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The solution to a second order linear ordinary differential equation with a non-homogeneous term that is a measure[†]

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(???)

We consider the solvability of the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 w'' + bw' - rw + h = 0, \tag{1}$$

inside an interval $\mathcal{J} \equiv \,]\alpha, \beta[$, where σ, b, r are given functions and h is a locally finite measure. This ODE is associated with the Hamilton-Jacobi-Bellman equations arising in the study of a wide range of stochastic optimisation problems. These problems are motivated by numerous applications and include optimal stopping, singular stochastic control and impulse stochastic control models in which the state process is given by a one-dimensional Itô diffusion. Under general conditions, we derive both analytic and probabilistic expressions for the solution to (1) that is required by the analysis of the relevant stochastic control models. We also establish a number of properties that are important for applications.

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1 Introduction

Let $\mathcal{J}$ be an open interval with left endpoint $\alpha \geq -\infty$ and right endpoint $\beta \leq \infty$, and let $\mathcal{B}(\mathcal{J})$ denote the Borel σ-algebra on $\mathcal{J}$. It is well known that a function $g : \mathcal{J} \to \mathbb{R}$ is the difference of two convex functions if and only if its second distributional derivative is a locally integrable measure on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$. Given such a function g, we denote by g'_- its left-hand first derivative, which

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is a function of finite variation, and we let

$$g''(dx) = g''_{\mathrm{ac}}(x)\, dx + g''_{\mathrm{s}}(dx) \tag{2}$$

be the Lebesgue decomposition of the second distributional derivative $g''(dx)$ into the measure $g''_{\mathrm{ac}}(x)\, dx$ that is absolutely continuous with respect to the Lebesgue measure and the measure $g''_{\mathrm{s}}(dx)$ that is mutually singular with the Lebesgue measure. Similarly, given a locally integrable measure h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$, we denote by

$$h(dx) = h_{\mathrm{ac}}(x)\, dx + h_{\mathrm{s}}(dx) \tag{3}$$

its Lebesgue decomposition.

Now, let $\sigma, b, r : \mathcal{J} \to \mathbb{R}$ be given Borel-measurable functions with $\sigma > 0$, and let h be a measure on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$. We aim at establishing general conditions on this data, under which there exists a solution w to the ODE (1) in the sense of distributions. In particular, we consider the solvability of (1) in the following sense.

Definition 1.1 A function $w : \mathcal{J} \to \mathbb{R}$ is a solution to the ODE (1) if it is the difference of two convex functions and $\mathcal{L}w = -h$, where $\mathcal{L}w$ is the measure on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ defined by

$$\mathcal{L}w(dx) = \frac{1}{2}\sigma^2(x) w''(dx) + b(x) w'_-(x)\, dx - r(x) w(x)\, dx. \tag{4}$$

Equivalently, a function $w : \mathcal{J} \to \mathbb{R}$ is a solution to the ODE (1) if it is the difference of two convex functions,

$$\frac{1}{2}\sigma^2(x) w''_{\mathrm{ac}}(x) + b(x) w'_-(x) - r(x) w(x) + h_{\mathrm{ac}}(x) = 0, \tag{5}$$

Lebesgue-a.e. in $\mathcal{J}$, and

$$w''_{\mathrm{s}}(dx) = -\frac{2}{\sigma^2(x)}\, h_{\mathrm{s}}(dx). \tag{6}$$

The ODE (1) arises in the study of several stochastic optimisation models with an expected discounted performance criterion over an infinite time horizon and with state process dynamics related to the one-dimensional Itô diffusion

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x \in \mathcal{J}, \tag{7}$$

where W is a standard one-dimensional Brownian motion. Such models have been motivated by numerous applications and arise in optimal stopping, singular stochastic control and impulse stochastic control; Alvarez [1–4], Beibel and Lerche [5], Beneš, Shepp and Witsenhausen [6], Davis and Zervos [9], Dayanik and Karatzas [10], Duckworth and Zervos [11], Guo and Shepp [13], Harrison and Taksar [14], Harrison and Taylor [15], Jacka [17, 18], Karatzas [19], Karatzas and Ocone [20], Karatzas, Ocone, Wang and Zervos [21], Karatzas and Sudderth [23], Ma [25], Øksendal [27,28], Salminen [31], Shreve, Lehoczky and Gavers [34], Shiryayev [32], Shiryaev and Peskir [33], and the references therein provide a highly incomplete list of such stochastic control problems. In the models studied in these references by means of dynamic programming and variational inequalities, special cases of the ODE (1) are considered. In these cases, the measure h, which is closely related to the associated payoff functionals, is assumed to be absolutely continuous with respect to the Lebesgue measure.

Apart from their independent theoretical interest, the results that we establish here are important for addressing stochastic control problems in which h is not absolutely continuous. Such a situation arises in the optimal stopping problem that aims at maximising the performance criterion

$$\mathbb{E}_x \left[\exp\left(-\int_0^\tau r(X_s)\, ds \right) g(X_\tau) \mathbf{1}_{\{\tau<\infty\}} \right] \tag{8}$$

over all stopping times τ. If g is C^1 with absolutely continuous first derivative and satisfies appropriate technical conditions, then an application of Itô's formula implies that this performance criterion is equal to

$$g(x) + \mathbb{E}_x \left[\int_0^\tau \exp\left(-\int_0^t r(X_s)\, ds \right) \tilde{\mathcal{L}} g(X_t)\, dt \right], \tag{9}$$

where

$$\tilde{\mathcal{L}} g = \frac{1}{2}\sigma^2 g'' + b g' - r g.$$

It follows that the problem of maximising (8) over τ is equivalent to maximising (9) over all stopping times τ. With regard to standard theory of optimal stopping, the Hamilton-Jacobi-Bellman (HJB) equation of this problem takes the form of the variational inequality

$$\max\left\{ \frac{1}{2}\sigma^2 w'' + b w' - r w + \tilde{\mathcal{L}} g,\ -w \right\} = 0.$$

Thus, one is faced with the solvability of (1).

This reformulation is not intended solely for providing a justification for this paper. Indeed, it turns out that deriving explicit solutions to special cases of this optimal stopping problem depends crucially on the properties of $\tilde{\mathcal{L}}g$ (see Alvarez [4, Corollary 4.2]). This situation is not confined to optimal stopping: Øksendal [27,28] used similar ideas to reformulate a number of singular control models to equivalent ones involving the function $\tilde{\mathcal{L}}g$ that he then solved. At this point, we are faced with the issue of extending these analyses to the cases that arise when g does not possess the regularity assumed above and $\tilde{\mathcal{L}}g$ is a measure as in (4) rather than a function. An example where such a generalisation becomes relevant is provided by the perpetual American butterfly spread option, the payoff function g of which is given by

$$g(x) = (x - K_1)^+ - 2(x - K_2)^+ + (x - K_3)^+,$$

for some constants $K_1 < K_2 < K_3$.

The solvability of the ODE (1) when h is a function satisfying appropriate integrability conditions has been extensively studied under general assumptions, and has been documented in several references, including Feller [12], Breiman [8], Mandl [26], Itô and McKean [16], Karlin and Taylor [24], Rogers and Williams [30], and Borodin and Salminen [7]. Our analysis here relies on the use of Itô calculus. For this reason, we restrict our attention to Itô diffusions such as the one given by (7) rather than more general diffusions, which may not be semimartingales. Also, we assume that the underlying diffusion X is non-explosive. Our results can be generalised with little effort to account for the cases that arise if the boundary points α, β are attainable (whether absorbing or reflecting), provided that we stay within a semimartingale framework. We have decided against addressing such a generalisation, partly because this would significantly complicate the exposition of our results and partly because most applications assume that the underlying state process is non-explosive.

At this point, it is of interest to make a comment on the relation of our results with the theory of additive functionals of Markov processes. To fix ideas, suppose that X is a Brownian motion and that $r(x) \equiv \lambda$, for some constant $\lambda > 0$. In this case, if D is a continuous additive functional of X, then $D = A^h$ for some measure h, where A^h is the process defined by (54) in Section 4 (see Revuz and Yor [29, Theorem X.2.9]). Furthermore, the λ-potential U_D^λ of D admits the expression

$$U_D^\lambda(x) := \mathbb{E}_x \left[\int_0^\infty e^{-\lambda t} \, dD_t \right] = \int_{\mathcal{J}} u^\lambda(x, s) \nu_D(ds), \tag{10}$$

where u^λ is the Green function defined by $u^\lambda(x,s) = (\sqrt{2\lambda})^{-1} \exp(-\sqrt{2\lambda}|x-s|)$

and ν_D is the Revuz measure of D (see Revuz and Yor [29, Theorem X.2.8]). Such representations can be extended to more general linear Markov processes including the ones that we consider here. One contribution of this paper is to provide an explicit expression for the right-hand side of (10) and to show that this satisfies the ODE (1) when X is an Itô diffusion such as the one given by (7).

The paper is organised as follows. In Section 2, we consider the Itô diffusion (7), we develop our assumptions and we review a number of well-known results on which our analysis depends. In Section 3, we establish an analytic expression for a special solution to (1), and we prove several analytic properties that this solution has. In Section 4, we establish a probabilistic expression for this special solution, and we show that it satisfies Dynkin's formula as well as the transversality condition, which are properties that are most important for the analysis of specific applications.

2 The associated Itô diffusion, assumptions and spaces of measures

We consider the Itô diffusion given by (7) and we make the following assumption.

Assumption 2.1 The functions $b, \sigma : \mathcal{J} \to \mathbb{R}$ are Borel-measurable and satisfy the following conditions:

$$\sigma^2(x) > 0, \quad \text{for all } x \in \mathcal{J} \equiv]\alpha, \beta[, \tag{11}$$

and

$$\int_{\bar\alpha}^{\bar\beta} \frac{1 + |b(s)|}{\sigma^2(s)}\, ds < \infty, \quad \text{for all } \alpha < \bar\alpha < \bar\beta < \beta. \tag{12}$$

Also, σ^2 is locally bounded, i.e.,

$$\sup_{s \in [\bar\alpha, \bar\beta]} \sigma^2(s) < \infty, \quad \text{for all } \alpha < \bar\alpha < \bar\beta < \beta. \tag{13}$$

Assumptions (11) and (12) are the non-degeneracy condition (ND)$'$ and the local integrability condition (LI)$'$ in Karatzas and Shreve [22, Section 5.5.C], respectively. These conditions are sufficient for the SDE (7) to have a weak solution that is unique in the sense of probability law up to a possible explosion time. In particular, given $c \in \mathcal{J}$, the scale function p and the speed measure

m, given by

$$p(x) = \int_c^x \exp\left(-2\int_c^s \frac{b(u)}{\sigma^2(u)}\,du\right) ds, \quad \text{for } x \in \mathcal{J}, \tag{14}$$

$$m(dx) = \frac{2}{\sigma^2(x)p'(x)}\,dx, \tag{15}$$

are well-defined. For future reference, we note that p satisfies the ODE

$$\frac{1}{2}\sigma^2(x)p''(x) + b(x)p'(x) = 0. \tag{16}$$

We also assume that the solution to (7) is non-explosive, i.e., the hitting time of the boundary $\{\alpha, \beta\}$ of the interval $\mathcal{J}$ is infinite with probability 1. With reference to the so-called Feller's test for explosions (see Theorem 5.5.29 in Karatzas and Shreve [22]), we therefore make the following assumption.

Assumption 2.2 If we define

$$l(x) = \int_c^x [p(x) - p(y)]\,m(dy), \quad \text{for } x \in \mathcal{J},$$

then $\lim_{x\downarrow\alpha} l(x) = \lim_{x\uparrow\beta} l(x) = \infty$.

To proceed further, we consider a weak solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ to the SDE (7). Given a point $a \in \mathcal{J}$, we denote by T_a the first hitting time of the set $\{a\}$, i.e.,

$$T_a = \inf\{t \geq 0 \mid \ X_t = a\}. \tag{17}$$

Our assumptions imply that

$$\mathbb{P}_x(T_a < \infty) > 0, \quad \text{for all } x, a \in \mathcal{J},$$

i.e., the diffusion X is *regular*. Also, we define

$$\Lambda_t = \int_0^t r(X_s)\,ds, \quad \text{for } t \geq 0.$$

The process Λ is well-defined thanks to the following assumption that we make.

Assumption 2.3 The function $r : \mathcal{J} \to]0, \infty[$ is Borel-measurable and locally bounded. Also, there exists $r_0 > 0$ such that $r(x) \geq r_0$, for all $x \in \mathcal{J}$.

In the presence of Assumptions 2.1, 2.2 and 2.3, the general solution to the homogeneous ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x \in \mathcal{J}. \tag{18}$$

exists in the classical sense and is given by

$$w(x) = A\phi(x) + B\psi(x), \tag{19}$$

for some constants $A, B \in \mathbb{R}$. The functions ϕ and ψ are C^1, their first derivatives are absolutely continuous functions,

$$0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0, \quad \text{for all } x \in \mathcal{J}, \tag{20}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0, \quad \text{for all } x \in \mathcal{J}, \tag{21}$$

and

$$\lim_{x \downarrow \alpha} \phi(x) = \lim_{x \uparrow \beta} \psi(x) = \infty. \tag{22}$$

These functions are unique, modulo multiplicative constants. Also, they satisfy

$$\phi(x) = \phi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right], \text{ for all } y < x, \tag{23}$$

and

$$\psi(x) = \psi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right], \text{ for all } x < y, \tag{24}$$

where T_y is the first hitting time of $\{y\}$ defined by (17) above.

Remark 2.4 Note that Assumption 2.3 implies

$$\lim_{t\to\infty} e^{-\Lambda_t}|g(X_t)|\mathbf{1}_{\{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}=\infty\}} = 0.$$

for all points $\bar{\alpha}, \bar{\beta} \in \mathcal{J}$ such that $\bar{\alpha} < x < \bar{\beta}$ and all locally bounded functions $g : \mathcal{J} \to \mathbb{R}$. We can therefore define $e^{-\Lambda_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}}}|g(X_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}})| = \lim_{t\to\infty} e^{-\Lambda_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}\wedge t}}|g(X_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}\wedge t})|$.

The following result, which we will need, is a straightforward consequence of the probabilistic expressions (23)–(23).

Lemma 2 Suppose that Assumptions 2.1–2.3 hold true, and fix any initial condition $x \in \mathcal{J}$ and any weak solution $\mathbb{S}_x$ to the SDE (7). Also, consider any strictly decreasing sequence (α_m) and any strictly increasing sequence (β_n) such that

$$\alpha_1 < x < \beta_1, \quad \lim_{m\to\infty} \alpha_m = \alpha \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \beta. \tag{25}$$

If $g : \mathcal{J} \to \mathbb{R}$ is a locally bounded function satisfying

$$\lim_{x\downarrow\alpha} \frac{|g(x)|}{\phi(x)} = \lim_{x\uparrow\beta} \frac{|g(x)|}{\psi(x)} = 0,$$

then

$$\lim_{m,n\to\infty} \mathbb{E}\left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}}} \left|g(X_{T_{\alpha_m}\wedge T_{\beta_n}})\right|\right] = 0.$$

Proof. Using (20)–(24), we calculate

$$\begin{aligned}
&\lim_{m,n\to\infty} \mathbb{E}\left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}}} \left|g(X_{T_{\alpha_m}\wedge T_{\beta_n}})\right|\right] \\
&\qquad \le \lim_{m,n\to\infty} \left\{|g(\alpha_m)|\,\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}}}\right] + |g(\beta_n)|\,\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\right]\right\} \\
&\qquad = \lim_{m\to\infty} \phi(x)\frac{|g(\alpha_m)|}{\phi(\alpha_m)} + \lim_{n\to\infty} \psi(x)\frac{|g(\beta_n)|}{\psi(\beta_n)} \\
&\qquad = 0,
\end{aligned}$$

and the result follows. □

If α is an *entrance* boundary point, i.e., if $\lim_{x\downarrow\alpha} \mathbb{P}_x(T_a < \infty) > 0$, for some $a \in \mathcal{J}$, then $\psi(\alpha) := \lim_{x\downarrow\alpha} \psi(x) > 0$, otherwise, if α is a *natural* boundary point, $\psi(\alpha) := \lim_{x\downarrow\alpha} \psi(x) = 0$. Similarly, if β is an entrance boundary point, then $\phi(\beta) := \lim_{x\uparrow\beta} \phi(x) > 0$, while, if β is a natural boundary point, $\phi(\beta) := \lim_{x\uparrow\beta} \phi(x) = 0$. Furthermore, the scale function p admits the expression

$$p'(x) = \frac{\phi(x)\psi'(x) - \phi'(x)\psi(x)}{C}, \quad \text{for all } x \in \mathcal{J}, \tag{26}$$

where

$$C = \phi(c)\psi'(c) - \phi'(c)\psi(c) > 0. \tag{27}$$

Apart from Lemma 2, all of these results concerning the functions ϕ and ψ can be found in several references, including Borodin and Salminen [7], Itô and McKean [16], and Rogers and Williams [30].

Now, we consider measures h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ such that

$$\int_{[\bar{\alpha},\bar{\beta}]} \frac{1}{\sigma^2(x)p'(x)}\,|h|(dx) < \infty, \quad \text{for all } \alpha < \bar{\alpha} < \bar{\beta} < \beta, \tag{28}$$

where $|h|$ is the total variation measure of h, and we fix any point $\gamma \in \mathcal{J}$.

Definition 2.5 The space $\mathcal{I}_{\phi,\psi}$ of (ϕ,ψ)-integrable measures is defined to be the set of all measures h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ such that

$$\int_{]\alpha,\gamma[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,|h|(ds) + \int_{[\gamma,\beta[} \frac{\phi(s)}{\sigma^2(s)p'(s)}\,|h|(ds) < \infty.$$

Definition 2.6 The space $\Sigma_{\phi,\psi}$ of (ϕ,ψ)-sumable measures is defined to be the set of all measures h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ satisfying (28) and such that the limits

$$\lim_{x\downarrow\alpha} \int_{]x,\gamma[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,h(ds) \quad \text{and} \quad \lim_{x\uparrow\beta} \int_{[\gamma,x[} \frac{\phi(s)}{\sigma^2(s)p'(s)}\,h(ds)$$

exist in $\mathbb{R}$.

Remark 2.7 It is worth noting that the definitions of $\mathcal{I}_{\phi,\psi}$ and $\Sigma_{\phi,\psi}$ do *not* depend on the choice of the point $\gamma \in \mathcal{J}$. Also, we plainly have $\mathcal{I}_{\phi,\psi} \subseteq \Sigma_{\phi,\psi}$. In fact, this inclusion is strict (see Example 2.8 below). However, we should note that, if h is a positive measure, then $h \in \Sigma_{\phi,\psi}$ if and only if $h \in \mathcal{I}_{\phi,\psi}$.

Example 2.8 Suppose that $\mathcal{J} =]0,\infty[$, $b \equiv 0$ and $\sigma(x) = \sqrt{2}x$, so that the Itô diffusion X is a geometric Brownian motion, and that $r(x) = 2$. In this context, we can see that

$$\phi(x) = x^{-1}, \quad \psi(x) = x^2 \quad \text{and} \quad p'(x) = 1, \tag{29}$$

where we have taken $c = 1$ for the definition (14) of the scale function p. Also, consider the measure h defined by

$$h(\Gamma) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \mathbf{1}_\Gamma\left(\frac{1}{2k+1}\right) - \sum_{k=1}^{\infty} \frac{1}{2k} \mathbf{1}_\Gamma\left(\frac{1}{2k}\right), \tag{30}$$

for $\Gamma \in \mathcal{B}(]0,\infty[)$. In view of (29), we calculate

$$\int_{]0,1]} \frac{2\psi(s)}{\sigma^2(s)p'(s)}\,|h|(ds) = |h|(]0,1]) = \sum_{k=0}^{\infty} \frac{1}{k} = \infty,$$

and

$$\lim_{x\downarrow 0} \int_{]x,1]} \frac{2\psi(s)}{\sigma^2(s)p'(s)}\,h(ds) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2.$$

However, combining these calculations with the fact that the support of the measure h is included in the interval $]0,1]$, we can see that $h \in \Sigma_{\phi,\psi}$ and that $h \notin \mathcal{I}_{\phi,\psi}$.

3 Analytic characterisation of a special solution to the ODE (1)

The purpose of this section is to study analytic properties of an appropriate special solution to the ODE (1). To this end, we consider a measure $h \in \Sigma_{\phi,\psi}$, and we define the function $Q_h : \mathcal{J} \to \mathbb{R}$ by

$$Q_h(x) = \phi(x) \int_{]\alpha,x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\,h(ds) + \psi(x) \int_{[x,\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\,h(ds). \quad (31)$$

The next result is concerned with showing that this function is a special solution to the ODE (1).

Proposition 3 Suppose that Assumptions 2.1–2.3 are satisfied. Given a measure $h \in \Sigma_{\phi,\psi} \supset \mathcal{I}_{\phi,\psi}$, the function Q_h defined by (31) is the difference of two convex functions,

$$(Q_h)'_-(x) = \phi'(x) \int_{]\alpha,x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\,h(ds) + \psi'(x) \int_{[x,\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\,h(ds), \quad (32)$$

and Q_h is a special solution to the ODE (1) in the sense of Definition 1.1. Furthermore, the operator $h \mapsto Q_h$ mapping $\Sigma_{\phi,\psi}$ into the set of all real-valued functions on $\mathcal{J}$ that are differences of two convex functions is positive, i.e.,

$$Q_h \geq 0, \quad \text{for all positive } h \in \Sigma_{\phi,\psi}, \quad (33)$$

and linear, i.e.,

$$Q_{a_1h_1+a_2h_2} = a_1Q_{h_1} + a_2Q_{h_2}, \quad \text{for all } a_1, a_2 \in \mathbb{R} \text{ and } h_1, h_2 \in \Sigma_{\phi,\psi}. \tag{34}$$

Proof. Recalling that h satisfies (28), we define the left-continuous function $H : \mathcal{J} \to \mathbb{R}$ by $H(\gamma) = 0$,

$$H(x) = -\int_{]x,\gamma[} \frac{2}{C\sigma^2(s)p'(s)}\, h(ds), \quad \text{if } x \in]\alpha, \gamma[, \tag{35}$$

and

$$H(x) = \int_{[\gamma,x[} \frac{2}{C\sigma^2(s)p'(s)}\, h(ds), \quad \text{if } x \in]\gamma, \beta[, \tag{36}$$

where the constant $C > 0$ is as in (27). Now consider any points $\bar{\alpha}, \bar{\beta} \in \mathcal{J}$ such that $\bar{\alpha} < \bar{\beta}$. Using the integration by parts formula, we calculate

$$-H(\bar{\alpha})\psi(\bar{\alpha}) - \int_{\bar{\alpha}}^{x} \psi'(s)H(s)\, ds = -H(x)\psi(x) + \int_{[\bar{\alpha},x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds), \tag{37}$$

$$H(\bar{\beta})\phi(\bar{\beta}) - \int_{x}^{\bar{\beta}} \phi'(s)H(s)\, ds = H(x)\phi(x) + \int_{[x,\bar{\beta}[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds). \tag{38}$$

If we define the function $\Theta_h^{\bar{\alpha},\bar{\beta}} : [\bar{\alpha}, \bar{\beta}] \to \mathbb{R}$ by

$$\begin{aligned} \Theta_h^{\bar{\alpha},\bar{\beta}}(x) = {} & \left[\int_{]\alpha,\bar{\alpha}[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds) - H(\bar{\alpha})\psi(\bar{\alpha})\right] \phi(x) \\ & + \left[\int_{[\bar{\beta},\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds) + H(\bar{\beta})\phi(\bar{\beta})\right] \psi(x) \\ & - \phi(x) \int_{\bar{\alpha}}^{x} \psi'(s)H(s)\, ds - \psi(x) \int_{x}^{\bar{\beta}} \phi'(s)H(s)\, ds, \end{aligned} \tag{39}$$

then these expressions imply

$$Q_h(x) = \Theta_h^{\bar{\alpha},\bar{\beta}}(x), \quad \text{for all } x \in [\bar{\alpha}, \bar{\beta}] \tag{40}$$

In view of (26) and the fact that H is left-continuous, we can see that the left-hand side first derivative of $\Theta_h^{\bar{\alpha},\bar{\beta}}$ is given by

$$\begin{aligned}(\Theta_h^{\bar{\alpha},\bar{\beta}})'_-(x) = {} & \left[\int_{]\alpha,\bar{\alpha}[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds) - H(\bar{\alpha})\psi(\bar{\alpha})\right]\phi'(x) \\ & + \left[\int_{[\bar{\beta},\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds) + H(\bar{\beta})\phi(\bar{\beta})\right]\psi'(x) - Cp'(x)H(x) \\ & - \phi'(x)\int_{\bar{\alpha}}^{x} \psi'(s)H(s)\, ds - \psi'(x)\int_{x}^{\bar{\beta}} \phi'(s)H(s)\, ds. \qquad (41)\end{aligned}$$

This calculation, (26), the integration by parts formulae (37)–(38) and the identity (40) imply (32). From this expression, we can also see that $(\Theta_h^{\bar{\alpha},\bar{\beta}})'_-$ is locally bounded and that the second distributional derivative of $\Theta_h^{\bar{\alpha},\bar{\beta}}$ is a measure. Furthermore, if

$$(\Theta_h^{\bar{\alpha},\bar{\beta}})''(dx) = (\Theta_h^{\bar{\alpha},\bar{\beta}})''_{\mathrm{ac}}(x)\, dx + (\Theta_h^{\bar{\alpha},\bar{\beta}})''_{\mathrm{s}}(dx) \qquad (42)$$

is the Lebesgue decomposition of the second distributional derivative $(\Theta_h^{\bar{\alpha},\bar{\beta}})''(dx)$ (see (2)), then we can calculate

$$\begin{aligned}(\Theta_h^{\bar{\alpha},\bar{\beta}})''_{\mathrm{ac}}(x) = {} & \left[\int_{]\alpha,\bar{\alpha}[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds) - H(\bar{\alpha})\psi(\bar{\alpha})\right]\phi''(x) \\ & + \left[\int_{[\bar{\beta},\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds) + H(\bar{\beta})\phi(\bar{\beta})\right]\psi''(x) \\ & - Cp''(x)H(x) - \frac{2h_{\mathrm{ac}}(x)}{\sigma^2(x)} \\ & - \phi''(x)\int_{\bar{\alpha}}^{x} \psi'(s)H(s)\, ds - \psi''(x)\int_{x}^{\bar{\beta}} \phi'(s)H(s)\, ds.\end{aligned}$$

and

$$(\Theta_h^{\bar{\alpha},\bar{\beta}})''_{\mathrm{s}}(dx) = -\frac{2}{\sigma^2(x)}\, h_{\mathrm{s}}(dx).$$

Now, it is straightforward to combine these expressions, (39) and (41) with the fact that the scale function p satisfies the ODE (16) and the fact that the functions ϕ, ψ are classical solutions to the homogeneous ODE (18) to see that $\Theta_h^{\bar{\alpha},\bar{\beta}}$ satisfies the ODE (1) inside the interval $]\bar{\alpha},\bar{\beta}[$. However, this observation (40) and the fact that $\bar{\alpha}$, $\bar{\beta}$ are arbitrary points in $\mathcal{J}$ imply that Q_h satisfies the ODE (1) inside $\mathcal{J}$ in the sense of Definition 1.1. Finally, (33) and (34) follow immediately from the definition of Q_h. □

Establishing conditions under which the solution to the ODE (1) that we have derived above is a monotone or a bounded function is an issue that is most important for applications. To this end, we define the positive measure $r_{\rm m}$ on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$ by

$$r_{\rm m}(dx) = r(x)\,dx, \tag{43}$$

and we consider the following definitions.

Definition 3.1 A measure h on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$ is r-increasing if, given any $x \in \mathcal{J}$, there exists a constant K_x such that the restriction of the measure $-h + K_x r_{\rm m}$ in $(]\alpha,x],\mathcal{B}(]\alpha,x]))$ and the restriction of the measure $h - K_x r_m$ in $([x,\beta[,\mathcal{B}([x,\beta[))$ both are positive measures.

The measure h is r-decreasing if $-h$ is r-increasing.

Definition 3.2 The r-supremum and the r-infimum of a measure h on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$ over a set $\Gamma \in \mathcal{B}(\mathcal{J})$ are defined by

$$\begin{aligned} r\text{-sup}_\Gamma\, h = \inf\bigl\{K \in \mathbb{R} \mid &\text{ the restriction of } -h + K r_{\rm m} \\ &\text{ in } (\Gamma,\mathcal{B}(\Gamma)) \text{ is a positive measure}\bigr\}, \\ r\text{-inf}_\Gamma\, h = \sup\bigl\{K \in \mathbb{R} \mid &\text{ the restriction of } h - K r_{\rm m} \\ &\text{ in } (\Gamma,\mathcal{B}(\Gamma)) \text{ is a positive measure}\bigr\}, \end{aligned}$$

with the usual conventions $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$.

Definition 3.3 A measure h on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$ is r-convergent in $\mathbb{R}$ at α if there exists a constant $r\text{-lim}_\alpha\, h \in \mathbb{R}$ such that, for all $\varepsilon > 0$, there exists $\alpha_\varepsilon \in \mathcal{J}$ such that the restrictions of the measures $h - (r\text{-lim}_\alpha\, h - \varepsilon)\, r_{\rm m}$ and $-h + (r\text{-lim}_\alpha\, h + \varepsilon)\, r_{\rm m}$ in $(]\alpha,\alpha_\varepsilon[,\mathcal{B}(]\alpha,\alpha_\varepsilon[)$ both are positive measures.

A measure h on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$ is r-convergent to ∞ at α, in which case we write $r\text{-lim}_\alpha\, h = \infty$, if, for all $K > 0$, there exists $\alpha_K \in \mathcal{J}$ such that the restriction of the measure $h - K r_{\rm m}$ in $(]\alpha,\alpha_K[,\mathcal{B}(]\alpha,\alpha_K[)$ is a positive measure.

A measure h on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$ is r-convergent to $-\infty$ at α, in which case we write $r\text{-lim}_\alpha\, h = -\infty$, if $r\text{-lim}_\alpha(-h) = \infty$.

The r-limits of a measure h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ at β are defined in a similar way.

Remark 3.4 below provides the intuition behind these definitions.

The following result is concerned with the issues considered above and with several other properties of the solution Q_h to the ODE (1) that are of interest in applications.

Proposition 4 Suppose that Assumptions 2.1–2.3 hold true, and consider any measure $h \in \Sigma_{\phi,\psi} \supset \mathcal{I}_{\phi,\psi}$. The function Q_h given by (31) satisfies

$$\lim_{x \downarrow \alpha} \frac{|Q_h(x)|}{\phi(x)} = \lim_{x \uparrow \beta} \frac{|Q_h(x)|}{\psi(x)} = 0, \tag{44}$$

and

$$r\text{-}\inf_{\mathcal{J}} h \leq Q_h(x) \leq r\text{-}\sup_{\mathcal{J}} h, \tag{45}$$

$$\phi(x)(Q_h)'_-(x) - \phi'(x) Q_h(x) = p'(x) \int_{[x,\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds), \tag{46}$$

$$\psi(x)(Q_h)'_-(x) - \psi'(x) Q_h(x) = -p'(x) \int_{]\alpha,x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds), \tag{47}$$

for all $x \in \mathcal{J}$. If r_{m} is the measure on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ defined by (43), then

$$Q_{r_{\mathrm{m}}}(x) = 1, \quad \text{for all } x \in \mathcal{J}. \tag{48}$$

If h is r-increasing (resp., r-decreasing) in the sense of Definition 3.1 and $|h|(\mathcal{J}) > 0$, then Q_h is strictly increasing (resp., strictly decreasing). Furthermore, if α (resp., β) is a natural boundary point and h is r-convergent at α (resp., at β) in the sense of Definition 3.3, then

$$\lim_{x \downarrow \alpha} Q_h(x) = r\text{-}\lim_{\alpha} h \quad \left(\text{resp., } \lim_{x \uparrow \beta} Q_h(x) = r\text{-}\lim_{\beta} h \right). \tag{49}$$

Proof. We can verify (46) and (47) by a straightforward calculation involving the definition (31) of Q_h and (26). To prove (48), we first note that (20), (21) and (26) imply

$$0 < \frac{\phi(x)\psi'(x)}{Cp'(x)} < 1 \quad \text{and} \quad 0 < -\frac{\phi'(x)\psi(x)}{Cp'(x)} < 1.$$

Combining these inequalities with (22), we can see that

$$\lim_{x\downarrow\alpha}\frac{\psi'(x)}{p'(x)}=\lim_{x\uparrow\beta}\frac{\phi'(x)}{p'(x)}=0. \tag{50}$$

Now, the fact that p' satisfies the ODE (16) and the fact that ϕ satisfies the ODE (18) imply

$$\begin{aligned}\frac{d}{dx}\left(\frac{\phi'(x)}{p'(x)}\right)&=\frac{2}{\sigma^2(x)p'(x)}\left[\frac{1}{2}\sigma^2(x)\phi''(x)+b(x)\phi'(x)\right]\\&=\frac{2r(x)\phi(x)}{\sigma^2(x)p'(x)}.\end{aligned}$$

Similarly, we can show that

$$\frac{d}{dx}\left(\frac{\psi'(x)}{p'(x)}\right)=\frac{2r(x)\psi(x)}{\sigma^2(x)p'(x)}.$$

In view of these calculations and the continuity of the functions ϕ', ψ' and p', we can see that

$$Q_{r_{\mathrm{m}}}(x)=\phi(x)\int_\alpha^x\frac{1}{C}\,d\left(\frac{\psi'(s)}{p'(s)}\right)+\psi(x)\int_x^\beta\frac{1}{C}\,d\left(\frac{\phi'(s)}{p'(s)}\right).$$

However, this expression, (50) and (26) imply (48).

To establish (45), suppose, without loss of generality, that $r\text{-}\inf_{\mathcal{J}} h>-\infty$, and let any $K\le r\text{-}\inf_{\mathcal{J}} h$. In view of Definition 3.2, $h(dz)-Kr(z)\,dz$ is a positive measure on $(\mathcal{J},\mathcal{B}(\mathcal{J}))$. Combining this observation and (48) with the linearity and positivity of $h\mapsto Q_h$ (see (33) and (34) in Proposition 3), we can see that

$$Q_h(x)-K=Q_{h-Kr_m}(x)\ge 0,\quad\text{for all } x\in\mathcal{J}.$$

It follows that $\inf_{x\in\mathcal{J}}Q_h(x)\ge r\text{-}\inf_{\mathcal{J}} h$. Similarly, we can show that $\sup_{x\in\mathcal{J}}Q_h(x)\le r\text{-}\sup_{\mathcal{J}} h$, and (45) follows.

Now, let us assume that the measure h is r-increasing and $|h|(\mathcal{J})>0$. Given $x\in\mathcal{J}$, let K_x be a constant as in Definition 3.1. Using (48), which implies that $(Q_{r_{\mathrm{m}}})'_-(x)\equiv 0$, the linearity of the operator $h\mapsto Q_h$ and (32), we can see that

$$\begin{aligned}(Q_h)'_-(x) &= \left(Q_h - Q_{K_x r_\mathrm{m}}\right)'_-(x) \\ &= \left(Q_{h-K_x r_\mathrm{m}}\right)'_-(x) \\ &= \phi'(x) \int_{]\alpha,x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\,(h - K_x r_\mathrm{m})(ds) \\ &\quad + \psi'(x) \int_{[x,\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\,(h - K_x r_\mathrm{m})(ds) \\ &> 0,\end{aligned}$$

the inequality following because the restriction of $-h + K_x r_\mathrm{m}$ in $(]\alpha, x], \mathcal{B}(]\alpha, x]))$ and the restriction of $h - K_x r_\mathrm{m}$ in $([x, \beta[, \mathcal{B}([x, \beta[))$ both are positive measures (see Definition 3.1), and because $\phi' < 0 < \psi'$. However, this calculation implies that Q_h is increasing.

To prove (44), we define

$$H_\psi(x) = -\int_{]x,\gamma]} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,h(ds), \quad \text{for } x \in]\alpha, \gamma[,$$

and we note that Definition 2.6 implies that

$$H_\psi(\alpha) := \lim_{x\downarrow\alpha} H_\psi(x) \quad \text{exists in } \mathbb{R}. \tag{51}$$

Using this definition and the integration by parts formula we calculate

$$\begin{aligned}&\frac{\psi(x)}{\phi(x)} \int_{]x,\gamma]} \frac{\phi(s)}{\sigma^2(s)p'(s)}\,h(ds) \\ &\quad = \frac{\psi(x)}{\phi(x)} \int_{]x,\gamma]} \frac{\phi(s)}{\psi(s)}\,dH_\psi(s) \\ &\quad = \frac{\psi(x)}{\phi(x)} \frac{\phi(\gamma)}{\psi(\gamma)} H_\psi(\gamma) - H_\psi(x) - \frac{\psi(x)}{\phi(x)} \int_x^\gamma H_\psi(s)\,d\frac{\phi(s)}{\psi(s)}.\end{aligned} \tag{52}$$

Now, consider any $y \in]\alpha, \gamma[$, and note that (20) and (21) imply

$$\frac{d}{dx}\frac{\psi(x)}{\phi(x)} = -\frac{\phi'(x)\psi(x) - \phi(x)\psi'(x)}{\phi^2(x)} < 0,$$

In light of this observation, we calculate

$$\begin{aligned}
&\limsup_{x\downarrow\alpha}\left[-\frac{\psi(x)}{\phi(x)}\int_x^\gamma H_\psi(s)\,d\frac{\phi(s)}{\psi(s)}\right]\\
&\le \limsup_{x\downarrow\alpha}\left[-\left(\frac{\psi(x)\phi(y)}{\phi(x)\psi(y)}-1\right)\sup_{s\in]\alpha,y]}H_\psi(s)-\frac{\psi(x)}{\phi(x)}\int_y^\gamma H_\psi(s)\,d\frac{\phi(s)}{\psi(s)}\right]\\
&=\sup_{s\in]\alpha,y]}H_\psi(s),
\end{aligned}$$

the equality following because $\lim_{x\downarrow\alpha}\psi(x)/\phi(x)=0$ (see (20)–(22)). By passing to the limit $y\downarrow\alpha$ in this inequality, we can see that

$$\limsup_{x\downarrow\alpha}\left[-\frac{\psi(x)}{\phi(x)}\int_x^\gamma H_\psi(s)\,d\frac{\phi(s)}{\psi(s)}\right]\le\limsup_{x\downarrow\alpha}H_\psi(x)=H_\psi(\alpha),$$

thanks to (51). Using symmetric arguments, we can also see that

$$\liminf_{x\downarrow\alpha}\left[-\frac{\psi(x)}{\phi(x)}\int_x^\gamma H_\psi(s)\,d\frac{\phi(s)}{\psi(s)}\right]\ge H_\psi(\alpha).$$

It follows that

$$\lim_{x\downarrow\alpha}\left[-\frac{\psi(x)}{\phi(x)}\int_x^\gamma H_\psi(s)\,d\frac{\phi(s)}{\psi(s)}\right]=H_\psi(\alpha)\equiv\lim_{x\downarrow\alpha}H_\psi(x).$$

However, this conclusion, (52) and the fact that $\lim_{x\downarrow\alpha}\phi(x)/\psi(x)$ imply

$$\lim_{x\downarrow\alpha}\frac{\psi(x)}{\phi(x)}\int_{]x,\gamma]}\frac{\phi(s)}{\sigma^2(s)p'(s)}\,h(ds)=0.$$

Combining this limit with the fact that

$$\lim_{x\downarrow\alpha}\int_{]\alpha,x[}\frac{2\psi(s)}{C\sigma^2(s)p'(s)}\,h(ds)=0$$

and the definition (31) of Q_h, we can see that $\lim_{x\downarrow\alpha}Q_h(x)/\phi(x)=0$, and (44) follows because $\phi>0$. Showing that $\lim_{x\uparrow\beta}|Q_h(x)|/\psi(x)=0$ involves similar arguments.

Finally, suppose that α is a natural boundary point, so that $\lim_{x\downarrow\alpha}\psi(x)=0$, and that h is r-convergent in $\mathbb{R}$ at α. Also, fix any $\varepsilon>0$, and let any $\alpha_\varepsilon\in\mathcal{J}$ such that the restriction of the measures $h-(r\text{-}\lim_\alpha h-\varepsilon)\,r_{\mathrm{m}}$ and

$-h + (r\text{-}\lim_\alpha h + \varepsilon)\, r_{\mathrm{m}}$ in $(]\alpha, \alpha_\varepsilon[, \mathcal{B}(]\alpha, \alpha_\varepsilon[))$ both are positive measures (see Definition 3.3). In this context, we calculate

$$
\begin{aligned}
&\limsup_{x\downarrow\alpha} Q_h(x) - r\text{-}\lim_\alpha h - \varepsilon \\
&\qquad = \limsup_{x\downarrow\alpha} Q_{h-(r\text{-}\lim_\alpha h+\varepsilon)r_{\mathrm{m}}}(x) \\
&\qquad = \limsup_{x\downarrow\alpha} \biggl[\phi(x) \int_{]\alpha,x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, \left[h - (r\text{-}\lim_\alpha h + \varepsilon)\, r_{\mathrm{m}}\right](ds) \\
&\qquad\quad + \psi(x) \int_{[x,\alpha_\varepsilon[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, \left[h - (r\text{-}\lim_\alpha h + \varepsilon)\, r_{\mathrm{m}}\right](ds) \\
&\qquad\quad + \psi(x) \int_{[\alpha_\varepsilon,\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, \left[h - (r\text{-}\lim_\alpha h + \varepsilon)\, r_{\mathrm{m}}\right](ds)\biggr] \\
&\qquad \le \limsup_{x\downarrow\alpha} \psi(x) \int_{[\alpha_\varepsilon,\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, \left[h - (r\text{-}\lim_\alpha h + \varepsilon)\, r_{\mathrm{m}}\right](ds) \\
&\qquad = 0.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\limsup_{x\downarrow\alpha} Q_h(x) \le r\text{-}\lim_\alpha h$. Similarly, we can show that $\liminf_{x\downarrow\alpha} Q_h(x) \ge r\text{-}\lim_\alpha h$, and thus establish that $\lim_{x\downarrow\alpha} Q_h(x) = r\text{-}\lim_\alpha h$. Also, we can prove that $\lim_{x\downarrow\alpha} Q_h(x) = r\text{-}\lim_\alpha h$ when $r\text{-}\lim_\alpha h = \infty$ or $r\text{-}\lim_\alpha h = -\infty$ using similar arguments. □

Remark 3.4 Definitions 3.1, 3.2 and 3.3 take very simple forms if the measure h is absolutely continuous, i.e., if $h_{\mathrm{s}} \equiv 0$ in the Lebesgue decomposition (3) of h. Indeed, in this case, h is r-increasing (resp., r-decreasing) if and only if h_{ac}/r is an increasing (resp., decreasing) function, and

$$
r\text{-}\sup_\Gamma h = \sup_{x\in\Gamma} \frac{h_{\mathrm{ac}}(x)}{r(x)} \quad \text{and} \quad r\text{-}\inf_\Gamma h = \inf_{x\in\Gamma} \frac{h_{\mathrm{ac}}(x)}{r(x)}.
$$

Also, h is r-convergent at α (resp., at β) if h_{ac}/r converges as x tends to α (resp., β), in which case,

$$
r\text{-}\lim_\alpha h = \lim_{x\downarrow\alpha} \frac{h_{\mathrm{ac}}(x)}{r(x)}, \quad \left(\text{resp., } r\text{-}\lim_\beta h = \lim_{x\uparrow\beta} \frac{h_{\mathrm{ac}}(x)}{r(x)}\right).
$$

At this point, it is worth noting that, if α (resp., β) is not a natural boundary point but an entrance one, then (49) is not necessarily true. We substantiate this claim by means of Example 4.2 that we develop in the next section because it requires the probabilistic representation of the function Q_h that we develop there.

Remark 3.5 Given a measure h satisfying (28), the function $Q_h^{\bar{\alpha},\bar{\beta}} : \mathcal{J} \to \mathbb{R}$ given by

$$Q_h^{\bar{\alpha},\bar{\beta}}(x) = \phi(x) \int_{]\bar{\alpha},x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds) + \psi(x) \int_{[x,\bar{\beta}[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds), \quad (53)$$

for some $\alpha < \bar{\alpha} < \bar{\beta} < \beta$, is well-defined even when $h \notin \Sigma_{\phi,\psi}$. Furthermore, we can use the same arguments as the ones in the proof of Proposition 3 to show that $Q_h^{\bar{\alpha},\bar{\beta}}$ is the difference of two convex functions and is a special solution to the ODE (1). However, such a solution does not, in general, satisfy (44) in Proposition 4 anymore. To see this claim, it suffices to consider a measure $h \in \Sigma_{\phi,\psi}$ and observe that

$$Q_h^{\bar{\alpha},\bar{\beta}}(x) = Q_h(x) - \phi(x) \int_{]\alpha,\bar{\alpha}]} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\, h(ds) - \psi(x) \int_{[\bar{\beta},\beta[} \frac{2\phi(s)}{C\sigma^2(s)p'(s)}\, h(ds).$$

4 Probabilistic characterisation of a special solution to the ODE (1)

Throughout this section, we assume that an initial condition $x \in \mathcal{J}$ and a weak solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ to the SDE (7) are fixed, and we make no further reference to this setting. We also assume that the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)$ satisfies the usual conditions. In this probabilistic setting, we denote by L^z the local time process of X at level $z \in \mathcal{J}$. In particular, we consider a modification of the local time family $(L_t^z;\ z \in]\alpha, \beta[,\ t \geq 0)$ such that the mapping $(z,t) \mapsto L_t^z$ is bicontinuous $\mathbb{P}_x$-a.s. (see Revuz and Yor [29, Theorem VI.1.7]).

Given a measure h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ satisfying (28), we define the finite variation process A^h by

$$A_t^h = \int_\alpha^\beta \frac{L_t^z}{\sigma^2(z)}\, h(dz). \quad (54)$$

The assumption that X is non-explosive implies that $\alpha < \inf_{u \leq t} X_u < \sup_{u \leq t} X_u < \beta$, for all $t \geq 0$. Combining this observation with the fact that the measure dL_t^z is supported on the set $\{X_t = z\}$ and the continuity of $z \mapsto L_t^z$, we can see that

$$A_t^{|h|} \leq \sup_{z \in \mathcal{J}} L_t^z \int_{\inf_{u \leq t} X_u}^{\sup_{u \leq t} X_u} \frac{1}{\sigma^2(z)} \, |h|(dz) < \infty,$$

the second inequality following thanks to (28), and the continuity and strict positivity of p'. It follows that A_t^h is well-defined for all $t \geq 0$. Also, we note that A^h is continuous, $A_0^h = 0$, and, if h is a positive measure, then A^h is an increasing process. These properties follow because $L_0^z = 0$ and the local time process L^z is increasing and continuous, for all $z \in \mathcal{J}$. For future reference, we also observe that the mapping $h \mapsto A^h$ is linear, i.e.,

$$A^{a_1 h_1 + a_2 h_2} = a_1 A^{h_1} + a_2 A^{h_2}, \tag{55}$$

for all $a_1, a_2 \in \mathbb{R}$ and all measures h_1, h_2 on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ satisfying (28).

Remark 4.1 It is worth noting that, if the measure h is absolutely continuous, i.e., $h_{\mathrm{s}} \equiv 0$ in the Lebesgue decomposition of h in (3), then the occupation times formula (see Revuz and Yor [29, Corollary VI.1.6]) implies that the process A^h admits the expression

$$A_t^h = \int_0^t h_{\mathrm{ac}}(X_u) \, du.$$

Before addressing the main results in the section, we prove the following preliminary result.

Lemma 5 Suppose that Assumptions 2.1–2.3 hold true. Consider a measure h on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ satisfying (28) and let $w : \mathcal{J} \to \mathbb{R}$ be a function satisfying the ODE (1) in the sense of Definition 1.1. Given any points $\bar{\alpha}, \bar{\beta} \in \mathcal{J}$ such that $\bar{\alpha} < x < \bar{\beta}$, and any $(\mathcal{F}_t)$-stopping time υ,

$$\mathbb{E}_x \left[e^{-\Lambda_{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge \upsilon}} w(X_{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge \upsilon}) \right] = w(x) - \mathbb{E}_x \left[\int_0^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge \upsilon} e^{-\Lambda_t} \, dA_t^h \right], \tag{56}$$

where $T_{\bar{\alpha}}$ and $T_{\bar{\beta}}$ are the first hitting times of the sets $\{\bar{\alpha}\}$ and $\{\bar{\beta}\}$, respectively, defined by (17).

Proof. Since the function w is the difference of two convex functions, the Itô-Tanaka formula (e.g., see Revuz and Yor [29, Theorem VI.1.5]) implies

$$w(X_t) = w(x) + \int_0^t b(X_u) w'_-(X_u) \, du + \frac{1}{2} \int_\alpha^\beta L_t^z \, w''(dz) + \int_0^t \sigma(X_u) w'_-(X_u) \, dW_u.$$

In view of the Lebesgue decomposition (42) of the measure w'' as in (3) and the occupation times formula, we can see that

$$\int_\alpha^\beta L_t^z w''_{\mathrm{ac}}(z)\, dz = \int_0^t \sigma^2(X_u) w''_{\mathrm{ac}}(X_u)\, du.$$

It follows that

$$\begin{aligned} w(X_t) = w(x) &+ \int_0^t \left[\frac{1}{2}\sigma^2(X_u) w''_{\mathrm{ac}}(X_u) + b(X_u) w'_-(X_u)\right] du \\ &+ \frac{1}{2}\int_\alpha^\beta L_t^z\, w''_{\mathrm{s}}(dz) + \int_0^t \sigma(X_u) w'_-(X_u)\, dW_u. \end{aligned}$$

Now, using the integration by parts formula for semimartingales and the fact that w satisfies the ODE (1) in the sense of Definition 1.1, we obtain

$$\begin{aligned} e^{-\Lambda_t} w(X_t) = w(x) &+ \frac{1}{2}\int_0^t e^{-\Lambda_u} d\int_\alpha^\beta L_u^z\, w''_{\mathrm{s}}(dz) + M_t \\ &+ \int_0^t e^{-\Lambda_u}\left[\frac{1}{2}\sigma^2(X_u) w''_{\mathrm{ac}}(X_u) + b(X_u) w'_-(X_u) - r(X_u) w(X_u)\right] du \\ = w(x) &- \int_0^t e^{-\Lambda_u} h_{\mathrm{ac}}(X_u)\, du - \int_0^t e^{-\Lambda_u}\, d\int_\alpha^\beta \frac{L_u^z}{\sigma^2(z)}\, h_{\mathrm{s}}(dz) + M_t, \end{aligned}$$

where M is the stochastic integral defined by

$$M_t = \int_0^t e^{-\Lambda_u} \sigma(X_u) w'_-(X_u)\, dW_u. \tag{57}$$

With regard to the Lebesgue decomposition of the measure h in (3), the occupation times formula and the definition (54) of the process A^h, we can see that

$$e^{-\Lambda_t} w(X_t) = w(x) - \int_0^t e^{-\Lambda_u}\, dA_u^h + M_t. \tag{58}$$

In view of Assumption 2.3 and the local boundedness of the functions w'_- and σ^2 (see also (13) in Assumption 2.1), we can see that the stopped process

$M^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v}$ has quadratic variation that satisfies

$$\begin{aligned}\mathbb{E}_x \left[\langle M^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v} \rangle_\infty \right] &= \mathbb{E}_x \left[\int_0^\infty \mathbf{1}_{\{t \le T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v\}} \left[e^{-\Lambda_t} \sigma(X_t) w'_-(X_t) \right]^2 dt \right] \\ &\le \frac{1}{2r_0} \sup_{x \in [T_{\bar{\alpha}}, T_{\bar{\beta}}]} \left[\sigma(x) w'_-(x) \right]^2 \\ &< \infty.\end{aligned}$$

It follows that $M^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v}$ is a uniformly square integrable martingale, so

$$M_\infty^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v} := \lim_{t \to \infty} M_t^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v} \quad \text{exists, } \mathbb{P}_x\text{-a.s.,}$$

and

$$\mathbb{E}_x \left[M_\infty^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v} \right] \equiv \mathbb{E}_x \left[M_{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge v} \right] = 0.$$

In light of this observation and Remark 2.4, we can see that (58) implies that

$$\lim_{t \to \infty} \int_0^{T_{\bar{\alpha}} \wedge T_{\bar{\beta}} \wedge t} e^{-\Lambda_u} \, dA_u^h \quad \text{exists, } \mathbb{P}_x\text{-a.s.,}$$

and that (56) holds true. □

The following result provides a probabilistic characterisation of the space $\mathcal{I}_{\phi,\psi}$.

Proposition 6 Suppose that Assumptions 2.1–2.3 hold true. A measure $h \in \Sigma_{\phi,\psi}$ belongs to $\mathcal{I}_{\phi,\psi}$ if and only if

$$\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} \, dA_t^{|h|} \right] < \infty, \tag{59}$$

where the increasing process $A^{|h|}$ is defined as in (54). Given any $h \in \mathcal{I}_{\phi,\psi}$,

$$R_h(x) := \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} \, dA_t^h \right] = Q_h(x). \tag{60}$$

Furthermore, given a measure $h \in \mathcal{I}_{\phi,\psi}$, the solution $Q_h \equiv R_h$ to the ODE (1) satisfies Dynkin's formula, i.e., given any $(\mathcal{F}_t)$-stopping time υ,

$$\mathbb{E}_x\left[e^{-\Lambda_\upsilon} Q_h(X_\tau)\mathbf{1}_{\{\upsilon<\infty\}}\right] = Q_h(x) - \mathbb{E}_x\left[\int_0^\upsilon e^{-\Lambda_t}\, dA_t^h\right], \tag{61}$$

$$\mathbb{E}_x\left[e^{-\Lambda_\upsilon} Q_h(X_\upsilon)\mathbf{1}_{\{\upsilon<\infty\}}\right] = \mathbb{E}_x\left[\int_\upsilon^\infty e^{-\Lambda_t}\, dA_t^h\right], \tag{62}$$

as well as the strong transversality condition, i.e., given any sequence (υ_n) of $(\mathcal{F}_t)$-stopping times such that $\lim_{n\to\infty}\upsilon_n = \infty$,

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\upsilon_n}}\left|Q_h(X_{\upsilon_n})\right|\mathbf{1}_{\{\upsilon_n<\infty\}}\right] = 0. \tag{63}$$

Proof. Combining the definition of $\mathcal{I}_{\phi,\psi}$ with the linear dependence of the function Q_h on the measure h (see (34)), which implies that $|Q_h| \le Q_{|h|}$, and the linear dependence of the stochastic process A^h on the measure h (see (55)), we can see that the result will follow if we prove it for positive h. Therefore, it suffices to develop the proof under the assumptions that $Q_h \ge 0$ and that A^h is an increasing process.

Consider any sequences (α_m) and (β_n) in $\mathcal{J}$ being as in Lemma 2. Given a measure $h \in \Sigma_{\phi,\psi} \supset \mathcal{I}_{\phi,\psi}$, not necessarily positive, we can see that the fact that Q_h satisfies the ODE (1) in the sense of Definition 1.1 and Lemma 5 with $\upsilon = \infty$ imply

$$\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}}} Q_h(X_{T_{\alpha_m}\wedge T_{\beta_n}})\right] = Q_h(x) - \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}} e^{-\Lambda_t}\, dA_t^h\right].$$

In view of (44) and Lemma 2, we can pass to the limits $m, n \to \infty$ to obtain

$$Q_h(x) = \lim_{m,n\to\infty} \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}} e^{-\Lambda_t}\, dA_t^h\right]. \tag{64}$$

Now, if h is a positive measure satisfying (28), then A^h is an increasing process and the monotone convergence theorem implies

$$\lim_{m,n\to\infty} \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}} e^{-\Lambda_t}\, dA_t^h\right] = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\, dA_t^h\right]. \tag{65}$$

Recalling that every positive measure $h \in \Sigma_{\phi,\psi}$ belongs to $\mathcal{I}_{\phi,\psi}$ (see Remark 2.7), we can see that, if the right hand side of (65) is equal to ∞, then $h \notin \mathcal{I}_{\phi,\psi}$ because, otherwise, (64) and (65) imply $Q_h(x) = \infty$, which contradicts the fact that Q_h is real-valued for all $h \in \Sigma_{\phi,\psi} \supset \mathcal{I}_{\phi,\psi}$. It follows that a positive measure in $\Sigma_{\phi,\psi}$ belongs to $\mathcal{I}_{\phi,\psi}$ if and only if (59) is true. Furthermore, if $h \in \mathcal{I}_{\phi,\psi}$ is a positive measure, then (64) and (65) imply (60).

To prove (61) and (62), we note that Lemma 5 yields

$$\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}\wedge v}} Q_h(X_{T_{\alpha_m}\wedge T_{\beta_n}\wedge v})\right] = Q_h(x) - \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}\wedge v} e^{-\Lambda_t}\, dA_t^h\right], \tag{66}$$

where (α_m) and (β_n) are any sequences as in Lemma 2. An application of the monotone convergence theorem implies

$$\lim_{m,n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_v} Q_h(X_v)\mathbf{1}_{\{v\le T_{\alpha_m}\wedge T_{\beta_n}\}}\right] = \mathbb{E}_x\left[e^{-\Lambda_v} Q_h(X_v)\mathbf{1}_{\{v<\infty\}}\right]$$

and

$$\lim_{m,n\to\infty} \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}\wedge v} e^{-\Lambda_t}\, dA_t^h\right] = \mathbb{E}_x\left[\int_0^{v} e^{-\Lambda_t}\, dA_t^h\right],$$

while Lemma 2 implies

$$\begin{aligned}\lim_{m,n\to\infty} \mathbb{E}_x & \left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}}} Q_h(X_{T_{\alpha_m}\wedge T_{\beta_n}})\mathbf{1}_{\{T_{\alpha_m}\wedge T_{\beta_n}<v\}}\right] \\ & \le \lim_{m,n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}}} Q_h(X_{T_{\alpha_m}\wedge T_{\beta_n}})\right] \\ & = 0. \end{aligned} \tag{67}$$

In view of these observations, we can pass to the limits $m, n \to \infty$ in (66) to obtain (61). Finally, (62) follows from (60) and (61), while (63) follows from (60), (61) and the monotone convergence theorem. □

When the measure h belongs to $\Sigma_{\phi,\psi}$ but not to $\mathcal{I}_{\phi,\psi}$, we do not have a nice probabilistic characterisation of the solution Q_h to the ODE (1) such as the one in the previous proposition. However, we should observe the following result that is a restatement of (64).

Lemma 7 Suppose that Assumptions 2.1–2.3 are satisfied, and consider any measure $h \in \Sigma_{\phi,\psi}$. Given a strictly decreasing sequence (α_m) and a strictly increasing sequence (β_n) satisfying (25),

$$Q_h(x) = \lim_{m,n\to\infty} \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}} e^{-\Lambda_t}\, dA_t^h\right], \tag{68}$$

where T_{α_m} and T_{β_n} are the first hitting times defined as in (17).

Finally, we can consider the following example that highlights the importance of the boundary points α and β classification to the validity of (49) in Proposition 4.

Example 4.2 Suppose that $\mathcal{J} =]0, \infty[$, and that $b(x) = \kappa(\theta - x)$ and $\sigma(x) = \sigma\sqrt{x}$, for some constants $\kappa, \theta, \sigma > 0$ such that $\kappa\theta - \frac{1}{2}\sigma^2 > 0$. The associated Itô diffusion X is the square-root mean-reverting process appearing in the Cox-Ingersoll-Ross interest rate model. Also, 0 is an entrance boundary point, and, if we choose $r(x) = r_0$, for some constant $r_0 > 0$, then Assumptions 2.1–2.3 are all satisfied. It is a standard exercise to calculate $\mathbb{E}_x[X_t] = \theta + (x - \theta)e^{-\kappa t}$. In view of this calculation, we can see that if h is the measure defined by $h(dx) = x\,dx$, then

$$\lim_{x \downarrow 0} Q_h(x) \equiv \lim_{x \downarrow 0} R_h(x) = \frac{\theta\kappa}{r_0(r_0 + \kappa)} > 0 = \lim_{x \downarrow 0} \frac{h_{\mathrm{ac}}(x)}{r(x)} \equiv r\text{-}\lim_0 h.$$

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