and (3.50), we obtain the estimates on $||\nabla v||(t)$ for $t \in [0, t_0]$, uniformly in $t_0 > 0$:

$$||\nabla v||^2(t) \leq B_1 \cdot B_2, \tag{3.51}$$

where

$$B_1 = C(||v(0)||^2\delta + ||F||_\infty^2(\delta^{-1} + \delta^{-2} + \nu^{-1}\delta^{-2}) + ||\nabla v(0)||^2),$$

and B_2 is:

$$\exp\{C\epsilon\delta^{-1}(\nu^{-3}||v(0)||^2 + \nu^{-3}\delta^{-2}||F||_\infty^2 + \nu^{-1})\times$$

$$(||v(0)||^2 + \delta^{-1}(\delta^{-1} + \delta^{-2})||F||_\infty^2) + C\delta^{-2}\nu^{-3}\},$$

where C is a positive constant independent of ν, δ, and t_0.

Notice that estimates (3.33) and (3.51) hold for the approximate Galerkin solutions independent of the x_2 diameters of the approximate domains. Using (3.33), (3.51) and (3.35) in (3.40), it is straightforward to obtain the bounds:

$$\int_0^t ||D^2v||^2(s)ds \leq L(t), \quad \int_0^t ||v_t||^2(s)ds \leq L(t), \tag{3.52}$$

for some continuous function L of t, independent of the approximate solutions and domains.

Passing to $n \to \infty$ in (3.29) in a standard way(see [22]), we see that the limiting vector function v is a unique strong solution to the system (3.24) and (3.25). Summarizing, we have:

Proposition 3.1. *Let $v(0) \in J_0(\Omega)$, and $F \in L^\infty([0, t_0]; L^2(\Omega))$, for some $t_0 > 0$. Then if $\nu > 2\pi$, there is a unique solution $v(t, x)$, $p(t, x)$ to system (3.24) and (3.25) such that:*

$$v \in C((0, t_0); J_0(\Omega)); \quad v_t, \ D^2v \ \in L^2((0, t_0); L^2(\Omega)). \tag{3.53}$$

Moreover, v attains initial data continuously in L^2, and v satisfies the estimates (3.33), (3.51), (3.52). If $t_0 = +\infty$, then (3.33) and (3.51) hold uniformly in $t \geq 0$.

3.3. Estimates on a Reaction-Diffusion Equation. We consider the reaction-diffusion equation (3.14) for T_1 with velocity u_1 a given function as described in Proposition 3.1. As usual, we introduce the moving frame coordinate $\xi = x_2 - c(t, \epsilon)$, $x_1 = x_1$, $t = t$. Equation (3.14) becomes:

$$T_{1,t} - c_0T_{1,\xi} - \Delta T_1 - g'(T_0(x_1, \xi))T_1 + u_0 \cdot \nabla T_1$$

$$= \epsilon c_1'(t, \epsilon)T_{1,\xi} - u_1 \cdot \nabla T_0 + c_1'T_{0,s} - \epsilon u_1 \cdot \nabla T_1 + \epsilon N(T_0, T_1, \epsilon) \equiv F_1. \tag{3.54}$$

Define the operator L:

$$(-L)T_1 = \Delta T_1 + c_0T_{1,\xi} - u_0 \cdot \nabla T_1 + g'(T_0(x_1, \xi))T_1, \tag{3.55}$$

with domain of definition $D(L) = \{T_1 \in H^2(\Omega) : T_{1,x_1}|_{\partial\Omega} = 0\}$. The operator L has a simple eigenvalue corresponding to the positive eigenfunction $T_{0,s}$, thanks to the monotonicity of the wave profile $T_{0,s} > 0$. For the bistable nonlinearity g, the arguments in [3] and [48] apply without using weighted spaces, and we have the following:

Lemma 3.1. *The operator L is sectorial [21] on $L^2(\Omega)$ with zero Neumann boundary condition; the spectrum of L stays inside a sector strictly in the right half plane except for a simple eigenvalue at zero corresponding to the eigenfunction $T_{0,s}(x_1, \xi)$. Operator L is invertible on the subspace:*

$$W \equiv \{u \in L^2(\Omega) : (u, T^\star_{0,s}) = 0, u_{x_1}|_{\partial\Omega} = 0\}, \tag{3.56}$$

where $T^\star_{0,s}$ is the positive nullfunction of the adjoint operator $L^\star$ in $L^2(\Omega)$ such that $(T_{0,s}, T^\star_{0,s}) = 1$. Moreover, the estimate:

$$||L^{-1}u||_{H^2} \leq C_\gamma||u||, \quad \forall \ u \in W, \tag{3.57}$$

for constant C_γ depending only on γ, the distance from the sector to the left half plane.

Lemma 3.1 implies that L is a generator of an analytical semigroup on W, and the usual fractional powers of $(-L)$ are well-defined, [21]. Equation (3.54) can be expressed as:

$$T_{1,t} = -LT_1 + F_1(T_1, c_1, \epsilon), \tag{3.58}$$

and will be solved for $T_1 \in W$ for all $t \geq 0$. For any given initial data $T_1(0) \in W$, let us write (3.58) into the related integral equation:

$$T_1(t) = e^{-Lt}T_1(0) + \int_0^t e^{-(t-s)L} F_1(T_1, c_1, \epsilon)(s)ds, \tag{3.59}$$

where we impose the condition:

$$(F_1, T^{\star}_{0,s}) = 0, \ \ \forall\, s \geq 0, \tag{3.60}$$

so that formula (3.59) provides bounded solutions for all time. In view of (3.54), we have:

$$c_1' = (\epsilon u_1 \cdot \nabla T_1 + u_1 \cdot \nabla T_0 - \epsilon c_1' T_{1,\xi} - \epsilon N(T_0, T_1, \epsilon), T^{\star}_{0,s}). \tag{3.61}$$

It follows from (3.58)-(3.61) that $T_1 \in W$ for all $t \geq 0$. The front equation (3.61) for c_1 is a nonlocal equation and the terms $u_1 \cdot \nabla T_1$ and $u_1 \cdot \nabla T_0$ reflect the strain effects of the fluid flows. In the passive case, $\epsilon = 0$, u_1 as a perturbation of the steady state u_0 will decay to zero when $t \to \infty$ if $\nu > 2\pi$ (see (3.32) with $F = 0$). Hence (3.61) implies that the front speed approaches an asymptotic constant value.

Now let us make a-priori estimates on solutions of the integral equation (3.59) in the space $L^{\infty}((0, t_0); W \cap H^1(\Omega))$ along with (3.61). Define:

$$m_{\alpha}(t_0) = \sup_{t \in (0,t_0)} ||L^{\alpha} T_1(t)||, \tag{3.62}$$

for all $T_1 \in W$, $\alpha \in [0, \frac{1}{2}]$, $t_0 > 0$. First we note that

$$\sup_{t \in [0,t_0]} ||e^{-Lt} L^{\alpha} T_1(0)|| \leq ||L^{\alpha} T_1(0)|| \leq C||T_1(0)||_{H^1},$$

where we use the fact that if $T_1 \in W$, $L^{\alpha} T_1 \in W$. By (3.60) and (3.61), we rewrite F_1 as:

$$F_1 = P_2(\epsilon c_1' T_{1,\xi}) + P_2(-u_1 \cdot \nabla T_0) + P_2(-\epsilon u_1 \cdot \nabla T_1) + \epsilon P_2 N, \tag{3.63}$$

where $P_2 \equiv Id - P_1$, and $P_1 u \equiv (u, T^{\star}_{0,s}) T_{0,s}$, i.e. P_2 is the projection from $L^2(\Omega)$ to W.

Applying L^{α}, $\alpha \in [0, \frac{1}{2}]$, to (3.59), we have:

$$||L^{\alpha} T_1||_2 \leq ||L^{\alpha} e^{-Lt} T_1(0)||_2 + \int_0^t ||L^{\alpha+\delta'} e^{-L(t-s)} L^{-\delta'} F_1||_2(s)ds,$$

$$\leq C||T_1(0)||_{H^1} + \int_0^t (t-s)^{-(\alpha+\delta')} e^{-\gamma(t-s)} ||L^{-\delta'} F_1||(s)ds, \tag{3.64}$$

where $\delta' \in (0, \frac{1}{2})$, $F_1 = F_1(T_1, c_1', \epsilon)$. By (3.63), we have:

$$||L^{-\delta'}F_1|| \leq \epsilon|c_1'|C_{\gamma,\delta'}||T_{1,\xi}|| + ||\nabla T_0||_\infty C_{\gamma,\delta'}||u_1|| \\ + \epsilon||L^{-\delta'}P_2(u_1\cdot\nabla T_1)|| + \epsilon C_{\gamma,\delta'}||N||, \quad (3.65)$$

for some positive constant $C_{\gamma,\delta'}$ depending only on γ and δ'.

We estimate:

$$\begin{aligned}
&||L^{-\delta'}P_2(u_1\cdot\nabla T_1)||\\
&= ||L^{-\delta'}(L+\gamma)^{\delta'}(L+\gamma)^{-\delta'}P_2(u_1\cdot\nabla T_1)||\\
&\leq ||L^{-\delta'}(L+\gamma)^{\delta'}||_{(L^2(W)\to L^2(W))}||(L+\gamma)^{-\delta'}P_2(u_1\cdot\nabla T_1)||\\
&\leq C_{\gamma,\delta'}||(L+\gamma)^{-\delta'}P_2(u_1\cdot\nabla T_1)||\\
&\leq C_{\gamma,\delta'}||(L+\delta')^{-\delta'}(-\Delta+\gamma)^{\delta'}||_{(L^2(\Omega)\to L^2(\Omega))}||(-\Delta+\gamma)^{-\delta'}P_2(u_1\cdot\nabla T_1)||\\
&\leq C_{\gamma,\delta'}||(-\Delta+\gamma)^{-\delta'}P_2(u_1\cdot\nabla T_1)||\\
&\leq C_{\gamma,\delta'}||(-\Delta+\gamma)^{-\delta'}u_1\cdot\nabla T_1-(u_1\cdot\nabla T_1,T_{0,s}^\star)(-\Delta+\gamma)^{-\delta'}T_{0,s}||\\
&\leq C_{\gamma,\delta'}||(-\Delta+\gamma)^{-\delta'}(u_1\cdot\nabla T_1)|| + C_{\gamma,\delta'}||u_1||\cdot||\nabla T_1||. \qquad (3.66)
\end{aligned}$$

By Lemma 2.1 of Kozono and Ogawa, [27], for $\delta' \in (0, \frac{1}{2})$, we have:

$$\begin{aligned}
||(-\Delta+\gamma)^{-\delta'}(u_1\cdot\nabla T_1)|| &\leq C_{\delta'}||(-\Delta)^{\frac{1}{2}-\delta'}u_1||\cdot||(-\Delta)^{\frac{1}{2}}T_1||,\\
&\leq C_{\delta'}||\nabla T_1||(||(-\Delta)^{\frac{1}{2}}u_1||+||u_1||),\\
&\leq C_{\delta'}||\nabla T_1||(||\nabla u_1||+||u_1||), \qquad (3.67)
\end{aligned}$$

for some constant $C_{\delta'}$ depending on δ'. It follows from (3.66) and (3.67) that:

$$||L^{-\delta'}P_2(u_1\cdot\nabla T_1)|| \leq C_{\gamma,\delta'}||u_1||_{H^1}\cdot||\nabla T_1||. \qquad (3.68)$$

Noticing that $|N| \leq C((T_1)^2+(T_1)^3)$, for C independent of T_1. Then the Gagliardo–Nirenberg inequality shows that:

$$\begin{aligned}
||T_1^2||_2 = ||T_1||_4^2 &\leq C||T_1||_2||\nabla T_1||_2,\\
||T_1^3|| = ||T_1||_6^3 &\leq C||T_1||_2||\nabla T_1||_2^2. \qquad (3.69)
\end{aligned}$$

Inequality (3.65) implies that

$$||L^{-\delta'}F_1|| \leq \epsilon|c_1'|C_{\gamma,\delta'}||T_{1,\xi}|| + ||\nabla T_0||_\infty C_{\gamma,\delta'}||u_1|| \\ +\epsilon C_{\gamma,\delta'}||u_1||_{H^1}\cdot||\nabla T_1|| + \epsilon C_{\gamma,\delta'}(||T_1||\cdot||\nabla T_1|| + ||\nabla T_1||^2||T_1||). \qquad (3.70)$$

Now, choose $\alpha = 0,\ \frac{1}{2}$, $\delta' \in (0, \frac{1}{2})$, we have from (3.64), (3.65) and (3.70) that:

$$m_{\frac{1}{2}} + m_0 \leq C||T_1(0)||_{H^1} + C_{\gamma,\delta'}[\epsilon|c_1'|_\infty m_{\frac{1}{2}} + ||\nabla T_0||_\infty||u_1||_\infty + \epsilon(||u_1||_\infty m_{\frac{1}{2}}$$

$$+||\nabla u_1||_\infty m_{\frac{1}{2}} + m_{\frac{1}{2}} m_0 + m_{\frac{1}{2}}^2 m_0)] \sup_{t\in[0,t_0]} \int_0^t ((t-s)^{-\delta'} + (t-s)^{-\delta'-\frac{1}{2}}) e^{-\gamma(t-s)} ds, \tag{3.71}$$

where $||\cdot||_\infty = \sup_{t\in[0,t_0]} ||\cdot||$, $|c'|_\infty = \sup_{t\in[0,t_0]} |c'|$; constants C and $C_{\gamma,\delta'}$ are independent of t_0.

Letting $M = m_{\frac{1}{2}} + m_0$, and lumping all the constants depending on γ, δ', we get:

$$M \le C||T_1(0)||_{H^1} + \epsilon C(|c_1'|M + (||u_1||_\infty + ||\nabla u_1||_\infty)M + M^2 + M^3) + C||u_1||_\infty, \tag{3.72}$$

where C depends on γ, δ' only. We get from (3.61) that

$$\begin{aligned} |c_1'| \le \epsilon ||T^\star_{0,s}||_\infty ||u_1|| \cdot ||\nabla T_1|| + C||u_1|| + \epsilon|c_1'| \cdot ||T_{1,\xi}|| \\ + \epsilon C(||T_1||^2 + ||T_1|| \cdot ||T_1||_4^2), \end{aligned}$$

or

$$|c_1'|_\infty \le \epsilon C||u_1||_\infty M + C||u_1|| + \epsilon|c_1'|_\infty M + \epsilon C(M^2 + M^3). \tag{3.73}$$

It is straightforward to verify that for small time t_0, the mapping defined by the right hand side of (3.59) on T_1 is a contraction in $L^\infty((0,t_0); W \cap H^1)$, which yields a unique mild solution. Parabolic regularity [46], then shows that it is a strong solution. We will consider long time solutions to (3.59) along with the Navier–Stokes equation in the next subsection. We summarise the above into:

Proposition 3.2. *The integral equation (3.59) along with (3.61) has a strong solution for* $t \in [0, t_0]$*, if* t_0 *is small enough. Moreover, the estimates (3.72) and (3.73) hold for the solution.*

3.4. Uniformly Bounded Solutions of the System. We turn to the solutions of system (3.12)-(3.15). Equation (3.12) can be written as:

$$u_{1,t} + u_0\cdot\nabla u_1 + u_1\cdot\nabla u_0 + \epsilon u_1\cdot\nabla u_1 = -\nabla\tilde{p}_1 + \nu\Delta u_1 + (T_0 - < T_0 >)\hat{x_2} + \epsilon T_1 \hat{x}_2, \tag{3.74}$$

where

$$< T_0 > = < T_0 > (x_2) = \frac{1}{2\pi}\int_0^{2\pi} T_0(x_1, x_2)dx_1,$$

$$\tilde{p}_1 = p_1 - \int_0^{x_2} < T_0 > dx_2.$$

It is obvious that $T_0 - < T_0 > \in L^\infty((0,\infty), L^2(\Omega))$. In the moving frame coordinate, (x_1, ξ, t), system (3.12)-(3.14) becomes:

$$\begin{aligned} u_{1,t} - c' u_{1,\xi} + u_0\cdot\nabla u_1 + u_1\cdot\nabla u_0 + \epsilon u_1\cdot\nabla u_1 = \\ - \nabla\tilde{p_1} + \nu\Delta u_1 + (T_0 - < T_0 >)\hat{x}_2 + \epsilon T_1\hat{x}_2, \end{aligned} \tag{3.75}$$

$$\nabla\cdot u_1 = 0, \tag{3.76}$$

$$T_{1,t} - c_0 T_{1,\xi} - \Delta T_1 - g'(T_0(\xi, x_1))T_1 + u_0\cdot\nabla T_1 = F_1. \tag{3.77}$$

Notice that the estimates (3.33) and (3.51) in Subsect. 3.2 remain the same in the moving frame coordinate for (3.75). Let us make an a-priori estimate of solutions to system (3.75)-(3.77) with initial data $u_1(0) \in J_0(\Omega)$, $T_1(0) \in W \cap H^1(\Omega)$. Define:

$$|u|_\infty \equiv \sup_{0<t<t_0} (||u||_2 + ||\nabla u||_2),$$

for any $t_0 > 0$, and $|c'|_\infty = \sup_{0<t<t_0} |c'|$.

Let us consider (3.75)-(3.77) in the space

$$V = \{(u, T, c) \in L^\infty((0, t_0), J_0 \times H^1) \times C^1[0, t_0]\colon |u|_\infty + |T|_\infty + |c'|_\infty < \infty\}.$$

The constant C below depends only on ν, γ, and δ'. By Propositions 3.1 and 3.2, we have:

$$|u_1|_\infty^2 \le ||u_1(0)||^2 + C(|T_0 - <T_0>|_\infty + \epsilon|T_1|_\infty)^2$$
$$+C[||u_1(0)||^2 + (|T_0 - <T_0>|_\infty + \epsilon|T_1|_\infty)^2 + ||\nabla u_1(0)||^2]\times$$
$$\exp\{\epsilon C(||u_1(0)||^2 + (|T_0 - <T_0>|_\infty + \epsilon|T_1|_\infty)^2 + 1)^2\}, \tag{3.78}$$

and

$$|T_1|_\infty \le C|T_1(0)|_{H^1} + \epsilon C(|c_1'|_\infty |T_1|_\infty$$
$$+|u_1|_\infty |T_1|_\infty + |T_1|_\infty^2 + |T_1|_\infty^3) + C|u_1|_\infty, \tag{3.79}$$

and

$$|c_1'|_\infty \le \epsilon C|u_1|_\infty |T_1|_\infty + C|u_1|_\infty + \epsilon|c_1'|_\infty |T_1|_\infty + \epsilon C(|T_1|_\infty^2 + |T_1|_\infty^3). \tag{3.80}$$

Taking the square root of (3.78) yields:

$$|u_1|_\infty \le ||u_1(0)||_{H^1} + C|T_0 - <T_0>|_\infty + \epsilon C|T_1|_\infty +$$
$$C(||u_1(0)||_{H^1} + |T_0 - <T_0>|_\infty + \epsilon|T_1|_\infty)\times$$
$$\exp\{\epsilon C(1 + ||u_1(0)||^2 + (|T_0 - <T_0>|_\infty + \epsilon|T_1|_\infty)^2)^2\}. \tag{3.81}$$

Set $K \equiv |u_1|_\infty + |T_1|_\infty$. To get rid of the last term on the right hand side of (3.79), let us multiply (3.81) by $C + 1$ and add the resulting inequality to (3.79) to find:

$$K \le C||u_1(0)||_{H^1} + C|T_0 - <T_0>|_\infty + \epsilon CK$$
$$+C(||u_1(0)||_{H^1} + |T_0 - <T_0>|_\infty + \epsilon K)\times$$
$$\exp\{\epsilon C(1 + ||u_1(0)||^2 + (|T_0 - <T_0>|_\infty + \epsilon K)^2)^2\}$$
$$+C||T_1(0)||_{H^1} + \epsilon C(K^2 + K^3 + |c_1'|_\infty K), \tag{3.82}$$

where

$$|c_1'|_\infty \le \epsilon CK^2 + CK + \epsilon|c_1'|_\infty K + \epsilon C(K^2 + K^3). \tag{3.83}$$

The above estimates on K remain the same for small time, and we can use the contraction mapping principle to construct local in time mild solutions u_1, T_1, c_1 in the space V for some pressure $\tilde{p}_1$. Standard regularity results for Navier–Stokes equations ([22])

and parabolic equations ([46]) then imply that the mild solutions are actually strong solutions. In particular, if in the definition of K or norm $|\cdot|_\infty$, we replace t_0 by t and t by τ, then K as a function of t is continuous for $t \in [0, t_0)$.

Proof of Theorem 3.1. Assume that $K = K(t) \leq A_0$, for $t \in [0, t_0)$, where t_0 is a positive time ensured by local existence, and A_0 is a constant to be properly chosen below (3.90). In particular, A_0 is independent of ϵ, $A_0 > 0$, $\epsilon A_0 < \frac{1}{2}$. We will show by a continuity argument that $K(t) \leq A_0$ for all $t \geq 0$ if ϵ is small enough.

We have from such a choice of A_0 and ϵ that:

$$|c_1'|_\infty \leq C[\epsilon K^2 + K + \epsilon K^3], \tag{3.84}$$

which implies via (3.82) that

$$\begin{aligned} K \leq\ & C||u_1(0)||_{H^1} + C|T_0 - <T_0>|_\infty + C\epsilon K + C||T_1(0)||_{H^1} \\ & + C(||u_1(0)||_{H^1} + |T_0 - <T_0>|_\infty + \epsilon K)\exp\{\epsilon C(1 + ||u_1(0)||^2 \\ & + |T_0 - <T_0>|_\infty^2 + \epsilon^2 K^2)^2\} + \epsilon C(K^2 + K^4) \end{aligned}$$

or

$$\begin{aligned} K \leq\ & C||u_1(0)||_{H^1} + C|T_0 - <T_0>|_\infty + C||T_1(0)||_{H^1} + \frac{1}{2}C\epsilon + \frac{C\epsilon}{2}K^2 \\ & + C(||u_1(0)||_{H^1} + |T_0 - <T_0>|_\infty + \epsilon K)\exp\{\epsilon C(1 + ||u_1(0)||^2 \\ & + |T_0 - <T_0>|_\infty^2 + \epsilon^2 K^2)^2\} + \epsilon C(K^2 + K^4), \end{aligned} \tag{3.85}$$

$$\begin{aligned} K \leq\ & C||u_1(0)||_{H^1} + C|T_0 - <T_0>|_\infty + C||T_1(0)||_{H^1} + \frac{1}{2}C\epsilon + C(||u_1(0)||_{H^1} + \\ & |T_0 - <T_0>|_\infty + \epsilon A_0)\exp\{\epsilon C(1 + ||u_1(0)||^2 + |T_0 - <T_0>|_\infty^2 + \epsilon^2 A_0^2)^2\} \\ & + \epsilon C(1 + A_0^2)K^2 \\ \equiv\ & K_0 + \epsilon C(1 + A_0^2)K^2. \end{aligned} \tag{3.86}$$

Since K is continuous in t, it follows from (3.86) that if

$$4K_0\epsilon C(1 + A_0^2) < 1, \tag{3.87}$$

then

$$K \leq 2K_0, \quad \text{for} \quad \mathrm{t} \in (0, \mathrm{t}_0), \tag{3.88}$$

if $K(0) \leq 2K_0$, which is true by our choice of K_0 with $C \geq 1$. To be consistent with our assumption of A_0, we have also:

$$2K_0 \leq A_0. \tag{3.89}$$

Now we choose:

$$\begin{aligned} A_0 = 4[& C||u_1(0)||_{H^1} + C|T_0 - <T_0>|_\infty + C||T_1(0)||_{H^1} \\ & + C(||u_0||_{H^1} + |T_0 - <T_0>|_\infty)]. \end{aligned} \tag{3.90}$$

Then there exists ϵ_0, depending only on $C = C(\nu, \gamma, \delta')$, $||u_1(0)||_{H^1}$, T_0, $||T_1(0)||_{H^1}$, such that if $\epsilon \in (0, \epsilon_0)$:

$$K_0 \leq \frac{A_0}{2},$$

$$4K_0\epsilon C(1 + A_0^2) \leq 2A_0\epsilon C(1 + A_0^2) < 1,$$

with $C \geq 1$, which implies that:

$$\epsilon A_0 < \frac{1}{2}.$$

Inequality (3.84) implies that

$$|c_1'|_\infty \leq CA_0 + \epsilon_0 CA_0^2 + \epsilon_0 A_0^3. \tag{3.91}$$

Since the above bounds on K and c' are independent of t_0, they are valid for all $t \geq 0$ by continuity . The rest of the theorem follows from standard regularity results for Navier–Stokes equations [22] and semilinear parabolic equations [46]. We finish the proof. □

Appendix

We would like to choose the α_k so that $F(\xi, \eta)$ is a positive function satisfying (2.13). Condition (2.13b) will be satisfied provided

$$(2n - k)\alpha_k > (k + 1)\alpha_{k+1}, \ \forall\ k = 0, 1, \ldots, 2n - 1. \tag{3.92}$$

Further let us suppose α_{2n-2} is large enough so that ($\lambda = (1 + \ell)^2/4\ell$)

$$4n\alpha_{2n-2}\alpha_{2n} > \lambda(2n - 1)\alpha_{2n-1}^2. \tag{3.93}$$

Now consider condition (2.13a):

$$2n(2n-1)\alpha_0\xi^{2n-2}\left(\sum_{k=0}^{2n-2}(k+1)(k+2)\alpha_{k+2}\xi^{2n-2-k}\eta^k\right) +$$
$$\left(\sum_{k=1}^{2n-2}(2n-k)(2n-k-1)\alpha_k\xi^{2n-2-k}\eta^k\right)\left(\sum_{k=0}^{2n-2}(k+1)(k+2)\alpha_{k+2}\xi^{2n-2-k}\eta^k\right)$$
$$> \lambda(2n-1)\left(\sum_{k=0}^{2n-2}(2n-k-1)(k+1)\alpha_{k+1}\xi^{2n-2-k}\eta^k\right)^2.$$

We can write this in the shorthand form

$$\alpha_0 B_1(\alpha_2, \ldots, \alpha_{2n}, 2n(2n-1)\alpha_{2n}r^{2n-2})$$
$$+ B_2(\alpha_1, \ldots, \alpha_{2n}, 4n(2n-1)\alpha_{2n-2}\alpha_{2n}r^{4n-4})$$
$$> \lambda(2n-1)B_3(\alpha_1, \ldots, \alpha_{2n-1}, (2n-1)^2\alpha_{2n-1}^2 r^{4n-4}),$$

where each of the B_i ($i = 1, 2, 3$) are the obvious polynomials where their last argument indicates the highest order term in $r \equiv \eta/\xi$. For the moment let us assume $B_1 \geq 0$ for all $\xi \in [-1, 1]$, $\eta \geq -1$. We show this is true below. Then, there exists $R \in \mathbb{R}_+$

independent of α_0, such that for all $|r| > R$, $B_2 > \lambda(2n-1)B_3$ (using condition (3.93)), i.e. such that (2.13a) is satisfied. Now suppose $|r| \leq R$, then we can clearly choose α_0 large enough to guarantee condition (2.13a).

Let k' denote all odd integers $0 < k' < 2n$. Then

$$\begin{aligned}
F(\xi,\eta) \geq & \left[\alpha_0 - \sum_{k'}\left(1-\frac{k'}{2n}\right)\frac{\alpha_{k'}}{\epsilon}\right]\xi^{2n} + \left[\alpha_{2n} - \sum_{k'}\left(\frac{k'}{2n}\right)\epsilon^{\frac{2n}{k'}-1}\alpha_{k'}\right]\eta^{2n} \\
& + \sum_{k'}\alpha_{k'}\left[\left(1-\frac{k'}{2n}\right)\frac{\xi^{2n}}{\epsilon} + \frac{k'}{2n}\epsilon^{\frac{2n}{k'}-1}\eta^{2n}\right] + \sum_{k'}\alpha_{k'}\xi^{2n-k'}\eta^{k'} \\
\geq & \left[\alpha_0 - \sum_{k'}\left(1-\frac{k'}{2n}\right)\frac{\alpha_{k'}}{\epsilon}\right]\xi^{2n} + \left[\alpha_{2n} - \sum_{k'}\frac{k'}{2n}\epsilon^{\frac{2n}{k'}-1}\alpha_{k'}\right]\eta^{2n} \\
= & \; c_1\xi^{2n} + c_2\eta^{2n}, \qquad (3.94)
\end{aligned}$$

where we choose ϵ small enough such that $c_2 > 0$ and then choose α_0 large enough so that $c_1 > 0$. Now recall that we need to demonstrate that for all $\xi \in [-1,1]$, $\eta \geq -1$, $B_1 = \sum_{k=0}^{2n-2}(k+1)(k+2)\alpha_{k+2}\xi^{2n-2-k}\eta^k \geq 0$. This is clear from an identical argument to that in (3.94), choosing α_2 large enough. We can easily choose the α_k ($k = 0, 1, \ldots, 2n$) such that the conditions (3.92), (3.93) and α_0, α_2 large enough are met.

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