Lecture #3

The Standardisation Theorem

1958 Curry/Feys for CL

1975 Plotkin for $\lambda\beta v$, $\lambda\beta$ via 'standard sequences'

1979 Mitschke for $\lambda\beta(\eta)$ via semistandardisation

1975 – 77 Lévy via labelled λ calculus
Barendregt
Hyland

<u>What does it say?</u>

if $M \twoheadrightarrow_{\beta} N$ then there exists a standard reduction $\sigma : M \twoheadrightarrow_{s} N$

where $\sigma : M_0 \xrightarrow{\Delta_0} M_1 \; \ldots \; M_i \xrightarrow{\Delta_i} M_{i+1} \longrightarrow \ldots \; M_n$

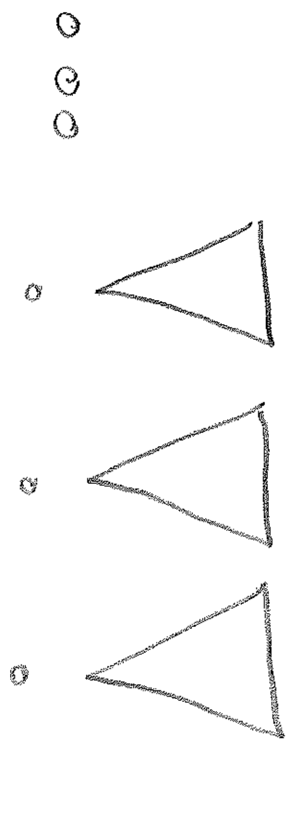
is <u>standard</u> (reducing the redex occurrence Δ_i at the i^{th} step)

if

$\forall i \, \forall j < i$ [Δ_i is <u>not</u> a residual of a redex to the left of Δ_j (relative to the given reduction from M_j to M_i)]

(†)

(Barendregt, 1984, Ch. 11 p 296 ; <u>nested parentheses</u> in original)



... Can we do
better ???

$$\boxed{\twoheadrightarrow_w \quad \text{weak-head reduction}}$$

let $\twoheadrightarrow_w$ be the <u>least</u> reflexive, transitive relation

- containing $\rightarrow_\beta$ $\qquad e \rightarrow_\beta e' \Rightarrow e \twoheadrightarrow_w e'$
- closed under application <u>on the right</u>

$$\frac{e \twoheadrightarrow_w e'}{e \cdot a \twoheadrightarrow_w e' \cdot a} \; (\cdot)$$

<u>NB</u> if $\lambda e \twoheadrightarrow_w e'$ then $e' \equiv \lambda e$ and there is <u>no</u> reduction

ditto. $x \twoheadrightarrow_w e' \Rightarrow e' \equiv x$ $\qquad$ 'weak head normal forms'

what about $e \cdot a$?

Plotkin 1975
Mitschke 1979
(Takahashi 1989/95)
McKinna/
Pollack 1993/99

say that $M \twoheadrightarrow_s N$ is <u>standard</u>
resp. $M \twoheadrightarrow_i N$ is <u>internal</u>

} <u>mutual</u> inductive def^n

if

$$\boxed{\dfrac{M \twoheadrightarrow_w P \twoheadrightarrow_i N}{M \twoheadrightarrow_s N}}$$

"<u>semi-standardisation</u>":- standard reduction factors into a <u>weak head</u> reduction followed by <u>internal</u> reduction

where

$$\dfrac{}{x \twoheadrightarrow_i x} \qquad \dfrac{M \twoheadrightarrow_s N}{\lambda M \twoheadrightarrow_i \lambda N}\,(\lambda)$$

<u>and</u>

$$\dfrac{M \twoheadrightarrow_i M' \qquad N \twoheadrightarrow_s N'}{MN \twoheadrightarrow_i M'N'}\,(\cdot)$$

<u>NB</u>

- in rule (λ), we revert to standard reduction under the outermost symbol λ
- in rule ($\cdot$), we <u>forbid</u> outermost $\twoheadrightarrow_w$ reduction on the <u>left</u> of an application

in this last rule :—

- suppose we wanted $MN \twoheadrightarrow_s M'N'$ from $M \twoheadrightarrow_s M'$, $N \twoheadrightarrow_s N'$

  then, inductively, we may suppose

  $M \twoheadrightarrow_w P \twoheadrightarrow_i M'$

- but then $MN \twoheadrightarrow_w PN$

- so provided we take $PN \twoheadrightarrow_i M'N'$

  then
  $$MN \twoheadrightarrow_w PN \twoheadrightarrow_i M'N'$$
  is a <u>plausible</u> factorisation of $MN \twoheadrightarrow M'N'$

<u>NB</u>

- $M \twoheadrightarrow_w N$ satisfies property $(\dagger)$, immediately
- if $P \twoheadrightarrow_i N$ satisfies $(\dagger)$, then so too does $M \twoheadrightarrow_w P \twoheadrightarrow_i N$
- $M \twoheadrightarrow_i N$ satisfies $(\dagger)$, by mutual induction

__actually__ in $MN \twoheadrightarrow_i M'N'$ from $M \twoheadrightarrow_i M'$, $N \twoheadrightarrow_s N'$

we could contract all the redexes in the sequence $N \twoheadrightarrow_s N'$ __before__ contracting the redexes in the sequence $M \twoheadrightarrow_i M'$

(and they would therefore violate the 'spatial' invariant (†))

so to be __yet more precise__ define

$\twoheadrightarrow_{li}$ 'left internal'

$\twoheadrightarrow_{ri}$ 'right internal'

$\twoheadrightarrow_i$ 'internal

(__mutually__ with one another, and hence also with $\twoheadrightarrow_s$)

by $\dfrac{M \twoheadrightarrow_i N}{MP \twoheadrightarrow_{li} NP}$, $\dfrac{P \twoheadrightarrow_s Q}{NP \twoheadrightarrow_{ri} NQ}$, $\dfrac{M \twoheadrightarrow_{li} P \twoheadrightarrow_{ri} N}{M \twoheadrightarrow_i N}$

(and $x \twoheadrightarrow_i x$ as before) then each of $\twoheadrightarrow_w$, $\twoheadrightarrow_{li}$, $\twoheadrightarrow_{ri}$, $\twoheadrightarrow_i$ and $\twoheadrightarrow_s$ satisfy property (†)

we're <u>almost</u> shown <u>that</u> if $M \twoheadrightarrow_\beta N$ then $M \twoheadrightarrow_s N$

(single step β) immediate, $M \to_\beta N \Rightarrow M \to_w N$ $\blacksquare$

(var) immediate, $x \twoheadrightarrow_w x \twoheadrightarrow_i x$, by (var) $\blacksquare$

(λ) immediate, $\twoheadrightarrow_i \subseteq \twoheadrightarrow_s$, by (var)

(app) we've seen this by way of motivating (app) for $\twoheadrightarrow_i$

so the only thing left to show is <u>TRANSITIVITY</u>

(and then we are done, with $\twoheadrightarrow_\beta$ as the least precongruence containing $\to_\beta$)

(actually we also need REFLEXIVITY for (β) – exercise!)

<u>transitivity</u> for $\twoheadrightarrow_s$ follows, provided:

$$\begin{tikzcd}
M \arrow[rr, two heads, "s"'] \arrow[dr, two heads, "w"'] & & P \arrow[rr, two heads, "s"'] \arrow[dr, two heads] & & N \\
& \cdot \arrow[ur, two heads, "i"'] \arrow[dr, dashed, two heads, "w"'] & & \cdot \arrow[ur, two heads, "i"'] & \\
& & \cdot \arrow[ur, dashed, two heads, "w\,i"'] & &
\end{tikzcd}$$

i.e that we can standardise

(factor into $\twoheadrightarrow_w \cdot \twoheadrightarrow_i$)

all composites of the form $\twoheadrightarrow_i \cdot \twoheadrightarrow_w$

<u>NB</u> we need to show simultaneously

$$\begin{cases} \cdot \ \twoheadrightarrow_i \cdot \twoheadrightarrow_w \ \subseteq \ \twoheadrightarrow_s \quad (*) \\ \cdot \ \twoheadrightarrow_i \cdot \twoheadrightarrow_i \ \subseteq \ \twoheadrightarrow_i \\ \cdot \ \twoheadrightarrow_s \cdot \twoheadrightarrow_s \ \subseteq \ \twoheadrightarrow_s \end{cases}$$

so by induction on $\twoheadrightarrow_w$

it suffices to show

(**) $\twoheadrightarrow_i \cdot \rightarrow_\beta \ \subseteq \ \twoheadrightarrow_s$ to conclude (*)

idea

- internal reduction $\twoheadrightarrow_i$
  <u>preserves</u>, <u>and reflects</u>, the shape of terms
- indeed, the property of <u>being a β redex</u> $(\overset{\nabla}{\circ})$

  if $M \twoheadrightarrow_i (\lambda R) \circ S$ then $M \equiv (\lambda P) \circ Q$

  where $\lambda P \twoheadrightarrow_i \lambda R$

  ie $P \twoheadrightarrow_s R$

  and $Q \twoheadrightarrow_s S$

- so to show $(**)$ $\quad P \twoheadrightarrow_s R \quad Q \twoheadrightarrow_s S$

  we need

$$\frac{P \twoheadrightarrow_s R \qquad Q \twoheadrightarrow_s S}{(\lambda P) \circ Q \to_\beta P[Q] \twoheadrightarrow_s R[S]}$$

$$\begin{tikzcd}
P[Q] \arrow[rr, two heads, "s"] \arrow[dr, two heads, "w"'] & & R[S] \\
& \circ \arrow[ur, two heads, "i"'] &
\end{tikzcd}$$

__finally__ $P \twoheadrightarrow_w W \twoheadrightarrow_i R$ from $P \twoheadrightarrow_s R$

so $P[Q] \twoheadrightarrow_w W[Q]$ (__exercise__)

so we may conclude the proof of (**), and hence of (*), and hence of transitivity of $\twoheadrightarrow_s$

__provided__

$$\frac{\boxed{P \twoheadrightarrow_{i/s} R} \quad Q \twoheadrightarrow_s S}{P[Q] \twoheadrightarrow_s R[S]}$$

one last __mutual__ induction on $\boxed{P \twoheadrightarrow_i R}$ resp. $\boxed{P \twoheadrightarrow_s R}$

__now__ $\twoheadrightarrow_s$ is a reflexive, transitive relation closed under (λ), $(\cdot)$

hence $M \twoheadrightarrow_\beta N \implies M \twoheadrightarrow_s N$

∎

more on normal forms

β normal forms may be inductively characterised

$$\frac{}{\mathrm{nf}_\beta\, x} \qquad \frac{\mathrm{nf}_\beta\, e}{\mathrm{nf}_\beta(\lambda e)} \qquad \frac{\mathrm{nf}_\beta\, e \quad \mathrm{nf}_\beta\, a}{\mathrm{nf}_\beta(e \cdot a)}\ (e \not\equiv \lambda e')$$

alternatively (neutral / normal)

$$\frac{}{\mathrm{ne}_\beta\, x} \qquad \frac{\mathrm{ne}_\beta\, e \quad \mathrm{nf}_\beta\, a}{\mathrm{ne}_\beta(e \cdot a)} \qquad \frac{\mathrm{ne}_\beta\, e}{\mathrm{nf}_\beta\, e} \qquad \frac{\mathrm{nf}_\beta\, e}{\mathrm{nf}_\beta(\lambda e)}$$

whnormal forms

$$\frac{}{\mathrm{whnf}_\beta\, x} \qquad \frac{}{\mathrm{whnf}_\beta(\lambda e)} \qquad \frac{\mathrm{whnf}_\beta\, e \quad (e \not\equiv \lambda e')}{\mathrm{whnf}_\beta(e \cdot a)}$$

alternatively define 'weakly neutral' and whnf_β (exercise)

Further remarks

- $\twoheadrightarrow_i$ preserves (and reflects) weak head normal forms

- normal forms are weak head normal forms

so we have as Corollaries

- leftmost reduction: if $M \twoheadrightarrow_\beta N$ and $nf_\beta N$
  then $\exists W.\ M \twoheadrightarrow_w W,\ whnf_\beta W,\ W \twoheadrightarrow_i N$

- quasi-normalisation: if $\nexists N.\ nf_\beta N,\ M \twoheadrightarrow_\beta N$ "M has no β-nf"
  then $\forall W.\ M \twoheadrightarrow_w W \Rightarrow \neg\, whnf_\beta W$

<u>a characterisation of</u>
<u>reduction to normal form</u> define, as usual $\xrightarrow{}\!\!\!\!\to_s^{nf}$, $\xrightarrow{}\!\!\!\!\to_i^{nf}$ mutually

by $M \xrightarrow{}\!\!\!\!\to_s^{nf} N \;=\; \exists W.\ \mathrm{whnf}\,W,\ M \xrightarrow{}\!\!\!\!\to_w W \xrightarrow{}\!\!\!\!\to_i^{nf} N$

and for applications $\dfrac{M \xrightarrow{}\!\!\!\!\to_i^{nf} M' \not\equiv \lambda P \qquad N \xrightarrow{}\!\!\!\!\to_i^{nf} N'}{MN \xrightarrow{}\!\!\!\!\to_i^{nf} M'N'}$ (otherwise as $\xrightarrow{}\!\!\!\!\to_i$)

then $\boxed{M \xrightarrow{}\!\!\!\!\to_s^{nf} N \ \text{ iff } \ M \xrightarrow{}\!\!\!\!\to_s N \ \text{ and } \ \mathrm{nf}_\beta\, N}$

the premise $M \xrightarrow{}\!\!\!\!\to_i^{nf} M' \not\equiv \lambda P$ is equivalent to

$M \xrightarrow{}\!\!\!\!\to_i M'$ and $\mathrm{ne}_\beta\, M'$

Also

- we have exactly analogous results for $\twoheadrightarrow_h$ <u>head</u> reduction satisfying

$$\frac{M \to_\beta N}{M \twoheadrightarrow_h N} \qquad \frac{M \twoheadrightarrow_h N}{MP \twoheadrightarrow_h NP} \qquad \frac{M \twoheadrightarrow_h N}{\lambda M \twoheadrightarrow_h \lambda N}$$

with appropriate adjustments to $\twoheadrightarrow_i$: $\dfrac{M \twoheadrightarrow_i N}{\lambda M \twoheadrightarrow_i \lambda N}$

- also for CL, taking $KMN \to_w M$, $SPQR \to_w (PR)(QR)$ as primitive $\twoheadrightarrow_w$ steps, together with 'obvious' $\twoheadrightarrow_i$ defn

- also for CL+Y, taking $Y \twoheadrightarrow_i Y$ and $YM \to_w M(YM)$ as primitive

<u>etc.</u>

- also for $\lambda_\beta + \Omega$ with $\Omega \twoheadrightarrow_i \Omega$, $\lambda\Omega \to_h \Omega$, $\Omega P \to_h \Omega$

FIN