Appendix A

The MacLaurin expansion of $(r^2+t^2+1)^{-1/2}$

The Taylor's series of a function $f(x,y)$ of two variables is

$$f(a+r,b+t)= f(a,b)+{}^{*}Df(a,b)+\frac{1}{2!}{}^{*}D^2 f(a,b)+\frac{1}{3!}{}^{*}D^3 f(a,b)...$$

Where

$${}^{*}D=\left(r\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}\right)$$

Now if a = b = 0 we obtain the MacLaurin series :

$$\begin{aligned}
f(r,t)= f(0,0)&+\left(r\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}\right)f(0,0)\\
&+\frac{1}{2!}\left(r^2\frac{\partial^2}{\partial x^2}+2rt\frac{\partial^2}{\partial x\partial y}+t^2\frac{\partial^2}{\partial y^2}\right)f(0,0)\\
&+\frac{1}{3!}\left(r^3\frac{\partial^3}{\partial x^3}+3r^2t\frac{\partial^3}{\partial x^2\partial y}+3rt^2\frac{\partial^3}{\partial x\partial y^2}+t^3\frac{\partial^3}{\partial y^3}\right)f(0,0)\\
&+\frac{1}{4!}\left(r^4\frac{\partial^4}{\partial x^4}+4r^3t\frac{\partial^4}{\partial x^3\partial y}+6r^2t^2\frac{\partial^4}{\partial x^2\partial y^2}+4rt^3\frac{\partial^4}{\partial x\partial y^3}+t^4\frac{\partial^4}{\partial y^4}\right)f(0,0)\\
&\;......
\end{aligned}\qquad\text{(A.1)}$$

To find the MacLaurin expansion of

$$f(x,y)=\left(x^2+y^2+1\right)^{-\frac{1}{2}}$$

we must first find the partial derivatives at (*x*,*y*) = (0,0)

$$\begin{aligned}
\frac{\partial}{\partial x}f(x,y)&=-\frac{1}{2}\left(x^2+y^2+1\right)^{-\frac{3}{2}}.2x\\
&=-x\left(x^2+y^2+1\right)^{-\frac{3}{2}}\\
\Rightarrow \frac{\partial}{\partial x}f(0,0)&=0
\end{aligned}\qquad\text{(A.2)}$$

$$\frac{\partial^2}{\partial x^2}f(x,y)=\frac{\partial}{\partial x}\left[-x\left(x^2+y^2+1\right)^{-\frac{3}{2}}\right]$$

$$= \left\{-\left(x^2+y^2+1\right)^{-\frac{3}{2}}\right\} + \left\{-x.-\frac{3}{2}\left(x^2+y^2+1\right)^{-\frac{5}{2}}.2x\right\}$$

$$= -\left(x^2+y^2+1\right)^{-\frac{3}{2}} + 3x^2\left(x^2+y^2+1\right)^{-\frac{5}{2}}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} f(0,0) = -1 \qquad \text{(A.3)}$$

$$\frac{\partial^3}{\partial x^3} f(x,y) = \frac{\partial}{\partial x}\left[-\left(x^2+y^2+1\right)^{-\frac{3}{2}} + 3x^2\left(x^2+y^2+1\right)^{-\frac{5}{2}}\right]$$

$$= \frac{3}{2}\left(x^2+y^2+1\right)^{-\frac{5}{2}}2x + \left\{6x\left(x^2+y^2+1\right)^{-\frac{5}{2}} + 3x^2.-\frac{5}{2}\left(x^2+y^2+1\right)^{-\frac{7}{2}}.2x\right\}$$

$$= 3x\left(x^2+y^2+1\right)^{-\frac{5}{2}} + 6x\left(x^2+y^2+1\right)^{-\frac{5}{2}} - 15x^3\left(x^2+y^2+1\right)^{-\frac{7}{2}}$$

$$= 9x\left(x^2+y^2+1\right)^{-\frac{5}{2}} - 15x^3\left(x^2+y^2+1\right)^{-\frac{7}{2}}$$

$$\Rightarrow \frac{\partial}{\partial x^3} f(0,0) = 0 \qquad \text{(A.4)}$$

$$\frac{\partial^4}{\partial x^4} f(x,y) = \frac{\partial}{\partial x}\left[9x\left(x^2+y^2+1\right)^{-\frac{5}{2}} - 15x^3\left(x^2+y^2+1\right)^{-\frac{7}{2}}\right]$$

$$= 9\left(x^2+y^2+1\right)^{-\frac{5}{2}} - 9x\frac{5}{2}\left(x^2+y^2+1\right)^{-\frac{7}{2}}2x$$

$$-45x^2\left(x^2+y^2+1\right)^{-\frac{7}{2}} + 15x^3\frac{7}{2}.\left(x^2+y^2+1\right)^{-\frac{9}{2}}.2x$$

$$= 9\left(x^2+y^2+1\right)^{-\frac{5}{2}} - 45x^2\left(x^2+y^2+1\right)^{-\frac{7}{2}}$$

$$-45x^2\left(x^2+y^2+1\right)^{-\frac{7}{2}} + 205x^4\left(x^2+y^2+1\right)^{-\frac{9}{2}}$$

$$= 9\left(x^2+y^2+1\right)^{-\frac{5}{2}} - 90x^2\left(x^2+y^2+1\right)^{-\frac{7}{2}} - 205x^4\left(x^2+y^2+1\right)^{-\frac{9}{2}}$$

$$\Rightarrow \frac{\partial^4}{\partial x^4} f(0,0) = 9 \qquad \text{(A.5)}$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y}\left[-x\left(x^2+y^2+1\right)^{-\frac{3}{2}}\right]$$

$$= -x.-\frac{3}{2}.\left(x^2+y^2+1\right)^{-\frac{5}{2}}.2y$$

$$= 3xy\left(x^2+y^2+1\right)^{-\frac{5}{2}}$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y} f(0,0) = 0 \qquad \text{(A.6)}$$

$$\frac{\partial^3}{\partial x^2 \partial y} f(x,y) = \frac{\partial}{\partial x}\left[3xy\left(x^2+y^2+1\right)^{-\frac{5}{2}}\right]$$

$$= 3y\left(x^2 + y^2 + 1\right)^{-\frac{5}{2}} - 3xy.\frac{5}{2}\left(x^2 + y^2 + 1\right)^{-\frac{7}{2}}.2x$$

$$= 3y\left(x^2 + y^2 + 1\right)^{-\frac{5}{2}} - 15x^2y\left(x^2 + y^2 + 1\right)^{-\frac{7}{2}}$$

$$\Rightarrow \frac{\partial^3}{\partial x^2 \partial y} f(0,0) = 0 \qquad \text{(A.7)}$$

$$\frac{\partial^4}{\partial x^3 \partial y} f(x,y) = \frac{\partial}{\partial y}\left[9x\left(x^2 + y^2 + 1\right)^{-\frac{5}{2}} - 15x^3\left(x^2 + y^2 + 1\right)^{-\frac{7}{2}}\right]$$

$$= \left[9x.-\frac{5}{2}.\left(x^2 + y^2 + 1\right)^{-\frac{7}{2}}.2y\right] - \left[15x^3.-\frac{7}{2}\left(x^2 + y^2 + 1\right)^{-\frac{9}{2}}.2y\right]$$

$$= -45xy\left(x^2 + y^2 + 1\right)^{-\frac{7}{2}} + 105x^3y\left(x^2 + y^2 + 1\right)^{-\frac{9}{2}}$$

$$\Rightarrow \frac{\partial^3}{\partial x^2 \partial y} f(0,0) = 0 \qquad \text{(A.8)}$$

$$\frac{\partial^4}{\partial x^2 \partial y^2} f(x,y) = \frac{\partial}{\partial y}\left[3y\left(x^2 + y^2 + 1\right)^{-\frac{5}{2}} - 15x^2y\left(x^2 + y^2 + 1\right)^{-\frac{7}{2}}\right]$$

$$= 3(x^2 + y^2 + 1)^{-\frac{5}{2}} - 15y^2(x^2 + y^2 + 1)^{-\frac{7}{2}}$$

$$- 15x^2(x^2 + y^2 + 1)^{-\frac{7}{2}} - 105x^2y^2(x^2 + y^2 + 1)^{-\frac{9}{2}}$$

$$\Rightarrow \frac{\partial^4}{\partial x^2 \partial y^2} f(0,0) = 3 \qquad \text{(A.9)}$$

Substituting the partial derivatives (A.2) to (A.9) in the MacLaurin expansion (A.1) gives:

$$f(r,t) = 1 + [r.0 + t.0] + \frac{1}{2!}\left[-r^2 + 2rt.0 - t^2\right]$$

$$+ \frac{1}{3!}\left[r^3.0 + 3r^2t.0 + 3rt^2.0 + t^3.0\right]$$

$$+ \frac{1}{4!}\left[r^4.9 + 4r^3t.0 + 6r^2t^2.0 + 4rt^3.0 + t^4.9\right]$$

$$+ \cdots\cdots$$

$$\Rightarrow \left(r^2 + t^2 + 1\right)^{-1/2} = 1 - \frac{1}{2!}\left(r^2 + t^2\right) + \frac{9}{4!}\left(r^2 + t^2\right)^2 \cdots\cdots$$