Lecture #2

The Church Rosser Theorem

1936 Church/Rosser for λI calculus

1958 Curry/Feys for CL

1965 Schroer for full β reduction

1967 Tait/Martin-Löf via parallel reduction

1975 Welch via super parallel reduction

What does it say?

• (DP)

'diamond property'

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, twoheadrightarrow] \arrow[dr, twoheadrightarrow] & \\ P \arrow[dr, twoheadrightarrow] & & Q \arrow[dl, twoheadrightarrow] \\ & \exists R & \end{tikzcd}$$

any two divergent reduction sequences may be brought back together

(for the cost of some more reduction)

R is the 'common reduct' of P, Q

corollaries

• (CR)

$$\begin{tikzcd}[column sep=small] M \arrow[dr, twoheadrightarrow] & \simeq & N \arrow[dl, twoheadrightarrow] \\ & \exists R & \end{tikzcd}$$

any two convertible terms have a common reduct

• if $M \simeq N$, N is in normal form then $M \twoheadrightarrow N$

β reduction

β normal forms

β equality / conversion

$$\boxed{\longrightarrow_\beta}$$

reserve this relation symbol
for the single 'top-level' instances

$$(\lambda e)\cdot a \longrightarrow_\beta e[a]$$

'redex' 'contractum'

"$(\lambda e)\cdot a$ contracts to $e[a]$"

$$\boxed{\longrightarrow\!\!\!\!\!\rightarrow_\beta}$$

let this be the **least** **precongruence*** containing $\rightarrow_\beta$.

* i.e.

- reflexive
- transitive
- closed under the term constructors(**)

$$(r)\ \frac{}{x \twoheadrightarrow_\beta x}\ (***)$$

$$(t)\ \frac{e_1 \twoheadrightarrow_\beta e \quad e \twoheadrightarrow_\beta e_2}{e_1 \twoheadrightarrow_\beta e_2}$$

$$(\beta)\ \frac{e_1 \rightarrow_\beta e_2}{e_1 \twoheadrightarrow_\beta e_2}$$

(**)

$$(\lambda)\ \frac{e_1 \twoheadrightarrow_\beta e_2}{\lambda e_1 \twoheadrightarrow_\beta \lambda e_2}$$

$$(\circ)\ \frac{e_1 \twoheadrightarrow_\beta e_2 \quad a_1 \twoheadrightarrow_\beta a_2}{e_1 \circ a_1 \twoheadrightarrow_\beta e_2 \circ a_2}$$

<u>NB</u>(***) $e \twoheadrightarrow_\beta e$ is an **admissible** property, by induction

$$\boxed{=_\beta,\ \simeq_\beta}$$

This is the <u>least</u> <u>congruence</u>(*) containing $\to_\beta$

(*) ie additionally consider an inference rule (s) $\dfrac{e \simeq_\beta e'}{e' \simeq_\beta e}$

<u>or</u> consider an additional rule $(\beta)'$ $\dfrac{e' \to_\beta e}{e \simeq_\beta e'}$

<u>or</u> define $\simeq_\beta$ by

$$\begin{tikzcd}[column sep=small]
e_1 \arrow[dr, two heads, "\beta"'] & \cdots & \bullet \arrow[dl, two heads] \arrow[dr, two heads] & \cdots & \bullet \arrow[dl, two heads] \arrow[dr, two heads] & \cdots & \bullet \arrow[dl, two heads] \arrow[dr, two heads] & \cdots & e_2 \arrow[dl, two heads] \\
& \bullet & & \bullet & & \bullet & & \bullet &
\end{tikzcd}$$

$\boxed{nf_\beta\, e}$

say that e is in β-normal form $nf_\beta\, e$ 'e in β-nf'

if • $e \twoheadrightarrow_\beta e' \Rightarrow e' \equiv e$ and reduction sequence is 'empty'

i.e. there is no subterm f of e such that $f \to_\beta f'$ in $e \twoheadrightarrow_\beta e'$

NB. if we consider $\omega = (\lambda x.xx)$ i.e. $\lambda((\text{var } 0) \cdot (\text{var } 0))$

then ω is in β-nf

• but if $\Omega = \omega \cdot \omega$ then $\Omega \twoheadrightarrow_\beta \Omega$ indeed $\Omega \to_\beta \Omega$

so Ω is not in β-nf indeed $\Omega \to_\beta \Omega \to_\beta \Omega \to \ldots$

so we've already left the realm of terminating expressions

why might Church-Rosser be <u>hard</u> to prove

problem #1

$$\begin{tikzcd} & \circ \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ {} & ? & {} \end{tikzcd}$$

$(\lambda e) \cdot a \longrightarrow_\beta e[a]$ seems OK

<u>but</u> 'a' is potentially repeated $0, 1, 2, \ldots$
many times in $e[a]$
<u>each</u> of which might subsequently reduce
(so we can't easily take 'contextual closure' of $\longrightarrow_\beta$)

problem #2

$$\begin{tikzcd} & \circ \arrow[dl, two heads, "\beta"'] \arrow[dr, two heads, "\beta"] & \\ {} & & {} \end{tikzcd}$$

<u>transitivity</u> rule kills you
almost before you can start doing anything
using proof by induction

problem #3 infinite reductions e.g. $\Omega \twoheadrightarrow_\beta \Omega \twoheadrightarrow \ldots$ mean you can't assume that $\twoheadrightarrow_\beta$ is <u>bounded</u>

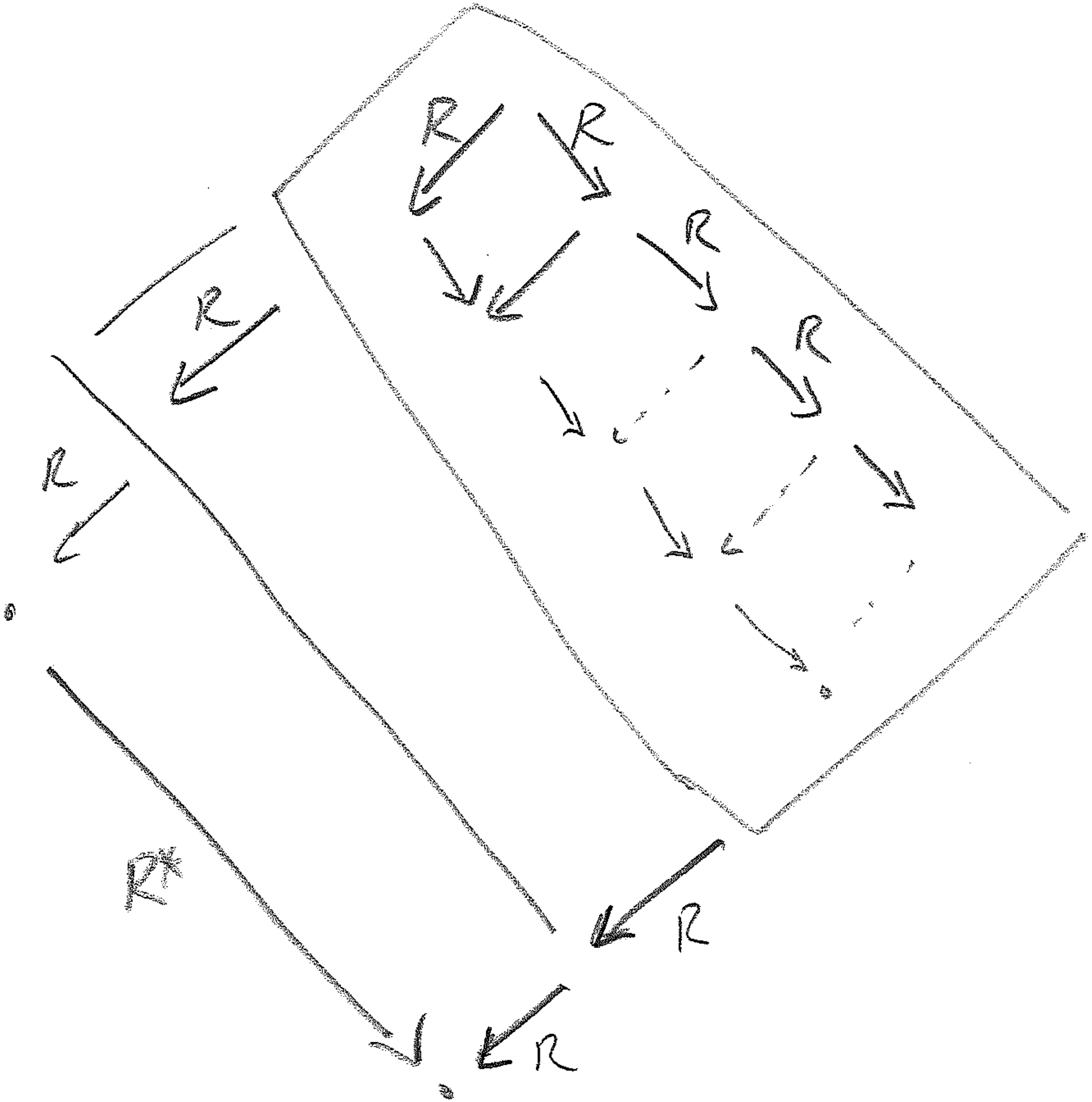
what we need

a relation R between terms such that

- $\longrightarrow_\beta \subseteq R$ so that R can do at least as much reduction
- $R^* \subseteq \twoheadrightarrow_\beta$ so that problematic transitivity in $\twoheadrightarrow_\beta$ can be 'tamed' by transitivity of R^*
- R is closed under 'contextual rules' for $(\lambda), (\cdot)$
- R satisfies DP

<u>lemma</u> R satisfies DP $\Rightarrow$ R^* satisfies DP

<u>lemma</u> R closed under $(\lambda), (\cdot)$ $\Rightarrow$ R^* closed under $(\lambda), (\cdot)$



R and R^* play nicely

'striplemma'

one solution to the problem

consider <u>two</u> relations R, S such that

- $S \subseteq R$
- $\longrightarrow_\beta \subseteq R$
- $R^* \subseteq \twoheadrightarrow_\beta$
- R closed under (λ) $(\circ)$

<u>together with</u> "triangle lemma" (Takahashi 1989/95; Lévy 1977; ...)

<u>if</u>

$$\begin{tikzcd} e_1 \arrow[r, twoheadrightarrow, "S"] \arrow[dr, twoheadrightarrow, "R"'] & e_2 \\ & e \end{tikzcd} \qquad (\dagger)$$

<u>then</u>

$$\begin{tikzcd} e_1 \arrow[rr, twoheadrightarrow, "S"] \arrow[dr, twoheadrightarrow, "R"'] & & e_2 \\ & {} \arrow[ur, dashed, twoheadrightarrow, "R"'] & \end{tikzcd}$$

(*) <u>lemma</u> if (R,S) satisfy $(\dagger)$ then R satisfies DP

(**) <u>provided</u> $\forall e \exists e'.\ e\, S\, e'$

<u>NB</u> (*) is a <u>partial correctness</u> statement

(**) the corresponding <u>termination</u> proof

(*) diamonds via triangles

$$
\begin{tikzcd}
& M \arrow[dl, two heads, "R"'] \arrow[dr, two heads, "R"] \arrow[dd, two heads, "S"] & \\
P \arrow[dr, dashed, "R"'] & & Q \arrow[dl, dashed, "R"] \\
& \bullet &
\end{tikzcd}
$$

Welch, 1975 ; Park, Aczel
Klop, van Raamsdonk

Say that (R) $e \Rrightarrow e'$ "e super-parallel reduces to e′"

(S) $e \Rrightarrow_d e'$ "e′ is a complete superdevelopment of e"

if

$$\overline{x \Rrightarrow x}\ (\mathrm{var})$$

$$\frac{e_1 \Rrightarrow e_2}{\lambda e_1 \Rrightarrow \lambda e_2}\ (\lambda)$$

$$\frac{e_1 \Rrightarrow e_2 \quad a_1 \Rrightarrow a_2}{e_1 \circ a_1 \Rrightarrow e_2 \circ a_2}\ (\circ)$$

plus for $R =_{def} \Rrightarrow$

$$\frac{e_1 \Rrightarrow \lambda e \quad a_1 \Rrightarrow a}{e_1 \circ a_1 \Rrightarrow e[a]}\ (\beta)$$

for $S =_{def} \Rrightarrow_d$
- above rule (β)
- in $(\circ)$ rule, forbid $e_2 \equiv \lambda e$

<u>idea</u>

- $S \subseteq R$ every rule defining S is an R-rule

- $\rightarrow_p \subseteq R$ by rule (p), plus proof of <u>reflexivity</u> (admissible by 'usual' reasoning)

- R <u>allows</u> reduction to proceed as far as the creation of a top level redex and then is <u>allowed</u> to contract it

- S <u>forces</u> contraction of all top level redexes created inductively as it proceeds

Termination

- $\forall e \, \exists e' \quad e \Rrightarrow_d e'$ induction on e; case analysis in $(\cdot)$ case

partial correctness

- $e_1 \overset{S}{\Rrightarrow}_d e'$, $e_1 \overset{R}{\Rrightarrow} e$ $\quad \Rightarrow \quad e \overset{R}{\Rrightarrow} e'$

idea the S reduction contracts at least as many redexes as R

so for the 'leftovers' in e, does one step of R suffice to mop up all the others?

answer yes (!)

proof induction on S
case analysis on R

lemma

$$\frac{e_1 \Rrightarrow e_2 \qquad a_1 \Rrightarrow a_2}{e_1[a_1] \Rrightarrow e_2[a_2]}$$

Substitutivity

Say that R is substitutive

$$\frac{e_1 \, R \, e_2 \qquad a_1 \, R \, a_2}{e_1[a_1] \; R \; e_2[a_2]}$$

strongly substitutive

$$\frac{e_1 \, R \, e_2 \qquad \sigma_1 \, [R] \, \sigma_2}{e_1[\sigma_1] \; R \; e_2[\sigma_2]}$$

where $[R]$ is the pairwise extension of R to σs

lemma R strongly substitutive $\Rightarrow$ R substitutive

lemma $\Rrightarrow$ is strongly substitutive (why?)