Lecture 1: Outside the category $\mathcal{C}$

$\mathcal{C}$

Lecture 2: Inside a category $\mathcal{P}$
Isomorphisms

In set theory, a set is defined by its elements.

\[ 3 = \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \]

\[ \text{\LARGE \Rightarrow } \text{Machine code mathematics.} \]

In category theory, what matters is the morphism in and out of an object.
(A is isomorphic to B iff)

\[ \text{In } \text{Set} \quad \exists \quad (A \cong B \text{ iff}) \]

\[ \exists \ f : A \rightarrow B \]

\[ \exists \ g : B \rightarrow A \]

\[ \text{st} \quad g \circ f = \text{id}_A \quad f \circ g = \text{id}_B \]

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A *Terminal object* in a category \( \text{C} \)

is an object \( T \) st.

for all objects \( X \in \text{Ob}(\text{C}) \), there is one and only one morphism \( X \rightarrow T \)

In \( \text{sets} \), \( \emptyset \times \emptyset \) is a terminal object

\( \emptyset \cup \emptyset \) is a terminal object

**Lemma:** In particular, if \( T \) \& \( T' \) are both terminal then \( T \cong T' \)
Proof

There is a map \( \mathbf{T} : \mathbf{T}' \to \mathbf{T} \).

There is also a map \( \mathbf{T}' : \mathbf{T} \to \mathbf{T}' \).

\( \mathbf{T}' \circ \mathbf{T} : \mathbf{T}' \to \mathbf{T}' \)

Because there is only one map from \( \mathbf{T}' \) to \( \mathbf{T} \) (because \( \mathbf{T}' \) is terminal).

\( \circ \circ \ \mathbf{T}' \circ \mathbf{T} = \mathbf{id}_{\mathbf{T}'} \)

Similarly \( \mathbf{T} \circ \mathbf{T}' = \mathbf{id}_{\mathbf{T}} \)

Other examples.

Logic

\( X \to \mathbf{T} \)

Monoids

\( (M, \cdot, e) \to (\mathbb{F} \times \mathbb{F}, 0, 1) \)
On what is a terminal object in Cop?

For every object X, there is a unique map in Cop: \( X \rightarrow \top \)

ie there is a unique map

In C from \( \top \rightarrow X \)

Ans: An initial object is an object \( I \) such that for every object \( X \)

there is a unique map \( I \rightarrow X \)

In Set, \( \emptyset \) is initial.

For \( \mathbb{N} \), \( (\mathbb{N}, 0, \times) \) is a monoid.
Definition product of sets

\[ A \times B = \{ (a, b) \mid a \in A, b \in B \} \]

Categorically, let \( C \) be an arbitrary category, let \( A, B \in \text{Hom}(C) \).

Their product \( A \times B \) is an object of \( C \) and

\[ \pi_1 : A \times B \rightarrow A \quad \text{data} \]
\[ \pi_2 : A \times B \rightarrow B \]

such that for every pair \((X, f, g) : (X \in \text{Hom}(C), f : X \rightarrow A, g : X \rightarrow B)\), there is a unique map \( \langle f, g \rangle : X \rightarrow A \times B \) such that

\[ \pi_2 \circ \langle f, g \rangle = g \]
\[ \pi_1 \circ \langle f, g \rangle = f \]
Lemma \[ \alpha: A \times B \xrightarrow{} B \times A \]

\[ \alpha = < \pi_2^{A \times B}, \pi_1^{A \times B} > \]

\[ \beta: B \times A \rightarrow A \times B \]

\[ \beta = < \pi_2^{B \times A}, \pi_1^{B \times A} > \]

\[ \beta \circ \alpha: A \times B \rightarrow A \times B \]

WTS \[ \beta \circ \alpha = id \]

Use uniqueness in defn of $A \times B$ to show this.
Products in \( C^{op} \)

Given objects \( A \) and \( B \) in \( C^{op} \), \( \hat{e} \) in \( C \),

their product in \( C^{op} \) is

1. An object \( A + B \) in \( C^{op} \)
2. projections \( \pi_1 : A + B \rightarrow A \) in \( C^{op} \)
   \( \pi_2 : A + B \rightarrow B \) in \( C^{op} \)

...more stuff...

\( \hat{e} \) An object \( A + B \) in \( C \)
\( \hat{e} \) a morphism \( A \overset{\hat{e}}{\rightarrow} A + B \)
\( \hat{e} \) a morphism \( B \overset{\hat{e}}{\rightarrow} A + B \)

\( \hat{e} \) a coproduct.

And Lemma A + B a B + A

Proof theorem holds for products

A product is symmetric