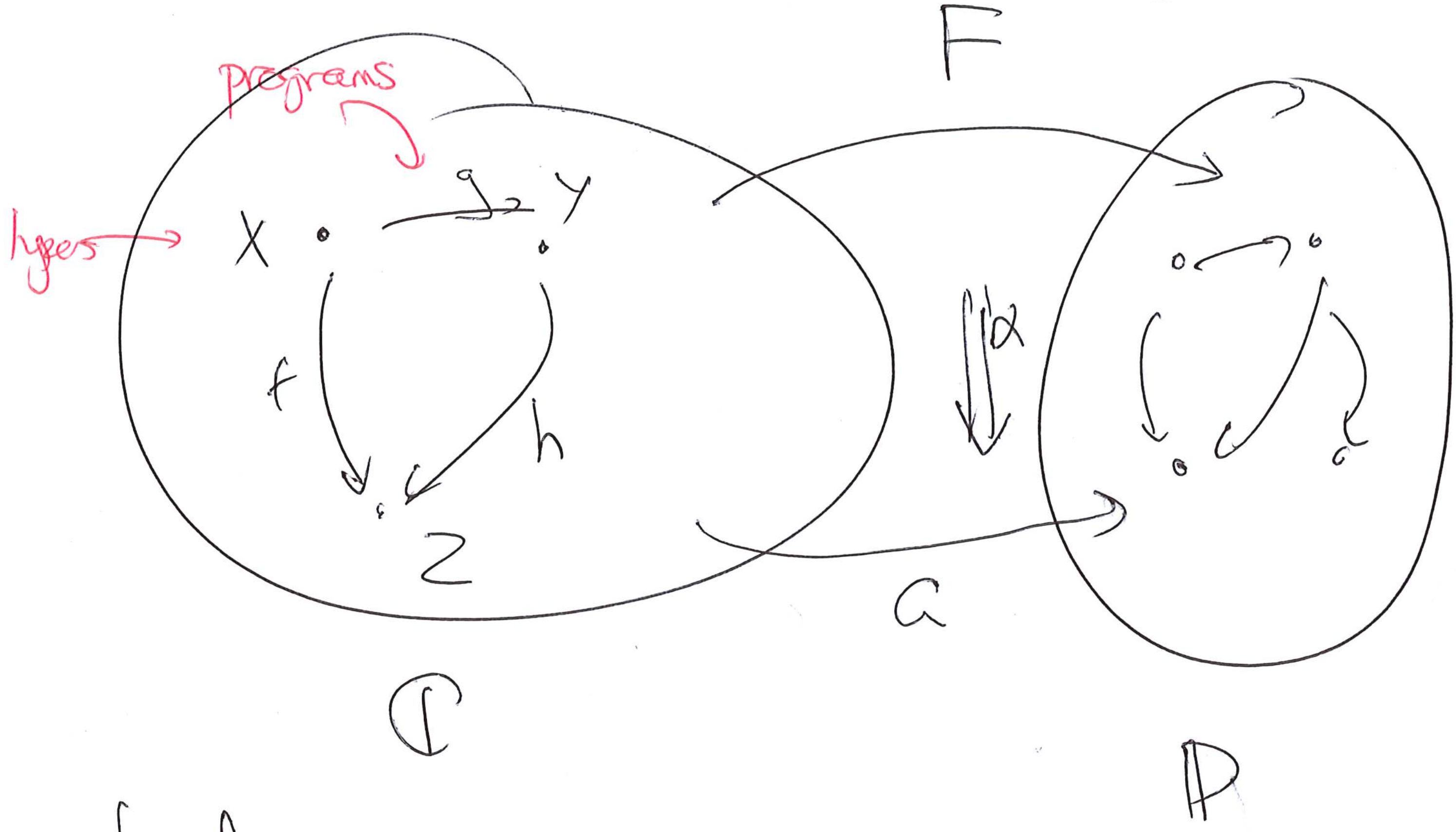
Category Theory II

Lecture 1 : Outside the category



Lecture 2 Inside a category

Isomorphisms

In set theory a set is defined by its elements.

$$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

$\Rightarrow$ Machine code mathematics

In category theory, what matters is the morphism in and out of an object.

(A is isomorphic to B iff)

In ~~Set~~ $\mathbb{C}$ ($A \simeq B$ iff) ③

$\exists\, f : A \longrightarrow B$

$\exists\, g : B \longrightarrow A$

st $\quad g \cdot f = id_A \qquad f \cdot g = id_B$

A Terminal object in a category $\mathbb{C}$

is an object T st.

forall objects $X \in |\mathbb{C}|$, there is one and only one morphism $X \xrightarrow{!} T$

In sets, $\{*\}$ is a terminal object

$\{\square\}$ is a terminal object

Lemma In particular if T & T' are both terminal then $T \simeq T'$

<u>Proof</u> There is a map $!_T : T' \longrightarrow T$.

There is also a map $!_{T'} : T \longrightarrow T'$

$$!_{T'} \circ !_T : T' \longrightarrow T'$$

Because there is only map from T' to T' (because T' is terminal)

$$\therefore \quad !_{T'} \circ !_T = id_{T'}$$

Similarly $\quad !_T \circ !_{T'} = id_T$

~~An~~ Other examples.

<u>Logic</u> $\quad X \longrightarrow T$

<u>Monoids</u> $\quad (M, \circ_M, e_M) \longrightarrow (\{*\}, \circ, e)$

Qn What is a terminal object in $\mathbb{C}^{op}$

for every object X, there is a unique map in $\mathbb{C}^{op}$ $X \longrightarrow \perp$

ie there is a unique map in $\mathbb{C}$ from $\perp \longrightarrow X$

Ans An initial object is an object $\perp$ such that for every object X there is a unique map $\perp \longrightarrow X$

In Set $\emptyset$ is initial

~~propo~~ False in logic

$(\{*\}, \circ, *)$ in monoids

Products

Definition product of sets

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$
$$= \{ (a,b,*) \mid a \in A,\ b \in B \}$$

Categorically Let $\mathbb{C}$ be an arbitrary category. Let $A, B \in |\mathbb{C}|$

Their product $A \times B$ is an object of $\mathbb{C}$ &

$$\pi_1 : A \times B \longrightarrow A$$
$$\pi_2 : A \times B \longrightarrow B$$

data

such that for every trip $\left(X \in |\mathbb{C}|,\ f : X \to A,\ g : X \to B \right)$

given any other similarly typed data

$$\begin{tikzcd}
 & X \arrow[dl, "g"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "f"] & \\
B & A \times B \arrow[l, "\pi_2"] \arrow[r, "\pi_1"'] & A
\end{tikzcd}$$

terminal amongst such data

there is a unique map $\langle f, g \rangle : X \to A \times B$

such that
$$\pi_2 \circ \langle f, g \rangle = g$$
$$\pi_1 \circ \langle f, g \rangle = f$$

Lemma

$$A \times B \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \times A \quad (\cong)$$

$\alpha : A \times B \longrightarrow B \times A$ — use this definition

$\alpha = \langle \pi_2^{A\times B}, \pi_1^{A\times B} \rangle$

$\beta : B \times A \longrightarrow A \times B$ — use $A \times B$ is a product

$\beta = \langle \pi_2^{B\times A}, \pi_1^{B\times A} \rangle$

$\beta \circ \alpha : A \times B \longrightarrow A \times B$

WTS $\beta \cdot \alpha = id$ use uniqueness in defn of $A \times B$ to show this

Products in $\mathbb{C}^{op}$

Given objects A & B in $\mathbb{C}^{op}$, ie in $\mathbb{C}$

their product in $\mathbb{C}^{op}$ is

(1) An object $A+B$ in $\mathbb{C}^{op}$

(2) projection $\pi_1 : A+B \longrightarrow A$ in $\mathbb{C}^{op}$

$\pi_2 : A+B \longrightarrow B$ in $\mathbb{C}^{op}$

& more stuff.

ie An object $A+B$ in $\mathbb{C}$

ie a morphism $A \xrightarrow{in_1} A+B$

a morphism $B \xrightarrow{in_2} A+B$

ie a coproduct.

And Lemma $A+B \cong B+A$

<u>Proof</u> theorem holds for products

& products are symmetric