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Category Theory

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Motivations

1. Compositionality

- Split large problems into smaller problems
&
put solutions together
to form ~~for~~ solutions for
the large problem

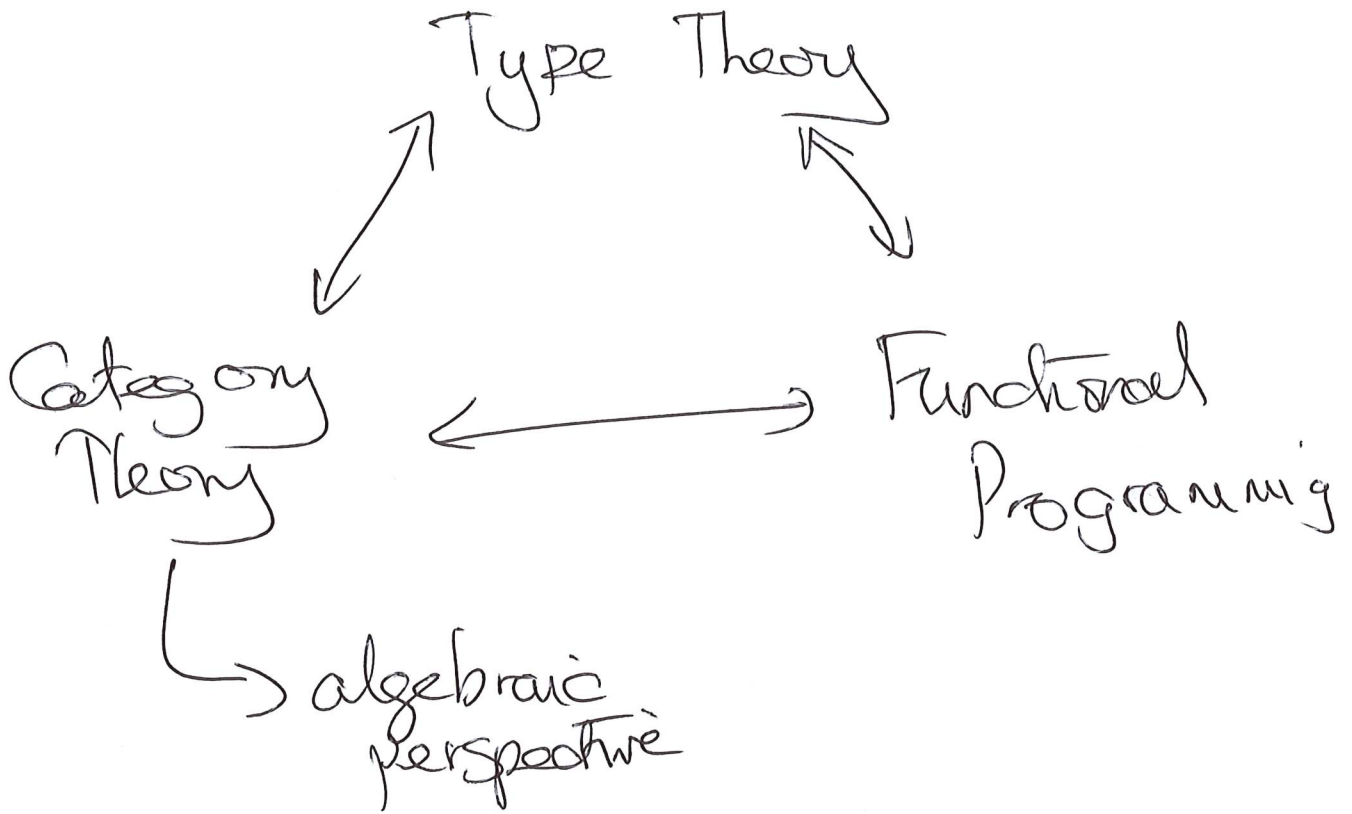
Compositionality \leftarrow Structure

Cat's Theory \equiv theory of structure

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1.2

Computational Trinitarianism



④ 1.3 Abstraction

Program 1. (i) A collection of types
 (ii) For any pair of types A, B
 $\text{Prog}(A, B)$ programs from A to B
 unit of comp $\rightarrow \text{id}_A \in \text{Prog}(A, A)$
 assoc $\rightarrow \circ : \text{Prog}(B, C) \times \text{Prog}(A, B) \rightarrow \text{Prog}(A, C)$

Logic 2. (i) A collection of propositions
 (ii) For any pair of props A & B
 $\text{Proofs}(A, B)$ proofs that $A \rightarrow B$
 unit for \circ $\rightarrow \text{id}_A \in \text{Proof}(A, A)$
 assoc $\rightarrow \circ : \text{Proof}(B, C) \times \text{Proof}(A, B) \rightarrow \text{Proof}(A, C)$
 Proof

~~1.2~~

3. Typically, a set theory has ^(S)

- (i) A collection of sets A, B
- (ii) for each pair of sets A, B ,
some functions $\text{Set}(A, B)$

unit for comp

(iii) $\text{id}_A \in \text{Set}(A, A)$

assoc \rightarrow

(iv) $\circ : \text{Set}(B, C) \times \text{Set}(A, B) \rightarrow \text{Set}(A, C)$

4. Algebra

A typical algebraic structure has, eg monoids

- (i) A collection of models M, M', \dots

- (ii) for any pair of models,

$\text{Hom}(M, M')$ of homomorphisms from M to M'

unit for \circ \rightarrow

(iii) $\text{id}_M \in \text{Hom}(M, M)$

\rightarrow associative

(iv) $\circ : \text{Hom}(M', M'') \times \text{Hom}(M, M') \rightarrow \text{Hom}(M, M'')$

Category Theory

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(1) Category

(2) Functor

lecture 1

(3) Natural Transformation

(4) Duality

(5) Limits

lecture 2

(6) ~~Adjunctions~~

(7) Monads

lecture 3

(8) Init algebras / final coalgebras

(9) Monoidal categories

(10) Fibrations

(11) HD Category Theory

Home Work

(12) Kan Extensions

(13) Enrichment

Def of Category

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A category \mathcal{C} consists of

(1) A collection of objects $|\mathcal{C}|$

X, Y, Z etc

(2) For every pair of objects X, Y ,
a collection of morphisms

$\mathcal{C}(X, Y)$

(3) For every object $X \in |\mathcal{C}|$, a
particular morphism $\text{id}_X \in \mathcal{C}(X, X)$

(4) For every triple of objects $X, Y, Z \in |\mathcal{C}|$
a composition operator

$$\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$$

such that (i) \circ is associative

$$f \circ (g \circ h) = (f \circ g) \circ h$$

(ii) id is a unit \bar{e}

$$f \circ \text{id} = f = \text{id} \circ f$$

Examples of Categories ⑧

- **Set** : Objects are sets
Morphisms are functions
- **Pos** : Objects are posets
Morphisms are monotone functions
- **Monoids** : Objects are monoids
Morphisms are monoid homomorphisms
- **Top** : Objects are topological spaces
Morphisms : continuous functions

- A useful way to define categories is

objects : Some structure of interest
morphisms : Structure preserving maps

- $\mathbb{0} = (\quad)$ $1 = (\bullet)$
 $2 = (\bullet \bullet)$ $\underline{2} = (\bullet \rightarrow \bullet)$
 $\omega = (\bullet \xrightarrow{0} \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \dots)$

Constructions / on Categories ①

Operations: Given a category \mathcal{C} ,
define some new categories

$$(i) \mathcal{C}^{op} : |\mathcal{C}^{op}| = |\mathcal{C}|$$

$$\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$$

Duality: Apply ~~new~~ definitions to
 \mathcal{C}^{op} to get new definitions

(ii) $\mathcal{C}^{\rightarrow}$: $|\mathcal{C}^{\rightarrow}|$ all the morphisms of \mathcal{C}

"arrows category"

$$\mathcal{C}^{\rightarrow}(f, g) \quad \begin{array}{ccc} X & \xrightarrow{h} & X' \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{k} & Y' \end{array}$$

(iii) $\mathcal{C} \times \mathcal{D}$ product category

$$|\mathcal{C} \times \mathcal{D}| = |\mathcal{C}| \times |\mathcal{D}|$$

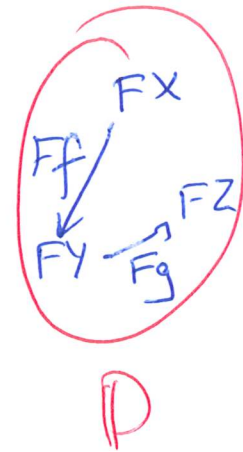
$$(\mathcal{C} \times \mathcal{D})((X, X'), (Y, Y')) = \mathcal{C}(X, Y) \times \mathcal{D}(X', Y')$$

Functors

(10)

Q What are the morphisms between categories

A The structure preserving maps



$$F : |C| \longrightarrow |D|$$

$$F_{x,y} : \forall x \forall y . C(x,y) \rightarrow D(Fx, Fy)$$

such that

$$F_{x,z}(f \circ g) = F_{x,z}(f) \circ F_{x,y}(g)$$

$$F_{x,x}(\text{id}_x) = \text{id}_{Fx}$$

Def Cat is the category whose objects are categories & morphisms are functors

Examples of functors

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(1) $\mathcal{U} : \text{Monoid} \rightarrow \text{Set}$

$$\begin{array}{ccc} (M, *, e) & \longmapsto & M \\ f \downarrow & & \downarrow f \\ (M', *, e') & \longmapsto & M' \end{array}$$

(2)

$\text{List} : \text{Set} \rightarrow \text{Monoids}$

$$\begin{array}{ccc} X & \longmapsto & \text{List}(X) \\ f \downarrow & & \downarrow \text{??} \\ Y & \longmapsto & \text{List}(Y) \end{array}$$

(3)

$\mathcal{C}(X, -) : \mathcal{C} \rightarrow \text{Set}$

$$\begin{array}{ccc} Y & \longmapsto & \mathcal{C}(X, Y) \\ f \downarrow & & \downarrow \text{??} \\ Z & \longmapsto & \mathcal{C}(X, Z) \end{array}$$

(4)

$\mathcal{C}(-, X) = \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

why \mathcal{C}^{op}

More functors

(12)

$$X : \text{Poset} \times \text{Poset} \longrightarrow \text{Poset}$$

$$X : \text{Set} \times \text{Set} \longrightarrow \text{Set}$$

$$X : \text{Monoid} \times \text{Monoid} \longrightarrow \text{Monoid}$$

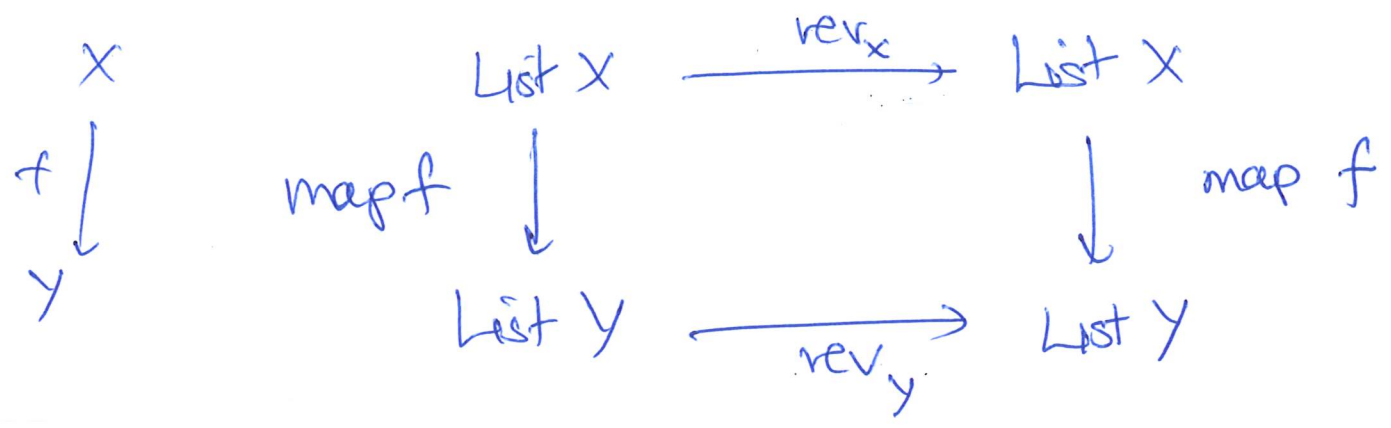
$$X : \text{Cat} \times \text{Cat} \longrightarrow \text{Cat}$$

↳ universal
structure via
category theory

Natural Transformations

NTs are simple ~~abstraction~~ abstraction of parametric polymorphism

$$\text{rev} : \forall X. \text{List } X \longrightarrow \text{List } X$$



~~Lemma~~
Definition NT

Given functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$

a natural transformation

$$\alpha : F \rightarrow G$$

consists of $\forall x \in |\mathcal{C}|$

a map $\alpha_x : Fx \rightarrow Gx$

such that for every map $f : x \rightarrow y$

$$\begin{array}{ccc}
 Fx & \xrightarrow{\alpha_x} & Gx \\
 Ff \downarrow & & \downarrow Gf \\
 Fy & \xrightarrow{\alpha_y} & Gy
 \end{array}$$

Show $\alpha : \mathcal{C} \rightarrow \mathcal{C} \rightarrow$

Examples

$$X \xrightarrow{[] } \text{List } X$$

$$\text{List } X \times (\text{List } X) \xrightarrow{\text{concat}} \text{List } X$$

$$\text{eval}_B = [A, B] \times A \longrightarrow B$$

$$f : X \longrightarrow X \times X$$

what equation
best
naturally
enforce

Def

Functor category

~~$\mathcal{D}^{\mathcal{C}}$~~ $\mathcal{D}^{\mathcal{C}}$

objects : functors F, G from \mathcal{C} to \mathcal{D}

Morphisms : ~~Actual~~

$\mathcal{D}^{\mathcal{C}}$ ~~$\mathcal{C}^{\mathcal{D}}$~~ $(F, G) =$ natural trans. from F to G