data Nat : Set where

  zero : Nat

  succ : Nat $\longrightarrow$ Nat

$$\text{~~Types~~} \xrightarrow{\text{prog}} \text{Type}$$

$$\left.\begin{array}{l} (X : \text{Set}) \xrightarrow{(f : \text{fn})} (Y : \text{Set}) \\ (X : \text{Cpo}) \xrightarrow[g : \text{cts fn}]{} (Y : \text{Cpo}) \\ (X : \text{Top}) \xrightarrow[g : \text{cts fn}]{} (Y : \text{Top}) \\ (X : \text{PER}) \xrightarrow[g : \text{realizer}]{} (Y : \text{PER}) \end{array}\right\}$$

$f \in \mathbb{C}(X, Y)$

g ~~[illegible]~~ $\top$

$\perp$

$\times$

$+$

$$\text{PL} \xrightarrow[\text{den sem}]{[\![\ ]\!]} \text{Cat}$$

Syntax   model

reconstruct / prediction

Chicken or Eggs?

Moxey [illegible]

Adjuctions

Given

$$\frac{2x \leq 5}{x \leq 5/2}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

<u>Defn</u> F is left adjoint to G

$F \dashv G : \mathbb{C} \longrightarrow \mathbb{D}$

iff

$$\phi^{-1} \circlearrowleft \frac{\mathbb{D}(FX, Y)}{\mathbb{C}(X, GY)} \circlearrowright \phi$$

natural in X & Y

Example , STLC

$$\llbracket [\,] \rrbracket = 1$$

$$\llbracket \Gamma, x : A \rrbracket = \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket$$

$$\llbracket \Gamma \vdash t : A \rrbracket \in \mathbb{C}(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket)$$

$$\mathrm{app}\ \ \frac{\mathbb{C}(\llbracket \Gamma \rrbracket \times \llbracket A \rrbracket, \llbracket B \rrbracket)}{\mathbb{C}(\llbracket \Gamma \rrbracket, \llbracket A \to B \rrbracket)}\ \ \lambda$$

$$\beta \quad \mathrm{app} \circ \lambda = \mathrm{id}$$

$$\eta \quad \lambda \circ \mathrm{app} = \mathrm{id}$$

$$\frac{\mathbb{C}(X \times A, B)}{\mathbb{C}(X, A \to B)}$$

$$- \times A \dashv A \to - \quad : \mathbb{C} \to \mathbb{C}$$

Eg STLC

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A . t : A \to B}$$

2

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

β $\quad (\lambda x . t)(u) = t[u/x]$

$$\lambda x . tx = t \qquad x \notin FV(t)$$

$$\frac{A \text{ type} \qquad B \text{ type}}{A \to B \text{ type}}$$

Another Example.

$$\begin{tikzcd}
\mathrm{Set} \arrow[rr, "\mathrm{List}"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathrm{Monoids} \arrow[ll, bend left, "U"]
\end{tikzcd}$$

$$\frac{\mathrm{Set}(X, UM)}{\mathrm{Monoid}(\mathrm{List}\,X \xrightarrow{g} M)}$$

use ~~to ex~~ as a definitional mechanism

<u>Now</u> <u>Cuteness Approaching</u>.

$$\forall X \forall Y \quad \frac{\mathbb{D}(FX, Y)}{\mathbb{C}(X, GY)}$$

choose $Y = FX$, id

$\mapsto \quad X \to GFY$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$Id \to GF : \mathbb{C} \to \mathbb{C}$

$FG \to Id : \mathbb{D} \to \mathbb{D}$

weak - equality.

Datatypes

<u>System T</u>

$$\mathrm{Nat} : \mathrm{Type}$$

$$\Gamma \vdash 0 : \mathrm{Nat} \qquad \frac{\Gamma \vdash n : \mathrm{Nat}}{\Gamma \vdash Sn : \mathrm{Nat}}$$

$$\Gamma \vdash u : X$$
$$\Gamma \vdash v : X \to X$$
$$\Gamma \vdash It(u,v) : \mathrm{Nat} \to X$$

$$It(u,v)\ 0 = u$$
$$It(u,v)\ (n+1) = v\ (It(u,v)\ n)$$

$$\begin{tikzcd}
1+\mathrm{Nat} \arrow[r, "{[0,S]}"] \arrow[d, "1+It(u,v)"'] \arrow[d, bend right=60, "1+h"'] & \mathrm{Nat} \arrow[d, "It(u,v)"] \arrow[d, bend left=60, "= h"] \\
1+X \arrow[r, "{[u,v]}"] & X
\end{tikzcd}$$

β-rules $\equiv$ commutation of this square

η-rule $\equiv$ Iterator is the unique map $N \to X$ making commutation

Add data types

IAS

$$\begin{tikzcd}
1+N \arrow[r, "{[0,\mathrm{succ}]}"] \arrow[d, "1+\mathrm{It}(u,v)"'] & N \arrow[d, "\mathrm{It}(u,v)"] \\
1+X \arrow[r, "{[u,v]}"'] & X
\end{tikzcd}$$

& It (u,v) is unique. β & η.

---- same style as $\bot, \top, \times, 0$ & $\to$

= universal props

Here is some stuff A
and for any other such stuff, B
Here is a map stuff $\underline{A} \to B$
$B \to \underline{A}$
which is unique
or the cat'y of stuff has $\bot$, $\top$

β $\quad \mathrm{It}(u,v)\ 0 = u$

$\mathrm{It}(u,v)\,(n+1) = v(\mathrm{It}(u,v)\, n)$

η $\quad$??

data ...

⋮

~~$f : N \to X$ where f zero $=$ ... f (succ n) $=$ ...~~

System T

$\equiv$ STLC +

Nat : Set

$\Gamma \vdash 0 : \mathrm{Nat}$

$\Gamma \vdash s : N \to N$

$\Gamma \vdash u : X$

$\Gamma \vdash v : X \to X$

$\Gamma \vdash \mathrm{It}(u,v) : N \to X$

Initial Algebra Semantics

$1 + \_ : \mathbb{C} \longrightarrow \mathbb{C}$ is a functor.

<u>Instead</u> consider an arbitrary functor

$$F : \mathbb{C} \longrightarrow \mathbb{C}$$

An Algebra for F is a morphism

$$(X, \; h : FX \longrightarrow X)$$

A morphism (X, h) to (Y, j)

$$\begin{tikzcd} FX \arrow[r, "h"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ FY \arrow[r, "j"'] & Y \end{tikzcd}$$

is a map $f : X \longrightarrow Y$ st the diagram commutes

This is a category F-Alg of F-algebras.

eg $FX = 1+X$ .... Nat

$FX = 1 + X \times X$ ... binary trees no data

$FX = A + X \times X$ ... binary trees data leaf from A

$FX = 1 + A \times X$ ... list of A

<u>Qn</u> What is the initial object in the cat/y of F-algebras.

$$\frac{X \text{ type} \vdash FX \text{ type}}{\mu F \text{ type}}$$

constructors

$$\begin{tikzcd} FX \arrow[r, "in"] \arrow[d, "F(\mathrm{fold}\, h)"'] & X \arrow[d, "\mathrm{fold}\, h"] \\ FY \arrow[r, "h"] & Y \end{tikzcd}$$

structure of the initial algebra

diagram commuting is β-rule

elimination rule

uniqueness ... η-rule

Final Coalgebra semantics. Dually

What is a (i) completion of finite

(ii) dual of finite

consider $A \times -$

What is the category of $A \times -$ -coalgebras

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \times X \arrow[d, "A \times f"] \\ Y \arrow[r] & A \times Y \end{tikzcd}$$

stream (A) = final $A \times -$ - coalgebra

ie $\dfrac{A \text{ type}}{\text{stream } A \text{ type}}$ $\langle hd, tl \rangle$: stream $A \to A \times$ str A

$unfold_{str} : \forall Y \, (Y \to A \times Y) \to Y \to \text{str } A$

$hd \, (unfold \, \phi) \, x = h \, x$

$tl \, (unfold \, \phi) \, x = unfold \, \phi \, (t \, x)$ + η.

Effects & Monadic Computation

$X \longrightarrow Y$ pure functions

$X \longrightarrow Y + Ex$ exceptions

$X \longrightarrow \mathbb{P}Y$ nondeterminism

$X \longrightarrow Pr\,Y$ probabilistic comp.

$S \times X \longrightarrow S \times Y$ stateful comp.

$X \longrightarrow [S, S \times Y]$

$X \longrightarrow (Y \to R) \to R$ continuations

Abstract Ex, $\mathbb{P}$, Pr, $[S, S \times \_\,]$

$(\_ \to R) \to R$

call these T

Idea

An T-effectful computation from X to Y

is an ~~effectful~~ pure computation $X \longrightarrow TY$

T needs extra structure to model

(1) the embedding of pure data into the effectful world

(2) to model composition of effectful comp —

(1) return $: \forall X . \ X \longrightarrow TX$

i.e. T is a functor

& a natural transform ~~$\eta : \mathrm{Id} \Rightarrow T$~~

$$\eta : \mathrm{Id} \to T$$

(II) Comp. of eff. computations

Given $X \longrightarrow TY$

$Y \longrightarrow TZ$

want $\overline{}$

$X \longrightarrow TZ$

<u>Assume</u> $\text{bind}_{Y,Z} : TY \longrightarrow (Y \rightarrow TZ) \longrightarrow TZ$

↳ gives composition

<u>or</u> equivalently

assume $\mu_X : T(TX) \longrightarrow TX$

ie a natural transform

$\mu : T^2 \longrightarrow T$

A monad on $\mathbb{C}$ is

a functor $T : \mathbb{C} \longrightarrow \mathbb{C}$

nat trans $\eta : \mathrm{Id} \longrightarrow T$

nat trans $\mu : T^2 \longrightarrow T$

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

$\sigma : F \to G$

$\sigma K : FK \to GK$

$(\sigma K)_X : FKX \to GKX$

$= \sigma_{KX}$

$K\sigma : KF \to KG$

$(K\sigma)_X : KFX \to KGX$

$= K\sigma_X$

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, equal, "\mathrm{id}"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}$$