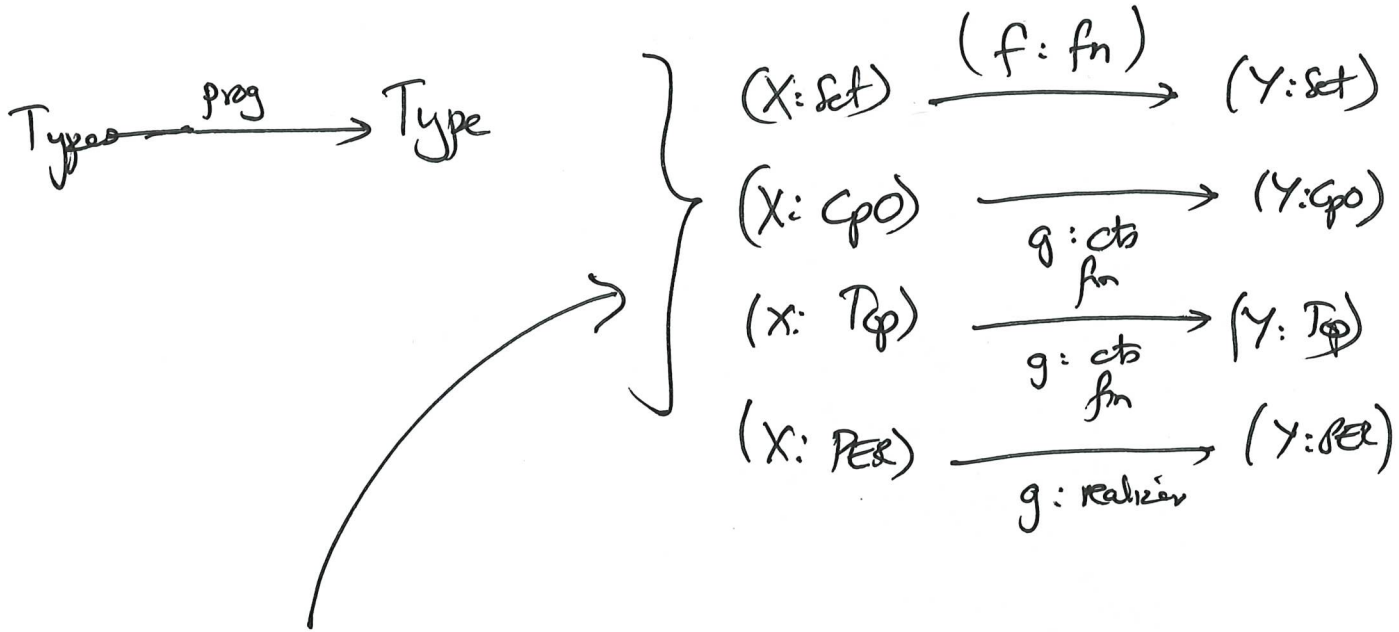


data Nat : Set where

①

zero : Nat

succ : Nat → Nat



$f \in \mathbb{Q} (x, y)$

$g \in T$
 \downarrow
 X
 $+$

PL $\xrightarrow{\llbracket \rrbracket}$ Cat
 den sem

Syntax $\xrightarrow{\text{model}}$ Model
 reductio/predctm

Checken or Eggs?

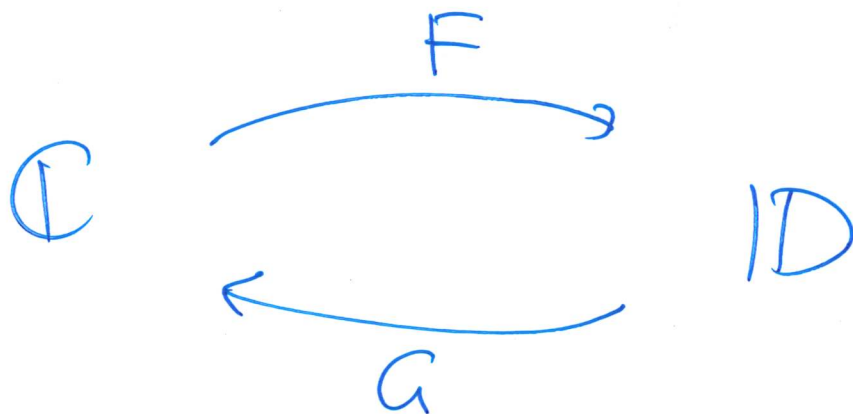
More then

Adjunctions

①

Given

$$\frac{2x \leq 5}{x \leq 5/2}$$



Defn

F is left adjoint to G

$$F \dashv G : \mathbb{C} \rightarrow \mathbb{D}$$

iff

$$\phi^{-1} \left(\frac{\mathbb{D}(FX, Y)}{\mathbb{C}(X, GY)} \right) \phi$$

natural in X & Y

Example, STLC

(3)

$$\llbracket [] \rrbracket = 1$$

$$\llbracket \Gamma, x:A \rrbracket = \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket$$

$$\llbracket \Gamma \vdash t:A \rrbracket \in \mathcal{C}(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket)$$

$$\text{app} \left(\frac{\mathcal{C}(\llbracket \Gamma \rrbracket \times \llbracket A \rrbracket, \llbracket B \rrbracket)}{\mathcal{C}(\llbracket \Gamma \rrbracket, \llbracket A \rightarrow B \rrbracket)} \right) \lambda$$

$$\beta \quad \text{app} \circ \lambda = \text{id}$$

$$\eta \quad \lambda \circ \text{app} = \text{id}$$

$$\frac{\mathcal{C}(X \times A, B)}{\mathcal{C}(X, A \rightarrow B)}$$

$$- \times A \rightarrow A \rightarrow : \mathcal{C} \rightarrow \mathcal{C}$$

Eg STLC

$$\frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x:A. t : A \rightarrow B}$$

(2)

$$\Gamma \vdash t : A \rightarrow B$$

$$\frac{\Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

$$\Gamma \vdash tu : B$$

β $(\lambda x. t)(u) = t[u/x]$

$$\lambda x. tx = t \quad x \notin FV(t)$$

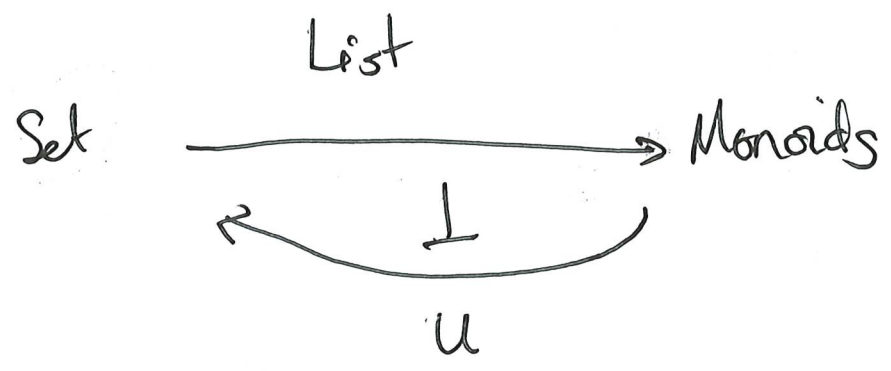
A type

B type

$A \rightarrow B$ type

Another Example.

(4)



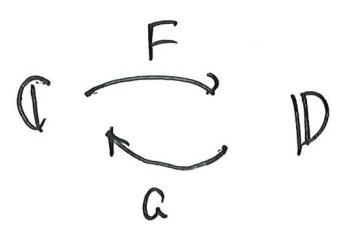
$$\frac{\text{Set } (X, UM)}{\text{Monoid } (\text{List } X \rightarrow M)}$$

use ~~for~~ as a definitional mechanism

Now Cubeness Approach

$$\forall X \forall Y \quad \frac{\mathbb{D}(FX, Y)}{\mathbb{D}(X, GY)}$$

choose $Y = FX$, id
 $\mapsto X \rightarrow GY$



$$\text{id} \rightarrow GF : C \rightarrow C$$

$$FG \rightarrow \text{id} : D \rightarrow D$$

weak - equality .

Datatypes

System T

Nat : Type

$\Gamma \vdash 0 : \text{Nat}$

$\frac{\Gamma \vdash n : \text{Nat}}{\Gamma \vdash S n : \text{Nat}}$

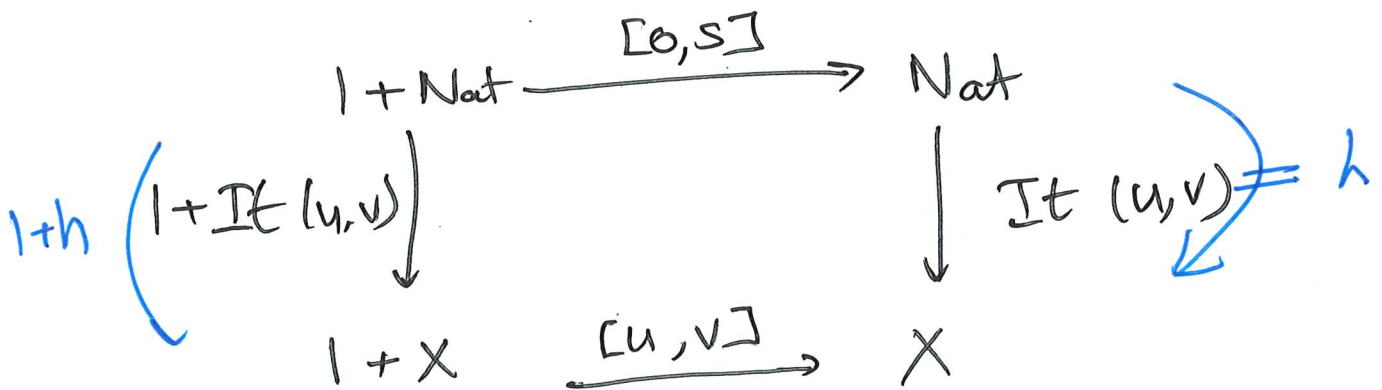
$\Gamma \vdash u : X$

$\Gamma \vdash v : X \rightarrow X$

$\Gamma \vdash \text{It}(u, v) : \text{Nat} \rightarrow X$

$\text{It}(u, v) 0 = u$

$\text{It}(u, v) (n+1) = v (\text{It}(u, v) n)$

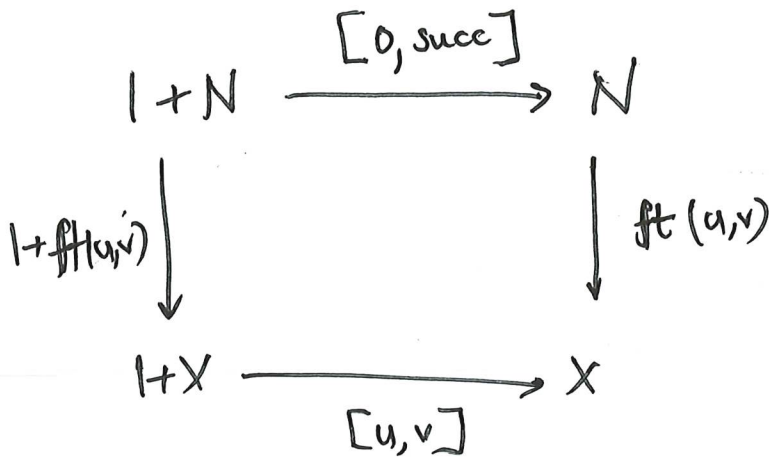


β -rules \equiv commutation of the square

η -rule \equiv If-expr is the unique map $\text{Nat} \rightarrow X$ making commutation

Add data types

IAS



& $f(u,v)$ is unique. β & η .

... same style as $\perp, \tau, x, 0$
& \rightarrow

= universal props

Here is some stuff A

and for any other such stuff, B

Here is a map stuff $\underline{A} \rightarrow \underline{B}$
 $\underline{B} \rightarrow \underline{A}$

which is unique

or the cat'y of stuff has \perp, τ

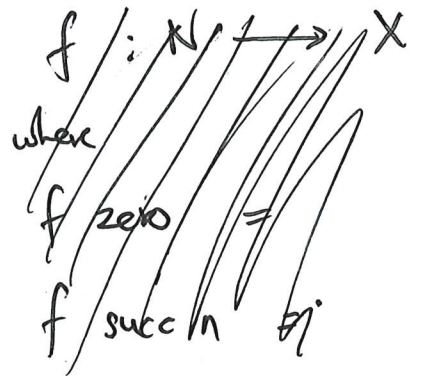
β $If(u,v) 0 = u$

$If(u,v) (f+1) = v(If(u,v))$

η ??

data ...

\vdots



System τ
 $\equiv \text{succ} +$

~~Net~~: set

$\vdash 0 : \text{Nat}$

$\vdash \tau : \text{Nat} \rightarrow \text{Nat}$

$\vdash u : X$

$\vdash f : X \rightarrow X$

$\vdash \text{If}(u,v)$
 $: \text{Nat} \rightarrow X$

Initial Algebra Semantics

$1 + _ : \mathbb{C} \rightarrow \mathbb{C}$ is a functor.

Instead consider an arbitrary functor

$$F : \mathbb{C} \rightarrow \mathbb{C}$$

An Algebra for F is a morphism

$$(X, h : FX \rightarrow X)$$

A morphism (X, h) to (Y, j)

$$\begin{array}{ccc} FX & \xrightarrow{h} & X \\ Ff \downarrow & \circlearrowleft & \downarrow f \\ FY & \xrightarrow{j} & Y \end{array}$$

is a map

$$f : X \rightarrow Y$$

st the diagram commutes

This is a category $F\text{-Alg}$ of F -algebras.

eg $FX = 1 + X \quad \dots \text{Nat}$

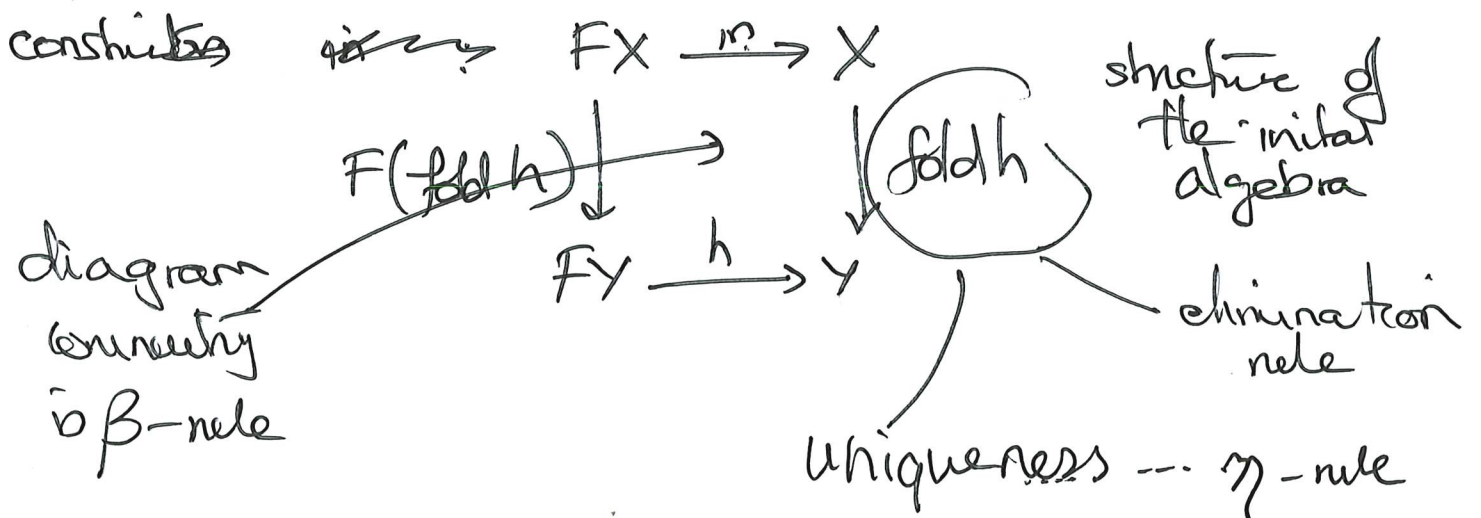
$FX = 1 + X \times X \quad \dots \text{binary trees}$
no data

$FX = A + X \times X \quad \dots \text{binary trees}$
data leaf
from A

$FX = 1 + A \times X \quad \dots \text{list of } A$

Qn What is the initial object in the category of F-algebras.

$X \text{ type } \vdash \quad FX \text{ type}$
 $\mu F \text{ type}$



Final Algebra semantics.

Dually

What is \rightarrow (i) completion of finite

(ii) dual of finite

consider

$Ax-$

What is the category of $Ax-$ -coalgebras

$$\begin{array}{ccc} X & \longrightarrow & AxX \\ f \downarrow & & \downarrow Ax f \\ Y & \longrightarrow & AxY \end{array}$$

$\text{streams}(A) = \text{final } Ax-$ coalgebra

ie $\frac{A \text{ type}}{\text{streams } A \text{ type}}$

$\langle \text{hd}, \text{tl} \rangle : \text{streams } A \rightarrow Ax \text{ streams } A$

$\text{unfold}_{\text{str}} : \forall Y (Y \rightarrow AxY) \rightarrow Y \rightarrow \text{streams } A$

$\text{hd}(\text{unfold } \phi) x = \text{hd } x$

$\text{tl}(\text{unfold } \phi) x = \text{unfold } \phi(\text{tl } x)$

+m)

Effects & Monadic Computation

$$X \longrightarrow Y$$

pure functions

$$X \longrightarrow Y + E_x$$

exceptions

$$X \longrightarrow \mathbb{P}Y$$

non-determinism

$$X \longrightarrow \text{Pr}Y$$

probabilistic comp.

$$S \times X \longrightarrow S \times Y$$

stateful comp.

$$X \longrightarrow [S, S \times Y]$$

$$X \longrightarrow (Y \rightarrow \mathbb{R} \rightarrow \mathbb{R})$$

continuations

Abstract

$E_x, \mathbb{P}, \text{Pr}, [S, S \times -]$

call these
+ \rightarrow

$$[- \rightarrow R] \rightarrow R$$

Idea

An T -effectful computation from X to Y

is an ~~effectful~~ pure computation $X \rightarrow TY$

T needs extra structure to

model

(1) the embedding of pure data into the effectful world

(2) to model composition of effectful comp —

(1) return

: $\forall X. X \rightarrow TX$

i.e. T is a functor
& a natural transformation ~~id~~ η

$\eta : Id \rightarrow T$

(ii) Comp. of eff. computation

$$\text{Given} \quad \begin{array}{l} X \longrightarrow TY \\ Y \longrightarrow TZ \end{array}$$

$$\text{want} \quad \frac{\quad}{X \longrightarrow TZ}$$

Assume $\text{bind} : TY \xrightarrow{Y, 2} (Y \rightarrow TZ) \rightarrow TZ$

↳ gives composition

or equivalently

assume $\mu_x : T(TX) \rightarrow TX$

i.e. a natural transform

$$\mu : T^2 \rightarrow T$$

A monad on \mathcal{C} is
 a functor $T : \mathcal{C} \rightarrow \mathcal{C}$

nat trans $\eta : \text{Id} \rightarrow T$

nat trans $\mu : T^2 \rightarrow T$

