Lecture #2

The Church Rosser Theorem

1936	<u>Church</u>/Rosser	for λI calculus
1958	Curry/<u>Feys</u>	for CL
1965	<u>Schroer</u>	for full β reduction
1967	<u>Tait</u>/Martin-Löf	via parallel reduction
1975	<u>Welch</u>	via <u>super</u> parallel reduction

What does it say?

- (DP) 'diamond property'

$$\begin{tikzcd} & M \arrow[dl, twoheadrightarrow] \arrow[dr, twoheadrightarrow] & \\ P \arrow[dr, twoheadrightarrow] & & Q \arrow[dl, twoheadrightarrow] \\ & \exists R & \end{tikzcd}$$

any two divergent reduction sequences may be brought back together (for the cost of some more reduction)

R is the 'common reduct' of P, Q

<u>corollaries</u>

- (CR)

$$\begin{tikzcd} M \arrow[dr, twoheadrightarrow] & \simeq & N \arrow[dl, twoheadrightarrow] \\ & \exists R & \end{tikzcd}$$

any two <u>convertible</u> terms have a common reduct

- if $M \simeq N$, N is in <u>normal form</u> then $M \twoheadrightarrow N$

β reduction

β normal forms

β equality / conversion

$$\longrightarrow_\beta$$

reserve this relation symbol
for the single 'top-level' instances

$$(\lambda e)\cdot a \longrightarrow_\beta e[a]$$

'redex' 'contractum'

"$(\lambda e)\cdot a$ contracts to $e[a]$"

$$\boxed{\longrightarrow\!\!\!\!\!\rightarrow_\beta}$$

Let this be the <u>least</u> <u>precongruence</u>* containing $\to_\beta$.

(r) $\dfrac{}{x \twoheadrightarrow_\beta x}$ (***)

(t) $\dfrac{e_1 \twoheadrightarrow_\beta e \twoheadrightarrow_\beta e_2}{e_1 \twoheadrightarrow_\beta e_2}$

(β) $\dfrac{e_1 \to_\beta e_2}{e_1 \twoheadrightarrow_\beta e_2}$

(**)

(λ) $\dfrac{e_1 \twoheadrightarrow_\beta e_2}{\lambda e_1 \twoheadrightarrow_\beta \lambda e_2}$

(•) $\dfrac{e_1 \twoheadrightarrow_\beta e_2 \quad a_1 \twoheadrightarrow_\beta a_2}{e_1 \cdot a_1 \twoheadrightarrow_\beta e_2 \cdot a_2}$

* i.e.

- reflexive
- transitive
- closed under the term constructors(**)

<u>NB</u>(***) $e \twoheadrightarrow_\beta e$ is an <u>admissible</u> property, by induction.

$$\boxed{=_\beta,\ \simeq_\beta}$$

this is the least congruence[(*)] containing $\rightarrow_\beta$

(*) ie additionally consider an inference rule (S) $\dfrac{e \simeq_\beta e'}{e' \simeq_\beta e}$

or consider an additional rule $(\beta)^{-1}$ $\dfrac{e' \rightarrow_\beta e}{e \simeq_\beta e'}$

or define $\simeq_\beta$ by

$$\begin{tikzcd}
e_1 \arrow[dr, two heads, "\beta"'] & \cdots & \cdot \arrow[dl, two heads, "\beta"] \arrow[dr, two heads, "\beta"'] & \cdots & \cdot \arrow[dl, two heads, "\beta"] \arrow[dr, two heads, "\beta"'] & \cdots & \cdot \arrow[dl, two heads, "\beta"] \arrow[dr, two heads, "\beta"'] & \cdots & e_2 \arrow[dl, two heads, "\beta"] \\
& \cdot & & \cdot & & \cdot & & \cdot &
\end{tikzcd}$$

$\boxed{\mathrm{nf}_\beta\, e}$

say that e is in β-normal form $\mathrm{nf}_\beta\, e$ 'e in β-nf'

if • $e \twoheadrightarrow_\beta e' \Rightarrow e' \equiv e$ and reduction sequence is 'empty'

ie. there is no subterm f of e such that $f \to_\beta f'$ in $e \twoheadrightarrow_\beta e'$

NB. • if we consider $\omega = (\lambda x.xx)$ i.e. $\lambda((\mathrm{var}\ 0) \cdot (\mathrm{var}\ 0))$

then ω is in β-nf

• but if $\Omega = \omega \cdot \omega$ then $\Omega \twoheadrightarrow_\beta \Omega$ indeed $\Omega \to_\beta \Omega$

so Ω is not in β-nf indeed $\Omega \twoheadrightarrow_\beta \Omega \twoheadrightarrow_\beta \Omega \twoheadrightarrow \ldots$

so we've already left the realm of terminating expressions

why might Church-Rosser be <u>hard</u> to prove

<u>problem #1</u>

$$\begin{array}{ccc} & \circ & \\ \swarrow_\beta & & \searrow_\beta \\ & ? & \end{array}$$

$(\lambda e) \cdot a \longrightarrow_\beta e[a]$ seems OK

<u>but</u> 'a' is potentially repeated 0, 1, 2, … many times in $e[a]$
<u>each</u> of which might subsequently reduce

(so we can't easily take 'contextual closure' of $\longrightarrow_\beta$)

<u>problem #2</u>

$$\begin{array}{ccc} & \circ & \\ \twoheadleftarrow_\beta & & \twoheadrightarrow_\beta \end{array}$$

<u>transitivity</u> rule kills you almost before you can start doing anything using proof by induction

<u>problem #3</u> infinite reductions eg. $\Omega \twoheadrightarrow_\beta \Omega \twoheadrightarrow \ldots$ mean you can't assume that $\twoheadrightarrow_\beta$ is <u>bounded</u>

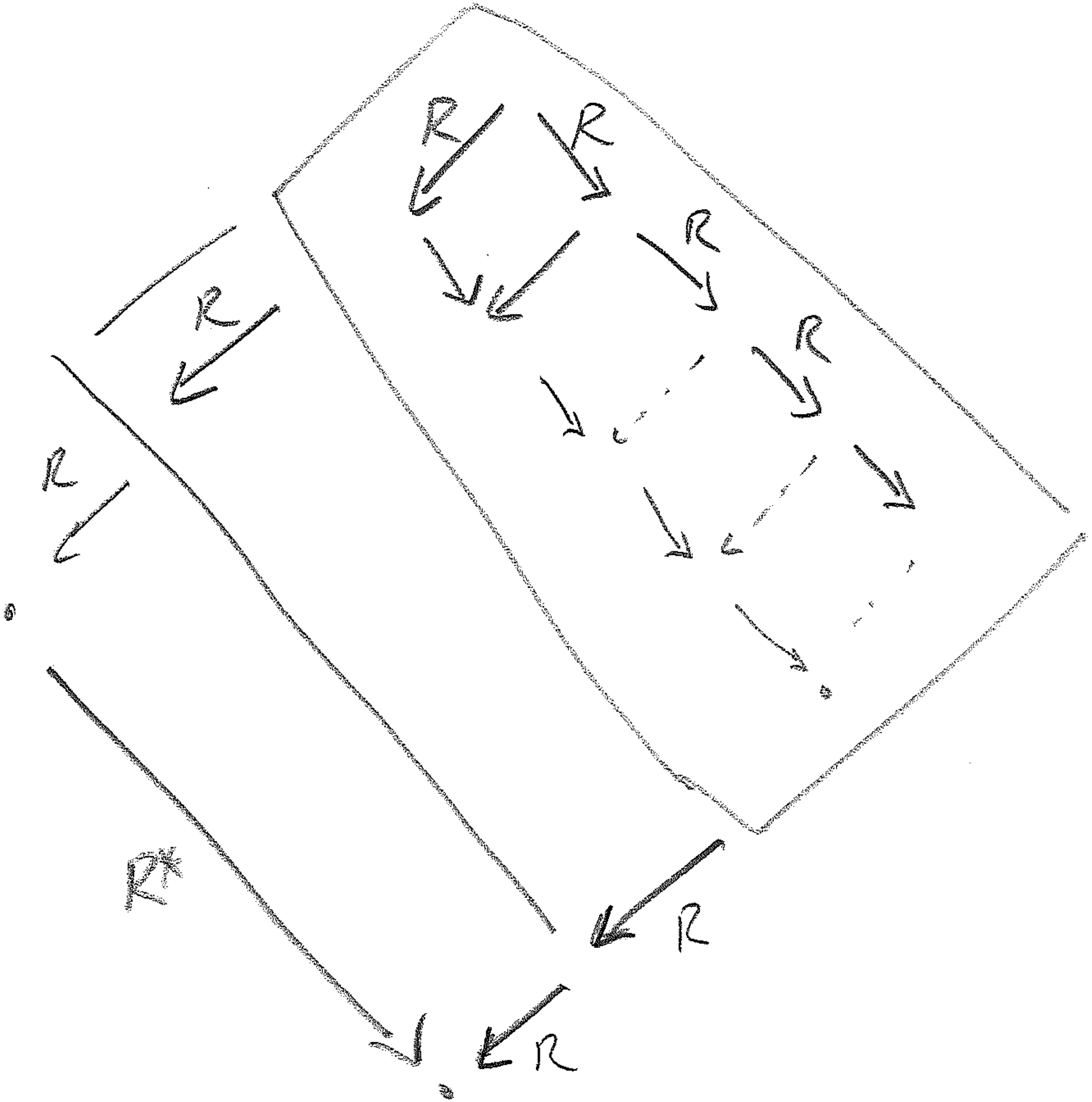
what we need

a relation R between terms such that

- $\to_\beta \subseteq R$ so that R can do at least as much reduction
- $R^* \subseteq \twoheadrightarrow_\beta$ so that problematic transitivity in $\twoheadrightarrow_\beta$ can be 'tamed' by transitivity of R^*
- R is closed under 'contextual rules' for $(\lambda), (\cdot)$
- R satisfies DP

<u>lemma</u> R satisfies DP $\Rightarrow$ R^* satisfies DP

<u>lemma</u> R closed under $(\lambda), (\cdot)$ $\Rightarrow$ R^* closed under $(\lambda), (\cdot)$



R and R* play nicely

'strip lemma'

one solution to the problem

consider <u>two</u> relations R, S such that

- $S \subseteq R$
- $\rightarrow_\beta \subseteq R$
- $R^* \subseteq \twoheadrightarrow_\beta$
- R closed under (λ) $(\cdot)$

<u>together with</u> "triangle lemma" (Takahashi 1989/95; Lévy 1977; ...)

<u>if</u>

$$\begin{tikzcd} e_1 \arrow[r, two heads, "S"] \arrow[dr, two heads, "R"'] & e_2 \\ & e \end{tikzcd} \qquad (\dagger)$$

<u>then</u>

$$\begin{tikzcd} e_1 \arrow[r, two heads, "S"] \arrow[dr, two heads, "R"'] & e_2 \\ & \cdot \arrow[u, two heads, dashed, "R"'] \end{tikzcd}$$

(*) <u>lemma</u> if (R, S) satisfy $(\dagger)$ then R satisfies DP

(**) <u>provided</u> $\forall e \exists e'.\ e \mathrel{S} e'$

<u>NB</u> (*) is a <u>partial correctness</u> statement

(**) the corresponding <u>termination</u> proof

(*) diamonds via triangles

$$
\begin{tikzcd}
 & M \arrow[dl, two heads, "R"'] \arrow[d, two heads, "S"] \arrow[dr, two heads, "R"] & \\
P \arrow[dr, dashed, "R"'] & & Q \arrow[dl, dashed, "R"] \\
 & \cdot &
\end{tikzcd}
$$

Welch, 1975 ; Park, Aczel
Klop, van Raamsdonk

Say that (R) $e \Rrightarrow e'$ "e super-parallel reduces to e'"

(S) $e \Rrightarrow_d e'$ "e' is a complete superdevelopment of e"

if $\dfrac{}{x \Rrightarrow x}$ (var) $\quad$ plus for $R =_{def} \Rrightarrow$ $\quad \dfrac{e_1 \Rrightarrow \lambda e \quad a_1 \Rrightarrow a}{e_1 \circ a_1 \Rrightarrow e[a]}$ (β)

$\dfrac{e_1 \Rrightarrow e_2}{\lambda e_1 \Rrightarrow \lambda e_2}$ (λ)

$\dfrac{e_1 \Rrightarrow e_2 \quad a_1 \Rrightarrow a_2}{e_1 \circ a_1 \Rrightarrow e_2 \circ a_2}$ ($\circ$)

for $S =_{def} \Rrightarrow_d$
- above rule (β)
- in ($\circ$) rule, <u>forbid</u> $e_2 \equiv \lambda e$

__idea__

- $S \subseteq R$ every rule defining S is an R-rule

- $\to_\rho \subseteq R$ by rule (ρ), plus proof of __reflexivity__ (admissible by 'usual' reasoning)

- R __allows__ reduction to proceed as far as the creation of a top level redex and then is __allowed__ to contract it

- S __forces__ contraction of all top level redexes created inductively as it proceeds

Termination

- $\forall e \, \exists e' \quad e \Rrightarrow_d e'$ induction on e; case analysis in $(\cdot)$ case

partial correctness

- $$\begin{tikzcd} e_1 \arrow[r, Rrightarrow, "S"] \arrow[d, Rrightarrow, "R"'] & e' \\ e & \end{tikzcd}$$ with $e_1 \overset{S}{\Rrightarrow}_d e'$, $\quad \Rightarrow \quad e \overset{R}{\Rrightarrow} e'$

<u>idea</u> the S reduction contracts <u>at least</u> as many redexes as R

so for the 'leftovers' in e, does <u>one</u> step of R suffice to mop up all the others?

<u>answer</u> yes (!)

<u>proof</u> induction on S
case analysis on R

<u>lemma</u>

$$\frac{e_1 \Rrightarrow e_2 \qquad a_1 \Rrightarrow a_2}{e_1[a_1] \Rrightarrow e_2[a_2]}$$

Substitutivity

Say that R is substitutive

$$\frac{e_1 \, R \, e_2 \qquad a_1 \, R \, a_2}{e_1[a_1] \, R \, e_2[a_2]}$$

strongly substitutive

$$\frac{e_1 \, R \, e_2 \qquad \sigma_1 \, [R] \, \sigma_2}{e_1[\sigma_1] \, R \, e_2[\sigma_2]}$$

where $[R]$ is the pointwise extension of R to σs

lemma R strongly substitutive $\Rightarrow$ R substitutive

lemma $\Rrightarrow$ is strongly substitutive (why?)