Lecture #2

The Church Rosser Theorem
1936 Church/Kosser for λI calculus
1958 Curry/Feys for CL
1965 Schroer for full reduction
1967 Tait/Martin-Löf via parallel reduction
1975 Welch via super parallel reduction
What does it say?

- (DP)
  - 'diamond property'

- (CR)
  - Corollaries

Any two divergent reduction sequences may be brought back together (for the cost of some unreduction).

R is the 'common reduct' of P, Q

M ∼ N

Any two convertible terms have a common reduct

if M ∼ N, N is in normal form then M →* N
\( \beta \) reduction
\( \beta \) normal forms
\( \beta \) equality / conversion
reserve this relation symbol for the single 'top-level' instances

\[(\lambda e) \cdot a \rightarrow \beta e[a]\]

'redux'  'contractum'

'(\lambda e) \cdot a$ contracts to $e[a]'
let this be the least precongruence containing $\rightarrow_{\beta}$.

* i.e.

- reflexive
- transitive
- closed under the term constructors

\[ (\ast) \quad (\beta) \quad (\ast\ast) \quad (\ast\ast\ast) \]

\[ e_1 \rightarrow_{\beta} e_2 \]

\[ x \rightarrow_{\beta} x \]

\[ e_1 \rightarrow_{\beta} e \rightarrow_{\beta} e_2 \]

\[ \lambda e_1 \rightarrow_{\beta} \lambda e_2 \]

\[ e_1 \rightarrow_{\beta} e_2 \rightarrow_{\beta} a_1 \rightarrow_{\beta} a_2 \]

\[ e_1 \rightarrow_{\beta} e_2 \rightarrow_{\beta} e_2 \cdot a_2 \]

\[ e \rightarrow_{\beta} e \text{ is an admissible property, by induction} \]
This is the least congruence containing \( \rightarrow^\beta \).

(\* ) ie additionally consider an inference rule (s) 

\[
\begin{array}{c}
\frac{e \leadsto^\rho e'}{e' \leadsto^\rho e} \\
\end{array}
\]

or consider an additional rule (\( \beta^1 \) ) 

\[
\begin{array}{c}
\frac{e \leadsto^\rho e'}{e' \leadsto^\rho e} \\
\end{array}
\]

or define \( \sim^\rho \) by 

\[
\begin{array}{c}
\vdots \quad \vdots \quad \vdots \\
\end{array}
\]
say that e is in \( \beta \)-normal form if \( e \in \beta\text{-nf} \)

\[\text{if } e \rightarrow_{\beta} e' \Rightarrow e' \equiv e \text{ and reduction sequence is 'empty'} \]

ie. there is no subterm \( f \) of \( e \) such that \( f \rightarrow f' \) in \( e \rightarrow_{\beta} e' \)

\(\text{NB. if we consider } w = (\lambda x.xx) \text{ i.e. } \lambda((\text{var} 0) \cdot (\text{var} 0)) \)

then \( w \) is in \( \beta\text{-nf} \)

\(\text{but if } \Omega = w \cdot w \text{ then } \Omega \rightarrow_{\beta} \Omega \text{ indeed } \Omega \rightarrow_{\beta} \Omega \)

so \( \Omega \) is not in \( \beta\text{-nf} \) indeed \( \Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \Omega \)

so we've already left the realm of terminating expressions
why might Church-Rosser be hard to prove

\[ \lambda \text{e}. a \rightarrow_\rho e[a] \quad \text{seems OK} \]

\[ \text{but 'a' is potentially repeated } 0, 1, 2, \ldots \]
\[ \text{many times in } e[a] \]
\[ \text{each of which might subsequently reduce} \]
\[ \text{(so we can't easily take 'contextual closure' } \Theta \rightarrow_\rho \text{)} \]

transitivity rule kills you

almost before you can start doing anything

using proof by induction

\[ \text{eq. } \Theta \rightarrow_\rho \Theta \rightarrow_\rho \ldots \]

\[ \text{mean you can't assume that } \rightarrow_\rho \text{ is bounded} \]

problem #1

\[ \rho \]

\[ \rho \]

\[ \rho \]

\[ \rho \]

problem #2

\[ \rho \]

\[ \rho \]

\[ \rho \]

\[ \rho \]

problem #3

\[ \text{infinite reductions} \]

\[ \text{eq. } \Theta \rightarrow_\rho \Theta \rightarrow_\rho \ldots \]
a relation \( R \) between terms such that

1. \( \rightarrow^\beta \subseteq R \) so that \( R \) can do at least as much reduction

2. \( R^* \subseteq \rightarrow^\beta \) so that problematic transitivity in \( \rightarrow^\beta \) can be 'tamed' by transitivity of \( R^* \)

3. \( R \) is closed under 'contextual rules' for \( (\lambda), (\cdot) \)

4. \( R \) satisfies DP

**Lemma** \( R \) satisfies DP \( \Rightarrow \) \( R^* \) satisfies DP

**Lemma** \( R \) closed under \( (\lambda), (\cdot) \) \( \Rightarrow \) \( R^* \) closed under \( (\lambda), (\cdot) \)
$R$ and $R^*$ play nicely

'stripe lemma'
Consider two relations \( R, S \) such that

1. \( S \subseteq R \)
2. \( \Rightarrow \beta \subseteq R \)
3. \( R^* \subseteq \Rightarrow \beta \)
4. \( R \) closed under \((\lambda), (\alpha)\)

Together with the "triangle lemma" (Takahashi 1989/95; Lévy 1977; ...)

\[
\begin{align*}
\text{if} & \\
& e_1 \xrightarrow{S} e_2 \\
\text{then} & \\
& e_1 \xrightarrow{R} e \\
& e_1 \xrightarrow{S} e_2 \\
& e_1 \xrightarrow{R} e \\
\end{align*}
\]

\((\dagger)\)

\((\star)\) Lemma: if \((R, S)\) satisfy \((\dagger)\), then \( R \) satisfies DP

\((\star\star)\) Provided \( \forall e \in \mathcal{E}, e \in S e' \)

NB: \((\star)\) is a partial correctness statement

\((\star\star)\) the corresponding termination proof
(1) diamonds via triangles
Say that $(R) \ e \equiv e' \ "e \ super-parallel \ reduces \ to \ e'\"$

$(\ast) \ e \equiv_{\lambda} e' \ "e \ is \ a \ complete \ superdevelopment \ of \ e'\"$

\[
\frac{x \equiv x \ (\forall x)}{\lambda e_1 \equiv \lambda e_2} \quad (\lambda) \quad \frac{e_1 \equiv e_2}{\lambda e_1 \equiv \lambda e_2}
\]

\[
\frac{e_1 \equiv e_2 \quad a_1 \equiv a_2}{e_1 \cdot a_1 \equiv e_2 \cdot a_2} \quad (\ast)
\]

\[
\frac{e_1 \equiv \lambda e \quad a_1 \equiv a}{e_1 \cdot a_1 \equiv e_2 \cdot a_2} \quad (\beta)
\]

\[
\frac{e_1 \equiv \lambda e \quad a_1 \equiv a}{e_1 \cdot a_1 \equiv e_2 \cdot a_2} \quad (\ast)
\]

for $S = a \equiv_{\forall x} \lambda e$, above rule $(\ast)$

\[
\text{in} \ (\ast) \ \text{rule, forbid } e_2 \equiv \lambda e
\]
idea

- $S \subseteq R$ every rule defining $S$ is an $R$-rule

- $\rightarrow \beta \subseteq R$ by w/e (\(\beta\)), plus proof of reflexivity
  (admissible by 'usual' reasoning)

- $R$ allows reduction to proceed as far as the creation of a top-level redex
  and then is allowed to contract it

- $S$ forces contraction of all top-level redexes
  created inductively as it proceeds
Termination

- \( \forall e \exists e' \, e \Rightarrow_d e' \) induction on \( e \); case analysis in \((\ast)\) case

- partial correctness

\[
\begin{align*}
\text{idea:} & \quad \text{the S-reduction contracts} \\
& \quad \text{at least as many reduces as} \ R \\
& \quad \text{so for the 'leftovers' in} \ e, \ \text{does} \\
& \quad \text{one step of} \ R \ \text{suffice to mop up} \\
& \quad \text{all the others?}
\end{align*}
\]

\[
\begin{align*}
& \quad \text{proof induction on} \ S \\
& \quad \text{case analysis on} \ R
\end{align*}
\]

- lemma

\[
\begin{align*}
e_1 \Rightarrow e_2 & \quad a_1 \Rightarrow a_2 \\
\hline
& \quad e_1[a_1] \Rightarrow e_2[a_2]
\end{align*}
\]
Say that \( R \) is substitutive.

\[
\frac{\text{e}, R \ e_2 \ a, R a_2}{\text{e}, [a_1] \ R \ e_2[a_2]}
\]

\[
\text{e}, R e_2 \ \sigma, [R] \sigma_2
\]

\[
\frac{\text{e}, [\sigma_1] \ R \ e_2 \ [\sigma_2]}{}
\]

where \([R]\) is the pointwise extension of \( R \) to \( \sigma \)s.

Lemma: \( R \) substitution \( \Rightarrow \) \( R \) strongly substitutive

Lemma: \( \Leftrightarrow \) is strongly substitutive (why?)