Structural Resolution meets Curry-Howard

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About myself

- 1998 – 2003 BSc+MSc degree in Logic, Moscow State University.
- 2004 – 2007 PhD in the University College Cork, Ireland.
- 2007 – 2008 First postdoc project with Yves Bertot in INRIA, France - on guardedness of corecursive functions in Coq. (I study coinduction/corecursion in different domains since then.)
- 2008 – 2011 EPSRC TCS research fellowship first in St Andrews, later transferred to the School of Computing in Dundee.
- 2010 – 2016 – Lecturer, Senior Lecturer, Reader in Computing, University of Dundee.
- 2016 – ?? – Associate Professor at Heriot-Watt University, Edinburgh.

Big Picture: Coinduction in Theorem Proving

- Pre-history is given be the studies of recursion and fixed points: Knaster, Tarski, Kleene, first half of XX century;
- 1994 Coquand and 1996 Gimenez – bring inductive/coinductive methods to Type theory and Coq;
- Nowadays, most mainstream Interactive Theorem Provers (ITP) have support for induction and coinduction:
  - Inductive and Coinductive types
  - Recursive and Corecursive functions
  - Inductive and Coinductive proof principles.

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It happens that type-theoretic setting gives just the right background for development of coinductive methods: pattern and co-pattern matching, laziness, productivity and guardedness, constructive view of proofs.

Coinduction in Automated Theorem Proving (ATP)

- Coinductive data types are as natural and common (in verification, CS) as inductive;
- There is a need to reason about infinite/cyclic computation
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Note: introduced MUCH later and in a limited form.

Bigger Goal and Motivation

1. Understand reasons why coinduction does not come to ATP as easily as to ITP;
2. Find new – more structural and more constructive – foundations for proof-search in ATP;
3. ... thereby introducing better (more natural and expressive) coinductive methods to ATP;
4. Bridge the gap between ITP and ATP, in which best sides of the two worlds are presented.

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My goal for today:

- illustrate and explain points 1-3 by means of (first-order) Horn Clause Logic and Resolution;
- invite discussion (and collaborators!)

Outline

Syntax of Horn-clause Logic

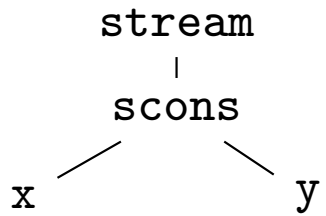
First-order signature Σ and terms, term-trees

- function symbols with arity;
- variables.

Example

- `stream` – arity 1
- `scons` – arity 2

Term-trees are trees over $\Sigma \cup V$, subject to branching $\approx$ arity:



Term Notation:

$\mathbf{Term}(\Sigma)$	Set of *finite* term trees over Σ
$\mathbf{Term}^{\infty}(\Sigma)$	Set of *infinite* term trees over Σ
$\mathbf{Term}^{\omega}(\Sigma)$	Set of *finite and infinite* term trees over Σ

$\mathbf{GTerm}(\Sigma)$, $\mathbf{GTerm}^{\infty}(\Sigma)$, $\mathbf{GTerm}^{\omega}(\Sigma)$ will denote sets of ground (variable free) terms.

Syntax of Horn-clause Logic

Horn Clauses

Given $A, B_1, \ldots, B_n \in \mathbf{Term}(\Sigma)$,

- a definite clause $A \leftarrow B_1, \ldots, B_k$
- a goal clause $\leftarrow B_1, \ldots, B_k$

Universal quantification is assumed.

A (definite) logic program is a finite set of definite clauses

... Gives us a Turing-complete programming language.

Example: lists of natural numbers

Example

```
        nat(0)  ←
     nat(s(x))  ←  nat(x)
     list(nil)  ←
list(cons(x,y))  ←  nat(x), list(y)
```

Logic Programming...

SLD resolution = Unification + Search

SLD-resolution

Program **NatList**:

Example

1.`nat(0)` $\leftarrow$
2.`nat(s(x))` $\leftarrow$ `nat(x)`
3.`list(nil)` $\leftarrow$
4.`list(cons(x,y))` $\leftarrow$

`nat(x), list(y)`

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SLD-resolution

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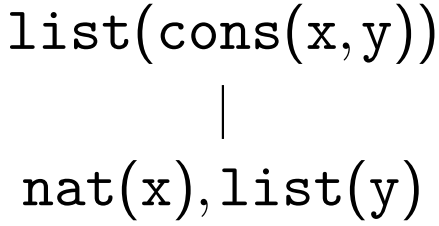

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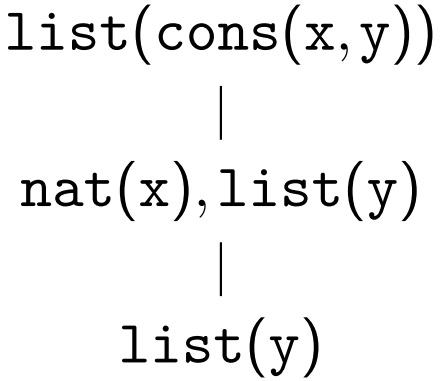

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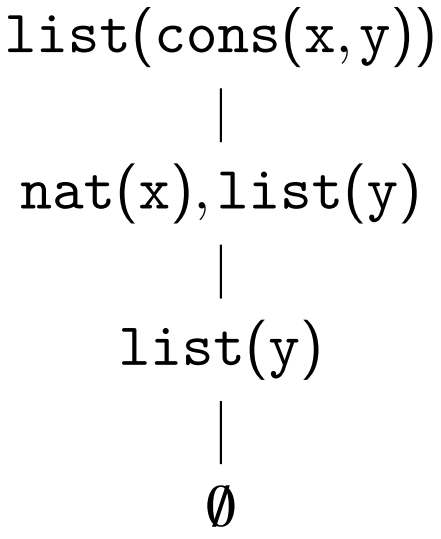

The answer is "Yes", `NatList` $\vdash$ *list*(*cons*(x, y)) if $x/0$, y/nil, but we can get more substitutions by backtracking.
SLD-refutation = finite successful SLD-derivation.

Horn Clauses in Type Class Inference (Haskell)

Equality type class

```
class Eq a where
(==) :: a -> a-> Bool.
```

Equality class instance declaration for datatype List

```
instance Eq x => Eq (List x) where...
```

Horn-Clause view of equality class instances for List and Char:

$$\begin{aligned} \kappa_1 : &\quad \mathrm{Eq}(x) \Rightarrow \mathrm{Eq}(\mathrm{List}(x)) \\ \kappa_2 : &\quad \Rightarrow \mathrm{Eq}(\mathrm{Char}) \end{aligned}$$

Outline

Big-Step Semantics of LP

Definition (Big step rule)

$$\frac{P \models \sigma(B_1), \ldots, P \models \sigma(B_n)}{P \models \sigma(A)},$$

for some grounding substitution σ, and $A \leftarrow B_1, \ldots B_n \in P$.

Example

Logic program *Nat*

1.`nat(0)` $\leftarrow$

2.`nat(s(x))` $\leftarrow$ `nat(x)`

Inductive Semantics of LP

Definition (Big step rule)

$$\frac{P \models \sigma(B_1), \ldots, P \models \sigma(B_n)}{P \models \sigma(A)},$$

for some grounding substitution σ, and $A \leftarrow B_1, \ldots B_n \in P$.

Definition

The *least Herbrand model* for P is the smallest set $M_P \subseteq \mathbf{GTerm}(\Sigma)$ *closed forward* under the rules.

Example

Taking the logic program *Nat*, we obtain the set $M_{Nat} = \{\texttt{nat(0)}, \texttt{nat(s(0))}, \texttt{nat(s(s(0)))}, \ldots\}$.

CoInductive Semantics of LP

Definition (Big step rule)

$$\frac{P \models \sigma(B_1), \ldots, P \models \sigma(B_n)}{P \models \sigma(A)},$$

for some grounding substitution σ, and $A \leftarrow B_1, \ldots B_n \in P$.

Definition

The *greatest complete Herbrand model* for P is the largest set $M_P^\omega \subseteq \mathbf{GTerm}^\omega(\Sigma)$ *closed backward* under the rules.

Example

M_{Nat}^ω will now be given by the set:
$\{\mathtt{nat(0)}, \mathtt{nat(s(0))}, \mathtt{nat(s(s(0)))}, \ldots\} \bigcup \{\mathtt{nat(s(s(...)))}\}$.

Coinductive programs

Some programs have only one natural interpretation:

Example

1.`bit(0)` $\leftarrow$
2.`bit(1)` $\leftarrow$
3.`stream(scons(x,y))` $\leftarrow$ `bit(x), stream(y)`

$M_{Stream} = \{$`bit(0)`$,$`bit(1)`$\}$
$M^{\omega}_{Stream} = \{$`bit(0)`$,$`bit(1)`$,$`stream(scons(0,scons(0, ...))`$,$
`stream(scons(1,scons(0, ...))`$,\ldots\}$

Outline

SLD-Resolution as a reduction system

Given a logic program P, and terms $t_1, \ldots, t_i, \ldots, t_n$ we define

- SLD-reduction: $P \vdash [t_1, \ldots, t_i, \ldots, t_n] \rightsquigarrow [\sigma(t_1), \ldots, \sigma(B_0), \ldots, \sigma(B_m), \ldots, \sigma(t_n)]$ if $A \leftarrow B_1, \ldots, B_m \in P$, and $t_i \sim_\sigma A$.

unifiers

$t \sim_\sigma t'$ denotes a unifier of t and t', i.e. $\sigma(t) = \sigma(t')$

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SLD-resolution is sound and complete, inductively [70s Apt, Van Emden, Kowalski]

- If $P \vdash t \rightsquigarrow^n \emptyset$, for some $t \in \mathbf{Term}(\Sigma)$, then there exists a substitution θ such that $\theta(t) \in M_P$.
- If $\theta(t) \in M_P$ for some grounding substitution θ, then there exists a reduction $P \vdash t \rightsquigarrow^n \emptyset$, with an answer σ such that there exists θ' making $\theta = \theta'\sigma$.

Coinductive soundness of SLD-resolution: "Computations at infinity" [80s, van Emden&Abdallah, Lloyd]

Definition

An infinite term t is *SLD-computable at infinity* with respect to a program P if there exist a finite term t' and an infinite fair SLD-derivation $G_0 = (? \leftarrow t'), G_1, G_2, \ldots G_k \ldots$ with mgus $\theta_1, \theta_2, \ldots \theta_k \ldots$ such that $d(t, \theta_k \ldots \theta_1(t')) \to 0$ as $k \to \infty$.

An SLD-derivation is *fair* if either it is finite, or it is infinite and, for every atom B appearing in some goal in the derivation, (a further instantiated version of) B is chosen within a finite number of steps.

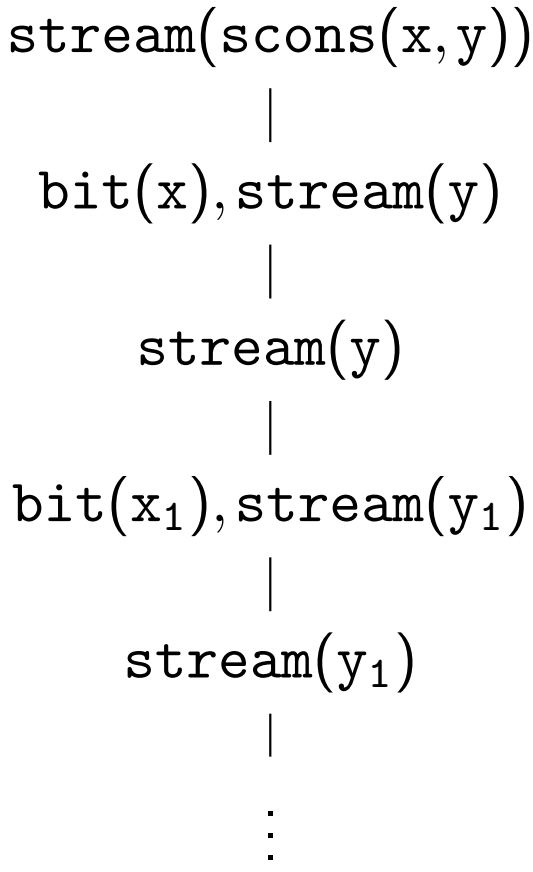
Example

Program **Stream**:

Example

1.bit(0) $\leftarrow$
2.bit(1) $\leftarrow$
3.stream(scons(x,y)) $\leftarrow$

bit(x), stream(y)



At infinity, a term scons(0, scons(0, ...)) is computed.

Computations at infinity are sound

Defining $C_P = \{t \in \mathbf{GTerm}^{\infty}(\Sigma) \mid$ there exists a term t' such that t is SLD-computable at infinity with respect to P by $t'\}$.

Theorem (Van Emden&Abdallah, Lloyd, 80s)

Given a $P \in \mathbf{LP}(\Sigma)$, $C_P \subseteq M_P^{\omega}$.

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Theorem (Van Emden&Abdallah, Lloyd, 80s)

Given a $P \in \mathbf{LP}(\Sigma)$, $C_P \subseteq M_P^{\omega}$.

A great insight:

This work was an early example of productive coinduction: requiring an infinite computation must produce an infinite term.

Two problems:

- Computations at infinity are not complete;
- They do not give rise to an implementable algorithm.

First Solution: CoLP [Gupta et al, 2007]

Program **Stream**:

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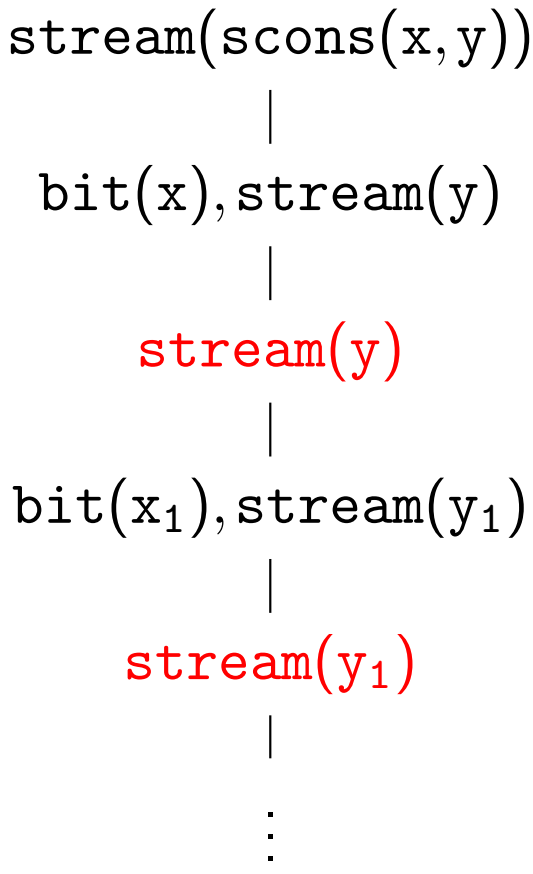


Look for a loop in resolvents: if it is found, close by circular unification: taking $\mathtt{stream}(\mathtt{y})$ and $\mathtt{stream}(\mathtt{scons}(0,\mathtt{y}_1))$ by circular unification gives $\mathtt{y} \mapsto \mathtt{scons}(0,\mathtt{y})$. Thus, we get a coinductive answer $\mathtt{x} \mapsto 0, \mathtt{y} \mapsto \mathtt{scons}(0,\mathtt{y})$.

Properties of CoLP

1. Sound relative to greatest complete Herbrand models;
2. Incomplete relative to greatest complete Herbrand models;
3. Neither sound nor complete relative to computations at infinity:

Example (Unsound)

Program:
$\mathtt{bad(x)} \leftarrow \mathtt{bad(x)}$
Will give rise to a reduction:
$\mathtt{bad(x)} \rightsquigarrow \mathtt{bad(x)} \rightsquigarrow \mathtt{bad(x)} \rightsquigarrow \ldots$
CoLP will conclude $bad(x)$ is entailed by the program, using loop detection. However, no infinite term is computed at infinity for this program. Its Herbrand model will not contain infinite terms, either.

Conclusion: CoLP's loop detection does not guarantee, on its own, productivity of computations.

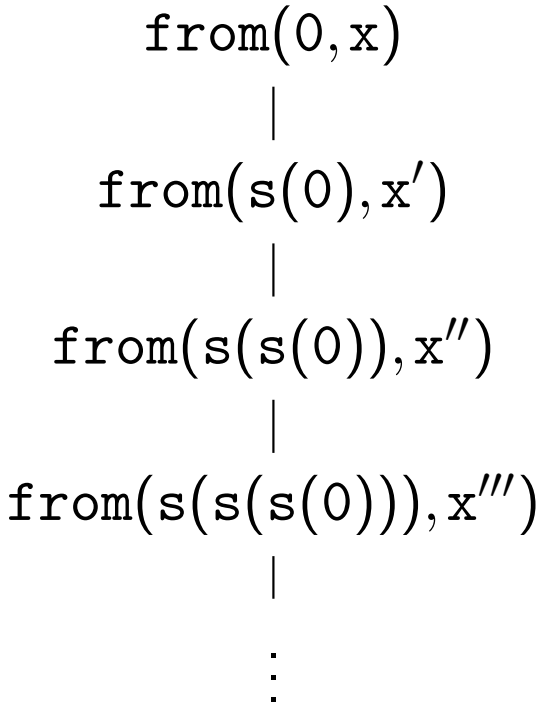
Incompleteness of CoLP for Irregular term trees

Program **From**:

Example

$1.\texttt{from(x,scons(x,y))} \leftarrow$

$\texttt{from(s(x),y)}$



No loop found, and CoLP does not terminate. But the term $\texttt{from(0,scons(0,scons(s(0),scons(s(s(0)),...))))}$ is computable at infinity.

A Fundamental Problem

Productivity theory for LP is absent:

- Operational Semantics of the 80s gives a notion of global productivity, but it is not implementable;
- Loop detection of CoLP is implementable but does not ensure global productivity;
- Notion of Observational Productivity is missing in both cases

Global and Observational Productivity elsewhere in Theorem Proving

In interactive theorem provers (Coq, Agda)
Values in co-inductive types are observationally productive when all observations of fragments made using recursive functions are guaranteed to be computable in finite time.

Observational Productivity

We call a function (observationally) productive, if, for any given input, it outputs a productive value.

Global and Observational Productivity elsewhere in Theorem Proving

Observational productivity is guaranteed by means of *guardedness checks* imposed on all corecursive functions:

```
CoInductive Stream A : Type :=
    SCons : A → Stream A → Stream A.
```

Observationally productive function:

```
CoFixpoint fib (n m : nat) :=SCons n (fib m (n+m)).
```

Observationally non-productive function:

```
CoFixpoint bad (f : A → A) (x : A) : Stream A :=
bad f (f x).
```

Observational productivity guarantees global productivity.

In search of a missing theory of productivity for ATP

- Direct borrowing of methods from ITP is hard:
  1. We have no types and type constructors to use as "guards";
  2. We have no pattern-matching.

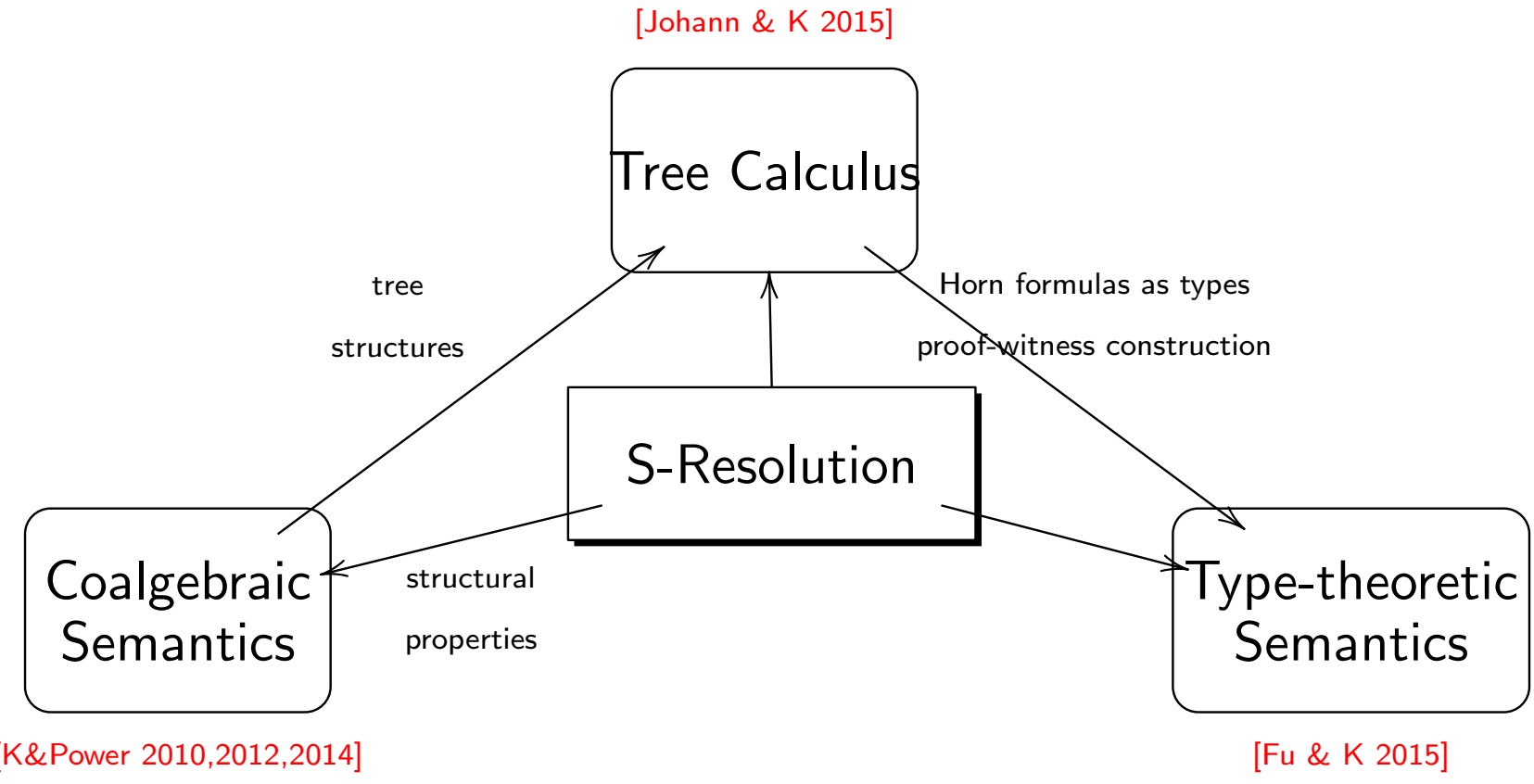
Is there a mysterious Missing productivity theory for LP and wider for ATP?

– Structural Resolution (also S-Resolution)

Outline

Structural Resolution:

a new operational semantics for productive coinductive proof search.



S-resolution reductions

matchers

$t \prec_\sigma t'$ denotes a matcher of t and t', i.e. $\sigma(t) = t'$

- SLD-reduction: $P \vdash [t_1, \ldots, t_i, \ldots, t_n] \rightsquigarrow [\sigma(t_1), \ldots, \sigma(B_0), \ldots, \sigma(B_m), \ldots, \sigma(t_n)]$ if $A \leftarrow B_1, \ldots, B_m \in P$, and $t_i \sim_\sigma A$.
- Term-Matching reduction: $P \vdash [t_1, \ldots, t_i, \ldots, t_n] \rightarrow [t_1, \ldots, \sigma(B_0), \ldots, \sigma(B_m), \ldots, t_n]$ if $A \leftarrow B_1, \ldots, B_m \in P$, and $A \prec_\sigma t_i$.

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- Substitutional reduction: $P \vdash [t_1, \ldots, t_i, \ldots, t_n] \hookrightarrow [\sigma(t_1), \ldots, \sigma(t_i), \ldots, \sigma(t_n)]$ if $A \leftarrow B_1, \ldots, B_m \in P$, and $t_i \sim_\sigma A$.
- S-resolution reduction: $P \vdash [\overline{t}] \rightarrow^\mu \circ \hookrightarrow^1 [\overline{t'}]$.

Then, $(P, \rightsquigarrow)$ is a reduction system that models SLD-reductions for P, and $(P, \rightarrow^\mu \circ \hookrightarrow^1)$ is a reduction system that models S-resolution reductions for P.

Example

```
1.bit(0) ←
2.bit(1) ←
3.stream(scons(x,y)) ← bit(x), stream(y)
```

1. SLD-resolution reduction:
   $[stream(x)] \rightsquigarrow [bit(x'), stream(y)] \rightsquigarrow [stream(y)] \rightsquigarrow$
   $[bit(x''), stream(y')] \rightsquigarrow \ldots$
2. Term-matching reduction: $[stream(x)] \nrightarrow$
3. S-resolution reduction:
   $[stream(x)] \hookrightarrow^1 [stream(scons(x', y))] \rightarrow^\mu$
   $[bit(x'), stream(y)] \hookrightarrow^1 [bit(0), stream(y)] \rightarrow^\mu [stream(y)] \hookrightarrow^1$
   $[stream(scons(x'', y'))] \rightarrow^\mu [bit(x''), stream(y')] \ldots$

Note how term-matching ($\approx$ pattern-matching) behaves for this coinductive program!

Productivity for free

Definition (Observational Productivity)

A logic program P is observationally productive if the reduction system $(P, \rightarrow)$ is strongly normalising, i.e. if every term-matching reduction is finite for P.

Example

All programs we have seen are productive with the exception of `Bad`. Program `From`, troublesome for CoLP's loop detection, is also productive, just as it would be in e.g. Coq.

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Benefits:

1. first-ever notion of observational productivity for LP;
2. simple and natural
3. more abstract than (and independent of) loop detection
4. Guarantees global productivity?

Lost completeness

By decomposing $\rightsquigarrow$ into $\rightarrow^{\mu} \circ \hookrightarrow^{1}$ we lost completeness of search (compared to SLD-resolution).

Example

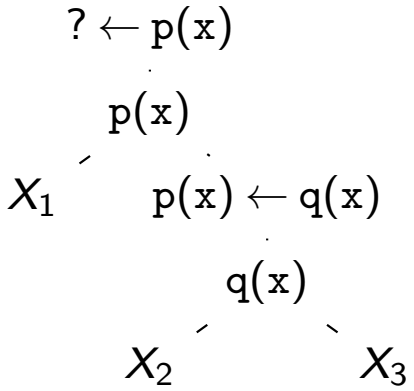
Consider the following program:
0. $\mathtt{p(c)} \leftarrow$
1. $\mathtt{p(X)} \leftarrow \mathtt{q(X)}$

We can show that $P \models \mathtt{p(c)}$, but $\mathtt{p(X)}$ will not result in a successful S-resolution reduction: $P \vdash \mathtt{p(X)} \rightarrow \mathtt{q(X)} \not\rightarrow$

Before we talk about coinductive properties of S-resolution, we must re-gain completeness of search.

Rewriting trees

A rewriting tree is a tree over $\mathbf{Term}(\Sigma) \cup \mathbf{Clause}(\Sigma) \cup Var_R$, subject to arity conditions: arity of an and-node is a number of clauses in the program, arity of an or-node is a number of terms in the clause body.



our example

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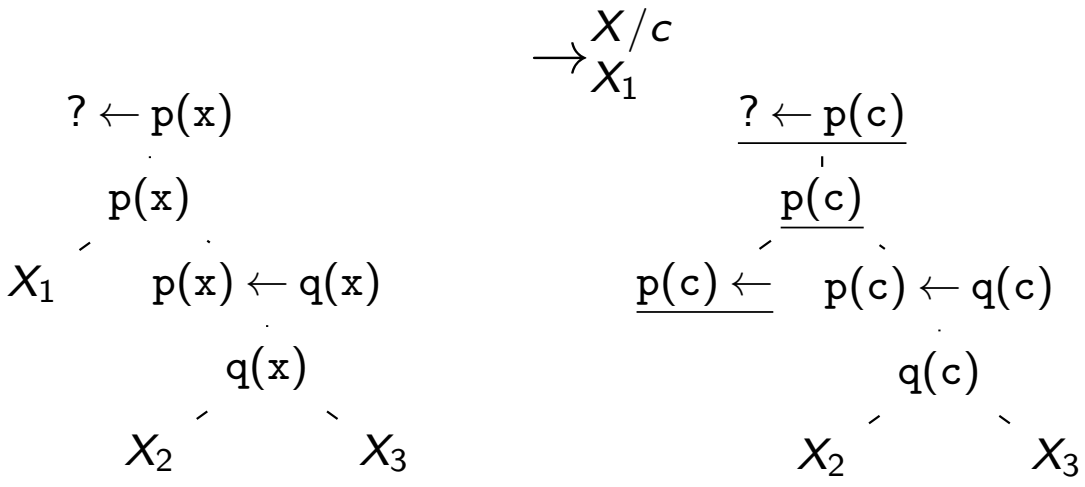
Rewriting tree substitutions and transitions



We call such transitions S-derivations

our example

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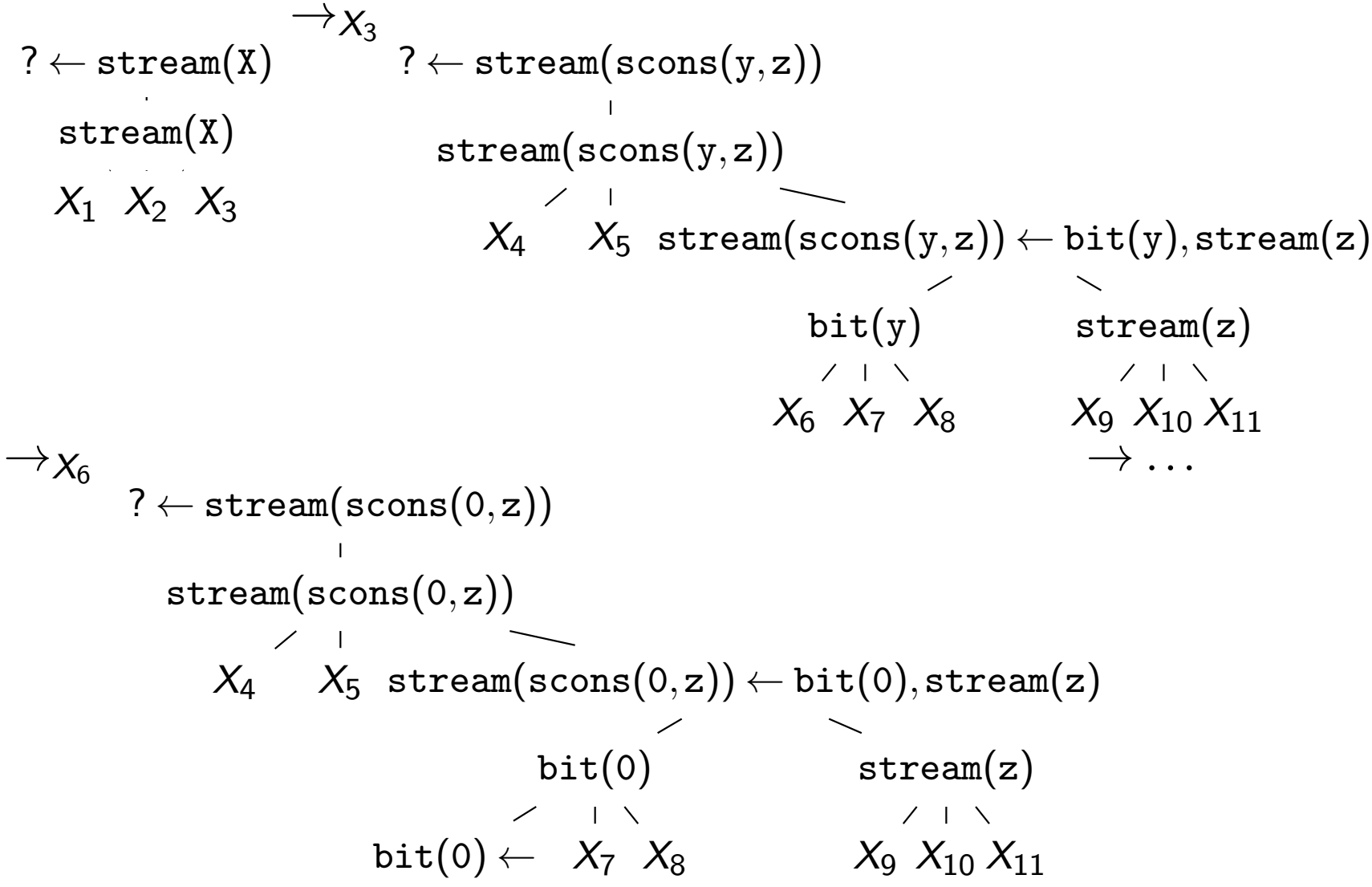
A coinductive productive S-derivation



We separate proof construction (done "vertically" by term matching) and proof search (done "horizontally" by unification)

Logic and Control

[Kowalski, 74]:

Logic Programming = Logic + Control

In fact, it was:

SLD-derivations = SLD-resolution + Searching strategies

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CoLP = SLD-resolution + Searching strategies + Loop detection

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We now have:

S-derivations = S-resolution + Tree Calculus

- S-resolution separates proof-construction from proof-search;
- Tree calculus makes it inductively sound and complete.
- Observational Productivity is a part of Logic (rather than Control)

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- S-resolution separates proof-construction from proof-search;
- Tree calculus makes it inductively sound and complete.
- Observational Productivity is a part of Logic (rather than Control)

We now resolve the coinductive properties

Global Productivity of S-Derivations

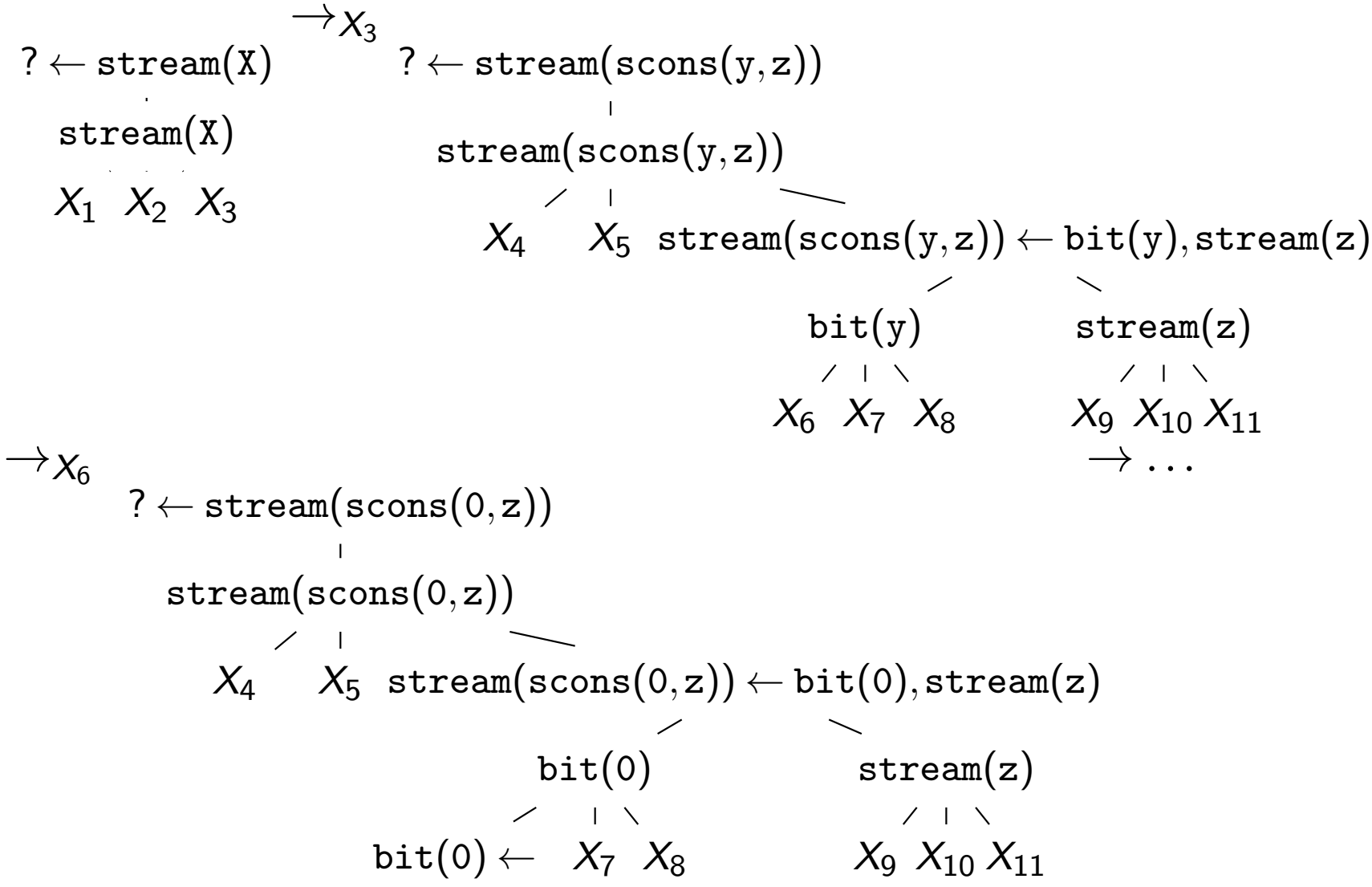
... assuming light inductive/coinductive typing Ty on predicates.

Definition

Let P be observationally productive, and let $t \in \mathbf{GTerm}^{\infty}(\Sigma)$. We say that $t' \in \mathbf{Term}(\Sigma)$ *finitely approximates* t if the following hold:

1. There is an infinite S-derivation $T_0 \to T_1 \to \ldots T_k \to \ldots$ with T_0 being a rewriting tree for t', and associated resolvents $\theta_1, \theta_2, \ldots \theta_k \ldots$ such that $d(t, \theta_k \ldots \theta_1(t')) \to 0$ as $k \to \infty$, and
2. (inductively-typed nodes are successfully closed infinitely often in the S-derivation).

Term t is *S-computable at infinity with respect to P and Ty* if there is a $t' \in \mathbf{Term}(\Sigma)$ such that t' finitely approximates t.



The term stream(scons(0, scons(0, ...))) is S-computable at infinity.

S-derivations and Global productivity of the 80s:

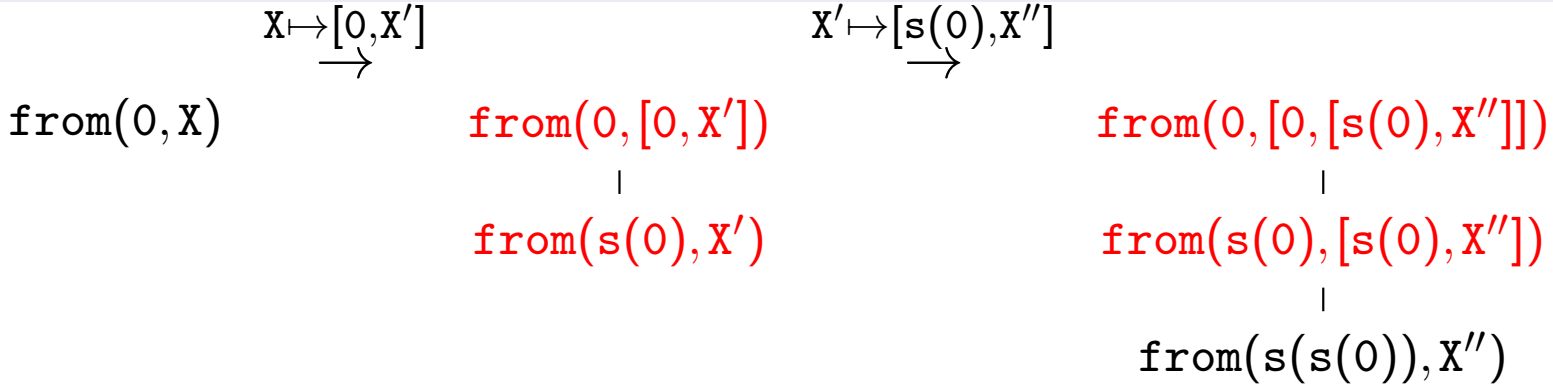
S-computations at infinity generalise computations at infinity

If a term t is SLD-computable at infinity, then it is S-computable at infinity.

How about finite proof-search procedure?

Coinductive Proof Principle

$$\mathtt{from}(\mathtt{x},\mathtt{scons}(\mathtt{x},\mathtt{y})) \leftarrow \mathtt{from}(\mathtt{s}(\mathtt{x}),\mathtt{y})$$



1. Form a coinductive hypothesis observing the second tree: $\mathtt{from}(\mathtt{s}(0),\mathtt{X}') \leftarrow \mathtt{from}(0,\mathtt{scons}(0,\mathtt{X}'))$
2. Apply it at the third tree (guarding the application by transition)
3. Close the coinductive proof by coinductive hypothesis application at the third tree.

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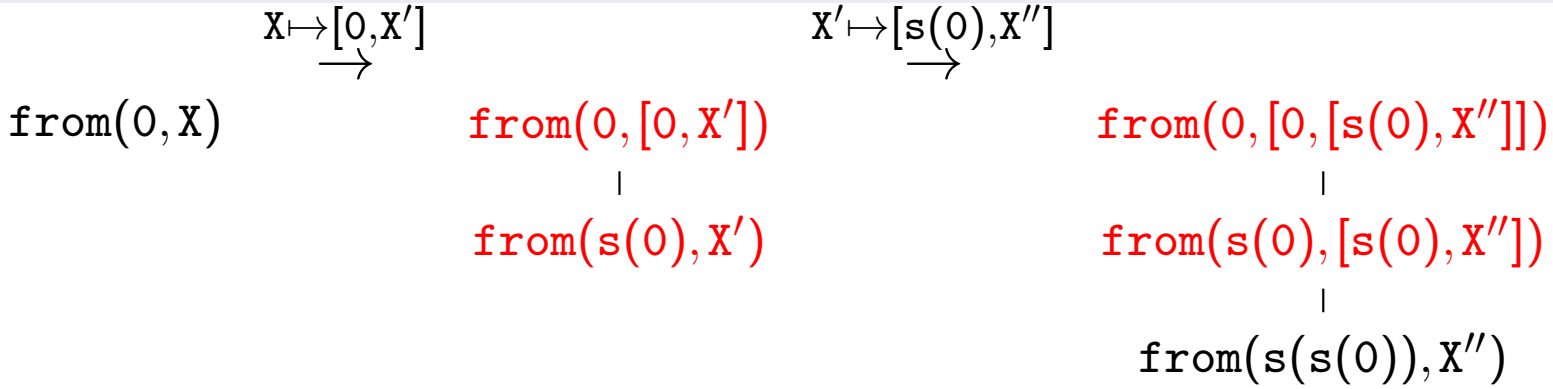

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General Coinductive Proof principle:

$$\frac{T \rightarrow \ldots \rightarrow T' \qquad T' \vdash_{CH} \theta(C)}{T \vdash_{CH} C}$$

Coinductive proof principle for S-resolution is...

...Sound and complete relative to S-computations at infinity

Given P is observationally productive, and $t \in \mathbf{Term}(\Sigma)$,
there exists a $t^* \in \mathbf{GTerm}^{\infty}(\Sigma)$ that is S-computable at infinity with respect to P and t
iff
there exists a coinductively observed proof for t.

...Sound relative to the greatest complete Herbrand models

Let P be observationally productive, and $t \in \mathbf{Term}(\Sigma)$. If there exists a coinductively observed proof for t, then there exists an infinite term $t^* \in M_P^{\omega}$ that is S-computable at infinity with respect to P and t.

Summary

- We have just given a coinductive proof-search method;
- that generalises computations at infinity from the 80s;
- but at the same time gives a coinductive proof principle and thus is implementable.
- It incorporates observational and global productivity on par with ITP;
- and generalises CoLP [Gupta, 2007]

Outline

Formalization of a Type System

Horn Formulas as Types

Proof evidence as Terms

Term $t \ ::= \ x \mid f(t_1, ..., t_n)$
Atomic Formula $A, B, C, D \ ::= \ P(t_1, ..., t_n)$
(Horn) Formula $F \ ::= \ A_1, ..., A_n \Rightarrow A$
Proof Term $p, e \ ::= \ \kappa \mid a \mid \lambda a.e \mid e \ e'$

$$\frac{}{\kappa : \forall \underline{x}.F} \ axiom \qquad \frac{e : F}{e : \forall \underline{x}.F} \ gen$$

$$\frac{e : \forall \underline{x}.F}{e : [\underline{t}/\underline{x}]F} \ inst \qquad \frac{e_1 : \underline{A} \Rightarrow D \quad e_2 : \underline{B}, D \Rightarrow C}{\lambda \underline{a}.\lambda \underline{b}.(e_2 \ \underline{b}) \ (e_1 \ \underline{a}) : \underline{A}, \underline{B} \Rightarrow C} \ cut$$

Soundness of SLD and term-matching reductions

- If $P \vdash \{A\} \rightsquigarrow^n \emptyset$, then there exists a proof $e : \forall \underline{x}. \Rightarrow \gamma A$.
- If $P \vdash \{A\} \rightarrow^n \emptyset$, then there exists a proof $e : \forall \underline{x}. \Rightarrow A$.

Example

κ_1 `nat(0)` $\leftarrow$
κ_2 `nat(s(x))` $\leftarrow$ `nat(x)`
κ_3 `list(nil)` $\leftarrow$
κ_4 `list(cons(x,y))` $\leftarrow$ `nat(x)`, `list(y)`

$\{\mathrm{list}(\mathrm{cons}(x, y))\} \rightsquigarrow \{\mathrm{nat}(\mathrm{x}), \mathrm{list}(y)\} \rightsquigarrow \{\mathrm{list}(y)\} \rightsquigarrow \emptyset$
yields a proof $(\lambda a.(\kappa_4\ a)\ \kappa_1)\ \kappa_3 : \mathrm{list}(\mathrm{cons}(0, nil))$
(β-reducible to $(\kappa_4 \kappa_3)\kappa_1 : \mathrm{list}(\mathrm{cons}(0, nil))$).

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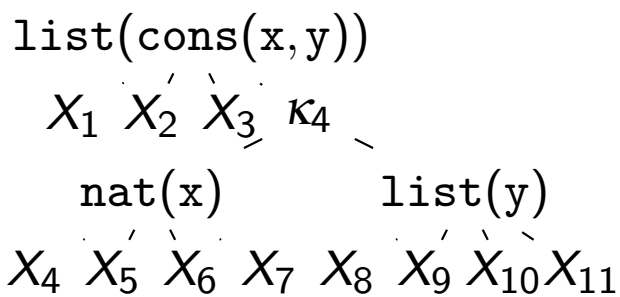
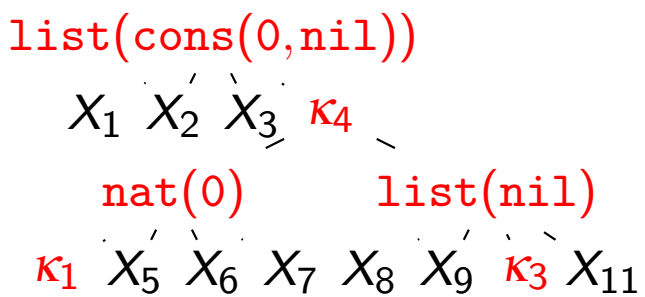
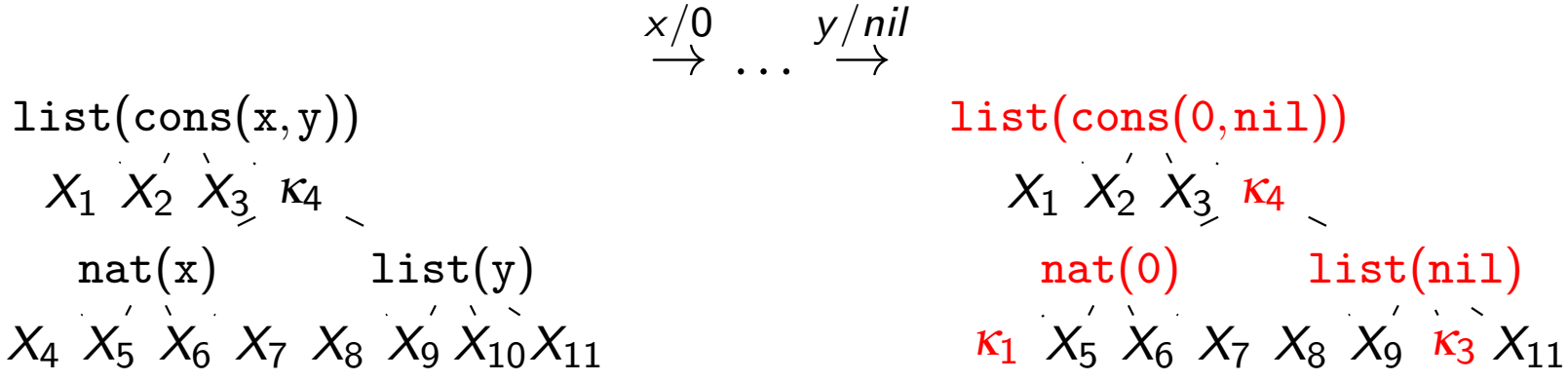
$\xrightarrow{x/0} \ \ldots \ \xrightarrow{y/nil}$





Structural resolution meets Curry-Howard

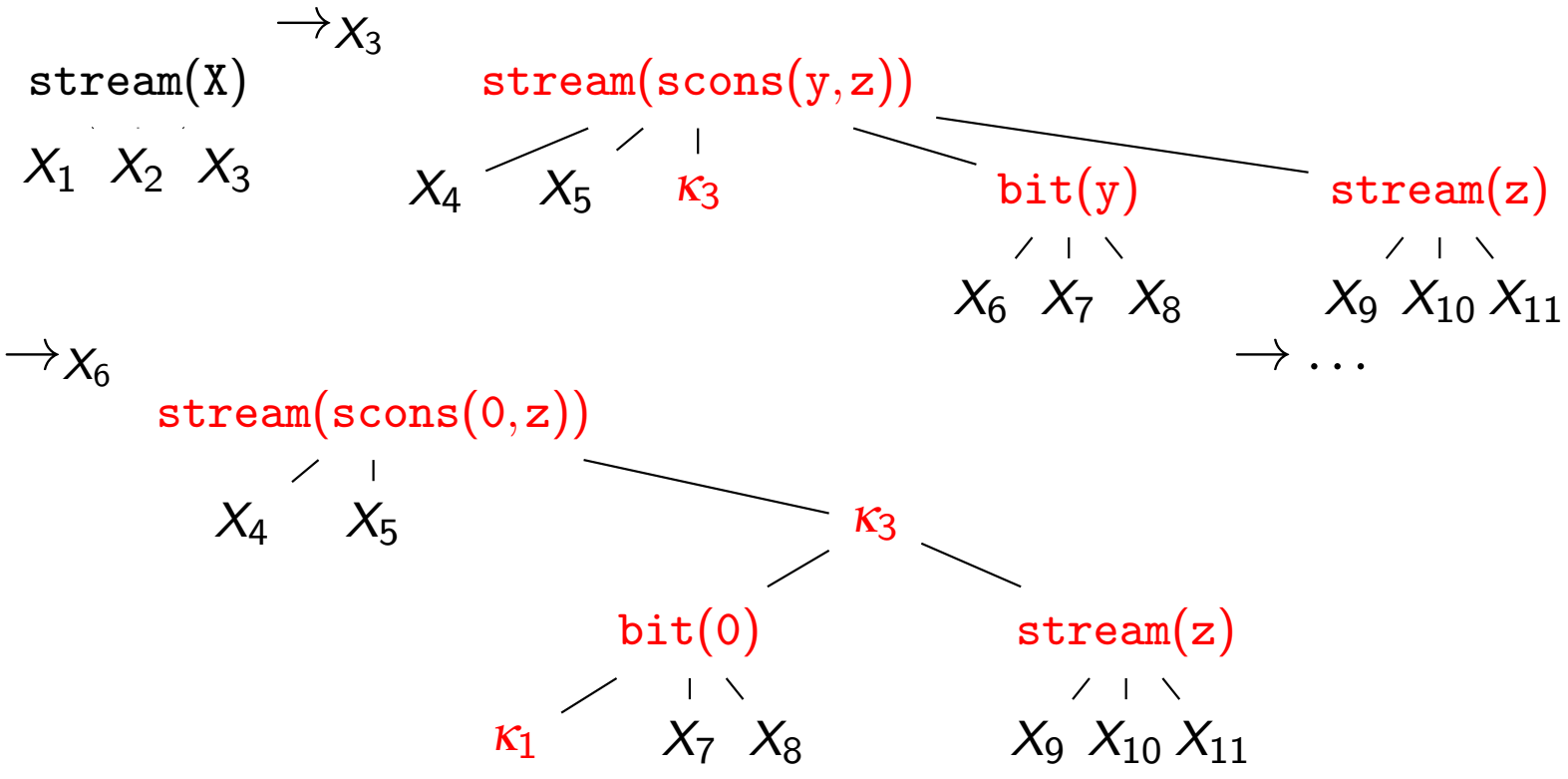
- Success rewriting trees always give a proof $e : A$, in which A is atomic root of the tree and e is a witness for clause applications.



$$(\kappa_4\,\kappa_3)\,\kappa_1 :: \mathrm{list}(\mathrm{cons}(0, \mathit{nil}))$$

Structural resolution meets Curry-Howard

- All open rewriting trees will correspond to a proof $e : B_1, \ldots, B_n \Rightarrow A$, where e is λ-term, A the root of that tree, and $B_1, \ldots, B_n$ – all open proof obligations.



$\lambda\alpha.(\kappa_3\alpha)\kappa_1 : \mathtt{stream(z)} \Rightarrow \mathtt{stream(scons(0,z))}$

Next step: fully formalise the coinductive proof principle.

Conclusions-1: Q and A

Q Why is it harder to implement coinduction in ATP than in ITP?

A ...The notions of proof and proof-search are badly separated.

A ... In absence of types and pattern-matching, we have fewer tools for structural analysis of programs and computations

Conclusions-1: Q and A

Q Why is it harder to implement coinduction in ATP than in ITP?

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A ... In absence of types and pattern-matching, we have fewer tools for structural analysis of programs and computations

Q Is it possible, in principle, to bridge the gap?

A Yes, by using structural resolution to separate proof and proof search components (term-matching and unification, rewriting trees and their transitions)

A Yes, by taking a Curry-Howard view on proof component of proof-search.

Current work

Applications of the above to Type Inference

To construct equality class instance for datatype list and int:

$$\begin{aligned} \kappa_1 : &\quad \mathrm{Eq}(x) \Rightarrow \mathrm{Eq}(\mathrm{list}(x)) \\ \kappa_2 : &\quad \Rightarrow \mathrm{Eq}(\mathrm{int}) \end{aligned}$$

When we call a query $\mathrm{Eq}(\mathrm{list}(\mathrm{int}))$, we can use LP-TM to construct a proof for $\mathrm{Eq}(\mathrm{list}(\mathrm{int}))$, which is $\kappa_2\ \kappa_1$, and then the evaluation of $\kappa_2\ \kappa_1$ will correspond to the process of evidence construction, thus yielding computational meaning of the proof.

Thank you!

S-resolution and CoALP webpage:
http://staff.computing.dundee.ac.uk/katya/CoALP/

S-resolution/CoALP authors, contributors, implementors:

- John Power
- Martin Schmidt
- Jonathan Heras
- Vladimir Komendantskiy
- Patty Johann
- Andrew Pond
- Peng Fu
- Frantisek Farka

We always welcome collaborators!
(Also currently have funding (and looking for) 1 PhD student to start working on it at Heriot-Watt.)