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Category Theory

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Motivations

1. Compositionality

- Split large problems into smaller problems
  &
  put solutions together to form ~~fol~~ solutions for the large problem

Compositionality $\leftarrow$ Structure

Catg Theory $\equiv$ theory of structure

1.2

Computational Trinitarianism

$$\begin{tikzcd} & \text{Type Theory} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{Category Theory} \arrow[rr, leftrightarrow] & & \text{Functional Programming} \end{tikzcd}$$

↳ algebraic perspective

④ 1.3 Abstraction

Program 1. (i) A collection of types
(ii) For any pair of types A, B
Prog (A, B) programs from A to B

unit of comp → $id_A \in$ Prog (A, A)

assoc → $\circ$: Prog $(B, C) \times$ Prog $(A, B) \longrightarrow$ Prog (A, C)

Logic 2. (i) A collection of propositions
(ii) For any pair of props A & B
Proofs (A, B) proofs that $A \rightarrow B$

unit for $\circ$ ~~3~~. → $id_A \in$ Proof (A, A)

assoc → $\circ$: Proof $(B, C) \times$ Proof $(A, B) \rightarrow$ Proof (A, C)

~~4~~.

~~[illegible]~~ 3. Typically, a set theory has

(i) A collection of sets A, B

(ii) for each pair of sets A, B, some functions $\mathrm{Set}(A,B)$

unit for comp (iii) $id_A \in \mathrm{Set}(A,A)$

assoc → (iv) $\circ : \mathrm{Set}(B,C) \times \mathrm{Set}(A,B) \to \mathrm{Set}(A,C)$

4. Algebra

A typical algebraic structure has, eg monoids ~~[illegible]~~

(i) A collection of models M, M' ...

(ii) for any pair of models, $\mathrm{Hom}(M,M')$ of homomorphisms from M to M'

unit for $\circ$ → (iii) $id_M \in \mathrm{Hom}(M,M)$

→ associative (iv) $\circ : \mathrm{Hom}(M',M'') \times \mathrm{Hom}(M,M') \to \mathrm{Hom}(M,M'')$

Category Theory

(1) Category
(2) Functor
(3) Natural Transformation
(4) Duality

lecture 1

(5) Limits
(6) Adjunctions

lecture 2

(7) Monads
(8) Init algebras / final coalgebras
(9) Monoidal categories

lecture 3

(10) fibrations
(11) HD Category theory
(12) Kan Extensions
(13) Enrichment

Home Work

Def of Category

A category $\mathbb{C}$ consists of

(1) A collection of objects $|\mathbb{C}|$

X, Y, Z etc

(2) For every pair of objects X, Y, a collection morphisms

$\mathbb{C}(X, Y)$

(3) For every object $X \in |\mathbb{C}|$, a particular morphism $id_X \in \mathbb{C}(X, X)$

(4) For every triple of objects $X, Y, Z \in |\mathbb{C}|$ a composition operator

$\circ : \mathbb{C}(Y, Z) \times \mathbb{C}(X, Y) \longrightarrow \mathbb{C}(X, Z)$

such that (i) $\circ$ is associative

$f \circ (g \circ h) = (f \circ g) \circ h$

(ii) id is a unit ie

$f \circ id = f = id \circ f$

Examples of Categories

- **Set** : Objects are sets
  morphisms are functions
- **Pos** : Objects are posets
  morphisms are monotone functions
- **Monoids** : Objects are monoids
  morphisms are monoid homomorphisms
- **Top** : Objects are topological space
  morphisms : continuous functions

– A useful way to define categories is

objects : Some structure of interest
morphisms : structure preserving maps

- $\mathbb{0} = (\quad)$ $\quad$ $\mathbb{1} = (\cdot)$

$2 = (\cdot \quad \cdot)$ $\quad$ $\underline{2} = (\cdot \longrightarrow \cdot)$

$\omega = (\underset{0}{\cdot} \longrightarrow \underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longrightarrow \underset{3}{\cdot} \quad \cdots)$

Constructions / Structure on Categories

Operations: Given a category $\mathbb{C}$, define some new categories

(i) $\mathbb{C}^{op}$: $|\mathbb{C}^{op}| = |\mathbb{C}|$

$$\mathbb{C}^{op}(X,Y) = \mathbb{C}(Y,X)$$

Duality : Apply ~~cons~~ definitions to $\mathbb{C}^{op}$ to get new definitions

(ii) $\mathbb{C}^{\to}$: $|\mathbb{C}^{\to}|$ all the morphisms of $\mathbb{C}$

"arrow category"

$\mathbb{C}^{\to}(f,g)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "k"'] & Y' \end{tikzcd}$$

(iii) $\mathbb{C} \times \mathbb{D}$ product category

$$|\mathbb{C} \times \mathbb{D}| = |\mathbb{C}| \times |\mathbb{D}|$$

$$(\mathbb{C} \times \mathbb{D})\big((X,X'), (Y,Y')\big) = \mathbb{C}(X,Y) \times \mathbb{D}(X',Y')$$

Functors

Q What are the morphisms between categories

A the structure preserving maps

$$F : |\mathbb{C}| \longrightarrow |\mathbb{D}|$$

$$F_{X,Y} : \forall X \forall Y .\ \mathbb{C}(X,Y) \rightarrow \mathbb{D}(FX, FY)$$

such that

$$F_{X,Z}(f \circ g) = F_{Y,Z}(f) \circ F_{X,Y}(g)$$

$$F_{X,X}(\mathrm{id}\, X) = \mathrm{Id}_{FX}$$

Def Cat is the category whose objects are categories & morphisms are functors

Examples of functors

(1) $U : \text{Monoid} \longrightarrow \text{Set}$

$$\begin{tikzcd}
(M, *, e) \arrow[d, "f"'] \arrow[r, mapsto] & M \arrow[d, "f"] \\
(M', *', e') \arrow[r, mapsto] & M'
\end{tikzcd}$$

(2) $\text{List} : \text{Set} \longrightarrow \text{Monoids}$

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[r, mapsto] & \text{List}(X) \arrow[d, "??"] \\
Y \arrow[r, mapsto] & \text{List}(Y)
\end{tikzcd}$$

(3) $\mathbb{C}(X, -) : \mathbb{C} \longrightarrow \text{Set}$

$$\begin{tikzcd}
Y \arrow[d, "f"'] \arrow[r, mapsto] & \mathbb{C}(X, Y) \arrow[d, "??"] \\
Z \arrow[r, mapsto] & \mathbb{C}(X, Z)
\end{tikzcd}$$

(4) $\mathbb{C}(-, X) : \mathbb{C}^{op} \longrightarrow \text{Set}$

Why $\mathbb{C}^{op}$

More functors

$$\times : \mathbf{Poset} \times \mathbf{Poset} \longrightarrow \mathbf{Poset}$$

$$\times : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$$

$$\times : \mathbf{Monoid} \times \mathbf{Monoid} \longrightarrow \mathbf{Monoid}$$

$$\times : \mathbf{Cat} \times \mathbf{Cat} \longrightarrow \mathbf{Cat}$$

↳ universal structure via category theory

Natural Transformations

__NTs__ are simple abstractions of parametric polymorphism

$$\mathrm{rev} : \forall X.\ \mathrm{List}\ X \longrightarrow \mathrm{List}\ X$$

$$\begin{tikzcd}
X \arrow[d, "f"'] \\
Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{List}\ X \arrow[r, "\mathrm{rev}_X"] \arrow[d, "\mathrm{map}\ f"'] & \mathrm{List}\ X \arrow[d, "\mathrm{map}\ f"] \\
\mathrm{List}\ Y \arrow[r, "\mathrm{rev}_Y"'] & \mathrm{List}\ Y
\end{tikzcd}$$

~~[illegible]~~

Definition NT

Given functors $F, G : \mathbb{C} \longrightarrow \mathbb{D}$

a natural transformation

$$\alpha : F \longrightarrow G$$

consists of $\forall X \in |\mathbb{C}|$

a map $\alpha_X : FX \longrightarrow GX$

such that for every map $f : X \to Y$

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}$$

Show $\alpha : \mathbb{C} \longrightarrow \mathbb{C}^{\rightarrow}$

Examples

$$X \xrightarrow{[\,]} \text{List } X$$

$$\text{List}(\text{List } X) \xrightarrow{\text{concat}} \text{List } X$$

$$\text{eval}_B : [A, B] \times A \longrightarrow B$$

what equation does naturality enforce

$$f : X \longrightarrow X \times X$$

Def Functor category $\cancel{\mathbb{C}^{\mathbb{D}}}$ $\mathbb{D}^{\mathbb{C}}$

objects: functors F, G from $\mathbb{C}$ to $\mathbb{D}$

morphisms: ~~Natural~~

$\mathbb{D}^{\mathbb{C}}$ $\cancel{\mathbb{C}^{\mathbb{D}}}$ (F, G) = natural trans from F to G