THE OPTIMAL TIMING OF INVESTMENT DECISIONS

by

Timothy Charles Johnson.

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Department of Mathematics,

King's College London,

The Strand,

London,

WC2R 2LS.

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I affirm that the work contained in this thesis is my own.

Timothy Charles Johnson,
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Abstract

This thesis addresses the problem of the optimal timing of investment decisions. A number of models are formulated and studied. In these, an investor can enter an investment that pays a dividend, and has the possibility to leave the investment, either receiving or paying a fee. The objective is to maximise the expected discounted cashflow resulting from the investor's decision making over an infinite time horizon. The initialisation and abandonment costs, the discounting factor, and the running payoffs are all functions of a state process that is modelled by a general one-dimensional positive Itô diffusion. Sets of sufficient conditions that lead to results of an explicit analytic nature are identified. These models have numerous applications in finance and economics.

To address the family of models that we study, we first solve the discretionary stopping problem that aims at maximising the performance criterion

$$\mathbb{E}_x \left[e^{-\int_0^\tau r(X_s) ds} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right]$$

over all stopping times τ, where X is a general one-dimensional positive Itô diffusion, r is a strictly positive function and g is a given payoff function. Our analysis, which leads to results of an explicit analytic nature, is illustrated by a number of special cases that are of interest in applications, and aspects of which have been considered in the literature and we establish a range of results that can provide useful tools for developing the solution to other stochastic control problems.

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1. INTRODUCTION

We formulate and solve a number of stochastic optimisation problems that are concerned with the optimal timing of investment decisions. In the course of our analysis we investigate the solvability of an ordinary differential equation, which plays a fundamental role in solving the associated Hamilton-Jacobi-Bellman (HJB) equations, and we solve a discretionary stopping problem, which is closely related to our investment models.

An investment is characterised by making a known payment in order to receive an unknown cashflow in the future. The holder of an investment may relinquish the cashflow once it has been initiated, for example, if the payoff of stopping is greater than the expected value of future cashflows. A simple example could be the decision to buy an equity, which has a transaction cost. Holding the equity gives the investor a dividends stream. If the investor feels that the equity is undervalued, and the current market price is less than the net present value of future dividends, they would buy the equity. Alternatively, if the investor held the equity and it was overvalued, they would sell it. The investor could repeat the process any number of times. Another example could be the decision to build a production facility, which will have a cost but will provide an income based on the demand for its product. At some point in the future the demand for the product may fall or, equivalently, more producers may enter the economy, and so the cashflow generated by the facility becomes negative. At this point, the investor may be tempted to abandon the production facility, which could incur a cost. This type of decision may be only possible once. As the investor is under no obligation to either initiate an investment project or to abandon an existing project, we describe the problem as discretionary.

The motivation for the thesis came about whilst the author was working in industry and so called “real options” models were being promoted, with the classical real options model being that introduced by McDonald and Siegel [MS86]. The problem considered aims to choose the stopping time τ that maximises

$$\mathbb{E}_x \left[(e^{-r\tau} X_\tau - K) \right] .$$

The model addresses the question of when is it optimal to initiate an investment project, the value of which is modelled by the state process X and initiating the project incurs a cost K, while r is a discounting rate. This model is closely related to the widely studied perpetual American call option, first considered by Samuelson [Sam65] and McKean [McK65]. When X is modelled by a geometric Brownian motion a result of this model is that an investor in a project would either act immediately, or, wait forever. This strategy is counter-intuitive to managers, whose gut feeling tells them there are some projects that should be initialised at some trigger level. The author was interested in whether generalising the payoff, state process dynamics and discounting would result in more intuitive results, and how cases where a stopping boundary, the boundary between stopping and continuation regions, could be identified from the problem data. Another limitation of the approach taken by McDonald and Siegel was that X represents the net present value (the discounted sum of all future cashflows) of the project. It would be preferable to develop models that represented the whole life of the investment, of initiation, running and abandonment. Once the partially reversible problem of initiating and then abandoning a project is solved in a general setting, it becomes a relatively straightforward exercise to address the reversible investment problem in a general setting.

Models relating to single entry and exit problems have been studied in the context of real options by various authors. For example, Paddock, Siegel and Smith [PSS88] and Dixit and Pindyck [DP94, Sections 6.3, 7.1] adopt an economics perspective. More recent works include Knudsen, Meister and Zervos [KMZ98], who generalise Dixit and Pindyck’s model, but consider the abandonment problem alone, and Duckworth and Zervos [DZ00], who extend Knudsen, Meister and Zervos to include the initialisation problem. In fact, the model studied by Duckworth and Zervos [DZ00] can be seen as an extension of a fundamental real options problem introduced by McDonald and Siegel [MS86], which is concerned with determining the value of a firm when there is an option to shut down. McDonald and Siegel [MS86] implicitly considered the payoff being the discounted future cashflows of the project, Knudsen, Meister and Zervos [KMZ98] explicitly model these payoffs in solving the abandonment problem, and Duckworth and Zervos [DZ00] combine the two approaches. All these papers assume that the underlying state process is represented by a geometric Brownian motion, and that the entry and exit costs as well as the discounting rate are all constants.

With regard problems involving sequential entries and exits, Brekke and Øksendal [BØ94] analyse a general model, without providing explicit results. Duckworth and Zervos [DZ01] consider the special case where the state process is represented by a geometric Brownian motion, the entry and exit costs as well as the discounting rate are constants, and they provide explicit results. Other authors, such as Lumley and Zervos [LZ01], Hodges [Hod04], Pham [Pha] and Wang [Wan05], consider related problems.

The objective of this thesis is to study general investment models with a view to obtaining results of an explicit nature. We consider models where the state process driving the economy is modelled by general one-dimensional Itô diffusions, rather than specific diffusions such as a geometric Brownian motion. We assume that the costs associated with taking or leaving the investment and the cashflow of the investment are deterministic functions of the state process. In addition, there is discounting, which again may be state dependent, and the investor has an infinite time horizon. Within this general context, we consider two investment models. In the first one, only one entry and/or exit into the investment may be made, this is closely related to "real options" problems were the decision is not reversible. In the second one, any number of entries and exits can be made and is more closely related to general investment problems. In addressing our objective, results are obtained that are useful not only in addressing optimal stopping problems but also in solving more general stochastic control problems.

In the models we study there is some stochastic process that represents the state of the economy, such as the price of an equity or the demand for a product. We model this state process, X, by the one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt \;+\; \sigma(X_t)\,dW_t, \quad X_0 = x > 0,$$

where W is a standard, one-dimensional Brownian motion, and b, σ are given deterministic functions such that $X_t > 0$, for all $t > 0$, with probability 1. Generalising the existing theory so that it can account for stochastic processes modelled by general Itô diffusions is motivated by a wide range of practical applications. These include financial and economic applications where the assets exist in equilibrium market conditions and tend to fluctuate about some long-term mean level, rather than, on average, grow or fall exponentially, as modelled by a geometric Brownian motion. Such assets include interest rates, exchange rates and commodities, where there is empirical evidence of mean-reversion (e.g., see Metcalf and Hassett [MH95] and Sarkar [Sar03]). Considering general Itô diffusions also facilitates modelling of non-financial applications, such as those found in biological systems.

Introducing state dependent discounting enables a more realistic modelling framework of decisions. In the context of financial decision-making, the discounting rate accounts for the time-value of money, for the associated investment's depreciation rate and for the likelihood of the investment's default. In view of this observation, discounting should reflect the effect of the economic environment on the likelihood of default of an investment project, which may well be related to the underlying asset's value or demand. Specifically, if a firm's income relies on the price of one product, they will find their borrowing costs higher if the price of that product falls. In a biological setting, state dependent discounting reflects the dependence of extinction likelihood on the environment.

State dependent payoffs are motivated by a number of applications. First, they allow for utility based decision making, which, apart from the work of Henderson and Hobson [HH02], and despite its fundamental importance, has hardly found its way into real options theory. Second, they allow for financial modelling based on non-monetary state processes, for example, when the underlying process is the demand for a product, which may be appropriate if the demand for a product can be modelled more easily than the product's price. Third, state-dependent payoff functions are useful when dealing with cases where inputs are a finite resource. For example, consider the case where a financier has decided to invest in a widget production facility, because the price of widgets is high. In this situation, one would expect there other financiers to be investing in other widget production facilities, and, if widget producers are a scarce resource, their cost may go up. Within this general framework, we identify the investment strategies that are optimal, depending on the dynamics of the state process as well as the structure of the payoff functions and the discounting factor.

To address the investment problems that we study, we first solve the discretionary stopping problem that aims to maximise

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad \text{where} \quad \Lambda_t = \int_0^t r(X_s) ds,$$

over all stopping times τ, where r is strictly positive and g is a given payoff function. This stopping problem is related to perpetual American options whose payoff is given by g. One of the attractive features of perpetual options is that one can obtain explicit analytic expressions for their values and they are important in the theory of finance because their prices provide upper bounds for the corresponding finite maturity options. In addition, our analysis provides the prices of perpetual American "power" options, which have been studied in discrete time by Novikov and Shiryaev [NS04], for a range of underlying asset price dynamics.

The theory of optimal stopping has numerous applications and has attracted the interest of numerous researchers. Important, older accounts of this theory include Shiryaev [Shi78], El-Karoui [EK79] and Krylov [Kry80] , while more recent contributions include Salminen [Sal85], Davis and Karatzas [DK94], Øksendal and Reikvam [ØR98], Beibel and Lerche [BL00], Guo and Shepp [GS01], Dayanik and Karatzas [DK03] and Alvarez [Alv04]. We solve the problem that we consider by means of dynamic programming techniques, specifically we employ Bellman's principle to identify a variational inequality, we verify that the solution to this equation identifies with the value function of our stopping problem and we provide a solution to the variational inequality by appealing to the so-called smooth pasting principle. This is an approach that differs from the one taken by, for example Beibel and Lerche [BL00], who use a martingale technique to identify the free-boundary between the continuation and stopping region, or Dayanik and Karatzas [DK03] and Alvarez [Alv04], who employ r-excessive functions.

In order to address our investment models in a general setting the solution of the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) = 0, \quad x \in]0, \infty[.$$

needs to be understood. This understanding is developed in Chapter 2. Many of the key results, in Section 2.4, were developed in the course of studying the discretionary stopping problem presented in Chapter 3, and can be best appreciated in hindsight. In particular, the importance of the transversality condition, introduced in Section 3.2, in enabling explicit results to be obtained cannot be understated.

Once we have the results from Chapter 2, we solve the discretionary stopping problem in Chapter 3. We solve this stopping problem under general assumptions on the underlying state process X, the payoff function g and the discounting rate r. We consider a number of special cases in Section 3.4.

The results of Chapter 3, in turn, can be applied in answering the investment problems in Chapter 4. We presented two types of investment problems, in Section 4.3 we addressed cases where there the structure of the problem prevents decisions from being reversed, and they could all be re-cast as versions of the discretionary stopping problem studied in the preceding chapter. In Section 4.4 we considered the case where decisions could be reversed. However, even if a decision can be reversed, it may not be optimal to reverse a decision, and in these circumstances the problems reduce to the problems studied in Section 4.3.

2. STUDY OF AN ORDINARY DIFFERENTIAL EQUATION

2.1 Introduction

In this chapter we study an ODE that plays a fundamental role in our analysis in the following chapters. The results of this chapter have applications not only in the field of optimal stopping but also in more general control problems.

In Section 2.4 we study the solution to the non-homogeneous ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) = 0, \quad x \in \,]0, \infty[, \tag{2.1}$$

which is related to the variational inequalities we solve in addressing our investment models. Prior to addressing the non-homogeneous ODE we investigate the solution to the associated homogeneous ODE in Section 2.3. In Section 2.2 we consider a positive one-dimensional Itô diffusion that is closely related with the ODE that we use in solving our problems in Chapters 3 and 4.

Most of the results presented here have been established by Feller [Fel52] and can be found in various forms in several references that include Breiman [Bre68], Mandl [Man68], Itô and McKean [IM74], Karlin and Taylor [KT81], Rogers and Williams [RW94] and Borodin and Salminen [BS02]. Our presentation, which is based on modern probabilistic techniques, has largely been inspired by Rogers and Williams [RW94, Sections V.3, V.5, V.7] and includes ramifications not found in the literature.

2.2 The properties of the underlying diffusion

We consider a stochastic system, the state process X of which satisfies is modelled by the positive, one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{2.2}$$

where W is a one-dimensional standard Brownian motion and $b, \sigma :]0, \infty[\to \mathbb{R}$ are given deterministic functions satisfying the conditions (ND)$'$ and (LI)$'$ in Karatzas and Shreve [KS91, Section 5.5C], and given in the following assumption.

Assumption 2.2.1 *The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ satisfy the following conditions:*

$$\sigma^2(x) > 0, \text{ for all } x \in]0, \infty[,$$
$$\text{for all } x \in]0, \infty[, \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(s)|}{\sigma^2(s)}\,ds < \infty.$$

This assumption guarantees the existence of a unique, in the sense of probability law, solution to (2.2) up to an explosion time. In particular, given $c > 0$, the *scale function* p_c and the *speed measure* $m_c(dx)$, given by

$$p_c(x) = \int_c^x \exp\left(-2\int_c^s \frac{b(u)}{\sigma^2(u)}\,du\right) ds, \quad \text{for } x > 0, \tag{2.3}$$
$$m_c(dx) = \frac{2}{\sigma^2(x) p_c'(x)}\,dx, \tag{2.4}$$

are well-defined. In what follows, we assume that the constant $c > 0$ is fixed.

We also assume that the diffusion X is *non-explosive* and the boundaries of the diffusion at zero and infinity are inaccessible. In particular, we impose the following assumption (see Karatzas and Shreve [KS91, Theorem 5.5.29]).

Assumption 2.2.2 *If we define*

$$u_c(x) = \int_c^x \Big[p_c(x) - p_c(y)\Big]\,m_c(dy), \tag{2.5}$$

then $\lim_{x \downarrow 0} u_c(x) = \lim_{x \to \infty} u_c(x) = \infty$.

2.3 The solution to the homogeneous ODE

The objective is to show that the general solution to the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x > 0. \tag{2.6}$$

is given by

$$w(x) = A\psi(x) + B\phi(x). \tag{2.7}$$

Here, $A, B \in \mathbb{R}$ are constants and the functions ϕ, ψ are defined by

$$\phi(x) = \begin{cases} 1/\mathbb{E}_c[e^{-\Lambda_{\tau_x}}], & \text{for } x < c, \\ \mathbb{E}_x[e^{-\Lambda_{\tau_c}}], & \text{for } x \geq c, \end{cases} \tag{2.8}$$

$$\psi(x) = \begin{cases} \mathbb{E}_x[e^{-\Lambda_{\tau_c}}], & \text{for } x < c, \\ 1/\mathbb{E}_c[e^{-\Lambda_{\tau_x}}], & \text{for } x \geq c. \end{cases} \tag{2.9}$$

respectively, with

$$\Lambda_t = \int_0^t r(X_s)ds.$$

Here, and in what follows, given a weak solution to (2.2), $\mathbb{E}_z$ represents an expectation with the diffusion starting at $X_0 = z$ and given a point $a \in]0, \infty[$, we denote by τ_a the first hitting time of $\{a\}$, i.e.,

$$\tau_a = \inf\{t \geq 0 | \, X_t = a\},$$

with the usual convention that $\inf \emptyset = \infty$.

Since X is continuous, a simple inspection of (2.8) (resp., (2.9)) reveals that ϕ (resp., ψ) is strictly decreasing (resp., increasing). Also, since X is non-explosive, these definitions imply

$$\lim_{x \downarrow 0} \phi(x) = \lim_{x \to \infty} \psi(x) = \infty.$$

We also need the following assumption.

Assumption 2.3.1 *The function $r :]0, \infty[\to]0, \infty[$ is locally bounded.*

One purpose of the following result is to show that the definitions of ϕ, ψ in (2.8), (2.9), respectively, do not depend, in a non-trivial way, on the choice of $c \in]0, \infty[$.

Lemma 2.3.1 *Suppose that Assumptions 2.2.1–2.3.1 hold. Given any $x, y \in]0, \infty[$ the functions ϕ, ψ defined by (2.8), (2.9), respectively, satisfy*

$$\phi(y) = \phi(x)\mathbb{E}_y[e^{-\Lambda_{\tau_x}}] \quad \textit{and} \quad \psi(x) = \psi(y)\mathbb{E}_x[e^{-\Lambda_{\tau_y}}], \quad \textit{for all } x < y. \tag{2.10}$$

Moreover, the processes $(e^{-\Lambda_t}\phi(X_t),\ t \geq 0)$ and $(e^{-\Lambda_t}\psi(X_t),\ t \geq 0)$ are both local martingales.

Proof. Given any points $a, b, c \in]0, \infty[$ such that $a < b < c$, we calculate

$$\begin{aligned}\mathbb{E}_a[e^{-\Lambda_{\tau_c}}] &= \mathbb{E}_a\left[e^{\Lambda_{\tau_b}}\, \mathbb{E}_a[e^{-(\Lambda_{\tau_c} - \Lambda_{\tau_b})} | \mathcal{F}_{\tau_b}]\right] \\ &= \mathbb{E}_a[e^{-\Lambda_{\tau_b}}]\mathbb{E}_b[e^{-\Lambda_{\tau_c}}],\end{aligned}$$

where the second equality follows thanks to the strong Markov property of X. In view of this result, given any $x < z < y$, the choice $a = x$, $b = z$ and $c = y$ yields

$$\mathbb{E}_x[e^{-\Lambda_{\tau_y}}] = \mathbb{E}_x[e^{-\Lambda_{\tau_z}}]\mathbb{E}_z[e^{-\Lambda_{\tau_y}}],$$

which, combined with the definition of ψ, implies the second identity in (2.10). We can verify the second identity in (2.10) when $x < y < z$ or $z < x < y$ as well as the second identity in (2.10) by appealing to similar arguments.

Now, given any initial condition x and any sequence (x_n) such that $x < x_1$ and $\lim_{n\to\infty} x_n = \sup]0, \infty[$, we observe that the second identity in (2.10) implies

$$\psi(X_t)\mathbf{1}_{\{t \leq \tau_{x_n}\}} = \psi(x_n)\mathbb{E}_{X_t}[e^{-\Lambda_{\tau_{x_n}}}]\mathbf{1}_{\{t \leq \tau_{x_n}\}}, \quad \text{for all } t \geq 0.$$

In view of this identity, we appeal to the strong Markov property of X, once again to calculate

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{\tau_{x_n}}}\psi(X_{\tau_{x_n}})\middle|\mathcal{F}_t\right] &= e^{-\Lambda_t}\psi(x_n)\mathbb{E}_x\left[e^{-(\Lambda_{\tau_{x_n}}-\Lambda_t)}\middle|\mathcal{F}_t\right]\mathbf{1}_{\{t<\tau_{x_n}\}} + e^{-\Lambda_{\tau_{x_n}}}\psi(x_n)\mathbf{1}_{\{\tau_{x_n}\leq t\}}\\ &= e^{-\Lambda_t}\psi(X_t)\mathbf{1}_{\{t<\tau_{x_n}\}} + e^{-\Lambda_{\tau_{x_n}}}\psi(x_n)\mathbf{1}_{\{\tau_{x_n}\leq t\}}\\ &= e^{-\Lambda_{(t\wedge\tau_{x_n})}}\psi(X_{t\wedge\tau_{x_n}}).\end{aligned}$$

However, this calculation and the tower property of conditional expectation implies that, given any times $s<t$,

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_{x_n}}}\psi(X_{t\wedge\tau_{x_n}})\middle|\mathcal{F}_s\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_{x_n}}}\psi(X_{t\wedge\tau_{x_n}})\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right]\\ &= e^{-\Lambda_{s\wedge\tau_{x_n}}}\psi(X_{s\wedge\tau_{x_n}}),\end{aligned}$$

which proves that $\left(e^{-\Lambda_t}\psi(X_t),\, t\geq 0\right)$ is a local-martingale. Proving that $\left(e^{-\Lambda_t}\phi(X_t),\, t\geq 0\right)$ is a local-martingale follows similar arguments. □

We can now prove the following result.

Theorem 2.3.1 *Suppose that Assumptions 2.2.1–2.3.1 hold. The general solution to the ordinary differential equation (2.6) exists in the* classical sense, *namely there exists a two dimensional space of functions that are C^1 with absolutely continuous first derivatives, and that satisfy (2.6) Lebesgue-a.e.. This solution is given by (2.7), where $A, B\in\mathbb{R}$ are constants and the functions ϕ, ψ are given by (2.8), (2.9), respectively. Moreover, ϕ is strictly decreasing, ψ is strictly increasing, and, if the drift $b\equiv 0$, then both ϕ and ψ are strictly convex.*

Proof. First, we recall that, given $l<x<m$,

$$\mathbb{P}_x(\tau_l<\tau_m)=\frac{p_c(x)-p_c(m)}{p_c(l)-p_c(m)} \tag{2.11}$$

(e.g., see Karatzas and Shreve [KS91, Proposition 5.5.22] or Rogers and Williams [RW94, Definition V.46.10]). Also in view of the second identity in (2.10), we can see that

$$\begin{aligned}\psi(x) &< \psi(m)\mathbb{E}_x\left[\mathbf{1}_{\{\tau_m<\tau_l\}}\right] \;+\; \psi(m)\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\,\mathbf{1}_{\{\tau_l<\tau_m\}}\right]\\ &= \psi(m)\mathbb{P}_x(\tau_m<\tau_l)+\psi(m)\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\middle|\mathcal{F}_{\tau_l}\right]\,\mathbf{1}_{\{\tau_l<\tau_m\}}\right].\end{aligned}$$

Now, since X has the strong Markov property we can see that

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\middle|\mathcal{F}_{\tau_l}\right]\mathbf{1}_{\{\tau_l<\tau_m\}} &= e^{-\Lambda_{\tau_l}}\mathbb{E}_x\left[e^{-\Lambda(\tau_m-\tau_l)}\middle|\mathcal{F}_{\tau_l}\right]\mathbf{1}_{\{\tau_l<\tau_m\}}\\ &= e^{-\Lambda_{\tau_l}}\frac{\psi(l)}{\psi(m)}\mathbf{1}_{\{\tau_l<\tau_m\}},\end{aligned}$$

with the last equality following thanks to (2.10). Combining these calculations we can see that

$$\begin{aligned}\psi(x) &< \psi(m)\mathbb{P}_x(\tau_m<\tau_l) \;+\; \psi(l)\mathbb{E}_x\left[e^{-\Lambda_{\tau_l}}\,\mathbf{1}_{\{\tau_l<\tau_m\}}\right]\\ &< \psi(m)\mathbb{P}_x(\tau_m<\tau_l) \;+\; \psi(l)\mathbb{P}_x(\tau_l<\tau_m).\end{aligned} \tag{2.12}$$

Now, let us assume that $b\equiv 0$, so that the diffusion X defined by (2.2) is in *natural scale*, in which case $p_c(x)=x-c$. Combining this fact with (2.11), it is straightforward to verify that

$$x = l\mathbb{P}_x(\tau_l<\tau_m)+m\mathbb{P}_x(\tau_m<\tau_l).$$

However, this calculation and (2.12) imply that ψ is strictly convex. In this case, we have also that

$$\mathbb{P}_x(\tau_l<\tau_m) = \frac{x-m}{l-m}. \tag{2.13}$$

Under the assumption that $b\equiv 0$, which implies that ψ is strictly convex, we can use the Itô-Tanaka and the occupation times formulae to calculate

$$\psi(X_t) - \int_0^t r(X_s)\psi(X_s)\,ds = \psi(x) + \int_{]0,\infty[} L_t^a \frac{1}{\sigma^2(a)} \left[\frac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right] + \int_0^t \psi'_-(X_s)\sigma(X_s)\,dW_s,$$

where ψ'_- is the left-hand-side first derivative of ψ, $\mu''(da)$ is the distributional second derivative of ψ, and L^a is the local time process of X at level a. With regard to the integration by parts formula, this implies

$$e^{-\Lambda_t}\psi(X_t) = \psi(x) + \int_0^t e^{-\Lambda_s}\,d\int_{]0,\infty[} L_s^a \frac{1}{\sigma^2(a)} \left[\frac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right] + \int_0^t e^{-\Lambda_s}\psi'_-(X_s)\,\sigma(X_s)\,dW_s.$$

Since $(e^{-\Lambda_t}\psi(X_t),\ t \geq 0)$ is a local-martingale (see Lemma 2.3.1), this identity implies that the finite variation process Q defined by

$$Q_t = \int_0^t e^{-\Lambda_s}d\int_{]0,\infty[} L_s^a \frac{1}{\sigma^2(a)} \left[\frac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right], \quad \text{for } t \geq 0,$$

is a local martingale. Since finite-variation local martingales are constant, it follows that $Q \equiv 0$, which implies

$$\int_{]0,\infty[} L_t^a\,\nu(da) = 0, \quad \text{for all } t \geq 0, \tag{2.14}$$

where the measure ν is defined by

$$\nu(da) = \frac{1}{2}\mu''(da) - \frac{r(a)\psi(a)}{\sigma^2(a)\psi''(a)}. \tag{2.15}$$

To proceed further, fix any points $l < a < m$, define

$$\tau_{l,m} = \inf\left\{t \geq 0 \,|\, X_t \notin\,]l,m[\right\},$$

and let (T_j) be a localising sequence for the local martingale $\int_0^{\cdot} \operatorname{sgn}(X_s - a)dX_s$. With regard to the definition of local times and Doob's optional sampling theorem, we can see that

$$\begin{aligned}\mathbb{E}_x\Big[\big|X_{\tau_{l,m}\wedge T_j} - a\big|\Big] &= |x-a| + \mathbb{E}_x\left[\int_0^{\tau_{l,m}\wedge T_j} \operatorname{sgn}(X_s - a)\, dX_s\right] + \mathbb{E}_x\Big[L^a_{\tau_{l,m}\wedge T_j}\Big] \\ &= |x-a| + \mathbb{E}_x\Big[L^a_{\tau_{l,m}\wedge T_j}\Big].\end{aligned}$$

However, passing to the limit using the dominated convergence theorem on the left hand side and the monotone convergence theorem on the right hand side, we can see that this identity implies

$$\begin{aligned}\mathbb{E}_x\Big[L^a_{\tau_{l,m}}\Big] &= \mathbb{E}_x\Big[\big|X_{\tau_{l,m}} - a\big|\Big] - |x-a| \\ &= \frac{(m-a)(x-l)}{m-l} + \frac{(a-l)(m-x)}{m-l} - |x-a|\,,\end{aligned} \tag{2.16}$$

the second equality following thanks to (2.13). Now, (2.14), the fact that $t \mapsto L^a_t$ increases on the set $\{t \geq 0 \,|\, X_t = a\}$ and Fubini's theorem, imply

$$\begin{aligned}0 = \mathbb{E}_x\Big[\int_{]0,\infty[} L^a_{\tau_{l,m}}\, \nu(da)\Big] &= \mathbb{E}_x\Big[\int_{]l,m[} L^a_{\tau_{l,m}}\, \nu(da)\Big] \\ &= \int_{]l,m[} \mathbb{E}_x\big[L^a_{\tau_{l,m}}\big]\, \nu(da).\end{aligned}$$

Combining this calculation with (2.16), it is a matter of algebraic calculation to verify that

$$\int_l^m h(a; l, x, m)\, \nu(da) = 0, \tag{2.17}$$

where $h(\cdot; l, x, m)$ is the *tent-like* function of height 1 defined by

$$h(a; l, x, m) = \begin{cases} (a-l)/(x-l), & \text{for } a \in [l, x], \\ (m-a)/(m-x), & \text{for } a \in [x, m]. \end{cases}$$

Now, fix any points $x_l < x_m$ in $]0, \infty[$ and let (l_j) and (m_j) be strictly decreasing and strictly increasing, respectively, sequences such that

$$l_1 < \frac{x_l + x_m}{2} < m_1, \quad \lim_{j \to \infty} l_j = x_l \quad \text{and} \quad \lim_{j \to \infty} m_j = x_m.$$

We can see that

$$\mathbf{1}_{]x_l, x_m[}(a) = \lim_{j \to \infty} H_j(a), \quad \text{for all } a \in]0, \infty[,$$

where the increasing sequence of functions (H_j) is defined by

$$\begin{aligned} H_j(a) = h &\left(a; x_l, \frac{x_l + x_m}{2}, x_m \right) \\ &+ \frac{x_l + x_m - 2l_j}{x_m - x_l} \, h \left(a; x_l, l_j, \frac{x_l + x_m}{2} \right) \\ &+ \frac{2m_j - (x_l + x_m)}{x_m - x_l} \, h \left(a; \frac{x_l + x_m}{2}, m_j, x_m \right), \quad \text{for } a \in]0, \infty[\text{ and } j \geq 1. \end{aligned}$$

Using the monotone convergence theorem and (2.17), it follows that

$$\nu(]x_l, x_m[) = \lim_{j \to \infty} \int_{x_l}^{x_m} H_j(a) \, \nu(da) = 0,$$

which proves that the signed measure ν assigns measure 0 to every open subset of $]0, \infty[$. However, this observation and the definition of ν in (2.15) imply that the total variation of ν is zero, and, therefore, $\mu''(da)$ is an absolutely continuous measure. It follows that there exists a function ψ'' such that

$$\mu''(da) = \psi''(a) \, da \quad \text{and} \quad \frac{1}{2} \sigma^2(a) \psi''(a) = r(a) \psi(a), \quad \text{Lebesgue-a.e..}$$

However, the second identity here shows that ψ is a classical solution to (2.6).

Now, let us consider the general case where the drift b does not vanish. In this case, we use Itô's formula to verify that, if $\bar{X} = p_c(X)$, then

$$d\bar{X}_t = \bar{\sigma}(\bar{X}_t)dW_t, \quad \bar{X}_0 = p_c(x),$$

where

$$\bar{\sigma}(\bar{x}) = p_c'\left(p_c^{-1}(\bar{x})\right)\sigma\left(p_c^{-1}(\bar{x})\right), \quad \text{for } \bar{x} \in]0, \infty[.$$

Since $\bar{X}$ is a diffusion in natural scale, the associated function $\bar{\psi}$ defined as in (2.9) is a classical solution of

$$\frac{1}{2}\sigma^2(a)\bar{\psi}''(a) - r(a)\bar{\psi}(a) = 0. \tag{2.18}$$

Now, recalling that p_c is twice differentiable in the classical sense, we can see that if we define $\tilde{\psi}(x) = \bar{\psi}(p_c(x))$ then

$$\begin{aligned}
\tilde{\psi}'(x) &= \bar{\psi}'\big(p_c(x)\big)p_c'(x), \\
\tilde{\psi}''(x) &= \bar{\psi}''\big(p_c(x)\big)\left[p_c'(x)\right]^2 + \bar{\psi}'\big(p_c(x)\big)p_c''(x).
\end{aligned}$$

However, combining these calculations with (2.18), we can see that $\tilde{\psi}$ satisfies the ODE (2.6).

To prove that $\tilde{\psi}$, namely the classical solution to (2.6), as constructed above, identifies with ψ defined by (2.9), we apply Itô's formula to $e^{-\Lambda(\tau_y\wedge T)}\tilde{\psi}(X_{\tau_y\wedge T})$, where $T > 0$ is a constant, and we use arguments similar to the ones employed in the proof of Theorem 3.3.1, to show that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau_y\wedge T}}\tilde{\psi}(X_{\tau_y\wedge T})\right] = \tilde{\psi}(x), \quad \text{for all } x < y.$$

Since $\psi > 0$ is increasing, the monotone and the dominated convergence theorems imply

$$\lim_{T\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_y\wedge T}}\tilde{\psi}(X_{\tau_y\wedge T})\right] = \tilde{\psi}(y)\mathbb{E}_x\left[e^{-\Lambda_{\tau_y}}\right], \quad \text{for all } x < y.$$

However, these calculations, show that $\tilde{\psi}$ satisfies the second identity in (2.10) and therefore identifies with ψ defined by (2.9). Proving all of the associated claims for ϕ follows similar reasoning. □

Remark 2.3.1 Although we have chosen to undertake this analysis for a positive diffusion, similar results can be obtained for a regular Itô diffusion with values in any interval $\mathcal{I}$ where $\mathcal{I} \subseteq \mathbb{R}$.

Using the fact that ϕ and ψ satisfy the ODE (2.6), it is a straightforward exercise to verify that the scale function, p_c, defined by (2.3) satisfies

$$p_c'(x) = \frac{\phi(x)\psi'(x) - \phi'(x)\psi(x)}{\mathcal{W}(c)}, \quad \text{for all } x > 0, \tag{2.19}$$

where $\mathcal{W}$ is the Wronskian of ϕ and ψ, defined by

$$\mathcal{W}(x) = \phi(x)\psi'(x) - \phi'(x)\psi(x)$$

and $\mathcal{W}(x) > 0$ for all $x > 0$.

2.4 Study of a non-homogeneous ordinary differential equation

We now study the non-homogeneous ODE (2.1),

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) = 0, \quad x \in]0, \infty[.$$

We need to impose the following assumptions, which are stronger than Assumption 2.2.1 and Assumption 2.3.1, respectively.

Assumption 2.2.1′ *The conditions of Assumption 2.2.1 hold true, and*

the function σ is locally bounded.

Assumption 2.3.1$'$ *The conditions of Assumption 2.3.1 hold true, and there exists a constant* r_0 *such that*

$$0 < r_0 \leq r(x) < \infty, \quad \textit{for all } x > 0. \tag{2.20}$$

We can now prove the following propositions.

Proposition 2.4.1 *Suppose that Assumption 2.2.1*$'$*, Assumption 2.2.2 and Assumption 2.3.1*$'$ *hold. The following statements are equivalent:*

(I) Given any initial condition $x > 0$ *and any weak solution* $\mathbb{S}_x$ *to (2.2),*

$$\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} |h(X_t)| \, dt \right] < \infty.$$

(II) There exists an initial condition $y > 0$ *and a weak solution* $\mathbb{S}_y$ *to (2.2) such that*

$$\mathbb{E}_y \left[\int_0^\infty e^{-\Lambda_t} |h(X_t)| \, dt \right] < \infty.$$

(III) Given any $x > 0$,

$$\int_0^x |h(s)| \, \psi(x) \, m(ds) < \infty \quad \textit{and} \quad \int_x^\infty |h(s)| \, \phi(x) \, m(ds) < \infty.$$

(IV) There exists $y > 0$ *such that*

$$\int_0^y |h(s)| \, \psi(x) \, m(ds) < \infty \quad \textit{and} \quad \int_y^\infty |h(s)| \, \phi(x) \, m(ds) < \infty.$$

If these conditions hold, then the function

$$R_h(x) = \frac{\phi(x)}{\mathcal{W}(c)} \int_0^x h(s)\psi(s) \, m(ds) + \frac{\psi(x)}{\mathcal{W}(c)} \int_x^\infty h(s)\phi(s) \, m(ds), \quad x \in]0, \infty[, \tag{2.21}$$

is well-defined, is twice differentiable in the classical sense and satisfies the ODE (2.1), Lebesgue-a.e.. In addition, R_h admits the expression

$$R_h(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_s} h(X_s)\, ds \right], \quad \textit{for all } x > 0. \tag{2.22}$$

Proof. Suppose that (IV) is true and let y be such that

$$C_1 := \int_0^y |h(s)|\psi(s)\, m(ds) < \infty \quad \text{and} \quad C_2 := \int_y^\infty |h(s)|\phi(s)\, m(ds) < \infty.$$

and so

$$\int_0^x |h(s)|\psi(s)\, m(ds), \quad \int_x^y |h(s)|\psi(s)\, m(ds) \leq C_1, \quad \text{for all } x \in \,]0, y[, \tag{2.23}$$

$$\int_y^x |h(s)|\phi(s)\, m(ds), \quad \int_x^\infty |h(s)|\phi(s)\, m(ds) \leq C_2, \quad \text{for all } x \in \,]y, \infty[. \tag{2.24}$$

Combining these inequalities with the fact that ψ is increasing and ϕ is decreasing, we can see that

$$\begin{aligned} \int_x^\infty |h(s)|\phi(s)\, m(ds) &= \int_x^y |h(s)|\phi(s)\, m(ds) + \int_y^\infty |h(s)|\phi(s)\, m(ds) \\ &\leq \frac{\phi(x)}{\psi(x)} \int_x^y |h(s)|\psi(s)\, m(ds) + C_2 \\ &\leq \frac{\phi(x)}{\psi(x)} C_1 + C_2 \\ &< \infty \end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\int_0^x |h(s)|\psi(s)\,m(ds) &= \int_0^y |h(s)|\psi(s)\,m(ds) + \int_y^x |h(s)|\psi(s)\,m(ds)\\ &\leq C_1 + \frac{\psi(x)}{\phi(x)}\int_y^x |h(s)|\phi(s)\,m(ds)\\ &\leq C_1 + \frac{\psi(x)}{\phi(x)}C_2\\ &< \infty. \end{aligned} \tag{2.26}$$

However, (2.23)–(2.26) imply that (III) holds. The reverse implication is obvious.

To proceed further, assume that $h = h_{\mathrm{B}}^+$ where h_{B}^+ is a positive and bounded measurable function. In this case, it is straightforward to verify that (I) and (II) are both satisfied. Since ϕ and ψ are continuous functions satisfying $\lim_{x\to\infty}\phi(x), \lim_{x\downarrow 0}\psi(x) < \infty$ and m is a locally finite measure, the function $R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}} :]0,\infty[\to \mathbb{R}_+$ given by (2.21), or, equivalently, by

$$\begin{aligned}R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(x) &= \frac{\phi(x)}{\mathcal{W}(c)}\mathbf{1}_{[1/k,k]}(x)\int_{1/k}^x h_{\mathrm{B}}^+(s)\psi(s)\,m(ds)\\ &\quad + \frac{\psi(x)}{\mathcal{W}(c)}\mathbf{1}_{[1/k,k]}(x)\int_x^k h_{\mathrm{B}}^+(s)\phi(s)\,m(ds), \end{aligned} \tag{2.27}$$

is well-defined and bounded for all $k > 1$. In light of the calculations

$$\begin{aligned}R'_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(x) &= \frac{\phi'(x)}{\mathcal{W}(c)}\int_0^x h_{\mathrm{B}}^+(s)\mathbf{1}_{[1/k,k]}(s)\psi(s)\,m(ds)\\ &\quad + \frac{\psi'(x)}{\mathcal{W}(c)}\int_x^\infty h_{\mathrm{B}}^+(s)\mathbf{1}_{[1/k,k]}(s)\phi(s)\,m(ds),\\ R''_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(x) &= \frac{\phi''(x)}{\mathcal{W}(c)}\int_0^x h_{\mathrm{B}}^+(s)\mathbf{1}_{[1/k,k]}(s)\psi(s)\,m(ds)\\ &\quad + \frac{\psi''(x)}{\mathcal{W}(c)}\int_x^\infty h_{\mathrm{B}}^+(s)\mathbf{1}_{[1/k,k]}(s)\phi(s)\,m(ds)\\ &\quad - \frac{2h_{\mathrm{B}}^+(x)}{\sigma^2(x)}\mathbf{1}_{[1/k,k]}(x) \end{aligned}$$

we can see that $R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}$ is twice differentiable in the classical sense, because this is true for the functions ψ and ϕ, and satisfies the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h_{\mathrm{B}}^{+}(x)\mathbf{1}_{[1/k,k]}(x) = 0, \tag{2.28}$$

Lebesgue-a.e. in $]0, \infty[$.

Now, fix any $k > 1$, any initial condition $x > 0$, any weak solution $\mathbb{S}_x$ to (2.2) and define the sequence of $(\mathcal{F}_t)$-stopping times (τ_l) by

$$\tau_l = \inf\left\{t \geq 0 \mid \ X_t \notin [1/l, l]\right\},$$

Using Itô's formula and the fact that $R_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}$ satisfies (2.28), we calculate

$$\begin{aligned} &e^{-\Lambda_{\tau_l \wedge T}} R_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(X_{\tau_l \wedge T}) + \int_0^{\tau_l \wedge T} e^{-\Lambda_s} h_{\mathrm{B}}^{+}(X_s)\mathbf{1}_{[1/k,k]}(X_s)\, ds \\ &= R_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(x) + M_t^{(l)}, \end{aligned} \tag{2.29}$$

where $M^{(l)}$ is defined by

$$M_t^{(l)} = \int_0^{\tau_l \wedge t} e^{-\Lambda_s}\sigma(X_s) R'_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(X_s)\, dW_s.$$

Combining the fact that $R'_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}$ is locally bounded, because $R_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}$ is continuous, with the the fact that σ is locally bounded from Assumption 2.2.1′, we can see that the quadratic variation of the local martingale $M_t^{(l)}$ satisfies

$$\begin{aligned} \mathbb{E}_x\left[\langle M^{(l)}\rangle_\infty\right] &= \int_0^\infty \mathbb{E}_x\left[1_{\{s \leq \tau_l\}}\left(e^{-\Lambda_s}\sigma(X_s)R'_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(X_s)\right)^2\right] ds \\ &\leq \sup_{y \in [1/l, l]}\left[\sigma(y)R'_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(y)\right]^2 \int_0^\infty \mathbb{E}_x\left[e^{-2\Lambda_s}\right] ds \\ &\leq \frac{1}{2r_0}\sup_{y \in [1/l, l]}\left[\sigma(y)R'_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(y)\right]^2 \\ &< \infty, \end{aligned}$$

the second inequality following as a consequence of (2.20) in Assumption 2.3.1′. This proves that $M^{(l)}$ is a martingale bounded in L^2, so, $\mathbb{E}_x\left[M_t^{(l)}\right] = 0$, for all $t \geq 0$. This observation and (2.29) imply

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau_l\wedge T}} R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(X_{\tau_l\wedge T}) + \int_0^{\tau_l\wedge T} e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\mathbf{1}_{[1/k,k]}\, ds\right] = R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(x). \tag{2.30}$$

Since $R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}$ is bounded, the dominated convergence theorem implies

$$\lim_{T\to\infty}\lim_{l\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{\tau_l\wedge T}} R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(X_{\tau_l\wedge T})\right] = 0,$$

while the monotone convergence theorem yields

$$\lim_{T\to\infty}\lim_{l\to\infty}\mathbb{E}_x\left[\int_0^{\tau_l\wedge T} e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\mathbf{1}_{[1/k,k]}(X_s)\, ds\right] = \mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\mathbf{1}_{[1/k,k]}(X_s)\, ds\right].$$

These limits and (2.30) imply

$$\mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\mathbf{1}_{[1/k,k]}(X_s)\, ds\right] = R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(x).$$

Recalling the definition of $R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}$ as in (2.27), we can pass to the limit $k \to \infty$ in this identity to obtain

$$R_{h_{\mathrm{B}}^+}(x) = \mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\, ds\right]. \tag{2.31}$$

Note that, since, h_{B}^+ plainly satisfies conditions (I) and (II), this identity also implies that h_{B}^+ satisfies conditions (III) and (IV).

Now assume that $h = h^+$, where h^+ is a positive measurable function. Using (2.31) with $h_{\mathrm{B}}^+ = h^+ \wedge n$, for $n \geq 1$, and applying the monotone convergence theorem, we can see that, given any initial condition $x > 0$ and any weak solution $\mathbb{S}_x$ to (2.2),

$$\mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_s} h^+(X_s)\, ds\right] = R_{h^+}(x), \tag{2.32}$$

where both sides may be equal to infinity. However, with reference to the definition of R_{h^+}, this proves that (I) and (III) are equivalent and that (II) and (IV) imply each other. Recalling the equivalence of (III) and (IV) that we proved above, it follows that statements (I)–(IV) are all equivalent. Furthermore, given any h satisfying (I)–(IV), we can immediately see that (2.32) implies (2.22) once we consider the decomposition $h = h^+ - h^-$ of h to its positive and its negative parts h^+ and h^-, respectively. □

The following result is concerned with a number of properties of the function R_h studied in the previous proposition.

Proposition 2.4.2 *Suppose that Assumption 2.2.1′, Assumption 2.2.2 and Assumption 2.3.1′ hold.. Let $h :]0, \infty[\to \mathbb{R}$ be a measurable function satisfying Conditions (I)–(IV) in Proposition 2.4.1. The function R_h given by (2.21) or (2.22) satisfies*

$$\lim_{x \downarrow 0} \frac{R_h(x)}{\phi(x)} = \lim_{x \to \infty} \frac{R_h(x)}{\psi(x)} = 0, \tag{2.33}$$

$$\inf_{x>0} \frac{h(x)}{r(x)} \leq R_h(x) \leq \sup_{x>0} \frac{h(x)}{r(x)}, \quad \text{for all } x > 0, \tag{2.34}$$

$$R_h'(x)\phi(x) - R_h(x)\phi'(x) = p_c'(x) \int_x^\infty h(s)\phi(s)\, m(ds), \quad \text{for all } x > 0, \tag{2.35}$$

$$R_h'(x)\psi(x) - R_h(x)\psi'(x) = -p_c'(x) \int_0^x h(s)\psi(s)\, m(ds), \quad \text{for all } x > 0, \tag{2.36}$$

if h/r is increasing (resp., decreasing), then R_h is increasing (resp., decreasing). Also,

$$R_r(x) = 1, \quad \text{for all } x > 0. \tag{2.37}$$

Furthermore, given a solution $\mathbb{S}_x = (\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P}_x, X, W)$ to (2.2) and an $(\mathfrak{F}_t)$-stopping *time* τ,

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} R_h(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = R_h(x) - \mathbb{E}_x\left[\int_0^\tau e^{-\Lambda_t} h(X_t)\,dt\right], \tag{2.38}$$

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} R_h(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = \mathbb{E}_x\left[\int_\tau^\infty e^{-\Lambda_t} h(X_t)\,dt\right], \tag{2.39}$$

while, if (τ_n) *is a sequence of stopping times such that* $\lim_{n\to\infty}\tau_n = \infty$, $\mathbb{P}_x$*-a.s., then*

$$\lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}\left|R_h(X_{\tau_n})\right|\mathbf{1}_{\{\tau_n<\infty\}}\right] = 0. \tag{2.40}$$

Proof. Fix a solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ to (2.2) and let τ be an $(\mathcal{F}_t)$-stopping time. Using the definition of Λ and the strong Markov property of X, we can see that (2.22) implies

$$\begin{aligned}
R_h(x) &= \mathbb{E}_x\left[\int_0^\tau e^{-\Lambda_t} h(X_t)\,dt + e^{-\Lambda_\tau}\mathbb{E}_x\left[\int_\tau^\infty e^{-(\Lambda_s-\Lambda_\tau)} h(X_s)\,ds \mid \mathcal{F}_\tau\right]\mathbf{1}_{\{\tau<\infty\}}\right] \\
&= \mathbb{E}_x\left[\int_0^\tau e^{-\Lambda_t} h(X_t)\,dt + e^{-\Lambda_\tau}\mathbb{E}_{X_\tau}\left[\int_0^\infty e^{-\Lambda_s} h(X_s)\,ds\right]\mathbf{1}_{\{\tau<\infty\}}\right] \\
&= \mathbb{E}_x\left[\int_0^\tau e^{-\Lambda_t} h(X_t)\,dt + e^{-\Lambda_\tau} R_h(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right],
\end{aligned}$$

which establishes (2.38). Also, this expression and (2.22) imply immediately (2.39), while (2.40) follows from the observation that $|R_h| \le R_{|h|}$ (note that h satisfies conditions (I)–(IV) of Proposition 2.4.1 if and only in $|h|$ does), (2.39) and the dominated convergence theorem.

Now, let $c > 0$ be the point that we used in (2.3) and (2.4) to define the scale function and the speed measure of the diffusion X. Given a solution $\mathbb{S}_c = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_c, X, W)$ to (2.2) and any $x > 0$, we denote by τ_x the first hitting time of $\{x\}$, and we note that $\lim_{x\downarrow 0}\tau_x = \lim_{x\to\infty}\tau_x = \infty$, $\mathbb{P}_c$-a.s., because the diffusion X is non-explosive by Assumption 2.2.2. In view of this observation and the fact that $r(x) \ge r_0 > 0$, for all $x > 0$, by Assumption 2.3.1′, we have that (2.40) implies

$$\lim_{x\downarrow 0}\mathbb{E}_c\left[e^{-\Lambda_{\tau_x}}\right] R_h(x) = \lim_{x\to\infty}\mathbb{E}_c\left[e^{-\Lambda_{\tau_x}}\right] R_h(x) = 0.$$

However, these limits and the definitions (2.8) and (2.9) of the functions ϕ and ψ imply (2.33). Also, a simple inspection of the ODE (2.1) reveals that if we set $h = r$ then R_r satisfies (2.37), noting that R_r is well-defined thanks to (2.20) in Assumption 2.3.1$'$. We can verify (2.35) and (2.36) by a straightforward calculation using the definition (2.21) of R_h and (2.19).

To proceed further, let us assume that h/r is increasing, let us fix any $x > 0$, and let us define $C_x = h(x)/r(x)$. In view of (2.38), the monotonicity of h/r, and the definition (2.9) of ψ we calculate

$$\begin{aligned} R_{h(\cdot)-C_x r(\cdot)}(x-\varepsilon) &= \mathbb{E}_{x-\varepsilon}\left[\int_0^{\tau_x} e^{-\Lambda_t}\left[h(X_t) - C_x r(X_t)\right] dt\right] + R_{h(\cdot)-C_x r(\cdot)}(x)\mathbb{E}_{x-\varepsilon}\left[e^{-\Lambda_{\tau_x}}\right] \\ &\leq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\psi(x-\varepsilon)}{\psi(x)}, \quad \text{for all } \varepsilon > 0, \end{aligned} \tag{2.41}$$

which shows that

$$\frac{R_{h(\cdot)-C_x r(\cdot)}(x) - R_{h(\cdot)-C_x r(\cdot)}(x-\varepsilon)}{\varepsilon} \geq \frac{R_{h(\cdot)-C_x r(\cdot)}(x)}{\psi(x)}\frac{\psi(x)-\psi(x-\varepsilon)}{\varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Recalling that $R_{h(\cdot)-C_x r(\cdot)}$ is C^1, we can pass to the limit $\varepsilon \downarrow 0$ in this inequality to obtain

$$R'_{h(\cdot)-C_x r(\cdot)}(x) \geq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\psi'(x)}{\psi(x)}. \tag{2.42}$$

Making a calculation similar to the one in (2.41) using the definition (2.8) of ϕ this time, we can see that

$$R_{h(\cdot)-C_x r(\cdot)}(x+\varepsilon) \geq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\phi(x+\varepsilon)}{\phi(x)}, \quad \text{for all } \varepsilon > 0.$$

Rearranging terms and passing to the limit $\varepsilon \downarrow 0$, we can see that this inequality implies

$$R'_{h(\cdot)-C_x r(\cdot)}(x) \geq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\phi'(x)}{\phi(x)}. \tag{2.43}$$

Recalling that the strictly positive function ψ is strictly increasing and that the strictly positive function ϕ is strictly decreasing, we can see that (2.42) implies $R'_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$ if $R_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$, while, (2.43) implies $R'_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$ if $R_{h(\cdot)-C_x r(\cdot)}(x) \leq 0$. Now, combining the inequality $R_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$ with the identities

$$\begin{aligned} R'_{h(\cdot)-C_x r(\cdot)}(x) &= \frac{\phi'(x)}{\mathcal{W}(c)} \int_0^x [h(s) - C_x r(s)]\, \psi(s)\, m(ds) \\ &\quad + \frac{\psi'(x)}{\mathcal{W}(c)} \int_x^\infty [h(s) - C_x r(s)]\, \phi(s)\, m(ds) \\ &= R'_h(x) - C_x R'_r(x) \\ &= R'_h(x) \end{aligned}$$

that follow from the definition (2.21) of R_h and (2.37), we can see that $R'_h(x) \geq 0$. However, since the point $x > 0$ has been arbitrary, this analysis establishes the claim that, if h/r is increasing, then R_h is increasing. Proving the claim corresponding to the case when h/r is decreasing follows similar symmetric arguments.

Finally, to show (2.34), let us assume that $\underline{h} := \inf_{x>0} h(x)/r(x) > -\infty$. In this case, we can use (2.37) and the representation (2.22) to calculate

$$\begin{aligned} R_h(x) - \inf_{x>0} \frac{h(x)}{r(x)} &= R_h(x) - R_{\underline{h}r(\cdot)}(x) \\ &= \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda t} \left[h(X_t) - \inf_{x>0} \frac{h(x)}{r(x)} r(X_t) \right] dt \right] \\ &\geq 0, \end{aligned}$$

which establishes the lower bound in (2.34). The upper bound in (2.34) can be established in exactly the same way, and the proof is complete. □

The following lemma gives result useful for practical applications.

Lemma 2.4.1 *Suppose that Assumption 2.2.1′, Assumption 2.2.2 and Assumption 2.3.1′ hold. Let* $h\ :]0,\infty[\to\mathbb{R}$ *be a measurable function satisfying Conditions (I)–(IV) in Proposition 2.4.1. If, in addition,* h/r *is increasing, then if* 0 *is a natural boundary (not an entrance boundary) then*

$$\lim_{x\downarrow 0} R_h(x) = \lim_{x\downarrow 0} \frac{h(x)}{r(x)}, \tag{2.44}$$

and if ∞ *is a natural boundary then*

$$\lim_{x\to\infty} R_h(x) = \lim_{x\to\infty} \frac{h(x)}{r(x)}. \tag{2.45}$$

Proof We have

$$\begin{aligned} R_h(x) &\leq \sup_{x>0} \frac{h(x)}{r(x)} \mathbb{E}_x\left[\int_0^\infty d\big(e^{-\Lambda_t}\big)\big)\right] \\ &= \sup_{x>0} \frac{h(x)}{r(x)} \end{aligned}$$

and

$$\begin{aligned} R_h(x) &\geq \inf_{x>0} \frac{h(x)}{r(x)} \mathbb{E}_x\left[\int_0^\infty d\big(e^{-\Lambda_t}\big)\big)\right] \\ &= \inf_{x>0} \frac{h(x)}{r(x)}. \end{aligned}$$

For $b < x$

$$\begin{aligned} R_h(x) &= \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + \mathbb{E}_x\left[\int_{\tau_b}^{\infty} h(X_t)e^{-\Lambda_t}dt\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\ &\geq \inf_{x>b}\frac{h(x)}{r(x)}\mathbb{E}_x\left[\int_0^{\tau_b} d\big(e^{-\Lambda_t}\big)\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\ &= \inf_{x>b}\frac{h(x)}{r(x)}\left(1-\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right]\right) + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\ &= \inf_{x>b}\frac{h(x)}{r(x)}\left(1-\frac{\phi(x)}{\phi(b)}\right) + R(b)\frac{\phi(x)}{\phi(b)} \end{aligned}$$

while for $x < b$

$$\begin{aligned} R_h(x) &= \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + \mathbb{E}_x\left[\int_{\tau_b}^{\infty} h(X_t)e^{-\Lambda_t}dt\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\ &\leq \sup_{x<b}\frac{h(x)}{r(x)}\mathbb{E}_x\left[\int_0^{\tau_b} d\big(e^{-\Lambda_t}\big)\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\ &\leq \sup_{x<b}\frac{h(x)}{r(x)}\left(1-\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right]\right) + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\ &= \sup_{x<b}\frac{h(x)}{r(x)}\left(1-\frac{\psi(x)}{\psi(b)}\right) + R(b)\frac{\psi(x)}{\psi(b)}. \end{aligned}$$

If ∞ is a natural boundary (not an entrance boundary) then $\lim_{x\to\infty}\phi(x) = 0$ and we can say

$$\begin{aligned} \lim_{x\to\infty} R_h(x) \leq &\limsup_{x\to\infty}\frac{h(x)}{r(x)} \\ \geq &\liminf_{x\to\infty}\frac{h(x)}{r(x)} \end{aligned}$$

and we have the result for the limit as x tends to infinity. Similarly if 0 is a natural boundary then $\lim_{x\to\infty}\psi(x) = 0$ and we can say

$$\begin{aligned}\lim_{x\downarrow 0} R_h(x) \leq\ &\limsup_{x\downarrow 0}\frac{h(x)}{r(x)}\\ \geq\ &\liminf_{x\downarrow 0}\frac{h(x)}{r(x)}\end{aligned}$$

and we have the result for the limit as x tends to zero. $\square$

Remark 2.4.1 In the cases where we do not have a natural boundary point, R_h does not converge to a value determined in a straightforward way by h and r in the limit. To see this, consider the case of the so-called square root mean reverting process, defined by

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t, \quad X_0 = x > 0,$$

where κ, θ and σ are positive constants satisfying $\kappa\theta - \frac{1}{2}\sigma^2 > 0$. Note that this diffusion has an entrance boundary point at zero. It is a standard exercise to calculate that

$$\begin{aligned}\mathbb{E}_x[X_t] &= \theta + (x-\theta)e^{-\kappa t},\\ \mathbb{E}_x\left[X_t^2\right] &= \left(\frac{\sigma^2\theta}{2\kappa} + \theta^2\right) + e^{-\kappa t}\left(2\theta(x-\theta) - 2\frac{\sigma^2\theta}{2\kappa} + \frac{\sigma^2 x}{\kappa}\right)\\ &\quad + e^{-2\kappa t}\left(\frac{\sigma^2\theta}{2\kappa} - \frac{\sigma^2 x}{\kappa} + (x-\theta)^2\right).\end{aligned}$$

Now, let us consider the following three cases for the payoff function, h:

$$h_1(x) = 0, \quad h_2(x) = x \quad \text{and} \quad h_3(x) = x^2.$$

and assume that r is a constant. In all these cases $\lim_{x\downarrow 0} h(x)/r(x) = 0$. However, we can see that

$$\begin{aligned}
\lim_{x \downarrow 0} R_{h_1}(x) &= 0, \\
\lim_{x \downarrow 0} R_{h_2}(x) &= \frac{\theta\kappa}{r(r+\kappa)} > 0, \\
\lim_{x \downarrow 0} R_{h_3}(x) &= \left(\frac{\sigma^2\theta}{2\kappa} + \theta^2 \right) \left[\frac{2\kappa^2}{r(r+\kappa)(r+2\kappa)} \right] > 0.
\end{aligned}$$

3. A DISCRETIONARY STOPPING PROBLEM

3.1 Introduction

In this chapter we address a discretionary stopping problem as a prelude to addressing our investment problems. We solve this stopping problem general conditions on the underlying diffusion, the payoff and the discounting. Also, we consider several special cases. Although all of the special cases of interest that we are aware of are associated with stochastic differential equations that have unique strong solutions, we adopt a weak formulation. Working within this more general framework, which involves no additional technicalities, has been motivated by the extra degrees of freedom that it offers relative to modelling and has a view to applications in stochastic control beyond optimal stopping.

Section 3.2 is concerned with a rigorous formulation of the stopping problem that we solve. In this section, we consider the case of the perpetual American call option which highlights an issue related to waiting forever, which is mentioned in the introduction in relation to McDonald and Siegel [MS86], namely that the problem data may not conform to standard economic theory. Combining the observations from this special case with the results of Section 2.4 we develop Assumption 3.2.1 that is sufficient for our problem to conform with applications in finance and economics.

In Section 3.3 we solve the discretionary stopping problem. With reference to the motivation of the thesis, we are particularly interested in identifying the nature of the stopping boundary given the problem data and in solving the discretionary stopping problem we distinguish cases based on the nature of the stopping boundary.

In Section 3.4, we address a number of special cases of interest. These cases involve a number of choices for the underlying state process X that have been considered in the literature, while the payoff functions identify with standard utility functions with the discount rate being assumed to be constant. In particular we consider the cases that arise when X is a geometric Brownian motion; a square-root mean-reverting process as in the Cox-Ingersoll-Ross interest rate model; and a geometric Ornstein-Uhlenbeck process, which has been proposed by Cortazar and Schwartz as a model for a commodity's price and has been used in population modelling.

3.2 Problem formulation and assumptions

We start by defining a stopping strategy.

Definition 3.2.1 Given an initial condition $x > 0$, a *stopping strategy* is any pair $(\mathbb{S}_x, \tau)$ such that $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution to (2.2) and τ is an $(\mathcal{F}_t)$-stopping-time. We denote by $\mathcal{S}_x$ the set of all such stopping strategies.

The objective is to maximise the performance criterion

$$J(\mathbb{S}_x, \tau) = \mathbb{E}_x\Big[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\Big],$$

where $\Lambda_t = \int_0^t r(X_s)\,ds$ and $g :]\,0,\infty[\to \mathbb{R}$ and $r :]0,\infty[\to]0,\infty[$ are given deterministic functions, over all stopping strategies $(\mathbb{S}_x, \tau) \in \mathcal{S}_x$. Accordingly we define the value function v by

$$v(x) = \sup_{(\mathbb{S}_x,\tau)\in\mathcal{S}_x} J(\mathbb{S}_x, \tau), \quad \text{for } x > 0.$$

Now, with a view to developing an understanding of the problem under consideration, we consider the case of a perpetual American call option written on an underlying asset, the stochastic dynamics of which are modelled by a geometric Brownian motion.

Lemma 3.2.1 *Suppose that X is a geometric Brownian motion, so that $b(x) = bx$ and $\sigma(x) = \sigma x$, for some constants b and σ, and $r(x) \equiv r > 0$, for some constant r. Suppose also that the payoff function is given by $g(x) = x - K$, where $K \geq 0$ is a constant. If $b > r$ (resp., $b < r$), then the process $(e^{-rt} X_t,\ t \geq 0)$ is a submartingale (resp., supermartingale) and $v(x) = \infty$ (resp., if $K = 0$, then $v(x) = x$).*

Proof. Given any initial condition $x > 0$,

$$e^{-rt} X_t = x e^{(b-r)t} e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}, \quad \text{for } t \geq 0.$$

Combining this observation with the fact that the process $(e^{-\frac{1}{2}\sigma^2 t + \sigma W_t},\ t \geq 0)$ is a martingale, we can see that all of the claims made are true. □

In the context of this lemma, we can see that $(\mathbb{S}_x, 0)$ is an optimal strategy if $K = 0$ and $b < r$. Given any $K \geq 0$, if $b - r > \frac{1}{2}\sigma^2$, then the stopping strategy $(\mathbb{S}_x^*, \tau^*)$, where $\mathbb{S}_x^*$ is a weak solution to (2.2) and

$$\tau^* = \inf\{t \geq 0 |\ W_t = -a\},$$

where $a > 0$ is any constant, provides an optimal strategy. Indeed, since $\tau^* < \infty$, $\mathbb{P}_x$-a.s., and $\mathbb{E}_x\left[\tau^*\right] = \infty$, this claim follows from the calculation

$$\begin{aligned}
\mathbb{E}_x[e^{-r\tau^*}(X_{\tau^*} - K)] &\geq x e^{-a\sigma} \mathbb{E}_x\left[e^{\left(b - r - \frac{1}{2}\sigma^2\right)\tau^*}\right] - K \\
&> x e^{-a\sigma}\left[1 + \left(b - r - \frac{1}{2}\sigma^2\right)\mathbb{E}_x[\tau^*]\right] - K \\
&= \infty.
\end{aligned}$$

When $b > r$ and $b - r < \frac{1}{2}\sigma^2$, we have not been able to find an optimal stopping strategy. As a matter of fact, we have been tempted to conjecture that there is no optimal stopping strategy in this case.

We note that, when $b > r$, which is associated with $v \equiv \infty$, and when $b = r$, which is a case that we have not associated with a conclusion,

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} g(X_t) \right] \equiv \lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} X_t \right] > 0,$$

for all initial conditions $x > 0$. In this case, the problem data does not satisfy the so-called *transversality* condition

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} g(X_t) \right] \equiv \lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} X_t \right] = 0.$$

Such a condition has a natural economic interpretation because it reflects the idea that one should expect that the present value of any asset should be equal to zero at the end of time, given that nobody can benefit by holding the asset after the end of time. We expect that problems in finance and economics should satisfy this condition.

Based on this observation and the results in Section 2.4, we impose the following assumption in order to be able to carry out our analysis of the problem.

Assumption 3.2.1 *The function $g :]0, \infty[\to \mathbb{R}$ is C^1 with an absolutely continuous first derivative. In addition, given any weak solution, $\mathbb{S}_x$ to (2.2) the function g satisfies*

$$\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} \left| \mathcal{L} g(X_t) \right| dt \right] < \infty, \quad \text{for all } x > 0, \tag{3.1}$$

where

$$\mathcal{L} g(x) := \frac{1}{2} \sigma^2(x) g''(x) + b(x) g'(x) - r(x) g(x).$$

Remark 3.2.1 If we define the measurable function h as

$$h(x) = -\mathcal{L} g(x), \quad \text{for } x > 0, \tag{3.2}$$

then, the function g satisfies the non-homogeneous ODE

$$\frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) + h(x) = 0, \quad x \in]0, \infty[.$$

Therefore, by identifying R_h with g and by Propositions 2.4.1 and 2.4.2 in Section 2.4, the following statements are true.

a. Given any weak solution, $\mathbb{S}_x$ to (2.2) and any $(\mathcal{F}_t)$-stopping time, τ, the function g satisfies

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] < \infty, \quad \text{for all } x > 0. \tag{3.3}$$

In addition, Dynkin's formula holds,

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = g(x) + \mathbb{E}_x\left[\left(\int_0^\tau e^{-\Lambda_t}\mathcal{L}g(X_t)dt\right)\mathbf{1}_{\{\tau<\infty\}}\right], \quad \text{for all } x > 0. \tag{3.4}$$

b. The function g satisfies the following *transversality condition.* Given any weak solution, $\mathbb{S}_x$, to (2.2), if (τ_n) is a sequence of stopping times such that $\lim_{n\to\infty}\tau_n = \infty$, $\mathbb{P}_x$-a.s., then

$$\lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}\left|g(X_{\tau_n})\right|\mathbf{1}_{\{\tau_n<\infty\}}\right] = 0. \tag{3.5}$$

c. With reference to the functions ϕ and ψ, defined by (2.8) and (2.9), respectively, and the scale function p_c defined by (2.3), the function g satisfies the following identities

$$\lim_{x\downarrow 0}\frac{g(x)}{\phi(x)} = \lim_{x\to\infty}\frac{g(x)}{\psi(x)} = 0, \tag{3.6}$$

$$g'(x)\phi(x) - g(x)\phi'(x) = -p_c'(x)\int_x^\infty \mathcal{L}g(s)\phi(s)\,m(ds), \quad \text{for all } x > 0, \tag{3.7}$$

$$g'(x)\psi(x) - g(x)\psi'(x) = p_c'(x)\int_0^x \mathcal{L}g(s)\psi(s)\,m(ds), \quad \text{for all } x > 0. \tag{3.8}$$

Remark 3.2.2 In relation to the case of a perpetual American option, studied above, we observe that

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s}|\mathcal{L}g(X_s)|\,ds\right] = |(b-r)|\int_0^\infty e^{(b-r)s}ds + K$$

and so the conditions laid out in Assumption 3.2.1 are not satisfied if $b \geq r$, which reflects some of the comments made earlier.

3.3 The solution to the discretionary stopping problem

In solving our stopping problem we shall employ the tools of dynamic programming and break our overall problem into a series of sub-problems. We do this by considering, without loss of generality, our options at time zero, which are either to *wait* or *stop*. Consider the case when we wait for a time Δt and then continue optimally, we expect that the value function v should satisfy the following inequality

$$v(x) \geq \mathbb{E}_x\left[e^{-\Lambda_{\Delta T}}v(X_{\Delta T})\right].$$

Using Itô's formula, dividing by Δt and taking the limit $\Delta t \downarrow 0$ yields

$$\frac{1}{2}\sigma^2(x)v''(x) + b(x)v'(x) - r(x)v(x) \leq 0.$$

Alternatively, we can stop, and so we expect that

$$v(x) \geq g(x).$$

We therefore expect that the value function v identifies with a solution w to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\left\{\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x),\ g(x) - w(x)\right\} = 0, \quad x > 0. \tag{3.9}$$

To develop our intuition of the problem in hand further, consider the case where the payoff is increasing. In this case we expect that there is a single boundary point, x^*, separating the continuation and stopping regions and we postulate that it is optimal to wait for as long as the state process X assumes values less than x^* and stop as soon as X hits the set $[x^*, \infty[$. With reference to the heuristic arguments given above, we therefore look for a solution w to (3.9) that satisfies

$$\frac{1}{2}\sigma^2(x)\, w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x < x^*, \tag{3.10}$$

$$g(x) - w(x) = 0, \quad \text{for } x \geq x^*. \tag{3.11}$$

Such a solution is given by

$$w(x) = \begin{cases} A\phi(x) + B\psi(x), & \text{if } x < x^*, \\ g(x), & \text{if } x \geq x^*, \end{cases}$$

where ϕ (resp., ψ) is the strictly decreasing (resp., increasing) function given by (2.8) (resp., (2.9)). In addition, we expect that the value function is positive and bounded near zero and so we must have $A = 0$. To specify the parameter B and x^*, we appeal to the so-called "smooth-pasting" condition of optimal stopping that requires the value function to be C^1, in particular, at the free boundary point x^*. This requirement yields the system of equations

$$B\psi(x^*) = g(x^*) \quad \text{and} \quad B\psi'(x^*) = g'(x^*),$$

which is equivalent to

$$B = \frac{g(x^*)}{\psi(x^*)} = \frac{g'(x^*)}{\psi'(x^*)} \quad \text{and} \quad q(x^*) = 0,$$

where q is defined by

$$q(x) = g(x)\psi'(x) - g'(x)\psi(x), \quad x > 0,$$

and we note that $q(x^*) = 0$ corresponds to a turning point of g/ψ, since $\psi(x) > 0$ for all $x > 0$.

To develop an intuition as to how g/ψ affects the stopping problem, consider a different approach to solving the problem. Instead of looking for the stopping time look for the stopping boundary, for example define the value function as

$$\begin{aligned}\tilde{v}(x) &= \sup_b \mathbb{E}_x\left[e^{-\Lambda_{\tau_b}} g(b)\right] \\ &= \sup_b \mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] g(b)\end{aligned}$$

Now, starting in the "continuation" region, so $x < x^*$ since g is increasing, we have

$$\mathbb{E}_x[e^{-\Lambda_{\tau_b}}] = \psi(x)/\psi(b)$$

and

$$\tilde{v}(x) = \sup_b \left(\frac{g(b)}{\psi(b)}\right) \psi(x).$$

This approach is adopted in Beibel and Lerche [BL00], and provides a clear explanation of why maxima of g/ψ or g/ϕ are of interest. Unfortunately, this approach is of limited value when $x > x^*$.

With these comments in mind, we now provide the following verification theorem, as a prelude to our main results. Note that the theorems in this section all rely on Assumptions 2.2.1′, 2.2.2, 2.3.1′, 3.2.1, which are, respectively, that the SDE has a weak solution, it is non-explosive, discounting is strictly positive and the payoffs satisfy the transversality conditions.

Theorem 3.3.1 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 3.2.1 hold. In addition, suppose that the HJB equation (3.9) has a solution w and $w \in C^1(]0,\infty[) \cap C^2(]0,\infty[\setminus S)$, where S is a set of a finite number of points, then the value function v defined in Section 3.2 identifies with w.*

Proof: Fix any initial condition $x > 0$ and any weak solution $\mathbb{S}_x$ to (2.2) and define

$$M_t = \int_0^t e^{-\Lambda_s}\sigma(X_s)w'(X_s)dW_s \tag{3.12}$$

$$L_t = \int_0^t e^{-\Lambda_s}\sigma(X_s)g'(X_s)dW_s. \tag{3.13}$$

In addition, we note that our Assumptions mean that Dynkin's and Itô's formulae imply

$$\mathbb{E}_x\left[L_\tau \mathbf{1}_{\{\tau<\infty\}}\right] = 0 \tag{3.14}$$

for any stopping time τ. Now, fix any initial condition $x > 0$ and any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{S}_x$, define

$$\tau_n = \inf\left\{t \geq 0 \,\middle|\, X_t \leq 1/n\right\}, \quad \text{for } n \geq 1/x^*.$$

and we have

$$M_t^{\tau_n} - L_t^{\tau_n} = \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} e^{-\Lambda_s}\sigma(X_s)\big(w'(X_s) - g'(X_s)\big)\, dW_s. \tag{3.15}$$

Given (3.9) and the nature of our problem imply that $w(x) = g(x)$ for all $x \geq x^*$ and that σ^2, w' and g' are all locally bounded and that r is strictly positive,

$$\begin{aligned}
\mathbb{E}_x\left[\langle M_t^{\tau_n} - L_t^{\tau_n}\rangle_\infty\right] &= \mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{\{s\leq\tau_n\}} e^{-2\Lambda_s}\sigma^2(X_s)\Big(w' - g'\Big)^2(X_s)\, ds\right] \\
&\leq \sup_{x\in[1/n,x^*]} \sigma^2(x)\big(w'(x) - g'(x)\big)^2 \mathbb{E}_x\left[\int_0^\infty e^{-2\Lambda_s}\, ds\right] \\
&\leq \sup_{x\in[1/n,x^*]} \frac{\sigma^2(x)\big(w'(x) - g'(x)\big)^2}{2r_0} \\
&< \infty.
\end{aligned}$$

With reference to [RY99, Chapter IV, Proposition 1.23], this implies that $\big\{(M_t - L_t), t < \infty\big\}$ is an L^2-bounded martingale. Therefore, by appealing to Doob's optional sampling theorem, it follows that $\mathbb{E}_x\left[\left(M_\tau^{\tau_n} - L_\tau^{\tau_n}\right)\mathbf{1}_{\{\tau\wedge\tau_n<\infty\}}\right] = 0$, which combined with (3.14) implies that $\mathbb{E}_x\left[M_\tau^{\tau_n}\mathbf{1}_{\{\tau\wedge\tau_n<\infty\}}\right] = 0$. Now, since $w \in C^1(]0,\infty[) \cap C^2(]0,\infty[\setminus \{x^*\})$ and w' is of bounded variation, we can use Itô's formula to calculate

$$e^{-\Lambda_{\tau\wedge\tau_n}} w(X_{\tau\wedge\tau_n})\mathbf{1}_{\{\tau\wedge\tau_n<\infty\}} = \ w(x) + \left(\int_0^{\tau\wedge\tau_n} e^{-\Lambda_s}\mathcal{L}w(X_s)\, ds + M_\tau^{\tau_n}\right)\mathbf{1}_{\{\tau\wedge\tau_n<\infty\}}, \tag{3.16}$$

adding the term $e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\tau_n<\infty\}}$ to both sides of this equation, taking expectations and given that w satisfies (3.9), we have

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\tau_n<\infty\}}\right] \le w(x) - w(1/n)\mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}\mathbf{1}_{\{\tau_n\le\tau<\infty\}}\right]. \tag{3.17}$$

Applying the dominated convergence theorem, given (3.3), implies

$$\lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\tau_n<\infty\}}\right] = \mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{3.18}$$

The fact that w remains bounded as x tends to 0 together with the fact that 0 is inaccessible imply

$$\lim_{n\to\infty} w(1/n)\mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}\mathbf{1}_{\{\tau_n\le\tau<\infty\}}\right] = 0. \tag{3.19}$$

In view of (3.18)–(3.19), (3.17) implies

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau\le\infty\}}\right] \le w(x),$$

which proves $v(x) \le w(x)$.

To prove the reverse inequality, given any $T > 0$, let $(\mathbb{S}_x^*, \tau^*)$ be the strategy considered in the statement of the theorem. By following the arguments that lead to (3.17) we can see that

$$\mathbb{E}_x\left[e^{-\Lambda^*_{\tau^*}} g(X^*_{\tau^*})\mathbf{1}_{\{\tau^*\le\tau^*_n\wedge T\}}\right] = w(x) - \mathbb{E}_x\left[e^{-\Lambda^*_{\tau^*_n}} w(1/n)\mathbf{1}_{\{\tau^*_n\le T<\tau^*\}}\right] - \mathbb{E}_x\left[e^{-\Lambda^*_T} w(X^*_T)\mathbf{1}_{\{T<\tau^*_n<\tau^*\}}\right].$$

This calculation and (3.18)–(3.19) imply

$$\mathbb{E}_x\left[e^{-\Lambda^*_{\tau^*}} g(X^*_{\tau^*})\mathbf{1}_{\{\tau^*\le\infty\}}\right] = w(x),$$

which proves $v(x) \ge w(x)$, and establishes the optimality of $(\mathbb{S}^*_x, \tau^*)$, and the proof is complete. □

The motivation for the thesis was to develop the theory of discretionary stopping such that we could understand under what conditions x^* exists in the interval $]0,\infty[$. The discussion around the perpetual American call, in Section 3.2, demonstrate that some problems do not conform to standard economic theory, and we impose Assumption 3.2.1 in order to restrict ourselves to those problems that do conform to standard economic theory. Given this verification theorem, we can now prove our main results, which we split into three theorems. Theorem 3.3.2 focuses on the two cases where immediately stopping or never stopping is optimal, and the stopping boundary, x^*, is not in the interval $]0,\infty[$. Theorem 3.3.3 focuses on the cases where there is a single, continuous, continuation region, separated from a single, continuous stopping region by a single point $x^* \in]0,\infty[$. Theorem 3.3.4 considers two cases when two stopping boundaries exist in $]0,\infty[$.

Theorem 3.3.2 (No stopping boundaries) *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 3.2.1 hold. We have the following solutions to the discretionary stopping problem formulated in Section 3.2 when there are no stopping boundaries in* $]0,\infty[$.

Case I. If $\mathfrak{L}g$ is positive for all $x>0$ then given any initial condition $x>0$, the value function v identifies with $w(x)=0$ and there is no admissible stopping strategy.

Case II. If $\mathfrak{L}g$ is negative for all $x>0$ given any initial condition $x>0$, the *value function v identifies with $w(x) = g(x)$ and the stopping strategy $(\mathbb{S}_x^*, 0) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (2.2) is optimal.*

Proof of Case I. The structure of this case implies that $x_\psi = \infty$, hence the HJB equation is equivalent to

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0 \qquad \text{for all } x \in]0, \infty[, \tag{3.20}$$

$$w(x) \geq g(x) \qquad \text{for all } x \in]0, \infty[. \tag{3.21}$$

Clearly, $w(x) = 0$ satisfies (3.20) and given the arguments preceding these theorems, we have $g(x) < 0$ for all x, and (3.21) is satisfied. Also, $w \in C^1(]0, \infty[) \cap C^2(]0, \infty[)$, and so appealing to Theorem 3.3.1, we complete the proof.

Proof of Case II. The structure of this case implies that $x_\psi = 0$, hence the HJB equation is equivalent to

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) \leq 0 \qquad \text{for all } x \in]0, \infty[, \tag{3.22}$$

$$w(x) = g(x) \qquad \text{for all } x \in]0, \infty[. \tag{3.23}$$

Clearly, $w(x) = g(x)$ satisfies (3.23) and given the arguments preceding these theorems, we have $\mathcal{L}g(x) < 0$ for all x, and (3.22) is satisfied. Since g satisfies Dynkin's formula, (3.4), and the transversality condition, (3.5), we can use a modification Theorem 3.3.1 to complete the proof. □

Remark 3.3.1 It is important to appreciate how the integrability condition, Assumption 3.2.1 impacts our problem. Firstly, it ensures our problem data satisfies the transversality condition, and so it conforms to standard economic theory. In addition we have (3.8), which implies that if $\mathcal{L}g$ is positive then g/ψ is increasing, and we have (3.6), that $\lim_{x\to\infty} g(x)/\psi(x) = 0$. These immediately imply that $g(x) < 0$ for all x, and, since our stopping is discretionary, we would never stop. In the case where $\mathcal{L}g$ is negative, then g/ψ is decreasing, and the corresponding consequence of

(3.6) is that $g(x) > 0$ for all x. We also note that (3.4) implies that

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] < g(x), \qquad \text{for all } x > 0,$$

and so it would be optimal to stop immediately.

For future reference, we associate an increasing g/ψ (resp., decreasing g/ϕ) with the continuation region, while a decreasing g/ψ (resp., increasing g/ϕ) is associated with stopping.

Theorem 3.3.3 (One stopping boundary) *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 3.2.1 hold. We have the following solutions to the discretionary stopping problem formulated in Section 3.2 when there is one stopping boundary in* $]0, \infty[$.

Case I. If g/ψ achieves a maximum for some $x_\psi \in]0, \infty[$, but g/ϕ does not, and $\mathcal{L}g(x) \leq 0$ for all $x \geq x_\psi$ then the value function v identifies with the function w defined by

$$w(x) = \begin{cases} B\psi(x), & \text{if } x < x_\psi, \\ g(x), & \text{if } x \geq x_\psi, \end{cases} \tag{3.24}$$

with $B = g(x_\psi)/\psi(x_\psi) > 0$ being the value of the maximum. Furthermore, given any initial condition $x > 0$, the stopping strategy $(\mathbb{S}^*_x, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}^*_x$ is a weak solution to (2.2) and

$$\tau^* = \inf\{t \geq 0 \mid X_t \geq x_\psi\},$$

is optimal.

Case II. If g/ϕ achieves a maximum for some $x_\phi \in]0, \infty[$, but g/ψ does not,and $\mathcal{L}g(x) \leq 0$ for all $x \leq x_\phi$, then the value function v identifies with the *function w defined by*

$$w(x) = \begin{cases} A\phi(x), & \text{if } x > x_\phi, \\ g(x), & \text{if } x \le x_\phi, \end{cases} \tag{3.25}$$

with $A = g(x_\phi)/\phi(x_\phi) > 0$ being the value of the maximum. Furthermore, given any initial condition $x > 0$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (2.2) and

$$\tau^* = \inf\{t \ge 0 \mid X_t \le x_\phi\},$$

is optimal.

Proof of Case I. Firstly, we note that since g/ψ achieves a maximum at $x = x_\psi$ and $\mathcal{L}g(x) \le 0$ for all $x > x_\psi$ then g/ψ is decreasing for all $x > x_\psi$. Given that (3.6) holds if there is a maximum of g/ψ at x_ψ its value is positive, and so $B > 0$.

To prove that w given by (3.24) satisfies the HJB equation (3.9), we need to show that

$$g(x) - w(x) \le 0, \quad \text{for } x > x_\psi, \tag{3.26}$$

$$\mathcal{L}w(x) \le 0, \quad \text{for } x \le x_\psi. \tag{3.27}$$

Using the fact that B is given by the value of the maximum, we can see that (3.26) is equivalent to

$$B = \frac{g(x_\psi)}{\psi(x_\psi)} \ge \frac{g(x)}{\psi(x)}, \quad \text{for all } x \le x_\psi,$$

which is true, given g/ψ has a maximum. Similarly, with regard to the structure of w, given by (3.24), (3.27) is equivalent to

$$\mathcal{L}g(x) \le 0, \quad \text{for all } x \ge x_\psi, \tag{3.28}$$

which is true by assumption.

To complete the proof, we apply Theorem 3.3.1.

Proof of Case II. This case is symmetric to Case I. We note that $\mathcal{L}g(x) \le 0$ for all $x \le x_\phi$ combined with the fact that g/ϕ achieves a maximum at $x = x_\phi$ ensures that g/ϕ is increasing for all $x < x_\phi$. Given (3.6) holds, then the maximum of g/ϕ at x_ϕ is positive, and so $A > 0$.

To prove that w given by (3.25) satisfies the HJB equation (3.9), we need to show that

$$\mathcal{L}w(x) \le 0, \quad \text{for } x \le x_\phi, \tag{3.29}$$

$$g(x) - w(x) \le 0, \quad \text{for } x > x_\phi. \tag{3.30}$$

With regard to the structure of w, given by (3.24), (3.29) is equivalent to

$$\mathcal{L}g(x) \le 0, \quad \text{for all } x \le x_\phi, \tag{3.31}$$

which is true by assumption. Similarly, using the fact that A is given by the value of the maximum, we can see that (3.30) is equivalent to

$$A = \frac{g(x_\phi)}{\phi(x_\phi)} \ge \frac{g(x)}{\phi(x)}, \quad \text{for all } x > x_\phi,$$

which is true, given g/ϕ has a maximum.

To complete the proof, we apply Theorem 3.3.1. □

Theorem 3.3.4 (Two stopping boundaries) *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 3.2.1 hold. We have the following solutions to the discretionary stopping problem formulated in Section 3.2 when there are two stopping boundaries in* $]0, \infty[$..

Case I. ***This statement is incorrect.*** *If g/ϕ achieves a maximum for some $x_\phi \in]0, \infty[$, and g/ψ achieves a maximum for some $x_\psi \in]x_\phi, \infty[$. If, in addition, $\mathcal{L}g(x) \le 0$ for all $x \le x_\phi$ and for all $x \ge x_\psi$ then the value* *function v identifies with the function w defined by*

$$w(x) = \begin{cases} g(x), & \text{if } x \leq x_\phi, \\ A\phi(x) + B\psi(x), & \text{if } x_\phi < x < x_\psi, \\ g(x), & \text{if } x \geq x_\psi, \end{cases} \tag{3.32}$$

with $A = g(x_\phi)/\phi(x_\phi) > 0$, and, $B = g(x_\psi)/\psi(x_\psi) > 0$ being the values of the maxima. Furthermore, given any initial condition $x > 0$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (2.2) and

$$\tau^* \;=\; \inf\{t \geq 0 \,|\; X_t \notin]x_\phi, x_\psi[\},$$

is optimal.

Case II. If g/ψ achieves a maximum for some $x_\psi \in]0, \infty[$, and g/ϕ achieves a maximum for some $x_\phi \in]x_\psi, \infty[$. If, in addition, $\mathcal{L}g(x) \leq 0$ for all $x \in [x_\psi, x_\phi]$, then the value function v identifies with the function w defined by

$$w(x) = \begin{cases} B\psi(x), & \text{if } x < x_\psi, \\ g(x), & \text{if } x_\psi \leq x \leq x_\phi, \\ A\phi(x), & \text{if } x > x_\phi, \end{cases} \tag{3.33}$$

with $B = g(x_\psi)/\psi(x_\psi) > 0$, and $A = g(x_\phi)/\phi(x_\phi) > 0$ being the values of the maxima. Furthermore, given any initial condition $x > 0$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (2.2) and

$$\tau^* \;=\; \inf\{t \geq 0 \,|\; X_t \in [x_\psi, x_\phi]\},$$

is optimal.

Proof of Case I. As above, we note that given the condition on $\mathcal{L}g$ and (3.6), then the maxima of g/ϕ and g/ψ are positive.

To prove that w given by (3.32) satisfies the HJB equation (3.9), we need to show that

$$\mathcal{L}w(x) \leq 0, \quad \text{for } x \leq x_\phi, \tag{3.34}$$
$$g(x) - w(x) \leq 0, \quad \text{for } x_\phi < x < x_\psi, \tag{3.35}$$
$$\mathcal{L}w(x) \leq 0, \quad \text{for } x \geq x_\psi. \tag{3.36}$$

We can see immediately that (3.34) and (3.36) hold, given the structure of w, given by (3.32), and the structure of $\mathcal{L}g$, given by assumption. We can see that (3.35) is equivalent to

$$g(x) \leq A\phi(x) + B\psi(x), \text{if } x_\phi < x < x_\psi. \tag{3.37}$$

This is true for all $g(x) \leq 0$ given $x \in]0, \infty[$ and A, B, ψ and ϕ are positive. If $g(x) > 0$, we can write (3.37) in two ways

$$\frac{g(x)}{\phi(x)} \leq \frac{g(x_\phi)}{\phi(x_\phi)} + \frac{g(x_\psi)}{\phi(x_\psi)}\frac{\psi(x)}{\phi(x)} \quad \text{or} \quad \frac{g(x)}{\psi(x)} \leq \frac{g(x_\phi)}{\phi(x_\phi)}\frac{\phi(x)}{\psi(x)} + \frac{g(x_\psi)}{\phi(x_\psi)}$$

which are true given that $g(x_\phi)/\phi(x_\phi)$ and $g(x_\psi)/\phi(x_\psi)$ are maxima and that all the parameters are positive.

To complete the proof, we apply a modification of Theorem 3.3.1.

Proof of Case II. As above, we note that given the condition on $\mathcal{L}g$ and (3.6), then the maxima of g/ϕ and g/ψ are positive.

To prove that w given by (3.33) satisfies the HJB equation (3.9), we need to show that

$$g(x) - w(x) \leq 0, \quad \text{for } x < x_\psi, \tag{3.38}$$
$$\mathcal{L}w(x) \leq 0, \quad \text{for } x_\psi \leq x \leq x_\phi, \tag{3.39}$$
$$g(x) - w(x) \leq 0, \quad \text{for } x > x_\phi. \tag{3.40}$$

Again, (3.39) is true given the structure of w and the assumption of $\mathcal{L}g$ in the interval $[x_\psi, x_\phi]$. Similarly we can show that (3.38) and (3.40) are true given that A and B are maxima.

To complete the proof, we apply a modification of Theorem 3.3.1. □

Remark 3.3.2 These results are similar to those derived, in a general sense, in Beibel and Lerche [BL00]. The main difference is that Beibel and Lerche consider the case where you look for a stopping boundary, with the assumption that you start in a continuation region. They do not investigate the nature of the problem beyond the stopping boundary. By doing this, we are able to propose Case II, which is not considered in Beibel and Lerche.

Theorem 3.3.3 depends on the nature of ϕ and ψ and whether g/ϕ or g/ψ achieve maxima, which are conditions not exogenous to the problem data. We have the following sufficient conditions on the existence of turning points of g/ψ and g/ϕ that satisfy the conditions in Theorem 3.3.3 and can be deduced from the problem data.

Lemma 3.3.1 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 3.2.1 hold and if $g(x) > 0$ for some $x \in]0, \infty[$. Then we have the following two cases*

(a) If

$$\mathcal{L}g(x) \begin{cases} > 0, & \textit{for } x < x_1, \\ < 0, & \textit{for } x > x_1, \end{cases} \quad x_1 > 0, \tag{3.41}$$

then g/ψ achieves a unique maximum for some $x_\psi \in]x_1, \infty[$ and x_ψ is the unique solution to $q_\psi(x) = 0$, where q_ψ is defined by

$$q_\psi(x) = p_c'(x) \int_0^x \mathcal{L}g(s)\psi(s)\, m(ds), \quad \textit{for all } x > 0. \tag{3.42}$$

(b) If

$$\mathfrak{L}g(x) \begin{cases} < 0, & \textit{for } x < x_2, \\ > 0, & \textit{for } x > x_2, \end{cases} \quad x_2 > 0. \tag{3.43}$$

then g/ϕ achieves a unique maximum for some $x_\phi \in]0, x_2[$ and x_ϕ is the unique solution to $q_\phi(x) = 0$, where q_ϕ is defined by

$$q_\phi(x) = p_c'(x) \int_x^\infty \mathfrak{L}g(s)\phi(s)\, m(ds), \quad \textit{for all } x > 0. \tag{3.44}$$

Proof of Case a. We have that g/ψ is increasing for $x < x_1$, given (3.8) and the fact that $\mathfrak{L}g$ is positive for $x < x_1$. Since $g(x) > 0$ for some $x \in]0, \infty[$, then for the same interval g/ψ is positive. Given (3.6), then g/ψ achieves at least one maximum. Given (3.41) and (3.8), this maximum is unique. Using (3.8), (3.42) is simply

$$q_\psi(x) = \psi^2(x) \frac{d}{dx} \left(\frac{g(x)}{\psi(x)} \right)$$

and noting $\psi(x) > 0$ for all $x > 0$, the proof is complete.

Proof of Case b. Similarly, we have that g/ϕ is decreasing for $x > x_2$, given (3.7) and the fact that $\mathfrak{L}g$ is positive for $x > x_2$. Since $g(x) > 0$ for some $x \in]0, \infty[$, then for the same interval g/ϕ is positive. Given (3.6), then g/ϕ achieves at least one maximum. Given (3.43) and (3.7), this maximum is unique. Given (3.41) and (3.8), this maximum is unique. Using (3.7), (3.44) is simply

$$q_\phi(x) = -\phi^2(x) \frac{d}{dx} \left(\frac{g(x)}{\phi(x)} \right)$$

and noting $\phi(x) > 0$ for all $x < \infty$, the proof is complete. □

Remark 3.3.3 We may be interested in cases where the discount function r has a hyperbolic term, to reflect the difficulty of borrowing if the economic environment is poor. In these cases we might be confronted with $\mathfrak{L}g$ making more than one crossing of $\mathfrak{L}g(x) = 0$. In these cases we have the following heuristic.

1. If $\mathcal{L}g(x) > 0$ for all $x < x_1$ and $\mathcal{L}g(x) < 0$ for all $x > x_2$, $x_1 \leq x_2$, then Case I of Theorem 3.3.3 is most likely to apply.

2. If $\mathcal{L}g(x) < 0$ for all $x < x_1$ and $\mathcal{L}g(x) > 0$ for all $x > x_2$, $x_1 \leq x_2$, then Case II of Theorem 3.3.3 is most likely to apply.

3. If $\mathcal{L}g(x) < 0$ for all $x < x_1$ and $\mathcal{L}g(x) < 0$ for all $x > x_2$, $x_1 \leq x_2$, then Case I of Theorem 3.3.4 could apply.

4. If $\mathcal{L}g(x) > 0$ for all $x < x_1$ and $\mathcal{L}g(x) > 0$ for all $x > x_2$, $x_1 \leq x_2$, then Case II of Theorem 3.3.4 could apply.

3.4 Special cases

We now consider a number of special cases of the general discretionary stopping problem that we studied in the previous section. Our aim here is to establish under what conditions the boundary, x^*, between the continuation and stopping region lies in $]0, \infty[$. This is in line with the motivation for the thesis discussed in the introduction.

We focus on cases where the discounting is constant, $r(x) = r > 0$. This is motivated by the fact that we can find explicit solutions to the relevant ODE. Remark 3.3.3 can provide insights into the cases where we have state dependent discounting. We consider state process dynamics that are widely used in finance and economics and payoffs associated with commonly used utility function. In particular, we investigate the situation when X is a *geometric Brownian motion*, in which case

$$b(x) = bx \quad \text{and} \quad \sigma(x) = \sigma x, \quad \text{for all } x > 0,$$

a *square-root mean-reverting* process, which arises when

$$b(x) = \kappa(\theta - x) \quad \text{and} \quad \sigma(x) = \sigma\sqrt{x}, \quad \text{for all } x > 0,$$

or a *geometric Ornstein-Uhlenbeck* process, in which case

$$b(x) = \kappa(\theta - x)x \quad \text{and} \quad \sigma(x) = \sigma x, \quad \text{for all } x > 0.$$

The payoff function, g, is given as

$$g(x) = \xi x^{\eta} - K \tag{3.45}$$

$$g(x) = \xi \ln(x + \eta) - K, \tag{3.46}$$

$$g(x) = \gamma\big(1 - \xi e^{-\eta x}\big), \tag{3.47}$$

where $\xi, \eta, \gamma > 0$ and $K \in \mathbb{R}$ are constants. For $\eta \in]0,1[$ and $K = 0$, the choice of g as in (3.45) identifies with a power utility function, while for $\eta \geq 1$, such a choice is associated with a perpetual American power option, discussed in the introduction. Choices of g as in (3.46) and (3.47) are associated with logarithmic utility and exponential utility functions, respectively.

The Itô diffusions under consideration have been well studied in the literature, and they all satisfy Assumptions 2.2.1′ and 2.2.2. In all cases, we assume that $r(x) \equiv r$, for some constant $r > 0$, so that Assumption 2.3.1′ is satisfied. So, our first objective is to establish under what conditions these state process dynamics and payoffs satisfy the transversality condition, which is true if Assumption 3.2.1 holds. Secondly, we wish to find the conditions under which $x^* \in]0, \infty[$. Our final objective in this section is to derive expressions for ϕ and ψ, once these are known, with the knowledge that $x^* \in]0, \infty[$, it is a straightforward exercise to identify the value of x^* and the value function.

Our choices of g are increasing and positive for some interval, so we would expect them to be associated with Case I of Theorem 3.3.3. Therefore, given that all our payoffs are positive for some x, we need to investigate under what conditions $\mathcal{L}g$ crosses the y-axis once, and we have the cases in Lemma 3.3.1. Clearly if our Assumptions hold for g, then they will hold for $-g$ with (3.43) replacing (3.41), and we would expect the choice of $-g$ to be associated with Case II of Theorem 3.3.3.

To show (3.1) of Assumption 3.2.1 is satisfied, it is sufficient to show that there exists a constant $j \geq 1$ such that

$$\big|\mathfrak{L}g(x)\big| \leq C(1+x^j), \quad \text{for all } x > 0, \tag{3.48}$$

$$\mathbb{E}\left[X_s^j\right] < \infty. \tag{3.49}$$

To see this, consider any weak solution, $\mathbb{S}_x$, to (2.2), and combine (2.20) from Assumption 2.3.1′, (3.48) with (3.49), fact that $X_t > 0$ and Fubini's Theorem, and so

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|\mathfrak{L}g(X_t)\big|dt\right] &\leq C\int_0^\infty e^{-\Lambda_t}dt + C\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}X_t^j dt\right] \\
&\leq \frac{C}{r_0} + C\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s}X_t^j dt\right] \\
&\leq \frac{C}{r_0} + C\int_0^\infty e^{-r_0 t}\mathbb{E}_x\left[X_t^j\right]dt \\
&< \infty.
\end{aligned}$$

It is a matter of calculation to verify that if g is as in (3.45) then j in (3.48) corresponds to η for a geometric Brownian motion and square root mean reverting diffusion and $\eta+1$ for an exponential and geometric Ornstein-Uhlenbeck diffusion. If g is as in (3.46), $j=1$ for all the diffusions under consideration, where as for g is as in (3.47), $\mathfrak{L}g(x) < C$ for all x. Hence, verification of Assumption 3.2.1 is a case of verifying (3.49) for these values of j. However, (3.49) holds for all the diffusions under consideration apart from geometric Brownian motion, as they all have finite moments of all orders.

We now turn our attention to identifying the functions ϕ and ψ that are associated with the four diffusions under consideration. It turns out that a number of the cases considered are related to Kummer's ordinary differential equation

$$zu''(z) + (\beta - z)u'(z) - \alpha u(z) = 0, \tag{3.50}$$

where $\alpha, \beta > 0$ are constants. Independent solutions to this ordinary differential equation can be expressed in terms of the confluent hypergeometric function ${}_1F_1$, defined by

$$
{}_1F_1(\alpha, \beta; z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\alpha)_m}{(\beta)_m} z^m,
$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$, and the function U, which is defined by

$$
U(\alpha, \beta; z) = \frac{\pi}{\sin \pi\beta} \left[\frac{{}_1F_1(\alpha, \beta; z)}{\Gamma(1+\alpha-\beta)\Gamma(\beta)} - z^{1-\beta} \frac{{}_1F_1(\alpha+1-\beta, 2-\beta; z)}{\Gamma(\alpha)\Gamma(2-\beta)} \right]
$$

(see Magnus, Oberhettinger and Soni [MOS66, Chapter VI] or Abramowitz and Stegun [AS72, Chapter 13]). We have, in addition, that

$$
\frac{d}{dz}\, {}_1F_1(\alpha, \beta; z) = \frac{\alpha}{\beta}\, {}_1F_1(\alpha+1, \beta+1; z) \quad \text{and} \quad \frac{d}{dz} U(\alpha, \beta; z) = -\alpha U(\alpha+1, \beta+1; z).
$$

For future reference, observe that for $\alpha, \beta > 0$, ${}_1F_1(\alpha, \beta; \cdot)$ is positive and strictly increasing on $]0, \infty[$, ${}_1F_1(\alpha, \beta; 0) = 1$ and $\lim_{z\to\infty} {}_1F_1(\alpha, \beta; z) = \infty$. Also, recalling the identity

$$
\frac{\pi}{\sin \pi\beta} = \Gamma(\beta)\Gamma(1-\beta),
$$

(see Magnus, Oberhettinger and Soni [MOS66, Chapter I] or Abramowitz and Stegun [AS72, 6.1.7]), we can see that

$$
U(\alpha, \beta; z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\, {}_1F_1(\alpha, \beta; z) - \frac{\Gamma(\beta)}{(1-\beta)\Gamma(\alpha)} z^{1-\beta}\, {}_1F_1(\alpha+1-\beta, 2-\beta; z).
$$

With regard to this expression, it is worth noting that, although the gamma function $x \mapsto \Gamma(x)$ has simple poles at $x = -m$, $m \in \mathbb{N}^*$, U is well defined and finite for $\beta = 2, 3, 4, \ldots$. Although we do not need this result in our analysis, it is worth noting that $\lim_{z\downarrow 0} U(\alpha, \beta; z) = \infty$ if $\beta > 1$. Also, for $\alpha > 0$ and $\beta > 1$, $U(\alpha, \beta; \cdot)$ is positive, strictly decreasing in $]0, \infty[$ and $\lim_{z\to\infty} U(\alpha, \beta; z) = 0$ (see Magnus, Oberhettinger and Soni [MOS66, Chapter VI] or Abramowitz and Stegun [AS72, Chapter 13]).

3.4.1 Geometric Brownian motion

Geometric Brownian motion is the most commonly used model in finance for the value of an asset. In this case, the state process dynamics are given by

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x > 0,$$

where b, σ are constants and the ODE associated with (3.9) is given by

$$\frac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{3.51}$$

We start by establishing the conditions under which (3.49) holds. Given

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t} X_t^j\,dt\right] = \int_0^\infty x \exp\left\{\left(jb - j\frac{1}{2}\sigma^2 + \frac{1}{2}j^2\sigma^2 - r\right)t\right\} \mathbb{E}\left[e^{\left(-\frac{1}{2}j^2\sigma^2 t + j\sigma W_t\right)}\right] dt$$

Assumption 3.2.1 is satisfied if

$$r > jb + \frac{1}{2}j(j-1)\sigma^2.$$

It is a straightforward, all be it tedious, exercise to verify that (3.41) is satisfied when

$$\begin{aligned}
&g \text{ is given by (3.45), } K > 0 \text{ and } r > \eta b + \frac{1}{2}\eta(\eta - 1)\sigma^2,\\
&g \text{ is given by (3.46) and } K > \xi \ln \eta,\\
&g \text{ is given by (3.47) and } \xi > 1.
\end{aligned}$$

The proof of the following well-known result is straightforward and omitted.

Lemma 3.4.1 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (3.51) are given by*

$$\psi(x) = x^n \quad \textit{and} \quad \phi(x) = x^m,$$

where the constants $m < 0 < n$ are defined by

$$m, n = \frac{1}{2} - \frac{b}{\sigma^2} \pm \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\,.$$

3.4.2 Square-root mean-reverting process

The diffusion X defined by

$$dX_t = \kappa(\theta - X_t)\,dt \;+\; \sigma\sqrt{X_t}\,dW_t, \quad X_0 = x > 0,$$

where κ, θ and σ are positive constants satisfying $\kappa\theta - \frac{1}{2}\sigma^2 > 0$ models the short rate in the Cox-Ingersoll-Ross interest rate model, and has attracted considerable interest in the theory of finance. Note that the assumption that $\kappa\theta - \frac{1}{2}\sigma^2 > 0$ is imposed because it is necessary and sufficient for X to be non-explosive, in particular for the hitting time of 0 to be infinite with probability 1. Also, the ODE associated with (3.9) takes the form

$$\frac{1}{2}\sigma^2 x w''(x) + \kappa(\theta - x)w'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{3.52}$$

We can verify that (3.41) is satisfied when

$$\begin{aligned}
&g \text{ is given by (3.45) and } K > 0,\\
&g \text{ is given by (3.46) and } K > \xi\left(\ln\eta - \frac{\kappa\theta}{\eta r}\right),\\
&g \text{ is given by (3.47) and } \xi > \frac{r}{r + \eta\kappa\theta}.
\end{aligned}$$

Lemma 3.4.2 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (3.52) are given by*

$$\psi(x) = {}_1F_1\left(\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa}{\sigma^2}x\right) \quad \textit{and} \quad \phi(x) = U\left(\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa}{\sigma^2}x\right).$$

Proof. Setting $y = 2\kappa x/\sigma^2$ and $h(y) = w(x)$, the ODE (3.52) becomes

$$yh''(y) + \left(\frac{2\kappa\theta}{\sigma^2} - y\right) h'(y) - \frac{r}{\kappa} h(y) = 0,$$

which is Kummer's equation for $\alpha = r/\kappa > 0$ and $\beta = 2\kappa\theta/\sigma^2 > 1$, the inequality as a consequence of the assumption that $\kappa\theta - \frac{1}{2}\sigma^2 > 0$, and the result follows. □

3.4.3 Geometric Ornstein-Uhlenbeck process

The diffusion X defined by

$$dX_t = \kappa(\theta - X_t)X_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{3.53}$$

where κ, θ and σ are positive constants, has been proposed by Cortazar and Schwartz [CS97] as a model for a commodity's price and has played a role in population modelling. The ordinary differential equation associated with (3.9) for this diffusion takes the form

$$\frac{1}{2}\sigma^2 x^2 w''(x) + \kappa(\theta - x)xw'(x) - rw(x) = 0, \quad \text{for } x > 0. \tag{3.54}$$

We can verify that (3.41) is satisfied when

$$\begin{aligned} &g \text{ is given by (3.45) and } K > 0,\\ &g \text{ is given by (3.46) and } K > \xi \ln \eta,\\ &g \text{ is given by (3.47) and } \xi > 1. \end{aligned}$$

Lemma 3.4.3 *The increasing function ψ and the decreasing function ϕ spanning the solution set to (3.54) are given by*

$$\psi(x) = x^n\,{}_1F_1\left(n, 2n + \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa x}{\sigma^2}\right) \quad \textit{and} \quad \phi(x) = x^n U\left(n, 2n + \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa x}{\sigma^2}\right),$$

where

$$n = \frac{1}{2} - \frac{\kappa\theta}{\sigma^2} + \sqrt{\left(\frac{\kappa\theta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\,.$$

Proof. Motivated by Dixit and Pindyck [DP94, Chapter 5, Section 5A], we consider a candidate for the solution to (3.54) of the form

$$w(x) = Ax^n f(x)$$

which results in

$$x^n f(x)\left[\frac{1}{2}\sigma^2 n(n-1) + \kappa\theta n - r\right] \\ + x^{n+1}\left[\frac{1}{2}\sigma^2 x f''(x)\big(\sigma^2 n + \kappa[\theta - x]\big) f'(x) - \kappa n f(x)\right] = 0.$$

This can be true for all $x > 0$ only if

$$\frac{1}{2}\sigma^2 n(n-1) + \kappa\theta n - r = 0, \tag{3.55}$$

and

$$\frac{1}{2}\sigma^2 x f''(x) + (\sigma^2 n + \kappa\theta - \kappa x) f'(x) - \kappa n f(x) = 0. \tag{3.56}$$

We note that the negative solution to (3.55) would result in choices for ψ and ϕ not having the required monotonicity properties. Choosing n to be the positive solution to (3.55), and setting $x = \sigma^2 y/(2\kappa)$ and $g(y) = f(x)$, we can see that (3.56) becomes

$$y g''(y) + \left(2n + \frac{2\kappa\theta}{\sigma^2} - y\right) g'(y) - n g(y) = 0,$$

which is Kummer's equation with $\alpha = n > 0$ and $\beta = 2n + 2\kappa\theta/\sigma^2 > 0$ and the expressions for ψ and ϕ in the statement follow.

Since x^n and ${}_1F_1$ are both increasing functions, the function ψ is plainly increasing. To see that ϕ is decreasing, we recall that

$$zU(a, b+1; z) = U(a-1, b; z) + (b-a)U(a, b; z)$$

(see Magnus, Oberhettinger and Soni [MOS66, Section 6.2]). Using this result, we calculate,

$$\frac{d}{dx}\phi(x) = -(\beta - \alpha + 1)nx^{n-1}U(\alpha + 1, \beta; x)$$

which is negative for if and only if $\beta > \alpha - 1$. However, we can see that $\beta > \alpha - 1$ if and only if

$$\frac{3}{2}\frac{\kappa\theta}{\sigma^2} + \sqrt{\left(\frac{\kappa\theta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > \frac{1}{2}\,,$$

which is true for all $\kappa, \theta, \sigma, r > 0$. $\square$

4. THE INVESTMENT PROBLEMS

4.1 Introduction

We now consider investment problems that involve both single entry and exit as well sequential entry and exit. Section 4.2 provides a formulation of the various investment problems studied.

In Section 4.3 we solve the single entry and exit problems. Specifically, we address the issues of the initialisation of a random cashflow, the abandonment of an existing cashflow and the initialisation and then the abandonment of a stochastic cashflow. We show that these problems can all be reduced to appropriate versions of the discretionary stopping problem studied in Chapter 3.

In Section 4.4, we study the sequential entry and exit decisions relying on intuition developed in Section 4.3. We start by investigating the case where multiple entry and exit decisions may define the optimal strategy. We then consider the simpler case where being "in" or "out" of the investment is optimal, whatever the value of the state process. We finish by considering the case, where although any number of entry and exit decisions are possible, the optimal strategy is, in fact, similar to the initialisation or abandonment strategies in Section 4.3.

The problems in Section 4.3 have been formulated to be irreversible, or in the case of the initialisation and then the abandonment of a cashflow, partially reversible. The problems in Section 4.4 have been formulated to be reversible, however switching back (reversing) may not form part of the optimal strategy.

4.2 Problem formulation and assumptions

We consider the following classes of decision strategies.

Definition 4.2.1 Given an initial condition $x > 0$, we define the following decision strategies, bearing in mind the definition of a stopping strategy given in Definition 3.2.1.

An *initialisation* strategy is any admissible stopping strategy $(\mathbb{S}_x, \tau_1) \in \mathcal{S}_x$.

An *abandonment* strategy is any admissible stopping strategy $(\mathbb{S}_x, \tau_0) \in \mathcal{S}_x$.

An admissible *initialisation and abandonment* strategy is any triplet $(\mathbb{S}_x, \tau_1, \tau_0)$ where $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution to (2.2) and τ_1, τ_0 are $(\mathcal{F}_t)$-stopping times such that $\tau_1 \le \tau_0$, $\mathbb{P}_x$-a.s.. We denote by $\mathcal{C}_x$ the family of all such admissible strategies.

An admissible *switching* strategy, is a pair $(\mathbb{S}_x, Z)$ where $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution to (2.2) and Z is an $\mathcal{F}_t$ adapted, finite variation, càdlàg process taking values in $\{0, 1\}$ with $Z_0 = z$. We denote by $\mathcal{Z}_{x,z}$ the set of all such control strategies.

We consider the following related optimisation problems. The first one, the *initialisation of a payoff flow problem*, can be regarded as determining the optimal time at which a decision maker should activate an investment project. In this context, each initialisation strategy $(\mathbb{S}_x, \tau_1) \in \mathcal{S}_x$ is associated with the performance criterion

$$J^{\mathrm{I}}(\mathbb{S}_x, \tau_1) = \mathbb{E}_x \left[\left(\int_{\tau_1}^{\infty} e^{-\Lambda_t} h(X_t)\, dt - e^{-\Lambda_{\tau_1}} g_1(X_{\tau_1}) \right) \mathbf{1}_{\{\tau_1 < \infty\}} \right]. \tag{4.1}$$

Here $h :]0, \infty[\to \mathbb{R}$ is a deterministic function modelling the payoff flow that the project yields after its initialisation, while $g_1 :]0, \infty[\to \mathbb{R}$ is a deterministic function modelling the cost of initialising the project and where $\Lambda_t := \int_0^t r(X_s) ds$, with $r :]0, \infty[\to]0, \infty[$, is a given deterministic discounting function. The objective of the decision maker is to maximise J^{I} over all initialisation strategies, $(\mathbb{S}_x, \tau_1) \in \mathcal{S}_x$. The resulting value function is defined by

$$v^{\mathrm{I}}(x) = \sup_{(\mathbb{S}_x, \tau_1) \in \mathcal{S}_x} J^{\mathrm{I}}(\mathbb{S}_x, \tau_1), \quad \text{for } x > 0. \tag{4.2}$$

The *abandonment of a payoff flow problem* aims at determining the optimal time at which a decision maker, who receives a payoff from an active investment project, should terminate the project. In this case, each abandonment strategy, $(\mathbb{S}_x, \tau_0) \in \mathcal{S}_x$, is associated with the performance index

$$J^{\mathrm{A}}(\mathbb{S}_x, \tau_0) = \mathbb{E}_x \left[\int_0^{\tau_0} e^{-\Lambda_t} h(X_t)\, dt - e^{-\Lambda_{\tau_0}} g_0(X_{\tau_0}) \mathbf{1}_{\{\tau_0 < \infty\}} \right]. \tag{4.3}$$

Here, the function $h :]0, \infty[\to \mathbb{R}$ is the same as in (4.1), while $g_0 :]0, \infty[\to \mathbb{R}$ is a deterministic function modelling the cost of abandoning the project. This problem's value function is given by

$$v^{\mathrm{A}}(x) = \sup_{(\mathbb{S}_x, \tau_0) \in \mathcal{S}_x} J^{\mathrm{A}}(\mathbb{S}_x, \tau_0), \quad \text{for } x > 0. \tag{4.4}$$

The *initialisation of a payoff flow with the option to abandon problem* is the combination of the previous two problems. It arises when a decision maker is faced with the requirement to optimally determine the time at which an investment project should be activated and, subsequently, the time at which the project should be abandoned. In this problem, the associated performance criterion is defined by

$$\begin{aligned} J^{\mathrm{IA}}(\mathbb{S}_x, \tau_1, \tau_0) = \mathbb{E}_x \Bigg[\left(-e^{-\Lambda_{\tau_1}} g_1(X_{\tau_1}) + \int_{\tau_1}^{\tau_0} e^{-\Lambda_t} h(X_t) dt \right) \mathbf{1}_{\{\tau_1 < \infty\}} \\ - e^{-\Lambda_{\tau_0}} g_0(X_{\tau_0}) \mathbf{1}_{\{\tau_0 \leq \infty\}} \Bigg], \end{aligned} \tag{4.5}$$

where the functions h, g_1 and g_0 are the same as in (4.1) and (4.3). The objective is to maximise this performance index over all admissible initialisation and abandonment strategies in $\mathcal{C}_x$, and the associated value function is defined by

$$v^{\mathrm{IA}}(x) = \sup_{(\mathbb{S}_x, \tau_1, \tau_0) \in \mathcal{C}_x} J^{\mathrm{IA}}(\mathbb{S}_x, \tau_1, \tau_0), \quad \text{for } x > 0. \tag{4.6}$$

The *sequential entry and exit problem* models the situation where the decision maker can choose between two deterministic payoffs of the state process, and there is no limit as to the number of times the decision maker can switch between the two payoffs, and the decision maker can reverse their decision. In this problem, the task of the decision maker is to select the times when the system should be switched. These sequential decisions form a control strategy and they are modelled by the process Z. For clarity, we shall differentiate the two payoffs by calling one "closed" and the other "open". If the system is open at time t then $Z_t = 1$, where as if it closed at time t then $Z_t = 0$. While operating in the open mode the system provides a running payoff given by a function $h_1 :]0, \infty[\to \mathbb{R}$ and while in closed mode the payoff is given by $h_0 :]0, \infty[\to \mathbb{R}$. The transition from one operating mode to the other is immediate. The transitions between open and closed modes are indicated by $(\Delta Z_s)^+ = \mathbf{1}_{\{Z_t - Z_{t-} = 1\}}$ with a cost given by the function $g_1 :]0, \infty[\to \mathbb{R}$. Switching between closed and open modes is given by $(\Delta Z_s)^- = \mathbf{1}_{\{Z_{t-} - Z_t = 1\}}$ and the associated cost is given by $g_0 :]0, \infty[\to \mathbb{R}$. Given this formulation, the objective of the decision maker is to maximise the performance criterion

$$\begin{aligned} J^{\mathrm{S}}(\mathbb{S}_x, Z) := \mathbb{E}_{x,z}\bigg[\int_0^\infty & e^{-\Lambda_t} \Big[Z_t h_1(X_t) + (1 - Z_t) h_0(X_t) \Big] dt \\ & - \sum_{0 \le s} e^{-\Lambda_s} \Big(g_1(X_s)(\Delta Z_s)^+ + g_0(X_s)(\Delta Z_s)^- \Big) \bigg] \end{aligned} \tag{4.7}$$

over all admissible switching strategies. Accordingly we define the value function v by

$$v^{\mathrm{S}}(x, z) := \sup_{(\mathbb{S}_x, Z) \in \mathcal{Z}_{x,z}} J^{\mathrm{S}}(\mathbb{S}_x, Z), \quad \text{for } x > 0. \tag{4.8}$$

In place of Assumption 3.2.1, used in the discretionary stopping problem, we impose the following conditions on the investment problems data.

Assumption 4.2.1 *The functions $g_1,\ g_0 :]0,\infty[\to \mathbb{R}$ are C^1 with absolutely continuous first derivatives and h and $h_1 - h_0$ are measurable. In addition:*

a. Given any weak solution, $\mathbb{S}_x$ to (2.2) the functions $g_1,\ g_0$ satisfy

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left|\mathfrak{L}g_0(X_t)\right|dt\right] < \infty, \quad \textit{for all } x > 0, \tag{4.9}$$

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left|\mathfrak{L}g_1(X_t)\right|dt\right] < \infty, \quad \textit{for all } x > 0, \tag{4.10}$$

where the operator $\mathfrak{L}$ is defined as in (2.6).

b. Given any weak solution, $\mathbb{S}_x$ to (2.2) the functions $h,\ h_1 - h_0$ satisfy

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left|h(X_t)\right|dt\right] < \infty, \quad \textit{for all } x > 0, \tag{4.11}$$

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left|h_1(X_t) - h_0(X_t)\right|dt\right] < \infty, \quad \textit{for all } x > 0, \tag{4.12}$$

These conditions are the integrability conditions which imply the transversality condition necessary for our problem data to conform to economic theory.

In the cases where there is the possibility of both entry and exit decisions, we impose the following assumption based on hindsight from our subsequent analysis.

Assumption 4.2.2 *The functions $g_1,\ g_0 :]0,\infty[\to \mathbb{R}$ satisfy the following conditions*

$$\mathfrak{L}\left(g_0 + g_1\right)(x) < 0 \quad \textit{for all } x > 0. \tag{4.13}$$

This condition ensures entry takes place before exit, it separates the entry and exit point and ensures the exit point is below the entry point.

In the case of the sequential entry and exit problem, in order to exclude the possibility of making a profit simply by switching, instantaneously, between open and closed modes, we have an additional assumption.

Assumption 4.2.3 *The functions $g_1,\ g_0 :]0,\infty[\to \mathbb{R}$ satisfy the following conditions*

$$g_0(x) + g_1(x) > 0, \quad \textit{for all } x > 0. \tag{4.14}$$

4.3 The single entry or exit investment problem

This section considers the two irreversible problems of initialising, with no subsequent abandonment, of a payoff flow (Subsection 4.3.1) and of abandoning, with no subsequent re-initialisation of a payoff flow (Subsection 4.3.2). The section ends with the semi-reversible problem of initialising a payoff with the option to subsequently abandon the payoff.

The theorems in this section rely on Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, which are, respectively, that the SDE has a weak solution, it is non-explosive, discounting is strictly positive and the payoffs satisfy the transversality conditions.

4.3.1 The initialisation of a payoff flow problem

Given Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 4.2.1, we have that the function R_h, defined in Proposition 2.4.1 with the representation

$$R_h(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s} h(X_s)\, ds\right], \quad \text{for all } x > 0,$$

is well-defined. Furthermore, we can use (2.39) of Proposition 2.4.2 to see that, given any initialisation strategy $(\mathbb{S}_x, \tau_1)$,

$$J^{\mathrm{I}}(\mathbb{S}_x, \tau_1) = \mathbb{E}_x\left[e^{-\Lambda_{\tau_1}}\left(R_h - g_1\right)(X_{\tau_1})\mathbf{1}_{\{\tau_1<\infty\}}\right].$$

It is easy to confirm that Assumption 4.2.1 means that all the assumptions of Theorems 3.3.2 and 3.3.3 are satisfied, so, with reference to Lemma 3.3.1, its solution is provided by the following result.

Theorem 4.3.1 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 4.2.1 hold, and consider the initialisation problem formulated in Section 4.2.*

Case I. *If $\mathcal{L}\left(R_h - g_1\right)(x) > 0$ for all $x > 0$ then given any initial condition $x > 0$, the value function is given by $v^{\mathrm{I}}(x) = 0$ and there is no admissible initialisation strategy.*

Case II. *If $\mathcal{L}\left(R_h - g_1\right)(x) < 0$ for all $x > 0$ then given any initial condition $x > 0$, the value function is given by $v^{\mathrm{I}}(x) = \left(R_h - g_1\right)(x)$ and the initialisation strategy $(\mathbb{S}^{\mathrm{I}}_x, 0) \in \mathcal{S}_x$, where $\mathbb{S}^{\mathrm{I}}_x$ is a weak solution to (2.2) is optimal.*

Case III. *If $\left(R_h - g_1\right)(x) > 0$ for some $x \in]0, \infty[$ and*

$$\mathcal{L}\left(R_h - g_1\right)(x) \begin{cases} > 0, & \text{for } x < x_1, \\ < 0, & \text{for } x > x_1, \end{cases} \quad x_1 > 0. \tag{4.15}$$

The value function v^{I} identifies with the function w^{I} defined by

$$w^{\mathrm{I}}(x) = \begin{cases} B\psi(x), & \text{if } x < x^{\mathrm{I}}_1, \\ \left(R_h - g_1\right)(x), & \text{if } x \geq x^{\mathrm{I}}_1, \end{cases} \tag{4.16}$$

with $x^{\mathrm{I}}_1 > 0$ being the unique solution to $q^{\mathrm{I}}(x) = 0$, where q^{I} is defined by

$$q^{\mathrm{I}}(x) = p'_c(x) \int_0^x \mathcal{L}\left(R_h - g_1\right)(s)\psi(s)\, m(ds), \quad \text{for all } x > 0, \tag{4.17}$$

and $B^{\mathrm{I}} > 0$ being given by

$$B^{\mathrm{I}} = \frac{\left(R_h - g_1\right)(x^{\mathrm{I}}_1)}{\psi(x^{\mathrm{I}}_1)} = \frac{\left(R_h - g_1\right)'(x^{\mathrm{I}}_1)}{\psi'(x^{\mathrm{I}}_1)}. \tag{4.18}$$

Furthermore, given any initial condition $x > 0$, the initialisation strategy $(\mathbb{S}^{\mathrm{I}}_x, \tau^{\mathrm{I}}_1) \in \mathcal{S}_x$, where $\mathbb{S}^{\mathrm{I}}_x$ is a weak solution to (2.2) and

$$\tau^{\mathrm{I}}_1 = \inf\{t \geq 0 \mid X_t \geq x^{\mathrm{I}}_1\},$$

is optimal.

4.3.2 The abandonment of a payoff flow problem

Now, (2.38) implies that, given any abandonment strategy $(\mathbb{S}_x, \tau_0)$,

$$J^{\mathrm{A}}(\mathbb{S}_x, \tau_0) = R_h(x) - \mathbb{E}_x \left[e^{-\Lambda_{\tau_0}} \left(R_h + g_0\right)(X_{\tau_0}) \mathbf{1}_{\{\tau_0 < \infty\}} \right]$$

and so

$$v^{\mathrm{A}}(x) = R_h(x) + \sup_{(\mathbb{S}_x, \tau_0) \in \mathcal{S}_x} \mathbb{E}_x \left[-e^{-\Lambda_{\tau_0}} \left(R_h + g_0\right)(X_{\tau_0}) \mathbf{1}_{\{\tau_0 < \infty\}} \right].$$

The structure of R_h implies that we associate this situation with Case II of Theorem 3.3.3, and so with reference to Lemma 3.3.1 its solution is provided by the following result.

Theorem 4.3.2 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′ and 4.2.1 hold, and consider the abandonment problem formulated in Section 4.2.*

Case I. *If $\mathcal{L}\left(R_h + g_0\right)(x) < 0$ or $\left(R_h + g_0\right)(x) \geq 0$ for all $x > 0$ then given any initial condition $x > 0$, the value function is given by $v^{\mathrm{A}}(x) = R_h(x)$ and there is no admissible abandonment strategy.*

Case II. *If $\mathcal{L}\left(R_h + g_0\right)(x) > 0$ for all $x > 0$ then given any initial condition $x > 0$, the value function is given by $v^{\mathrm{A}}(x) = -g_0(x)$ and the abandonment strategy $(\mathbb{S}_x^{\mathrm{A}}, 0) \in \mathcal{S}_x$, where $\mathbb{S}_x^{\mathrm{A}}$ is a weak solution to (2.2) is optimal.*

Case III. *If $\left(R_h + g_0\right)(x) < 0$ for some $x \in]0, \infty[$ and*

$$\mathcal{L}\left(R_h + g_0\right)(x) \begin{cases} > 0, & \text{for } x < x_0, \\ < 0, & \text{for } x > x_0, \end{cases} \quad x_0 > 0. \tag{4.19}$$

The value function v^{A} identifies with the function w^{A} defined by

$$w^{\mathrm{A}}(x) = \begin{cases} -g_0(x), & \text{if } x \leq x_0^{\mathrm{A}} < x_0, \\ A\phi(x) + R_h(x), & \text{if } x > x_0^{\mathrm{A}}, \end{cases} \tag{4.20}$$

with $x_0^{\mathrm{A}} > 0$ being the unique solution to $q^{\mathrm{A}}(x) = 0$, where q^{A} is defined by

$$q^{\mathrm{A}}(x) = p_c'(x) \int_x^{\infty} \mathcal{L}\left(R_h + g_0\right)(s)\phi(s)\, m(ds), \quad \text{for all } x > 0, \tag{4.21}$$

and $A^{\mathrm{A}} > 0$ being given by

$$-A^{\mathrm{A}} = \frac{\left(R_h + g_0\right)(x_0^{\mathrm{A}})}{\phi(x_0^{\mathrm{A}})} = \frac{\left(R_h + g_0\right)'(x_0^{\mathrm{A}})}{\phi'(x_0^{\mathrm{A}})}. \tag{4.22}$$

Furthermore, given any initial condition $x > 0$, the abandonment strategy $(\mathbb{S}_x^{\mathrm{A}}, \tau_0^{\mathrm{A}}) \in \mathcal{S}_x$, where $\mathbb{S}_x^{\mathrm{A}}$ is a weak solution to (2.2) and

$$\tau_0^{\mathrm{A}} \;=\; \inf\{t \geq 0 \mid X_t \leq x_0^{\mathrm{A}}\},$$

is optimal.

4.3.3 The initialisation of a payoff flow with the option to abandon problem

With regard to the initialisation of a payoff flow with the option to abandon problem, recall that the performance criterion is given by

$$\begin{aligned} J^{\mathrm{IA}}(\mathbb{S}_x, \tau_1, \tau_0) = \; & \mathbb{E}_x\Bigg[e^{-\Lambda_{\tau_1}}\Big(R_h(X_{\tau_1}) - \mathbb{E}_{X_{\tau_1}}\Big[\big(e^{-\Lambda_{\tau_0}}\big(R_h + g_0\big)(X_{\tau_0})\big)\,\mathbf{1}_{\{\tau_0<\infty\}}\Big] \\ & - g_1(X_{\tau_1})\Big)\mathbf{1}_{\{\tau_1<\infty\}}\Bigg]. \end{aligned}$$

To address this problem we start by observing that the optimal strategy should never involve initialisation followed immediately by abandonment. This will be the case if the initialisation point, which we denote by x_1^{IA}, is below the abandonment point, which we denote by x_0^{IA}. With reference to Theorems 4.3.1 and 4.3.2, we have a number of possibilities. Firstly, if Case I of Theorem 4.3.1 applies we would never initialise, which corresponds to $x_1^{\mathrm{IA}} = \infty$, and the problem does not arise. For Case II of Theorem 4.3.2, characterised by $\mathcal{L}\left(R_h + g_0\right)(x) > 0$ for all $x > 0$, we note that Assumption 4.2.2 means that we cannot have $\mathcal{L}\left(R_h - g_1\right)(x) < 0$ for any x, and so Cases II–III of Theorem 4.3.1 are precluded. If we have Case III of Theorem 4.3.1 and Case I of Theorem 4.3.2, then initialisation is optimal but we would never abandon. In this case we can repose the initialisation and abandonment problem as a pure initialisation problem,

$$\sup_{(\mathbb{S}_x,\tau_1,\tau_0)\in\mathcal{C}_x} J^{\mathrm{IA}}(\mathbb{S}_x,\tau_1,\tau_0) = \sup_{(\mathbb{S}_x,\tau_1)\in\mathcal{S}_x} J^{\mathrm{I}}(\mathbb{S}_x,\tau_1).$$

Hence, both initialisation and abandonment should only be optimal if we have Case III of Theorem 4.3.1 and Case III of Theorem 4.3.2. These cases are characterised, in part, by (4.15) and (4.19). With regard to these definitions and Assumption 4.2.2, we will have that $x_0 < x_1$. Also, recall that the performance criterion associated with abandonment is given and by

$$J^{\mathrm{A}}(\mathbb{S}_x,\tau_0) = R_h(x) - \mathbb{E}_x\Big[\left(e^{-\Lambda_{\tau_0}}\left(R_h + g_0\right)(X_{\tau_0})\right)\mathbf{1}_{\{\tau_0<\infty\}}\Big]$$

and the optimal strategy would be to abandon if X hits the stopping region $]0, x_0^{\mathrm{A}}]$, With $x_0^{\mathrm{A}} < x_0$. Hence

$$\begin{aligned}
v^{\mathrm{IA}}(x) &= \sup_{(\mathbb{S}_x,\tau_1,\tau_0)\in\mathcal{C}_x} J^{\mathrm{IA}}(\mathbb{S}_x,\tau_1,\tau_0) \\
&= \sup_{(\mathbb{S}_x,\tau_1,\tau_0)\in\mathcal{C}_x} \mathbb{E}_x\left[\left(e^{-\Lambda_{\tau_1}}\left(J^{\mathrm{A}}(\mathbb{S}_{X_{\tau_1}},\tau_0) - g_1(X_{\tau_1})\right)\right)\mathbf{1}_{\{\tau_1<\infty\}}\right] \\
&= \sup_{(\mathbb{S}_x,\tau_1)\in\mathcal{S}_x} \mathbb{E}_x\left[\left(e^{-\Lambda_{\tau_1}}\left(v^{\mathrm{A}} - g_1\right)(X_{\tau_1})\right)\mathbf{1}_{\{\tau_1<\infty\}}\right],
\end{aligned}$$

which equates to an optimal stopping problem with a payoff of $v^{\mathrm{A}}(x) - g_1(x)$. With reference to Case (a) of Lemma 3.3.1, we require $v^{\mathrm{A}}(x) - g_1(x) > 0$ for some $x \in \,]0, \infty[$ and

$$\mathcal{L}(v^{\mathrm{A}} - g_1)(x) \begin{cases} > 0, & \text{for } x < x_1, \\ < 0, & \text{for } x > x_1, \end{cases} \qquad x_1 > 0.$$

with

$$v^{\mathrm{A}}(x) = \begin{cases} -g_0(x), & \text{if } x \le x_0^{\mathrm{A}}, \\ A^{\mathrm{A}}\phi(x) + R_h(x), & \text{if } x > x_0^{\mathrm{A}}. \end{cases}$$

Given that we have $x_0^{\mathrm{A}} < x_0 < x_1$, we note that

$$v^{\mathrm{A}}(x) - g_1(x) = \begin{cases} -(g_0(x) + g_1(x)), & \text{if } x \le x_0^{\mathrm{A}}, \\ A^{\mathrm{A}}\phi(x) + R_h(x) - g_1(x), & \text{if } x > x_0^{\mathrm{A}}. \end{cases}$$

$$\mathcal{L}(v^{\mathrm{A}} - g_1)(x) = \begin{cases} -\mathcal{L}(g_0 + g_1)(x), & \text{if } x \le x_0^{\mathrm{A}}, \\ \mathcal{L}(R_h - g_1)(x), & \text{if } x > x_0^{\mathrm{A}}. \end{cases}$$

Based on these observations we have the following theorem.

Theorem 4.3.3 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1 and 4.2.2 hold, and consider the initialisation and abandonment problem formulated in Section 4.2. Suppose, in addition that*

$$\mathcal{L}\left(R_h + g_0\right)(x) \begin{cases} > 0, & \text{for } x < x_0, \\ < 0, & \text{for } x > x_0, \end{cases} \quad x_0 > 0, \tag{4.23}$$

$$\mathcal{L}\left(R_h - g_1\right)(x) \begin{cases} > 0, & \text{for } x < x_1, \\ < 0, & \text{for } x > x_1, \end{cases} \quad x_1 > 0, \tag{4.24}$$

$$R_h(x) + g_0(x) < 0 \text{ and } R_h(x) - g_1(x) > 0 \quad \text{for some } x > 0. \tag{4.25}$$

In this case, the optimal strategy will involve initialisation of the payoff flow and abandonment at a later time. The value function v^{IA} *identifies with* w^{IA} *where*

$$w^{\mathrm{IA}}(x) = \begin{cases} B^{\mathrm{IA}}\psi(x), & \text{if } x < x_1^{\mathrm{IA}}, \\ A^{\mathrm{A}}\phi(x) + R_h(x) - g_1(x), & \text{if } x \geq x_1^{\mathrm{IA}}, \end{cases} \quad \text{where } x_1^{\mathrm{IA}} > x_0^{\mathrm{A}}. \tag{4.26}$$

Here, A^{A} *and* x_0^{A} *are as defined in Theorem 4.3.2, with* $x_1^{\mathrm{IA}} > 0$ *being the unique solution to* $q^{\mathrm{IA}}(x) = 0$*, where* q^{IA} *is defined by*

$$q^{\mathrm{IA}}(x) = p_c'(x)\left[\mathcal{W}(c)A^{\mathrm{A}} + \int_0^x \mathcal{L}\left(R_h - g_1\right)(s)\psi(s)\, m(ds)\right], \quad \text{for all } x > 0, \tag{4.27}$$

and $B^{\mathrm{IA}} > 0$ *given by*

$$B^{\mathrm{IA}} = \frac{\left(A^{\mathrm{A}}\phi + R_h - g_1\right)(x_1^{\mathrm{IA}})}{\psi(x_1^{\mathrm{IA}})} = \frac{\left(A^{\mathrm{A}}\phi + R_h - g_1\right)'(x_1^{\mathrm{IA}})}{\psi'(x_1^{\mathrm{IA}})}. \tag{4.28}$$

Furthermore, given any initial condition $x > 0$*, the initialisation strategy* $(\mathbb{S}_x^{\mathrm{IA}}, \tau_1^{\mathrm{IA}}) \in \mathcal{S}_x$*, where* $\mathbb{S}_x^{\mathrm{IA}}$ *is a weak solution to (2.2) and*

$$\tau_1^{\mathrm{IA}} = \inf\{t \geq 0 \mid X_t \geq x_1^{\mathrm{IA}}\},$$

combined with the abandonment strategy $(\mathbb{S}_x^{\text{IA}}, \tau_0^{\text{IA}}) \in \mathcal{S}_x$, *where* $\mathbb{S}_x^{\text{IA}}$ *is a weak solution to (2.2) and*

$$\tau_0^{\text{IA}} \;=\; \inf\{t > \tau_1^{\text{IA}} \mid X_t \le x_0^{\text{IA}}\},$$

is optimal.

Proof: We first note that the payoff $A\phi(x) + R_h(x) - g_1(x)$ satisfies all the assumptions associated with Theorem 4.3.1 and the payoff $R_h + g_0$ satisfies all the conditions of Theorem 4.3.2. In addition

$$\begin{aligned}
\mathcal{W}(c)A^{\text{A}} &= \mathcal{W}(c)\frac{A^{\text{A}}\phi(x_0^{\text{A}})\psi'(x_0^{\text{A}}) - A^{\text{A}}\phi'(x_0^{\text{A}})\psi(x_0^{\text{A}})}{\mathcal{W}(x_0^{\text{A}})} \\
&= -\,\mathcal{W}(c)\frac{(R_h + g_0)\,(x_0^{\text{A}})\,\psi'(x_0^{\text{A}}) - (R_h + g_0)'\,(x_0^{\text{A}})\,\psi(x_0^{\text{A}})}{\mathcal{W}(x_0^{\text{A}})} \\
&= \int_0^{x_0^{\text{A}}} \mathcal{L}(R_h - g_0)(s)\psi(s)\,m(ds)
\end{aligned}$$

and so

$$\begin{aligned}
q^{\text{IA}}(x) &= p_c'(x)\left[\mathcal{W}(c)A^{\text{A}} - \int_0^x \mathcal{L}\left(R_h - g_1\right)(s)\psi(s)\,m(ds)\right] \\
&= p_c'(x)\left[\int_0^{x_0^{\text{A}}} \mathcal{L}(R_h - g_0)(s)\psi(s)\,m(ds) - \int_0^x \mathcal{L}(R_h - g_1)(s)\psi(s)\,m(ds)\right] \\
&= p_c'(x)\left[\int_0^{x_0^{\text{A}}} -\mathcal{L}(g_0 + g_1)\psi(s)\,m(ds) + \int_{x_0^{\text{A}}}^x \mathcal{L}(R_h - g_1)(s)\psi(s)\,m(ds)\right].
\end{aligned}$$

The structure of $\mathcal{L}(R_h - g_1)(x)$, given by (4.24), and $\mathcal{L}(g_0 + g_1)$, from (4.13) of Assumption 4.2.2, combined with the fact that $\mathcal{W}(c) > 0$ for any choice of c, means that $q^{\text{IA}}(x) = 0$ has a unique solution and in addition $x_0^{\text{A}} < x_1^{\text{IA}}$. Consequently, $\tau_0^{\text{IA}} > \tau_1^{\text{IA}}$. □

Remark 4.3.1 Comparing (4.26) and (4.16) we can see that $v^{\text{IA}} > v^{\text{I}}$. This conforms with the intuition that a project that can be abandoned after initialisation has more value than one that cannot. What is less intuitive, but is in line with "real options" theory, is that by comparing (4.27) and (4.17) we can see that $x_1^{\mathrm{IA}} \geq x_1^{\mathrm{I}}$, and you would wait longer to initialise a project that can be abandoned.

4.4 The sequential entry and exit investment problem

We now consider the cases where the decision maker is allowed to reverse their decisions. We start in Subsection 4.4.1 by considering the case where switching between the "open" and "closed" modes, depending on the state process, is the optimal strategy. In Subsection 4.4.2 we turn our attention to the cases where it is optimal to switch to the open or closed modes for specific values of the state process but not to switch back, while in Subsection 4.4.3 it is optimal to operate either in the open or closed modes for all values of the state process. The cases in Subsections 4.4.2–4.4.3 reduce to ones studied above in Section 4.3 and Section 3.3, however here the problem data mean that switching does not form part of the optimal strategy, rather than the problem being formulated to prevent switching.

We can see that the performance criterion $J^{\mathrm{S}}(\mathbb{S}_x, Z)$ is equivalent to

$$J^{\mathrm{S}}(\mathbb{S}_x, Z) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t} h_0(X_t)dt\right] + \tilde{J}^{\mathrm{S}}(\mathbb{S}_x, Z)$$

where

$$\tilde{J}^{\mathrm{S}}(\mathbb{S}_x, Z) = \mathbb{E}_{x,z}\left[\int_0^\infty e^{-\Lambda_t} Z_t h(X_t)dt - \sum_{0\leq s} e^{-\Lambda_s}\Big(g_1(X_s)(\Delta Z_s)^+ + g_0(X_s)(\Delta Z_s)^-\Big)\right]$$

and $h(x) = h_1(x) - h_0(x)$. Hence,

$$\begin{aligned} v^{\mathrm{S}}(z,x) &= R_{h_0}(x) + \tilde{v}^{\mathrm{S}}(z,x) \\ \tilde{v}^{\mathrm{S}}(z,x) &= \sup_{\mathbb{S}_x, Z\in\mathcal{Z}_{x,z}} \tilde{J}^{\mathrm{S}}(\mathbb{S}_x, Z), \quad \text{for } x > 0, z \in \{0,1\} \end{aligned}$$

and we focus our attention on identifying $\tilde{v}^{\mathrm{S}}(z,x)$, since $R_{h_0}(x)$ is a deterministic function.

In order to understand the form of our variational inequality, we consider the case where the system starts in the closed mode, $Z = 0$, and our options are either to *wait* or *switch*. If we wait for a time Δt and then continue optimally, we expect that the value function v should satisfy the following inequality

$$\tilde{v}^{\mathrm{S}}(0,x) \geq \mathbb{E}_x \left[e^{-\Lambda_{\Delta T}} \tilde{v}^{\mathrm{S}}(0, X_{\Delta T}) \right],$$

using Itô's formula, dividing by Δt and taking the limit $\Delta t \downarrow 0$, yields

$$\frac{1}{2}\sigma^2(x)\tilde{v}^{\mathrm{S}}_{xx}(0,x) + b(x)\tilde{v}^{\mathrm{S}}_x(0,x) - r(x)\tilde{v}^{\mathrm{S}}(0,x) \leq 0.$$

Alternatively, we can switch, and so we expect that the value function of staying closed is at least as good as the value function of being open minus the cost of opening, so

$$\tilde{v}^{\mathrm{S}}(0,x) \geq \tilde{v}^{\mathrm{S}}(1,x) - g_1(x).$$

If we start in the open mode and $Z = 1$ then similar arguments yield

$$\frac{1}{2}\sigma^2(x)\tilde{v}^{\mathrm{S}}_{xx}(1,x) + b(x)\tilde{v}^{\mathrm{S}}_x(1,x) - r(x)\tilde{v}^{\mathrm{S}}(1,x) + h(x) \leq 0,$$

if we wait, or, we can switch, and so we expect that

$$\tilde{v}^{\mathrm{S}}(1,x) \geq \tilde{v}^{\mathrm{S}}(0,x) - g_0(x).$$

Therefore, rearranging these expressions and generalising for z, we expect that the value function, $\tilde{v}^{\mathrm{S}}$, identifies with a solution, w, of the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \max \Big\{ &\frac{1}{2}\sigma^2(x) w_{xx}(z,x) + b(x) w_x(z,x) - r(x) w(z,x) + z h(x), \\ &w(1-z,x) - w(z,x) - z g_0(x) - (1-z) g_1(x) \Big\} = 0. \end{aligned} \tag{4.29}$$

For future reference, note that the theorems in this section all rely on Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, which are, respectively, that the SDE has a weak solution, it is non-explosive, discounting is strictly positive and the payoffs satisfy the transversality conditions.

4.4.1 The case when switching is optimal

Given the intuition developed in the study of the initialisation, abandonment and initialisation and abandonment cases, we start by considering the switching case where the optimal strategy is to switch from the "open" mode to "closed" for all $x \leq x_0^{\mathrm{S}}$, and optimal to switch from the "closed" mode to the "open" mode for all $x \geq x_1^{\mathrm{S}}$. With reference to standard heuristic arguments that explain the structure of (4.29), we look for a solution w to (4.29) that satisfies,

$$w(0,x) - w(1,x) - g_0(x) = 0, \qquad \text{for } x \leq x_0^{\mathrm{S}}, \tag{4.30}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x) + b(x)w_x(1,x) - r(x)w(1,x) + h(x) = 0, \qquad \text{for } x > x_0^{\mathrm{S}}, \tag{4.31}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(0,x) + b(x)w_x(0,x) - r(x)w(0,x) = 0, \qquad \text{for } x < x_1^{\mathrm{S}}, \tag{4.32}$$

$$w(1,x) - w(0,x) - g_1(x) = 0, \qquad \text{for } x \geq x_1^{\mathrm{S}}. \tag{4.33}$$

Such a solution is given by

$$w(1,x) = \begin{cases} B^{\mathrm{S}}\psi(x) - g_0(x), & \text{if } x \leq x_0^{\mathrm{S}}, \\ A^{\mathrm{S}}\phi(x) + R_h(x), & \text{if } x > x_0^{\mathrm{S}}. \end{cases} \tag{4.34}$$

$$w(0,x) = \begin{cases} B^{\mathrm{S}}\psi(x), & \text{if } x < x_1^{\mathrm{S}}, \\ A^{\mathrm{S}}\phi(x) + R_h(x) - g_1(x), & \text{if } x \geq x_1^{\mathrm{S}}. \end{cases} \tag{4.35}$$

To specify the parameters A^{S}, B^{S}, x_0^{S} and x_1^{S}, we appeal to the so-called "smooth-pasting" condition of optimal stopping that requires the value function to be C^1, in particular at the free boundary points x_0^{S} and x_1^{S}. This requirement yields the system of equations

$$\begin{aligned} A^{\mathrm{S}}\phi(x_0^{\mathrm{S}}) + R_h(x_0^{\mathrm{S}}) &= B^{\mathrm{S}}\psi(x_0^{\mathrm{S}}) - g_0(x_0^{\mathrm{S}}), \\ A^{\mathrm{S}}\phi'(x_0^{\mathrm{S}}) + R_h'(x_0^{\mathrm{S}}) &= B^{\mathrm{S}}\psi'(x_0^{\mathrm{S}}) - g_0'(x_0^{\mathrm{S}}), \end{aligned}$$

and

$$\begin{aligned} A^{\mathrm{S}}\phi(x_1^{\mathrm{S}}) + R_h(x_1^{\mathrm{S}}) - g_1(x_1^{\mathrm{S}}) &= B^{\mathrm{S}}\psi(x_1^{\mathrm{S}}), \\ A^{\mathrm{S}}\phi'(x_1^{\mathrm{S}}) + R_h'(x_1^{\mathrm{S}}) - g_1'(x_1^{\mathrm{S}}) &= B^{\mathrm{S}}\psi'(x_1^{\mathrm{S}}). \end{aligned}$$

From these expressions we can see that

$$\begin{aligned} A^{\mathrm{S}} &= \frac{(g_0+R_h)'(x_0^{\mathrm{S}})\,\psi(x_0^{\mathrm{S}}) - (g_0+R_h)(x_0^{\mathrm{S}})\,\psi'(x_0^{\mathrm{S}})}{\mathcal{W}(c)p_c'(x_0^{\mathrm{S}})} \\ &= -\frac{(g_1-R_h)'(x_1^{\mathrm{S}})\,\psi(x_1^{\mathrm{S}}) - (g_1-R_h)(x_1^{\mathrm{S}})\,\psi'(x_0^{\mathrm{S}})}{\mathcal{W}(c)p_c'(x_1^{\mathrm{S}})} \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} B^{\mathrm{S}} &= \frac{(g_0+R_h)'(x_0^{\mathrm{S}})\,\phi(x_0^{\mathrm{S}}) - (g_0+R_h)(x_0^{\mathrm{S}})\,\phi'(x_0^{\mathrm{S}})}{\mathcal{W}(c)p_c'(x_0^{\mathrm{S}})} \\ &= -\frac{(g_1-R_h)'(x_1^{\mathrm{S}})\,\phi(x_1^{\mathrm{S}}) - (g_1-R_h)(x_1^{\mathrm{S}})\,\phi'(x_0^{\mathrm{S}})}{\mathcal{W}(c)p_c'(x_1^{\mathrm{S}})}. \end{aligned} \tag{4.37}$$

Combining (4.36)–(4.37) with the identities (2.35)–(2.36), noting that $\mathcal{W}(c) > 0$ for any choice of $c > 0$, we have the system of equations

$$\begin{aligned} q_0^{\mathrm{S}}(x_0^{\mathrm{S}}, x_1^{\mathrm{S}}) = 0, \\ q_1^{\mathrm{S}}(x_0^{\mathrm{S}}, x_1^{\mathrm{S}}) = 0, \end{aligned}$$

where

$$q_0^{\mathrm{S}}(u,v) = \int_u^\infty \mathcal{L}(g_0+R_h)(s)\phi(s)m(ds) + \int_v^\infty \mathcal{L}(g_1-R_h)(s)\phi(s)m(ds) \tag{4.38}$$

and

$$q_1^{\mathrm{S}}(u,v) = -\left[\int_0^u \mathcal{L}(g_0+R_h)(s)\psi(s)m(ds) + \int_0^v \mathcal{L}(g_1-R_h)(s)\psi(s)m(ds)\right]. \tag{4.39}$$

With these observations in mind we have the following theorem.

Theorem 4.4.1 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, 4.2.2 and 4.2.3 hold, and consider the switching problem formulated in Section 4.2. Suppose, in addition that*

$$\mathcal{L}\left(g_0+R_h\right)(x) \begin{cases} >0, & \text{for } x<x_0, \\ <0, & \text{for } x>x_0, \end{cases} \quad x_0>0, \tag{4.40}$$

$$\mathcal{L}\left(g_1-R_h\right)(x) \begin{cases} <0, & \text{for } x<x_1, \\ >0, & \text{for } x>x_1, \end{cases} \quad x_1>0, \tag{4.41}$$

$$R_h(x)+g_0(x)<0 \text{ and } R_h(y)-g_1(y)>0 \quad \text{for some } x,y>0. \tag{4.42}$$

The value function $\tilde{v}^{\mathrm{S}}$ identifies with w defined by (4.34)–(4.35) with A^{S}, $B^{\mathrm{S}}>0$, being given by (4.36)–(4.37), respectively, and $0<x_0^{\mathrm{S}}<x_1^{\mathrm{S}}$ being the unique solutions to $q_0^{\mathrm{S}}(y,z)=0$ and $q_1^{\mathrm{S}}(y,z)=0$, where q_0^{S}, q_1^{S} are defined by (4.38)–(4.39), respectively.

Furthermore, define the $\mathcal{F}_t$-adapted, finite variation, càdlàg control process Z^{S}, taking values in $\{0,1\}$, as being switched from the closed state to the open state ($Z_{s-}^{\mathrm{S}}=0$ to $Z_s^{\mathrm{S}}=1$) at stopping times given by

$$\tau_1^{\mathrm{S}} = \inf\{t\geq s\geq 0 \mid X_t\geq x_1^{\mathrm{S}} \text{ and } Z_s^{\mathrm{S}}=Z_t^{\mathrm{S}}=0\},$$

while Z^{S} is switched from the open state to the closed state ($Z_{s-}^{\mathrm{S}}=1$ to $Z_s^{\mathrm{S}}=0$) at stopping times given by

$$\tau_0^{\mathrm{S}} = \inf\{t\geq s\geq 0 \mid X_t\leq x_0^{\mathrm{S}} \text{ and } Z_s^{\mathrm{S}}=Z_t^{\mathrm{S}}=1\}.$$

Given any initial condition $x > 0$ and $z \in \{0, 1\}$, the switching strategy $(\mathbb{S}^{\mathrm{S}}_x, Z^{\mathrm{S}}) \in \mathcal{Z}_{x,z}$, where $\mathbb{S}^{\mathrm{S}}_x$ is a weak solution to (2.2) and Z^{S} with $Z^{\mathrm{S}}_0 = z$ is optimal.

Proof. *Step 1.* We begin by proving that the opening point, x^{S}_1, and closing point, x^{S}_0, are unique with $x^{\mathrm{S}}_0 < x^{\mathrm{S}}_1$. We start by making the trivial observation that (4.40) and (4.41) and Assumption 4.2.2 imply that $x_0 < x_1$. Now, we show that (4.39) defines uniquely a mapping $l :]0, \infty[\to]0, \infty[$ such that

$$q^{\mathrm{S}}_1(u, l(u)) = 0 \quad \text{ and } \quad l(u) > u.$$

First, fix any $0 < u < \infty$ such that

$$q^{\mathrm{S}}_1(u, u) = \; - \int_0^u \mathcal{L}(g_0 + g_1)(s)\psi(s)m(ds)$$

which is positive given (4.13) of Assumption 4.2.2. Also

$$\frac{\partial}{\partial v} q^{\mathrm{S}}_1(u, v) = \; - \frac{2\mathcal{L}(g_1 - R_h)(v)\psi(v)}{\sigma^2(v)\mathcal{W}(c)p'_c(v)}$$

which is positive for $x < x_1$ and negative for $x > x_1$, and so if there is $l(u) > u$ such that $q^{\mathrm{S}}_1(u, l(u)) = 0$ then it is unique. To show that $l(u)$ exists, combine the fact that $\lim_{x\to\infty}(g_1 - R_h)(x)/\psi(x) = 0$, which is a consequence of (4.10) and the identities in Proposition 2.4.2, with (4.41) and (4.42) to see that

$$\lim_{x\to\infty} \int_0^x \mathcal{L}(g_1 - R_h)(s)\psi(s)m(ds) = \infty,$$

and so, for any $u < \infty$,

$$\lim_{v\to\infty} q^{\mathrm{S}}_1(u, v) < 0.$$

Now, observe that

$$\begin{aligned}
\lim_{u\downarrow 0} q^{\mathrm{S}}_1(u, l(u)) &= \; - \int_0^{l(u)} \mathcal{L}(g_1 - R_h)(s)\psi(s)m(ds) \\
&= \; - \frac{1}{p'_c(l(u))} q^{\mathrm{I}}(l(u))
\end{aligned}$$

where q^{I} is given by (4.17). Since we define $l(u)$ by $q_1^{\mathrm{S}}(u,l(u))=0$, with reference to Theorem 4.3.1, which tells us that $q^{\mathrm{I}}(x)=0$ at $x=x_1^{\mathrm{I}}$, we can see that $\lim_{u\downarrow 0} l(u)=x_1^{\mathrm{I}}>x_1>0$ and consequently $x_1^{\mathrm{S}}>x_1$. For future reference, observe that

$$\begin{aligned}
dq_1^{\mathrm{S}}(u,l(u)) &= \frac{\partial}{\partial u}q_1^{\mathrm{S}}(u,l(u))du+\frac{\partial}{\partial v}q_1^{\mathrm{S}}(u,l(u))dv \\
&= 0
\end{aligned}$$

and so we have that

$$\begin{aligned}
l'(u) = &\; -\frac{\frac{\partial}{\partial u}q_1^{\mathrm{S}}(u,l(u))}{\frac{\partial}{\partial v}q_1^{\mathrm{S}}(u,l(u))} \\
= &\; -\frac{\mathfrak{L}(g_0+R_h)(u)}{\mathfrak{L}(g_1-R_h)(l(u))}\frac{\psi(u)}{\psi(l(u))}\frac{\sigma^2(l(u))p_c'(l(u))}{\sigma^2(u)p_c'(u)}.
\end{aligned} \tag{4.43}$$

Now consider

$$\begin{aligned}
q_0^{\mathrm{S}}(u,l(u)) &= \int_u^\infty \mathfrak{L}(g_0+R_h)(s)\phi(s)m(ds)+\int_{l(u)}^\infty \mathfrak{L}(g_1-R_h)(s)\phi(s)m(ds) \\
&= \int_u^{l(u)} \mathfrak{L}(g_0+R_h)(s)\phi(s)m(ds)+\int_{l(u)}^\infty \mathfrak{L}(g_0+g_1)(s)\phi(s)m(ds)
\end{aligned}$$

and noting that $\mathfrak{L}(g_0+g_1)(x)\leq 0$ by Assumption 4.2.2, and $\mathfrak{L}(g_0+R_h)(x)<0$ for $x>x_0$ by (4.40), we have that

$$\lim_{u\to\infty} q_0^{\mathrm{S}}(u,l(u))<0.$$

Using (4.43), observe that

$$\begin{aligned}
\frac{\partial}{\partial u}q_0^{\mathrm{S}}(u,l(u)) = &\; -\mathfrak{L}(g_0+R_h)(u)\phi(u)m(du)+\mathfrak{L}(g_1-R_h)(l(u))\phi(l(u))m(dl(u))l'(u) \\
= &\; -\mathfrak{L}(g_0+R_h)(u)\psi(u)m(du)\left[\frac{\phi(u)}{\psi(u)}-\frac{\phi(l(u))}{\psi(l(u))}\right].
\end{aligned}$$

Given

$$\frac{d}{dx}\left(\frac{\phi(x)}{\psi(x)}\right)=-\frac{\psi'(x)\phi(x)-\psi(x)\phi'(x)}{\psi^2(x)}<0$$

we have that $\frac{\partial}{\partial u} q_0^{\mathrm{S}}(u, l(u))$ has the sign of $-\mathcal{L}(g_0 + R_h)(u)$. Using this with the fact that $\lim_{u\to\infty} q_0^{\mathrm{S}}(u, l(u)) < 0$, there will be a unique $x_0^{\mathrm{S}} < x_0$ such that $q_0^{\mathrm{S}}(x_0^{\mathrm{S}}, l(x_0^{\mathrm{S}})) = 0$ if $\lim_{u\downarrow 0} q_0(u)$ is positive. Noting that $\lim_{u\downarrow 0} l(u) = x_1^{\mathrm{I}} > x_1$ we have

$$\begin{aligned}
\lim_{u\downarrow 0} \int_{l(u)}^{\infty} \mathcal{L}(g_1 - R_h)(s)\phi(s)m(ds) &= \int_{x_1^{\mathrm{I}}}^{\infty} \mathcal{L}(g_1 - R_h)(s)\phi(s)m(ds) \\
&= \frac{(g_1 - R_h)(x_1^{\mathrm{I}})\,\phi'(x_1^{\mathrm{I}}) - (g_1 - R_h)'(x_1^{\mathrm{I}})\phi(x_1^{\mathrm{I}})}{p_c'(x_1^{\mathrm{I}})} \\
&= \frac{-B^{\mathrm{I}}\psi(x_1^{\mathrm{I}})\phi'(x_1^{\mathrm{I}}) + B^{\mathrm{I}}\psi'(x_1^{\mathrm{I}})\phi(x_1^{\mathrm{I}})}{p_c'(x_1^{\mathrm{I}})} \\
&= B^{\mathrm{I}}\mathcal{W}(c)
\end{aligned}$$

where $\mathcal{W}(c) > 0$ for any choice of $c > 0$ and $B^{\mathrm{I}} > 0$ is the co-efficient defined in (4.18) and is positive given (4.41) and (4.42). Hence

$$\begin{aligned}
\lim_{u\downarrow 0} q_0^{\mathrm{S}}(u, l(u)) &= \lim_{u\downarrow 0} \int_{u}^{\infty} \mathcal{L}(g_0 + R_h)(s)\phi(s)m(ds) + B^{\mathrm{I}}\mathcal{W}(c) \\
&> \lim_{u\downarrow 0} \int_{u}^{\infty} \mathcal{L}(g_0 + R_h)(s)\phi(s)m(ds) \\
&= \lim_{u\downarrow 0} \frac{1}{p_c'(u)} q^{\mathrm{A}}(u)
\end{aligned}$$

where $q^{\mathrm{A}}(u)$ is defined by (4.21). Since R_h satisfies (2.33) and g_0 satisfies (3.6), we have that $\lim_{u\downarrow 0}(g_0 + R_h)(u)/\phi(u) = 0$. Combined with (4.40) and (4.42), we have that $\lim_{u\downarrow 0} q^{\mathrm{A}}(u)$ is positive and so $\lim_{u\downarrow 0} q_0(u)$ is positive. As a result,

$$x_0^{\mathrm{S}} < x_0 < x_1 < x_1^{\mathrm{S}} \tag{4.44}$$

and x_0^{S}, x_1^{S} are unique.

Step 2. We now confirm that A^{S} and B^{S} are positive. Combining (4.36) with

(2.36) and (3.8) we have either,

$$\begin{aligned}\mathcal{W}(c)A^{\mathrm{S}} &= \int_0^{x_0^{\mathrm{S}}} \mathcal{L}(R_h + g_0(s))\psi(s)\, m(ds) \\ &\geq 0\end{aligned} \tag{4.45}$$

with the final inequalities following from (4.40) and (4.44), or,

$$\mathcal{W}(c)A^{\mathrm{S}} = \ - \int_0^{x_1^{\mathrm{S}}} \mathcal{L}(g_1 - R_h)(s)\psi(s)\, m(ds). \tag{4.46}$$

Similarly (4.37) with (2.35) and (3.7) yield

$$\begin{aligned}\mathcal{W}(c)B^{\mathrm{S}} &= \int_{x_1^{\mathrm{S}}}^{\infty} \mathcal{L}\left(g_1 - R_h\right)(s)\phi(s)\, m(ds) \\ &\geq \ 0\end{aligned}$$

with the final inequality following from (4.41) and (4.44). Hence, both A^{S} and B^{S} are positive.

Step 3. We now show that (4.34) and (4.35) satisfy (4.29). To do this they also need to satisfy the following inequalities;

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x) + b(x)w_x(1,x) - r(x)w(1,x) + h(x) \leq 0, \qquad \text{for } x \leq x_0^{\mathrm{S}}, \tag{4.47}$$

$$w(0,x) - w(1,x) - g_0(x) \leq 0, \qquad \text{for } x > x_0^{\mathrm{S}}, \tag{4.48}$$

$$w(1,x) - w(0,x) - g_1(x) \leq 0, \qquad \text{for } x < x_1^{\mathrm{S}}, \tag{4.49}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(0,x) + b(x)w_x(0,x) - r(x)w(0,x) \leq 0, \qquad \text{for } x \geq x_1^{\mathrm{S}}. \tag{4.50}$$

We note that (4.34) satisfies (4.47) given $\mathcal{L}(g_0 + R_h)(x) > 0$ for $x < x_0$ by (4.40) the fact that $x_0^{\mathrm{S}} < x_0$ while (4.35) satisfies (4.50) given $x_1^{\mathrm{S}} > x_1$ and (4.41).

For $x > x_0^{\mathrm{S}}$, (4.48) becomes

$$\frac{A^{\mathrm{S}}\phi(x_0^{\mathrm{S}}) + R(x_0^{\mathrm{S}}) + g_0(x_0^{\mathrm{S}})}{\psi(x_0^{\mathrm{S}})} \leq \frac{A^{\mathrm{S}}\phi(x) + R(x) + g_0(x)}{\psi(x)}$$

and so we require $A^{\mathrm{S}}\phi(x) + R(x) + g_0(x)/\psi(x)$ to be increasing for $x > x_0^{\mathrm{S}}$. With reference to (2.36), (3.8), (4.39) and (4.46), we have that

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{A^{\mathrm{S}}\phi(x) + R(x) + g_0(x)}{\psi(x)}\right] \\
&= \frac{[A^{\mathrm{S}}\phi'(x) + R'(x) + g_0'(x)]\psi(x) - [A^{\mathrm{S}}\phi(x) + R(x) + g_0(x)]\psi'(x)}{\psi^2(x)} \\
&= \frac{p_c'(x)}{\psi^2(x)}\left[\int_0^x \mathcal{L}(g_0 + R_h)(s)\psi(s)m(ds) - \mathcal{W}(c)A^{\mathrm{S}}\right] \\
&= \frac{p_c'(x)}{\psi^2(x)}\left[\int_0^x \mathcal{L}(g_0 + R_h)(s)\psi(s)m(ds) + \int_0^{x_1^{\mathrm{S}}} \mathcal{L}(g_1 - R_h)(s)\psi(s)m(ds)\right] \\
&= -\frac{p_c'(x)}{\psi^2(x)} q_1^{\mathrm{S}}(x, x_1^{\mathrm{S}}).
\end{aligned}$$

For $x \in]x_0^{\mathrm{S}}, x_1^{\mathrm{S}}[$, (4.48) is satisfied since $q_1^{\mathrm{S}}(y,z) < 0$ for $y > x_0^{\mathrm{S}}$ and $z = x_1^{\mathrm{S}}$. For $x > x_1^{\mathrm{S}}$, (4.48) is satisfied given (4.13) of Assumption 4.2.2 and (4.40).

For $x < x_1^{\mathrm{S}}$, (4.49) becomes

$$\frac{A^{\mathrm{S}}\phi(x) + R(x) - g_1(x)}{\psi(x)} \leq \frac{A^{\mathrm{S}}\phi(x_1^{\mathrm{S}}) + R(x_1^{\mathrm{S}}) - g_1(x_1^{\mathrm{S}})}{\psi(x_1^{\mathrm{S}})}$$

and so we require $A^{\mathrm{S}}\phi(x) + R(x) - g_1(x)/\psi(x)$ to be increasing in $]x_0^{\mathrm{S}}, x_1^{\mathrm{S}}]$. Using (2.36), (3.8), (4.39) and (4.45) note that

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{A^{\mathrm{S}}\phi(x) + R(x) - g_1(x)}{\psi(x)}\right] \\
&= \frac{[A^{\mathrm{S}}\phi'(x) + R'(x) - g_1'(x)]\psi(x) - [A^{\mathrm{S}}\phi(x) + R(x) - g_1(x)]\psi'(x)}{\psi^2(x)} \\
&= -\frac{p_c'(x)}{\psi^2(x)}\left[\mathcal{W}(c)A^{\mathrm{S}} + \int_0^x \psi(s)\mathcal{L}(g_1 - R_h)(s)m(ds)\right] \\
&= -\frac{p_c'(x)}{\psi^2(x)}\left[\int_0^{x_0^{\mathrm{S}}} \psi(s)\mathcal{L}(g_0 + R_h)(s)m(ds) + \int_0^x \psi(s)\mathcal{L}(g_1 - R_h)(s)m(ds)\right] \\
&= \frac{p_c'(x)}{\psi^2(x)} q_1^{\mathrm{S}}(x_0^{\mathrm{S}}, x).
\end{aligned}$$

Given $q_1^{\mathrm{S}}(y,z) > 0$ for $y = x_0^{\mathrm{S}}$ and $z < x_1^{\mathrm{S}}$, (4.49) is satisfied for $x \in]x_0^{\mathrm{S}}, x_1^{\mathrm{S}}[$. For

$x \leq x_0^{\mathrm{S}}$, (4.49) is satisfied given (4.13) of Assumption 4.2.2 and (4.41).

Step 4. To verify that the solution w to the HJB equation (4.29) that we have constructed identifies with the value function v of the optimal stopping problem, we fix any initial condition $x > 0$ and any weak solution $\mathbb{S}_x$ to (2.2). Define an arbitrary control strategy, Z, by picking arbitrary times at which to switch between the open and closed modes. We can now define a sequence of $(\mathcal{F}_t)$-stopping times (τ_m) by

$$\begin{aligned} \tau_1 &= \inf\{t \geq 0 \,\big|\, Z_t \neq z\} \\ \tau_{m+1} &= \inf\{t > \tau_m \,\big|\, Z_t \neq Z_{t-}\}. \end{aligned}$$

In addition, given any $T > 0$, fix any initial condition $x > 0$ and any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{S}_x$, define

$$\tau_n = \inf\left\{t \geq 0 \,\big|\, X_t \notin [1/n, n]\right\}, \quad \text{for } n \geq 1.$$

Now, since $w \in C^1(]0,\infty[) \cap C^2(]0,\infty[\setminus \{x_0^{\mathrm{S}}, x_1^{\mathrm{S}}\})$ and w' is of bounded variation, we can use the Itô-Tanaka formula, to calculate

$$\begin{aligned} e^{-\Lambda_{t\wedge\tau_n\wedge T}} w(Z_{t\wedge\tau_n\wedge T}, X_{t\wedge\tau_n\wedge T}) = w(z,x) + M_t^{T,n} \\ + \int_0^{t\wedge\tau_n\wedge T} e^{-\Lambda_s} \left(\mathfrak{L}w(Z_s, X_s)\right) ds \\ + \sum_{0<s\leq t\wedge\tau_n\wedge T} e^{-\Lambda_s} \left[w(Z_s, X_s) - w(Z_{s-}, X_s)\right] \end{aligned} \tag{4.51}$$

where

$$\begin{aligned} M_t^{T,n} &:= \int_0^{t\wedge\tau_n\wedge T} e^{-\Lambda_s} \left(\sigma(X_s) w_x(Z_s, X_s)\right) dW_s \\ &= L_t^{T,n} + \tilde{M}_t^{T,n} \end{aligned}$$

and

$$L_t^{T,n} := \int_0^{t\wedge\tau_n\wedge T} e^{-\Lambda_s}\sigma(X_s)\Big(\big(B^{\mathrm{S}}\psi'(X_s)\mathbf{1}_{\{X_s\le x_0^{\mathrm{S}}\}} + A^{\mathrm{S}}\phi'(X_s)\mathbf{1}_{\{X_s>x_0^{\mathrm{S}}\}}\big)\mathbf{1}_{\{Z_s=1\}} + \big(B^{\mathrm{S}}\psi'(X_s)\mathbf{1}_{\{X_s<x_1^{\mathrm{S}}\}} + A^{\mathrm{S}}\phi'(X_s)\mathbf{1}_{\{X_s\ge x_1^{\mathrm{S}}\}}\big)\mathbf{1}_{\{Z_s=0\}}\Big)dW_s.$$

Given that g_0, g_1 and R_h all satisfy Dynkin's formula, we have that $\mathbb{E}_{x,z}[\tilde{M}_t^{T,n}] = \mathbb{E}_{x,z}[M_t^{T,n} - L_t^{T,n}] = 0$. With reference to Itô's isometry, the continuity of ψ' and ϕ' and Assumptions 2.2.1′ and 2.3.1′, we can see that

$$\begin{aligned}
&\mathbb{E}_{x,z}\left[\left(L_T^{T,n}\right)^2\right] \\
&= \int_0^T e^{-2\Lambda_t}\sigma^2(X_t)\Bigg(\Big(B^{\mathrm{S}}\psi'(X_t)\mathbf{1}_{\{X_t\le x_0^{\mathrm{S}}\}} + A^{\mathrm{S}}\phi'(X_t)\mathbf{1}_{\{X_t>x_0^{\mathrm{S}}\}}\Big)\mathbf{1}_{\{Z_t=1\}} \\
&\qquad + \Big(B^{\mathrm{S}}\psi'(X_t)\mathbf{1}_{\{X_t<x_1^{\mathrm{S}}\}} + A^{\mathrm{S}}\phi'(X_t)\mathbf{1}_{\{X_t\ge x_1^{\mathrm{S}}\}}\Big)\mathbf{1}_{\{Z_t=0\}}\Bigg)^2\mathbf{1}_{\{s\le\tau_n\}}\,dt \\
&= \int_0^T e^{-2\Lambda_t}\sigma^2(X_t)\Bigg(\Big(\big(B^{\mathrm{S}}\psi'(X_t)\big)^2\mathbf{1}_{\{X_t\le x_0^{\mathrm{S}}\}} + \big(A^{\mathrm{S}}\phi'(X_t)\big)^2\mathbf{1}_{\{X_t>x_0^{\mathrm{S}}\}}\Big)\mathbf{1}_{\{Z_t=1\}} \\
&\qquad + \Big(\big(B^{\mathrm{S}}\psi'(X_t)\big)^2\mathbf{1}_{\{X_t<x_1^{\mathrm{S}}\}} + \big(A^{\mathrm{S}}\phi'(X_t)\big)^2\mathbf{1}_{\{X_t\ge x_1^{\mathrm{S}}\}}\Big)\mathbf{1}_{\{Z_t=0\}}\Bigg)\mathbf{1}_{\{s\le\tau_n\}}\,dt \\
&\le \int_0^T e^{-2\Lambda_t}\sigma^2(X_t)\Big(\big(2B^{\mathrm{S}}\psi'(X_t)\big)^2\mathbf{1}_{\{X_t\le x_1^{\mathrm{S}}\}} + \big(2A^{\mathrm{S}}\phi'(X_t)\big)^2\mathbf{1}_{\{X_t>x_0^{\mathrm{S}}\}}\Big)\mathbf{1}_{\{s\le\tau_n\}}\,dt \\
&\le \sup_{x<x_1^{\mathrm{S}}}\Big(2B^{\mathrm{S}}\psi'(x)\sigma(x)\Big)^2\int_0^T e^{-2\Lambda_t}dt + \sup_{x>x_0^{\mathrm{S}}}\Big(2A^{\mathrm{S}}\phi'(x)\sigma(x)\Big)^2\int_0^T e^{-2\Lambda_t}dt \\
&< \infty.
\end{aligned}$$

This calculation shows that $L^{T,n}$ is a square-integrable martingale. Therefore by appealing to Doob's optional sampling theorem it follows that $\mathbb{E}_{x,z}\big[L_t^{T,n}\big] = 0$, and consequently $\mathbb{E}_{x,z}[M_t^{T,n}] = 0$. In view of this observation, we can add

$$\int_0^{\tau_n\wedge T} e^{-\Lambda_t}Z_t h(X_t)dt - \sum_{0\le t\le\tau_n\wedge T} e^{-\Lambda_t}\Big(g_1(X_t)(\Delta Z_t)^+ + g_0(X_t)(\Delta Z_t)^-\Big)$$

to both sides of (4.51), on taking expectations and given that w satisfies (4.29), we have

$$\begin{aligned}
&\mathbb{E}_{x,z}\left[\int_0^{\tau_n\wedge T} e^{-\Lambda_t} Z_t h(X_t)dt - \sum_{0\le t\le \tau_n\wedge T} e^{-\Lambda_t}\Big(g_1(X_t)(\Delta Z_t)^+ + g_0(X_t)(\Delta Z_t)^-\Big)\right] \\
&\le w(z,x) - \mathbb{E}_{x,z}\left[e^{-\Lambda_{\tau_n\wedge T}} w(Z_{\tau_n\wedge T}, X_{\tau_n\wedge T})\right]. \qquad (4.52)
\end{aligned}$$

The dominated convergence theorem gives

$$\lim_{T\to\infty} \mathbb{E}_{x,z}\left[e^{-\Lambda_{\tau_n\wedge T}} w(Z_{\tau_n\wedge T}, X_{\tau_n\wedge T})\right] = \mathbb{E}_{x,z}\left[e^{-\Lambda_{\tau_n}} w(Z_{\tau_n}, X_{\tau_n})\right]. \qquad (4.53)$$

Noting that

$$\begin{aligned}
&\mathbb{E}_{x,z}\left[e^{-\Lambda_{\tau_n}} w(Z_{\tau_n}, X_{\tau_n})\right] \\
&= \mathbb{E}_{x,z}\Bigg[e^{-\Lambda_{\tau_n}}\Bigg(\bigg(\Big(B^{\mathrm{S}}\psi(X_{\tau_n}) - g_0(X_{\tau_n})\Big)\mathbf{1}_{\{X_{\tau_n}\le x_0^{\mathrm{S}}\}} \\
&\qquad\qquad + \Big(A^{\mathrm{S}}\phi(X_{\tau_n}) + R_h(X_{\tau_n})\Big)\mathbf{1}_{\{X_{\tau_n}> x_0^{\mathrm{S}}\}}\bigg)\mathbf{1}_{\{Z_{\tau_n}=1\}} \\
&\qquad + \bigg(\Big(B^{\mathrm{S}}\psi(X_{\tau_n})\Big)\mathbf{1}_{\{X_{\tau_n}< x_1^{\mathrm{S}}\}} \\
&\qquad\qquad + \Big(A^{\mathrm{S}}\phi(X_{\tau_n}) + R_h(X_{\tau_n}) - g_1(X_{\tau_n})\Big)\mathbf{1}_{\{X_{\tau_n}\ge x_1^{\mathrm{S}}\}}\bigg)\mathbf{1}_{\{Z_{\tau_n}=0\}}\Bigg)\Bigg]
\end{aligned}$$

and given R_h, g_0 and g_1 all satisfy the transversality condition, (2.40) of Proposition

2.4.2, and that $A^{\mathrm{S}}\phi$ and $B^{\mathrm{S}}\psi$ are positive, we have that

$$\begin{aligned} 0 \le \lim_{n\to\infty} \mathbb{E}_{x,z}\left[e^{-\Lambda_{\tau_n}} w(Z_{\tau_n}, X_{\tau_n})\right] &\le \lim_{n\to\infty} \mathbb{E}_{x,z}\Bigg[e^{-\Lambda_{\tau_n}}\Big(A^{\mathrm{S}}\phi(X_{\tau_n})\mathbf{1}_{\{X_{\tau_n} > x_0^{\mathrm{S}}\}} \\ &\qquad\qquad + B^{\mathrm{S}}\psi(x)\mathbf{1}_{\{X_{\tau_n} < x_1^{\mathrm{S}}\}}\Big)\Bigg] \\ &\le \Big(A^{\mathrm{S}}\phi(x_0^{\mathrm{S}}) + B^{\mathrm{S}}\psi(x_1^{\mathrm{S}})\Big)\lim_{n\to\infty} \mathbb{E}_{x,z}\left[e^{-\Lambda_{\tau_n}}\right] \\ &= 0 \end{aligned} \tag{4.54}$$

with the last equality following as a consequence of Assumption 2.3.1′. Given (4.11) of Assumption 4.2.1b and the continuity of g_0, g_1 and Assumption 2.3.1′, the dominated convergence theorem gives

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}_{x,z}\left[\int_0^{\tau_n\wedge T} e^{-\Lambda_t} Z_t h(X_t)dt - \sum_{0\le t\le \tau_n\wedge T} e^{-\Lambda_t}\Big(g_1(x_1^{\mathrm{S}})(\Delta Z_t)^+ + g_0(x_0^{\mathrm{S}})(\Delta Z_t)^-\Big)\right] \\ &= \mathbb{E}_{x,z}\left[\int_0^{\infty} e^{-\Lambda_t} Z_t h(X_t)dt - \sum_{0\le t\le \infty} e^{-\Lambda_t}\Big(g_1(X_t)(\Delta Z_t)^+ + g_0(X_t)(\Delta Z_t)^-\Big)\right]. \end{aligned} \tag{4.55}$$

In view of (4.53)–(4.55), (4.52) implies

$$\mathbb{E}_{x,z}\left[\int_0^{\infty} e^{-\Lambda_t} Z_t h(X_t)dt - \sum_{0\le t\le \infty} e^{-\Lambda_t}\Big(g_1(x_1^{\mathrm{S}})(\Delta Z_t)^+ + g_0(x_0^{\mathrm{S}})(\Delta Z_t)^-\Big)\right] \le w(z,x), \tag{4.56}$$

which proves $\tilde{J}^{\mathrm{S}}(\mathbb{S}_x, Z) \le w(z,x)$.

To prove that $\tilde{v}(z,x) = w(z,x)$ for the optimal strategy proposed in the statement of the theorem, let $(\mathbb{S}_x^{\mathrm{S}}, Z^{\mathrm{S}}) \in \mathcal{Z}_{x,z}$ be the switching strategy considered in the statement of the theorem. By following the arguments that lead to (4.56) we can see that

$$\mathbb{E}_{x,z}\left[\int_0^\infty e^{-\Lambda_t} Z_t^{\mathrm{S}} h(X_t)dt - \sum_{0\le t\le\infty} e^{-\Lambda_t}\Big(g_1(x_1^{\mathrm{S}})(\Delta Z_t^{\mathrm{S}})^+ + g_0(x_0^{\mathrm{S}})(\Delta Z_t^{\mathrm{S}})^-\Big)\right] = w(z,x).$$

□

4.4.2 The case when one mode is optimal for certain values of the state process

We now consider the case where the optimal strategy is to operate either in the "open" mode for all x, but it is only optimal to switch from the "closed" mode to the "open" mode for $x \ge x_1^{\mathrm{S}}$, or, to operate in the "closed" mode for all x, but it is only optimal to switch from the "open" mode to the "closed" mode for $x \le x_0^{\mathrm{S}}$. Since the optimal strategy will not result in a reversing of the actions taken, these cases are versions of those presented in Theorems 4.3.1 and 4.3.2.

For the case where, once in the open mode, it is never optimal to switch to the closed mode, we look for a solution w to (4.29) that satisfies

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x) + b(x)w_x(1,x) - r(x)w(1,x) + h(x) = 0, \quad \text{for all } x, \tag{4.57}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(0,x) + b(x)w_x(0,x) - r(x)w(0,x) = 0, \quad \text{for } x < x_1^{\mathrm{S}}, \tag{4.58}$$

$$w(1,x) - w(0,x) - g_1(x) = 0, \quad \text{for } x \ge x_1^{\mathrm{S}}. \tag{4.59}$$

Such a solution is given by

$$w(1,x) = R_h(x) \tag{4.60}$$

$$w(0,x) = \begin{cases} B^{\mathrm{S}}\psi(x), & \text{if } x < x_1^{\mathrm{S}}, \\ R_h(x) - g_1(x), & \text{if } x \ge x_1^{\mathrm{S}}. \end{cases} \tag{4.61}$$

Noting that $w(0,x)$, defined by (4.61), identifies with $w^{\mathrm{I}}(x)$, defined by (4.16) of Theorem 4.3.1, we can state the following theorem.

Theorem 4.4.2 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, 4.2.2 and 4.2.3 hold, and consider the switching problem formulated in Section 4.2. Suppose, in addition that*

$$(R_h - g_1)(x) > 0, \quad \text{for some } x \in]0, \infty[, \tag{4.62}$$

$$\mathcal{L}(R_h - g_1)(x) \begin{cases} > 0, & \text{for } x < x_1, \\ < 0, & \text{for } x > x_1, \end{cases} \quad x_1 > 0, \tag{4.63}$$

$$\mathcal{L}(R_h + g_0)(x) < 0, \quad \text{for all } x. \tag{4.64}$$

Then, $x_1^{\mathrm{S}} > 0$ is the unique solution to $q_1^{\mathrm{S}}(x) = 0$, where q_1^{S} is defined by

$$q_1^{\mathrm{S}}(x) = \mathcal{W}(c) p_c'(x) \int_0^x \mathcal{L}(R_h - g_1)(s) \psi(s)\, m(ds), \quad \text{for all } x > 0,$$

and $B^{\mathrm{S}} > 0$ being given by

$$B^{\mathrm{S}} = \frac{(R_h - g_1)(x_1^{\mathrm{S}})}{\psi(x_1^{\mathrm{S}})} = \frac{(R_h - g_1)'(x_1^{\mathrm{S}})}{\psi'(x_1^{\mathrm{S}})}. \tag{4.65}$$

The value function $\tilde{v}^{\mathrm{S}}$ identifies with w defined by (4.60)–(4.61). Furthermore, given any initial condition $x > 0$ and $z \in \{0, 1\}$, the control strategy $(\mathbb{S}_x^{\mathrm{S}}, Z^{\mathrm{S}}) \in \mathcal{Z}_{x,z}$, where $\mathbb{S}_x^{\mathrm{S}}$ is a weak solution to (2.2) and if $z = 0$ then $Z^{\mathrm{S}} = 1$ for all $t \geq \tau_1^{\mathrm{S}}$,

$$\tau_1^{\mathrm{S}} = \inf\{t \geq 0 \mid X_t \geq x_1^{\mathrm{S}}\},$$

where as if $z = 1$ then $Z^{\mathrm{S}} = 1$ for all t is optimal.

Proof: The statements regarding x_1^{S} and B^{S} are a consequence of Theorem 4.3.1 and (4.62)–(4.63). In addition to w, given by (4.60)–(4.61), satisfying (4.57)–(4.59), we need to show that that it satisfies the following inequalities

$$w(0,x) - w(1,x) - g_0(x) \le 0, \qquad \text{for all } x > 0, \tag{4.66}$$

$$w(1,x) - w(0,x) - g_1(x) \le 0, \qquad \text{for } x < x_1^{\mathrm{S}}, \tag{4.67}$$

$$\frac{1}{2}\sigma^2(x) w_{xx}(0,x) + b(x) w_x(0,x) - r(x) w(0,x) \le 0, \qquad \text{for } x \ge x_1^{\mathrm{S}}. \tag{4.68}$$

For (4.66), we have two distinct inequalities

$$B^{\mathrm{S}}\psi(x) - R_h(x) - g_0(x) \le 0, \qquad \text{if } x < x_1^{\mathrm{S}}, \tag{4.69}$$

$$-\big(g_0(x) + g_1(x)\big) \le 0, \qquad \text{if } x \ge x_1^{\mathrm{S}}, \tag{4.70}$$

and (4.70) is true given (4.14) of Assumption 4.2.3. Given that B^{S} is defined by (4.65), (4.69) can be written as

$$\frac{(R_h + g_0)(x)}{\psi(x)} \ge \frac{(R_h - g_1)(x_1^{\mathrm{S}})}{\psi(x_1^{\mathrm{S}})}, \qquad \text{for all } x < x_1^{\mathrm{S}},$$

which is true at x_1^{S}, since $g_0(x) > -g_1(x)$, for all x, by (4.14) of Assumption 4.2.3. To see that (4.69) is true for all $x < x_1^{\mathrm{S}}$, observe that

$$\frac{d}{dx}\left(\frac{(R_h + g_0)(x)}{\psi(x)}\right) = \frac{p_c'(x)}{\psi^2(x)} \int_0^x \mathcal{L}\left(R_h + g_0\right)(s)\psi(s)m(ds)$$

and given $\mathcal{L}(R_h + g_0) < 0$ for all x, by (4.64), the condition is satisfied. Similarly, we can write (4.67) as

$$\frac{(R_h - g_1)(x)}{\psi(x)} \le \frac{(R_h - g_1)(x_1^{\mathrm{S}})}{\psi(x_1^{\mathrm{S}})}, \qquad \text{for } x < x_1^{\mathrm{S}}.$$

Consider

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(R_h-g_1)(x)}{\psi(x)}\right) &= \frac{(R_h-g_1)'(x)\,\psi(x)-(R_h-g_1)(x)\,\psi'(x)}{\psi^2(x)} \\
&= \frac{p_c'(x)}{\psi^2(x)}\int_0^x \mathcal{L}\left(R_h-g_1\right)(s)\psi(s)\,m(ds) \\
&= \frac{q_1^{\mathrm{S}}(x)}{\psi^2(x)}
\end{aligned}$$

and given that $q_1^{\mathrm{S}}(x)>0$ for $x<x_1^{\mathrm{S}}$, we have that $(R_h-g_1)(x)/\psi(x)$ is increasing for $x<x_1^{\mathrm{S}}$ and hence (4.67) is satisfied. Recalling that, as a consequence of Theorem 4.3.1, $x_1^{\mathrm{S}}>x_1$, (4.61) satisfies (4.68) under (4.63).

Having confirmed that w, defined by (4.60) and (4.61), satisfies (4.29), we need to confirm that the solution w equates with the value function $\tilde{v}^{\mathrm{S}}$. To do this we can use similar arguments to those used in developing (4.56) of the proof of Theorem 4.4.1. We can show that for an arbitrary control strategy $\tilde{J}^{\mathrm{S}}(\mathbb{S}_x,Z)\le w(z,x)$, while adopting the optimal strategy defined in the statement of the theorem gives $w=\tilde{v}^{\mathrm{S}}$.

□

We now consider the case where the optimal strategy is to operate in the "closed" mode for all x, but it is only optimal to switch from the "open" mode to the "closed" mode for $x<x_0^{\mathrm{S}}$.

Again, with reference to standard heuristic arguments that explain the structure of (4.29), we look for a solution w to (4.29) that satisfies

$$\frac{1}{2}\sigma^2(x)w_{xx}(0,x)+b(x)w_x(0,x)-r(x)w(0,x)=0, \quad \text{for all } x>0 \tag{4.71}$$

$$w(0,x)-w(1,x)-g_0(x)=0, \quad \text{for } x\le x_0^{\mathrm{S}}, \tag{4.72}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x)+b(x)w_x(1,x)-r(x)w(1,x)+h(x)=0, \quad \text{for } x>x_0^{\mathrm{S}}. \tag{4.73}$$

Such a solution is given by

$$w(0,x) = 0 \tag{4.74}$$

$$\text{and} \qquad w(1,x) = \begin{cases} -g_0(x), & \text{if } x < x_0^{\mathrm{S}}. \\ A\phi(x) + R_h(x), & \text{if } x \geq x_0^{\mathrm{S}}. \end{cases} \tag{4.75}$$

Noting that $w(1,x)$, defined by (4.75), identifies with $w^{\mathrm{A}}(x)$, defined by (4.20) of Case III in Theorem 4.3.2, and we can state the following theorem.

Theorem 4.4.3 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, 4.2.2 and 4.2.3 hold, and consider the switching problem formulated in Section 4.2. Suppose, in addition that*

$$(R_h + g_0)(x) \;\; < 0, \quad \textit{for some } x \in \,]0,\infty[, \tag{4.76}$$

$$\mathcal{L}(R_h + g_0)(x) \begin{cases} > 0, & \textit{for } x < x_0, \\ < 0, & \textit{for } x > x_0, \end{cases} \quad x_0 > 0, \tag{4.77}$$

$$\mathcal{L}(R_h - g_1)(x) \;\; < 0, \quad \textit{for all } x. \tag{4.78}$$

Then, $x_0^{\mathrm{S}} > 0$ is the unique solution to $q_0^{\mathrm{S}}(x) = 0$, where q_0^{S} is defined by

$$q_0^{\mathrm{S}}(x) = p_c'(x) \int_x^\infty \mathcal{L}(R_h + g_0)(s)\phi(s)\, m(ds), \quad \textit{for all } x > 0,$$

and $A^{\mathrm{S}} > 0$ being given by

$$-A^{\mathrm{S}} = \frac{(R_h + g_0)(x_0^{\mathrm{S}})}{\phi(x_0^{\mathrm{S}})} = \frac{(R_h + g_0)'(x_0^{\mathrm{S}})}{\phi'(x_0^{\mathrm{A}})}. \tag{4.79}$$

The value function $\tilde{v}^{\mathrm{S}}$ identifies with w defined by (4.74)–(4.75). Furthermore, given any initial condition $x > 0$ and $z \in \{0,1\}$, the control strategy $(\mathbb{S}_x^{\mathrm{S}}, Z^{\mathrm{S}}) \in \mathcal{Z}_{x,z}$, where $\mathbb{S}_x^{\mathrm{S}}$ is a weak solution to (2.2) and if $z = 1$ then $Z^{\mathrm{S}} = 0$ for all $t \geq \tau_0^{\mathrm{S}}$,

$$\tau_0^{\mathrm{S}} = \inf\{t \geq 0 \mid X_t \leq x_0^{\mathrm{S}}\},$$

where as if $z = 0$ then $Z^{\mathrm{S}} = 0$ for all t is optimal.

Proof: The statements regarding x_0^{S} and A^{S} are a consequence of Case III in Theorem 4.3.2 and (4.76)–(4.77). In addition to w, given by (4.74)–(4.75), satisfying (4.71)–(4.73), we need to show that that it satisfies the following inequalities

$$w(1,x) - w(0,x) - g_1(x) \le 0, \quad \text{for all } x > 0, \tag{4.80}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x) + b(x)w_x(1,x) - r(x)w(1,x) + h_1(x) \le 0, \quad \text{for } x \ge x_0^*, \tag{4.81}$$

$$w(0,x) - w(1,x) - g_0(x) \le 0, \quad \text{for } x > x_0^*. \tag{4.82}$$

For (4.80), we have two distinct inequalities

$$-\big(g_0(x) + g_1(x)\big) \le 0, \qquad \text{if } x < x_0^{\mathrm{S}}, \tag{4.83}$$

$$A^{\mathrm{S}}\phi(x) + R_h(x) - g_1(x) \le 0, \qquad \text{if } x \ge x_0^{\mathrm{S}}, \tag{4.84}$$

and (4.83) is true given (4.14) of Assumption 4.2.3. Given that A^{S} is defined by (4.79), (4.84) can be written as

$$-\frac{(R_h + g_0)(x_0^{\mathrm{S}})}{\phi(x_0^{\mathrm{S}})} \le -\frac{(R_h - g_1)(x)}{\phi(x)}, \qquad \text{for all } x \ge x_0^{\mathrm{S}},$$

which is true, noting that $g_1(x_0^{\mathrm{S}}) > -g_0(x_0^{\mathrm{S}})$ by (4.14) of Assumption 4.2.3. Noting

$$\frac{d}{dx}\left(\frac{(R_h - g_1)(x)}{\phi(x)}\right) = \frac{p_c'(x)}{\phi^2(x)}\int_0^x \mathfrak{L}\left(R_h - g_1\right)(s)\phi(s)m(ds)$$

and given $\mathfrak{L}\left(R_h - g_1\right) < 0$ for all x, by (4.78), this ensures the condition is satisfied. Similarly, we can write (4.82) as

$$\frac{(R_h + g_0)(x_0^{\mathrm{S}})}{\phi(x_0^{\mathrm{S}})} \le \frac{(R_h + g_0)(x)}{\phi(x)}, \qquad \text{for } x > x_0^{\mathrm{S}}.$$

Consider

$$\begin{aligned}\frac{d}{dx}\left(\frac{(R_h+g_0)(x)}{\phi(x)}\right) &= \frac{(R_h+g_0)'(x)\,\phi(x)-(R_h+g_0)(x)\,\phi'(x)}{\phi^2(x)}\\ &= \frac{p_c'(x)}{\phi^2(x)}\int_0^x \mathfrak{L}\left(R_h+g_0\right)(s)\phi(s)\,m(ds)\\ &= -\frac{q_0^{\mathrm{S}}(x)}{\phi^2(x)}\end{aligned}$$

and given that $q_0^{\mathrm{S}}(x)<0$ for $x>x_0^{\mathrm{S}}$, we have that $(R_h+g_0)(x)/\phi(x)$ is increasing for $x>x_0^{\mathrm{S}}$ and hence (4.82) is satisfied. Recalling that, as a consequence of Theorem 4.3.1, $x_0^{\mathrm{S}}<x_0$, (4.75) satisfies (4.81) under (4.77).

Having confirmed that w, defined by (4.74) and (4.75), satisfies (4.29), we need to confirm that the solution w equates with the value function $\tilde{v}^{\mathrm{S}}$. To do this we again use similar arguments to those used in developing (4.56) of the proof of Theorem 4.4.1. We can show that for an arbitrary control strategy $\tilde{J}^{\mathrm{S}}(\mathbb{S}_x,Z)\le w(z,x)$, while adopting the optimal strategy defined in the statement of the theorem yields $w=\tilde{v}^{\mathrm{S}}$.

□

4.4.3 The case when one mode is optimal for all values of the state process

We consider two cases, where it is always optimal for the system to be operated in the "open" mode for all values of the state process, or, it is optimal to always have the system operated in the "closed" mode.

In the case where it is always optimal to operate in the open mode ($Z_t=1$ for all t), we look for a solution w to (4.29) that satisfies

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x)+b(x)w_x(1,x)-r(x)w(1,x)+h(x)=0,\quad \text{for all } x>0, \tag{4.85}$$

$$w(1,x)-w(0,x)-g_1(x)=0,\quad \text{for all } x>0. \tag{4.86}$$

Such a solution is given by

$$w(1,x) = R_h(x) \tag{4.87}$$

$$w(0,x) = w(1,x) - g_1(x). \tag{4.88}$$

Theorem 4.4.4 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, 4.2.2 and 4.2.3 hold, and consider the switching problem formulated in Section 4.2. Suppose, in addition that*

$$\mathcal{L}\left(g_1 - R_h\right)(x) > 0, \quad \textit{for all } x > 0. \tag{4.89}$$

The value function $\tilde{v}^{\mathrm{S}}$ identifies with w defined by (4.87)–(4.88). Furthermore, given any initial condition $x > 0$ and $z \in \{0,1\}$, the control strategy $(\mathbb{S}_x^{\mathrm{S}}, Z^{\mathrm{S}}) \in \mathcal{Z}_{x,z}$, where $\mathbb{S}_x^{\mathrm{S}}$ is a weak solution to (2.2) and $Z^{\mathrm{S}} = 1$ for all $t > 0$, is optimal.

Proof: We start by confirming that w, defined by (4.87) and (4.88), satisfies the HJB equation (4.29). In addition to (4.85)–(4.86) they also need to satisfy the following inequalities

$$\frac{1}{2}\sigma^2(x)w_{xx}(0,x) + b(x)w_x(0,x) - r(x)w(0,x) \leq 0, \quad \text{for all } x > 0, \tag{4.90}$$

$$w(0,x) - w(1,x) - g_0(x) \leq 0, \quad \text{for all } x > 0. \tag{4.91}$$

Noting that $w(0,x) = R_h(x) - g_1(x)$, simple substitution of (4.89) into (4.90) and (4.14) of Assumption 4.2.3 into (4.91) show that these are satisfied.

Having confirmed that w, defined by (4.87) and (4.88), satisfy (4.29), we can use similar arguments to those used in developing (4.56) of the proof of Theorem 4.4.1 to show that for an arbitrary control strategy $\tilde{J}^{\mathrm{S}}(\mathbb{S}_x, Z) \leq w(z,x)$, while adopting the optimal strategy defined in the statement of the theorem we have $w = \tilde{v}^{\mathrm{S}}$. □

Similarly, in the case where it is always optimal to operate in the closed mode

($Z = 0$ for all t), we look for a solution w to (4.29) that satisfies

$$\frac{1}{2}\sigma^2(x)w_{xx}(0,x) + b(x)w_x(0,x) - r(x)w(0,x) = 0, \quad \text{for all } x > 0, \tag{4.92}$$

$$w(0,x) - w(1,x) - g_0(x) = 0, \quad \text{for all } x > 0. \tag{4.93}$$

Such a solution is given by

$$w(0,x) = 0 \tag{4.94}$$

$$w(1,x) = w(0,x) - g_0(x). \tag{4.95}$$

Theorem 4.4.5 *Suppose that Assumptions 2.2.1′, 2.2.2, 2.3.1′, 4.2.1, 4.2.2 and 4.2.3 hold, and consider the switching problem formulated in Section 4.2. Suppose, in addition that*

$$\mathcal{L}(R_h + g_0)(x) \geq 0, \quad \text{for all } x > 0. \tag{4.96}$$

The value function $\tilde{v}^{\mathrm{S}}$ identifies with w defined by (4.94)–(4.95). Furthermore, given any initial condition $x > 0$ and $z \in \{0,1\}$, the control strategy $(\mathbb{S}_x^{\mathrm{S}}, Z^{\mathrm{S}}) \in \mathcal{Z}_{x,z}$, where $\mathbb{S}_x^{\mathrm{S}}$ is a weak solution to (2.2) and $Z^{\mathrm{S}} = 0$ for all $t > 0$, is optimal.

Proof: For (4.94) and (4.95) to satisfy (4.29) they also need to satisfy the following inequalities.

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x) + b(x)w_x(1,x) - r(x)w(1,x) + h(x) \leq 0, \quad \text{for all } x > 0, \tag{4.97}$$

$$w(1,x) - w(0,x) - g_1(x) \leq 0, \quad \text{for all } x > 0. \tag{4.98}$$

Substitution of (4.96) into (4.97) and (4.14) of Assumption 4.2.3 into (4.98) show that these are satisfied.

Similar arguments used in developing (4.56) of the proof of Theorem 4.4.1 show that for an arbitrary control strategy $\tilde{J}^{\mathrm{S}}(\mathbb{S}_x, Z) \leq w(z,x)$, while adopting the optimal strategy defined in the statement of the theorem we have $w = \tilde{v}^{\mathrm{S}}$. □

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