Lecture #3

The Standardisation Theorem

1958 Curry / Feys for CL

1975 Plotkin for $\lambda\beta_v$, $\lambda\beta$ via 'standard sequences'

1979 Mitschke for $\lambda\beta(\eta)$ via semistandardisation

1975 – 77 Lévy
Barendregt
Hyland
via labelled λ calculus

<u>What does it say?</u>

if $M \twoheadrightarrow_{\beta} N$ then there exists a standard reduction $\sigma : M \twoheadrightarrow_s N$

where $\sigma : M_0 \xrightarrow{\Delta_0} M_1 \;\dots\; M_i \xrightarrow{\Delta_i} M_{i+1} \longrightarrow \dots\; M_n$

is <u>standard</u> (reducing the redex occurrence Δ_i at the i^{th} step)

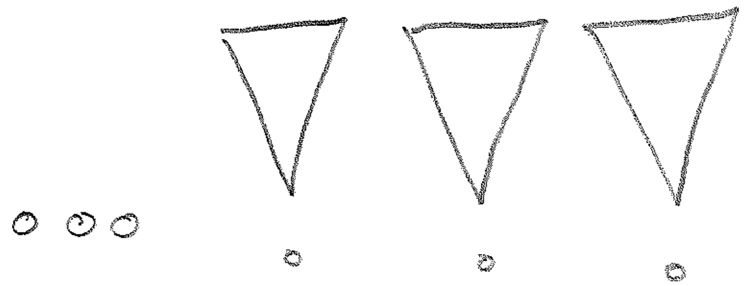
if

$\forall i \, \forall j < i$ [Δ_i is <u>not</u> a residual of a redex

(†) to the left of Δ_j

(relative to the <u>given</u> reduction from M_j to M_i)]

(Barendregt, 1984, Ch. 11 p 296; <u>nested parentheses</u> in original)



... Can we do

better ???

$$\longrightarrow\!\!\!\!\rightarrow_w$$

weak-head reduction

let $\twoheadrightarrow_w$ be the <u>least</u> reflexive, transitive relation

- containing $\rightarrow_\beta$ $\qquad e \rightarrow_\beta e' \Rightarrow e \twoheadrightarrow_w e'$
- closed under application <u>on the right</u>

$$\frac{e \twoheadrightarrow_w e'}{e \cdot a \twoheadrightarrow_w e' \cdot a} \; (\cdot)$$

<u>NB</u> if $\lambda e \twoheadrightarrow_w e'$ then $e' \equiv \lambda e$ and there is <u>no</u> reduction

ditto. $x \twoheadrightarrow_w e' \Rightarrow e' \equiv x$ $\qquad$ 'weak head normal forms'

what about $e \cdot a$?

Plotkin 1975
Mitschke 1979
(Takahashi 1989/95)
McKinna/
Pollack 1993/99

say that $M \twoheadrightarrow_s N$ is <u>standard</u>
resp. $M \twoheadrightarrow_i N$ is <u>internal</u>
} <u>mutual</u> inductive def^n

if
$$\boxed{\dfrac{M \twoheadrightarrow_w P \twoheadrightarrow_i N}{M \twoheadrightarrow_s N}}$$

"<u>semi-standardisation</u>":- standard reduction factors into a <u>weak head</u> reduction followed by <u>internal</u> reduction

where
$$\dfrac{}{x \twoheadrightarrow_i x} \qquad \dfrac{M \twoheadrightarrow_s N}{\lambda M \twoheadrightarrow_i \lambda N}\ (\lambda)$$

<u>and</u>
$$\dfrac{M \twoheadrightarrow_i M' \quad N \twoheadrightarrow_s N'}{MN \twoheadrightarrow_i M'N'}\ (\cdot)$$

<u>NB</u>
- in rule (λ), we revert to standard reduction under the outermost symbol λ
- in rule $(\cdot)$, we <u>forbid</u> outermost $\twoheadrightarrow_w$ reduction on the <u>left</u> of an application

in this last rule :–

- suppose we wanted $MN \twoheadrightarrow_s M'N'$ from $M \twoheadrightarrow_s M'$, $N \twoheadrightarrow_s N'$

  then, inductively, we may suppose

  $M \twoheadrightarrow_w P \twoheadrightarrow_i M'$

- but then $MN \twoheadrightarrow_w PN$

- so provided we take $PN \twoheadrightarrow_i M'N'$

  then

  $$MN \twoheadrightarrow_w PN \twoheadrightarrow_i M'N'$$

  is a <u>plausible</u> factorisation of $MN \twoheadrightarrow M'N'$

<u>NB</u>

- $M \twoheadrightarrow_w N$ satisfies property $(\dagger)$, immediately
- if $P \twoheadrightarrow_i N$ satisfies $(\dagger)$, then so too does $M \twoheadrightarrow_w P \twoheadrightarrow_i N$
- $M \twoheadrightarrow_i N$ satisfies $(\dagger)$, by mutual induction

<u>actually</u> in $MN \twoheadrightarrow_i M'N'$ from $M \twoheadrightarrow_i M'$, $N \twoheadrightarrow_s N'$

we could contract all the redexes in the sequence $N \twoheadrightarrow_s N'$

<u>before</u> contracting the redexes in the sequence $M \twoheadrightarrow_i M'$

(and they would therefore violate the 'spatial' invariant (+))

so to be <u>yet more precise</u> define $\twoheadrightarrow_{li}$ 'left internal'

$\twoheadrightarrow_{ri}$ 'right internal'

(<u>mutually</u> with one another, and hence also with $\twoheadrightarrow_s$) $\twoheadrightarrow_i$ 'internal'

by $\dfrac{M \twoheadrightarrow_i N}{MP \twoheadrightarrow_{li} NP}$, $\dfrac{P \twoheadrightarrow_s Q}{NP \twoheadrightarrow_{ri} NQ}$, $\dfrac{M \twoheadrightarrow_{li} P \twoheadrightarrow_{ri} N}{M \twoheadrightarrow_i N}$

(and $x \twoheadrightarrow_i x$ as before) then each of $\twoheadrightarrow_w$, $\twoheadrightarrow_{li}$, $\twoheadrightarrow_{ri}$, $\twoheadrightarrow_i$ and $\twoheadrightarrow_s$ satisfy property (+)

we've <u>almost</u> shown that if $M \twoheadrightarrow_\beta N$ then $M \twoheadrightarrow_s N$

(single step β) immediate, $M \to_\beta N \Rightarrow M \twoheadrightarrow_w N$ $\square$

(var) immediate, $x \twoheadrightarrow_w x \twoheadrightarrow_i x$, by (var) $\square$

(λ) immediate, $\twoheadrightarrow_i \;\subseteq\; \twoheadrightarrow_s$, by (var)

(app) we've seen this by way of motivating (app) for $\twoheadrightarrow_i$

so the only thing left to show is <u>TRANSITIVITY</u>

(and then we are done, with $\twoheadrightarrow_\beta$ as the least precongruence containing $\to_\beta$)

(actually we also need REFLEXIVITY for (β) – exercise!)

<u>transitivity</u> for $\xrightarrow{\quad}\!\!\!\!\!\twoheadrightarrow_s$ follows, provided:

$$\begin{tikzcd}[column sep=small]
M \arrow[rr, ->>, "s"'] \arrow[dr, ->>, "w"'] & & P \arrow[rr, ->>, "s"'] \arrow[dr, ->>, "w"] & & N \\
& \cdot \arrow[ur, ->>, "i"'] \arrow[dr, dashed, ->>, "w"'] & & \cdot \arrow[ur, ->>, "i"'] & \\
& & \cdot \arrow[ur, dashed, ->>, "i"'] & &
\end{tikzcd}$$

i.e that we can standardise

(factor into $\twoheadrightarrow_w \cdot \twoheadrightarrow_i$)

all composites of the form $\twoheadrightarrow_i \cdot \twoheadrightarrow_w$

<u>NB</u> we need to show simultaneously

$$\left\{\begin{array}{ll} \cdot \; \twoheadrightarrow_i \cdot \twoheadrightarrow_w \;\subseteq\; \twoheadrightarrow_s & (*) \\ \cdot \; \twoheadrightarrow_i \cdot \twoheadrightarrow_i \;\subseteq\; \twoheadrightarrow_i & \\ \cdot \; \twoheadrightarrow_s \cdot \twoheadrightarrow_s \;\subseteq\; \twoheadrightarrow_s & \end{array}\right.$$

so by induction on $\twoheadrightarrow_w$ it suffices to show

$(**)\quad \twoheadrightarrow_i \cdot \longrightarrow_\beta \;\subseteq\; \twoheadrightarrow_s$ to conclude $(*)$

idea

- internal reduction $\twoheadrightarrow_i$

  preserves, and reflects, the shape of terms

- indeed, the property of being a β redex (!)

  if $M \twoheadrightarrow_i (\lambda R)\circ S$ then $M \equiv (\lambda P)\circ Q$

  where $\lambda P \twoheadrightarrow_i \lambda R$

  ie $P \twoheadrightarrow_s R$

  and $Q \twoheadrightarrow_s S$

- so to show (**)

  we need

$$\frac{P \twoheadrightarrow_s R \qquad Q \twoheadrightarrow_s S}{\begin{tikzcd} (\lambda P)\circ Q \arrow[r, "\beta"'] & P[Q] \arrow[rr, two heads, "s"] \arrow[dr, two heads, "w"'] & & R[S] \\ & & \circ \arrow[ur, two heads, "i"'] & \end{tikzcd}}$$

finally $\quad P \twoheadrightarrow_w W \twoheadrightarrow_i R \quad$ from $\quad P \twoheadrightarrow_s R$

so $\quad P[Q] \twoheadrightarrow_w W[Q] \quad$ (exercise)

so we may conclude the proof of (**), and hence of (*), and hence of transitivity of $\twoheadrightarrow_s$

provided
$$\frac{\boxed{P \twoheadrightarrow_{i/s} R} \quad Q \twoheadrightarrow_s S}{P[Q] \twoheadrightarrow_s R[S]}$$

one last mutual induction on $\boxed{P \twoheadrightarrow_i R}$ resp. $\boxed{P \twoheadrightarrow_s R}$

now $\twoheadrightarrow_s$ is a reflexive, transitive relation closed under (λ), ($\cdot$)

hence $\quad M \twoheadrightarrow_\beta N \implies M \twoheadrightarrow_s N$

more on normal forms

β normal forms may be inductively characterised

$$\frac{}{nf_\beta\, x} \qquad \frac{nf_\beta\, e}{nf_\beta(\lambda e)} \qquad \frac{nf_\beta\, e \quad nf_\beta\, a}{nf_\beta(e \cdot a)} \; (e \not\equiv \lambda e')$$

alternatively (neutral/normal)

$$\frac{}{ne_\beta\, x} \qquad \frac{ne_\beta\, e \quad nf_\beta\, a}{ne_\beta(e \cdot a)} \qquad \frac{ne_\beta\, e}{nf_\beta\, e} \qquad \frac{nf_\beta\, e}{nf_\beta(\lambda e)}$$

whnormal forms

$$\frac{}{whnf_\beta\, x} \qquad \frac{}{whnf_\beta(\lambda e)} \qquad \frac{whnf_\beta\, e \quad (e \not\equiv \lambda e')}{whnf_\beta(e \cdot a)}$$

alternatively define 'weakly neutral' and $whnf_\beta$ (exercise)

Further remarks

- $\longrightarrow\!\!\!\!\rightarrow_i$ preserves (and reflects) weak head normal forms

- normal forms are weak head normal forms

so we have as Corollaries

- leftmost reduction: if $M \longrightarrow\!\!\!\!\rightarrow_\beta N$ and $\mathrm{nf}_\beta N$

  then $\exists W.\ M \longrightarrow\!\!\!\!\rightarrow_w W,\ \mathrm{whnf}_\beta W,\ W \longrightarrow\!\!\!\!\rightarrow_i N$

- quasi-normalisation: if $\nexists N.\ \mathrm{nf}_\beta N,\ M \longrightarrow\!\!\!\!\rightarrow_\beta N$ "M has no β-nf"

  then $\forall W.\ M \longrightarrow\!\!\!\!\rightarrow_w W \Rightarrow \neg\, \mathrm{whnf}_\beta W$

a <u>characterisation</u> of <u>reduction to normal form</u>

define, as usual $\xrightarrow{\quad}\!\!\!\!\to{}^{nf}_{s}$ and $\xrightarrow{\quad}\!\!\!\!\to{}^{nf}_{i}$ mutually

by $M \twoheadrightarrow^{nf}_{s} N \;=\; \exists W.\ \mathrm{whnf}\, W,\ M \twoheadrightarrow_{w} W \twoheadrightarrow^{nf}_{i} N$

and for applications $$\frac{M \twoheadrightarrow^{nf}_{i} M' \not\equiv \lambda P \qquad N \twoheadrightarrow^{nf}_{i} N'}{MN \twoheadrightarrow^{nf}_{i} M'N'}$$ (otherwise as $\twoheadrightarrow_{i}$)

then $\boxed{M \twoheadrightarrow^{nf}_{s} N \quad \text{iff} \quad M \twoheadrightarrow_{s} N \ \text{and}\ \mathrm{nf}_{\beta}\, N}$

the premise $M \twoheadrightarrow^{nf}_{i} M' \not\equiv \lambda P$ is equivalent to

$M \twoheadrightarrow_{i} M'$ and $\mathrm{ne}_{\beta}\, M'$

Also

- we have exactly analogous results for $\longrightarrow\!\!\!\rightarrow_h$ <u>head</u> reduction satisfying

$$\frac{M \longrightarrow_\beta N}{M \longrightarrow\!\!\!\rightarrow_h N} \qquad \frac{M \longrightarrow\!\!\!\rightarrow_h N}{MP \longrightarrow\!\!\!\rightarrow_h NP} \qquad \frac{M \longrightarrow\!\!\!\rightarrow_h N}{\lambda M \longrightarrow\!\!\!\rightarrow_h \lambda N}$$

with appropriate adjustments to $\longrightarrow\!\!\!\rightarrow_i$: $\dfrac{M \longrightarrow\!\!\!\rightarrow_i N}{\lambda M \longrightarrow\!\!\!\rightarrow_i \lambda N}$

- also for CL, taking $KMN \longrightarrow_w M$, $SPQR \longrightarrow_w (PR)(QR)$ as primitive $\longrightarrow\!\!\!\rightarrow_w$ steps, together with 'obvious' $\longrightarrow\!\!\!\rightarrow_i$ defn

- also for CL+Y, taking $Y \longrightarrow\!\!\!\rightarrow_i Y$ and $YM \longrightarrow_w M(YM)$ as primitive

etc.

- also for $\lambda_\beta + \Omega$ with $\Omega \longrightarrow\!\!\!\rightarrow_i \Omega$, $\lambda\Omega \longrightarrow_h \Omega$, $\Omega P \longrightarrow_h \Omega$

FIN