Lecture #3

The Standardisation Theorem
1958 Curry / Feys for CL

1975 Plotkin for $\lambda^v$, $\lambda^v$ via 'standard sequences'

1979 Mitschke for $\lambda^p$ via semi-standardisation

1975 - 77 Lévy via labelled $\lambda$-calculus

Barendregt

Hyland
what does it say?

if \( M \xrightarrow{p} N \) then there exists a standard reduction \( \delta: M \xrightarrow{\delta} N \)

where \( \delta: M_0 \xrightarrow{\Delta_0} M_1 \quad ... \quad M_i \xrightarrow{\Delta_i} M_{i+1} \quad ... \quad M_n \)

is standard (reducing the redex occurrence \( \Delta_i \) at the \( i^{th} \) step)

if \( \forall i \forall j < i \left[ \Delta_i \text{ is not a residual of a redex to the left of } \Delta_j \right] \)

(†)

(-relative to the given reduction from \( M_j \) to \( M_i \))

(Barandregt, 1984, Ch. 11 p 296 ; nested parentheses in original)
... Can we do better ???
let $\rightarrow^\omega$ be the least reflexive, transitive relation

- containing $\rightarrow^\rho$ 
  $e \rightarrow^\rho e' \Rightarrow e \rightarrow^\omega e'$

- closed under application on the right 
  $e \rightarrow^\omega e'$

\[ e = a \quad \text{and} \quad (\ast) \]

\[ e = a \rightarrow^\omega e' \cdot a \]

NB if $\lambda e \rightarrow^\omega e'$ then $e' = \lambda e$ and there is no reduction

ditto. $x \rightarrow^\omega e' \Rightarrow e' = x$ "weak head normal forms"

what about $e = a$?
say that $M \to_{s} N$ is standard

resp. $M \to_{i} N$ is internal

\[\begin{array}{c}
M \to_{s} \lambda P \to_{s} \lambda N \\
\hline
M \to_{s} N
\end{array}\]

\[\begin{array}{c}
M \to_{s} N \\
\hline
\lambda M \to_{s} \lambda N
\end{array}\] (\(\lambda\))

where $x \to_{i} x$

and $M \to_{i} M' \quad \lambda N \to_{s} \lambda N'$

\[\begin{array}{c}
M \to_{i} M' \\
\hline
MN \to_{i} \lambda M'N'
\end{array}\] (\(\cdot\))

"Semi-standardization": standard reduction factors into a weak head reduction followed by internal reduction

NB in rule (\(\lambda\)), we revert to standard reduction under the outermost symbol \(\lambda\)

in rule (\(\cdot\)), we factor outermost \(\to_{s}\) reduction on the left off an application
in this last rule: Suppose we wanted \( MN \rightarrow_{s} M'N' \) from \( M \rightarrow_{s} M' \) and \( N \rightarrow_{s} N' \), then, inductively, we may suppose
\[ M \rightarrow_{w} P \rightarrow_{w} M' \]

- but then \( MN \rightarrow_{w} PN \)
- so provided we take \( PN \rightarrow_{w} M'N' \) then
\[ MN \rightarrow_{w} PN \rightarrow_{w} M'N' \]
is a plausible factorisation of \( MN \rightarrow_{w} M'N' \)

**NB**
- \( M \rightarrow_{w} N \) satisfies property (†), immediately
- if \( P \rightarrow_{w} iN \) satisfies (†), then so too does \( M \rightarrow_{w} P \rightarrow_{w} iN \)
- \( M \rightarrow_{w} iN \) satisfies (†), by mutual induction
in $M \to^i M', N \to^s N'$, we could contract all the redexes in the sequence $N \to^s N'$ before contracting the redexes in the sequence $M \to^i M'$ (and they would therefore violate the `spatial' invariant ($\dagger$)).

So to be yet more precise define $\to^{li}$ `left internal', $\to^{ri}$ `right internal', $\to^i$ `internal'.

(mutually with one another, and hence also with $\to_s$)

by $M \to^i N$, $P \to^s Q$, $M \to^{li} P \to^{ri} N$.

(and with all before) then each of $\to^i$, $\to^{li}$, $\to^{ri}$, $\to^s$ satisfy property ($\dagger$).
we're almost shown that if \( M \rightarrow^\beta N \) then \( M \rightarrow^\alpha N \)

(simple step) immediate, \( M \rightarrow^\beta N \Rightarrow M \rightarrow^\alpha N \) 

(var) immediate, \( x \rightarrow^\alpha x \rightarrow^\beta x \), by (var)

(\lambda) immediate, \( \rightarrow^\alpha \subseteq \rightarrow^\beta \), by (var)

(app) we've seen this by way of motivating (app) for \( \rightarrow^\beta \);
so the only thing left to show is TRANSITIVITY

(and then we are done, with \( \rightarrow^\beta \) as the least precongruence containing \( \rightarrow^\beta \))

(actually we also need REFLEXIVITY for (\beta) - exercise!)
transitivity for $\rightarrow_s$

forms, provided:

i.e that we can standardise

(factor into $\rightarrow_w \cdot \rightarrow_i$)

all composites of the form $\rightarrow_i \cdot \rightarrow_w$

NB we need to show simultaneously

so by induction on $\rightarrow_w$

it suffices to show

$$\star \star \quad \rightarrow_i \circ \to \beta = \rightarrow_s$$

to conclude $\star$
idea. internal reduction \( \rightarrow \Rightarrow \)

preserves, and reflects, the shape of terms

- indeed, the property of being a \( \beta \) redex (\( \triangledown \))

if \( M \rightarrow_{s} (\lambda R) \cdot S \), then \( M \equiv (\lambda P) \cdot Q \)

where \( \lambda P \rightarrow_{s} \lambda R \)

\( \Rightarrow P \rightarrow_{s} R \)

and \( Q \rightarrow_{s} S \)

- so to show (\( \ast \ast \)) \( P \rightarrow_{s} R \quad Q \rightarrow_{s} S \)

we need

\( (\lambda P) \cdot Q \rightarrow_{p} P[Q] \rightarrow_{s} R[S] \)
finally $P \rightarrow_{w} W \rightarrow_{i} R$ from $P \rightarrow_{s} R$

so $P[Q] \rightarrow_{w} W[Q]$ (exercise)

So we may conclude the proof of $(\ast \ast)$, and hence of $(\ast)$, and hence of transitivity of $\rightarrow_{s}$

provided $P \rightarrow_{i/s} R \text{ Q } \rightarrow_{s} S$ are last mutual

induction on $P \rightarrow_{i} R$ resp. $P \rightarrow_{s} R$

now $\rightarrow_{s}$ is a reflexive, transitive relation closed under $(\lambda), (\cdot)$

hence $M \rightarrow_{\beta} N \Rightarrow M \rightarrow_{s} N$
more on normal forms

\[\begin{align*}
\text{normal forms} & \quad \frac{nf_p e}{nf_p x} \quad \frac{nf_p e}{nf_p (\lambda e)} \\
\text{may be inductively} & \quad \frac{nf_p e}{nf_p (e \cdot a)} \\
\text{characterised}
\end{align*}\]

\[\begin{align*}
\text{alternatively} & \quad \frac{ne_p e}{ne_p x} \quad \frac{ne_p e}{ne_p (e \cdot a)} \\
\text{(neutral/normal)} & \quad \frac{ne_p e}{ne_p (e \cdot a)} \\
\end{align*}\]

\[\begin{align*}
\text{whinormal forms} & \quad \frac{whnf_p e}{whnf_p x} \quad \frac{whnf_p e}{whnf_p (\lambda e)} \\
\text{alternatively} & \quad \frac{whnf_p e}{whnf_p (e \cdot a)} \\
\text{define 'weakly neutral' and whnf_p (exercise)}
\end{align*}\]
Further remarks

- \( \rightarrow^* \) preserves (and reflects) weak head normal forms

- Normal forms are weak head normal forms

So we have as Corollaries

- Leftmost reduction
  
  If \( M \rightarrow^* \beta N \) and \( \text{nfr} \) \( N \)
  
  Then \( \exists W. \ M \rightarrow^* \omega W \), \( \text{whnfr} \) \( W \), \( W \rightarrow^* \gamma N \)

- Quasi-normalisation

  If \( \not\exists N. \text{nfr} N \), \( M \rightarrow^* \beta N \) "\( M \) has no \( \beta \)-nf"

  Then \( \forall W. \ M \rightarrow^* \omega W \rightarrow \bot \text{whnfr} W \)
A characterization of reduction to normal form define, as usual mutually

\[ M \rightarrow^{nf}_s N = \exists W. \text{whnf} W, M \rightarrow^{nf}_w W \rightarrow^{nf}_i N \]

and for applications

\[ M \rightarrow^{nf}_i M' \neq \lambda P \quad N \rightarrow^{nf}_i N' \]

then

\[ M \rightarrow^{nf}_s N \text{ iff } M \rightarrow^{nf}_s N \text{ and } nf \text{ p } N \]

the premise \( M \rightarrow^{nf}_i M' \neq \lambda P \) is equivalent to

\[ M \rightarrow^{nf}_i M' \text{ and } nf \text{ p } M' \]
Also, we have exactly analogous results for $\Rightarrow^h_\ast$ head reduction satisfying:

\[
\begin{align*}
M & \Rightarrow^h_\ast N \\
M & \Rightarrow^h_\ast N \\
M & \Rightarrow^h_\ast N \\
M & \Rightarrow^h_\ast N \\
\lambda M & \Rightarrow^h_\ast \lambda N \\
\lambda M & \Rightarrow^h_\ast \lambda N
\end{align*}
\]

with appropriate adjustments to $\Rightarrow_\ast$:

\[
\begin{align*}
M & \Rightarrow_\ast N \\
M & \Rightarrow_\ast N \\
\lambda M & \Rightarrow_\ast \lambda N
\end{align*}
\]

- also for CL, taking $KMN \Rightarrow_\ast M$, $SPQR \Rightarrow_\ast (PR)(QR)$ as primitive $\Rightarrow_\ast^0$ steps, together with 'diamonds' $\Rightarrow_\ast^0$:

- also for CL+Y, taking $Y \Rightarrow_\ast Y$ and $YM \Rightarrow_\ast M(YM)$ as primitive:

etc.

- also for $\lambda + \Omega$ with $\Omega \Rightarrow_\ast \Omega$, $\lambda \Omega \Rightarrow_\ast \Omega$, $\Omega P \Rightarrow_\ast \Omega$.
FIN