HERIOT-WATT UNIVERSITY
BSc IN ACTUARIAL SCIENCE

Life Insurance Mathematics A (F70LA)
Tutorial Problems

1. Show that, under the uniform distribution of deaths, for integer x and $0 < s < 1$:

$$Pr[T_x \leq s \mid T_x \leq 1] = s$$

(It then follows that T_x, given $T_x \leq 1$ has a uniform distribution on the interval $(0, 1)$.)

2. Explain why the *AM92* mortality table includes select mortality while the *English Life Table No. 15 - males* does not.

3. Consider the following select mortality table with a select period of 3 years:

x	$q_{[x]}$	$q_{[x-1]+1}$	$q_{[x-2]+2}$	q_x
70	0.040	0.070	0.090	0.10
71	0.044	0.077	0.099	0.11
72	0.048	0.084	0.108	0.12
73	0.052	0.091	0.117	0.13

   (a) Calculate the values of ${}_2q_{[70]}$ and ${}_{1|2}q_{[70]+1}$.

   (b) Complete the missing elements in the following life table:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}
70				10000
71				

   (c) Calculate the values of $d_{[70]}$, $d_{[70]+1}$, $d_{[70]+2}$.

4. A life table with a two-year select period is being constructed. Ultimate values of l_x will be the same as those for ELT 15 Males (ELT 12 Males), while $q_{[x]} = 0.3q_x$ and $q_{[x]+1} = 0.6q_{x+1}$. Calculate the value of $l_{[70]}$.

5. (a) An endowment assurance with term n years and sum assured £1 pays £1 at time
$$\begin{cases} K_x + 1 & \text{if } K_x < n, \\ n & \text{if } K_x \geq n. \end{cases}$$
Explain why it is appropriate for a life office to use select life tables for endowment assurances.

   (b) Explain why in the A1967-70 mortality table the ratio $q_{[60]}/q_{60}$ is much smaller than $q_{[20]}/q_{20}$.

   (c) For a particular select life table with select period $s = 2$ years, we are given $l_{40} = 100000$, $p_{39} = 0.93$, $p_{38} = 0.94$, as well as $q_{[x]} = 0.5q_x$ and $q_{[x-1]+1} = 0.8q_x$. Calculate $l_{[38]}$.

6. (a) Calculate the expected present value and the variance of the present value of a benefit of £100 payable to a life currently aged 30 on survival to age 60. There is no payment if they die before age 60.
Basis: Mortality; ELT12 Males. Interest; 5% per annum.

(b) If a company sells 1,000 identical policies, how much capital should the company put aside to have 95% confidence that it will meet the liabilities from that portfolio? State any assumptions that you make.

7. Evaluate, on the basis of ELT12 Males mortality and 4% per annum interest, the following:

   (a) Expected value of the present value of a benefit of £2,000 paid at the end of year of death of a life aged 45 if they die within 3 years. There is no payment on survival.

   (b) Expected present value of a benefit of £10,000 payable to a life aged 40 on survival to age 65.

   (c) $A^1_{35:\overline{5}|}$

   (d) $A_{35:\overline{20}|^{\frac{1}{}}}$

   (e) ${}^2A^1_{35:\overline{5}|}$

   (f) ${}^2A_{35:\overline{20}|^{\frac{1}{}}}$.

8. Using the tables find the values of the following:

   (a) $A_{35:\overline{25}|}$ on the basis of AM92 Ultimate mortality and 4% interest.

   (b) $A^1_{35:\overline{25}|}$ on the basis of AM92 Ultimate mortality and 4% interest.

9. Given that the force of mortality is given by

$$\mu_x = \begin{cases} 0.001 & \text{for} \quad x \leq 40 \\ 0.001 + 0.01(x - 40) & \text{for} \quad x > 40 \end{cases}$$

   and the interest rate is 6.5% per annum effective, calculate the E.P.V. of a pure endowment benefit of £1,000, with a term of 30 years payable to a life currently aged 30.

10. On the basis of AM92 Ultimate mortality and 6% p.a. interest, evaluate

    (a) the expected present value of a benefit of £1 payable to a life currently aged 28 on survival to age 65 with no payment on prior death.

    (b) the expected present value of a benefit of £1 payable at the end of year of death of a life currently aged 28, if the death occurs within 37 years. There is no benefit on survival.

11. A woman aged 60 purchases a policy with a benefit of £1,000 payable at the end of year of death if death occurs within 5 years. The mortality basis is a(55) Ultimate.

    (a) Assuming interest of 6% per annum, evaluate the expected value and the variance of the present value of this benefit.

    (b) Assuming interest of 7% per annum, evaluate the expected value and the variance of the present value of this benefit.

12. Using the tables find the values of the following:

    (a) A_{55} on the basis of A1967-70 Ultimate mortality and 4% interest.

    (b) A_{60} on the basis of AM92 Ultimate mortality and 6% interest.

    (c) $\ddot{a}_{60}$ on the basis of A1967-70 Ultimate mortality and 6% interest.

    (d) a_{58} on the basis of AM92 Ultimate mortality and 4% interest.

    (e) A_{65} on the basis of a(55) Males Ultimate mortality and 4% interest.

    (f) A_{60} on the basis of a(55) Males Ultimate mortality and 6% interest.

    (g) $\ddot{a}_{60}$ on the basis of AM92 Ultimate mortality and 6% interest.

    (h) a_{68} on the basis of a(55) Males Ultimate mortality and 4% interest.

13. Calculate the expected present value of a benefit of a series of payments of amount £1 per annum payable annually in advance to a life currently aged 45 and paid for at most 5 years. The basis for the calculation is ELT15 Females mortality and 4% interest per annum effective.

14. Evaluate $\ddot{a}_{35:\overline{5}|}$ on the basis ELT12 Males mortality and 4% per annum interest. On the same basis evaluate $a_{35:\overline{5}|}$.

15. For the following use the values and/or results from Question 3.

    (a) Using $i = 0.06$ calculate the value of:

        i. $\ddot{a}_{[70]:\overline{3}|}$

        ii. $A_{[70]:\overline{3}|}$

    (b) Let $K_{[x]}$ be the curtate future lifetime of $[x]$ (that is, $K_{[x]} = \lfloor T_{[x]} \rfloor$, the integer part of $T_{[x]}$). Suppose the function $g(k)$ equals $100,000v^{k+1}$ if $k = 0, 1, 2$ and $100,000v^3$ if $k \geq 3$.

    At $i = 0.06$, what is $E[g(K_{[70]})]$?

    What is $Var[g(K_{[70]})]$?

    (c) Suppose $h(k)$ equals v^{k+1} if $k = 0, 1, 2$ and 0 otherwise.

    At $i = 0.06$, what is $E[h(K_{[70]})] = A^{1}_{[70]:\overline{3}|}$?

    What is $E[h(K_{[70]})^2] = A^{1}_{[70]:\overline{3}|}$ at $i' = 0.1236$?

    (d) A term assurance policy has been issued to a life aged 70 and subject to the select mortality table above. The policy pays a sum assured of 100,000 at the end of the year of death provided this occurs during the next three years.

    What is the expected present value of the benefit under this policy (with $i = 0.06$)?

    What is the variance of the present value of the benefit?

(e) Explain why the variance in part (d) is much higher than that for the endowment policy in part (b).

16. Evaluate $\ddot{a}_{35:\overline{5}|}$ on the basis A1967–70 Ultimate mortality and 4% per annum interest. On the same basis evaluate $a_{35:\overline{5}|}$.

17. Find the expected value of the present value of a benefit of a series of payments of amount £1 per annum payable annually in advance to a life currently aged 45 and paid for at most 30 years. The basis for the calculation is A1967–70 Ultimate mortality and 4% per annum interest.

18. Evaluate the following on the basis of A1967–70 Ultimate mortality and 4% per annum interest

    (a) $\ddot{a}_{35}$, $\ddot{a}_{35:\overline{15}|}$, ${}_{10|}\ddot{a}_{35:\overline{15}|}$.
    (b) a_{35} , $a_{35:\overline{15}|}$, ${}_{10|}a_{35:\overline{15}|}$.
    (c) A_{40}, $A^{\ 1}_{35:\overline{10}|}$, $A^{\ 1}_{35:\overline{20}|}$, $A_{35:\overline{10}|}$, $A_{35:\overline{20}|}^{\ 1}$.

19. Evaluate the following on the basis of a(55) Males Ultimate mortality and 8% per annum interest

    (a) $\ddot{a}_{65}$, $\ddot{a}_{65:\overline{10}|}$, ${}_{10|}\ddot{a}_{65:\overline{15}|}$.
    (b) a_{68}, $a_{60:\overline{15}|}$, ${}_{10|}a_{60:\overline{15}|}$.
    (c) A_{70}, $A^{\ 1}_{65:\overline{10}|}$, $A^{\ 1}_{65:\overline{20}|}$, $A_{65:\overline{10}|}$, $A_{65:\overline{20}|}^{\ 1}$.

20. A man aged 50 wishes to purchase a policy with a sum assured of £100,000. On the basis of AM92 Ultimate or Select mortality and 4% per annum interest, calculate the expected present value of the benefit if the policy is

    (a) a whole life policy (Ultimate),
    (b) a pure endowment policy with a term of 30 years (Ultimate),
    (c) a pure endowment policy with a term of 10 years (Select),
    (d) a term assurance policy with a term of 30 years (Ultimate),
    (e) a term assurance policy with a term of 10 years (Select),
    (f) an endowment assurance policy with a term of 30 years (Ultimate),
    (g) an endowment assurance policy with a term of 10 years (Select),
    (h) an endowment assurance policy with a term of 10 years deferred for 10 years (Ultimate).

    Assume that any death benefits are paid at the end of year of death.

21. A man aged 45 purchases a policy with a benefit of £10,000 payable at the end of year of death if death occurs within 25 years. In return the man will pay premiums of amount $£P$ per annum payable annually in advance throughout his lifetime for at most 15 years.

    On the basis of AM92 Ultimate mortality and 4% per annum interest:

    (a) Calculate the expected present value of the benefit.
    (b) Calculate the expected present value of the premiums.
    (c) Find the value of P that makes the E.P.V. of the benefit equal the E.P.V. of the premiums.

22. On the basis of A1967-70 ($i = 4\%$), evaluate and interpret the following pairs

    (a) $A_{60:\overline{5}|}$ and $A_{[60]:\overline{5}|}$,

    (b) $\ddot{a}_{60:\overline{5}|}$ and $\ddot{a}_{[60]:\overline{5}|}$.

    Evaluate $\ddot{a}_{[37]+1:\overline{18}|}$ for $i = 4\%$.

23. A 30–year old man purchases a benefit of annual payments of £100 payable in advance from the age of 60, and payable for as long as he is alive.

Interest:	6% per annum
Mortality:	Before age 60: A1967-70 Ultimate
	Age 60 and above: a(55) Males, Ultimate.

    Determine the EPV of the benefit.

24. (a) Calculate the annual premium for the following policy:

    A whole of life assurance to a life aged 45 at issue and a sum assured of £50,000.

    Level premiums are payable annually in advance for life.

    Premium basis:

Mortality	A1967-70 Select	
Interest	6%	
Expenses	Initial	£200
	Renewal	£50 at the time of payment of the second and each subsequent premium

    Profit criterion: $E[PV \text{ profit}] = 0$.

    (b) Repeat your calculations assuming that premiums are payable for at most 15 years.

25. Calculate the amount of the annuity under the following policy:

    A level annuity payable annually in arrears for life or a maximum of 20 years if the policyholder survives from the date of issue. The single premium for this contract is £150,000 and has been paid by a life currently aged 60.

    Premium basis

Mortality	a(55) select - females	
Interest	4%	
Expenses	Initial	£150
	Claims	£25 at the time of each annuity payment
Profit criterion:	$E[PV \text{ profit}] = 0$.	

26. A life office is about to issue a deferred annuity contract to a man aged 30. The policyholder will pay level premiums annually in advance ceasing after 35 years or on earlier death. At age 65 an annuity will commence of £5,000 per annum payable annually in advance for life.

    Calculate the annual premium.

    Premium basis:

Mortality	AM92 Select	
Interest	4%	
Expenses	Initial	£125 plus 25% of the first premium
	Renewal	£20 plus 2% of each premium at the time of the second and each subsequent premium
	Claims	£15 at the time of each anuity payment
Profit criterion:	$E[PV$ profit$] = £200$.	

27. Calculate the amount of the level annual premium under the following deferred annuity policy. The policy is to be sold to a man aged 30. Premiums will be payable annually in advance for at most 20 years. An annuity of £20,000 per annum payable quarterly in advance commencing at age 60. If the policyholder dies before age 60 then all premiums paid will be returned without interest.

    Premium basis:

Mortality:	A1967-70	Select
Interest:	4%	
Expenses:	Initial:	£100 plus 50% of the annual premium
	Renewal:	£60 at the time of the second and each subsequent premium
	Claims:	£150 at the end of the year of death before age 60
		£10 at the time of each annuity payment
		£50 on death after age 60 at the end of the year of death

    Profit criterion: $E[PV_\pi] = £400$

28. Repeat your calculations from Exercise 27 assuming that the annuity is guaranteed for 5 years (provided the policyholder reaches age 60), and that the mortality assumption is A1967-70 Select up to age 60 and a(55) ultimate, males after age 60.

29. Calculate the values of the following financial functions assuming A1967-70 mortality and 4% interest:

    (a) $(IA)_{[60]}$

    (b) $(IA)^{1}_{[60]:\overline{20}|}$

    (c) $(IA)_{[60]:\overline{20}|}$

    (d) $(I\ddot{a})_{[60]}$

    (e) $(I\ddot{a})_{[60]:\overline{20}|}$

    (f) $\ddot{a}_{[40]:\overline{35}|}$

    (g) $a_{[40]:\overline{35}|}$

    (h) $\ddot{a}^{(12)}_{[40]:\overline{35}|}$

    (i) $a^{(12)}_{[40]:\overline{35}|}$

    (j) $\bar{a}_{[40]:\overline{35}|}$

30. Use the relationship in compound interest that $(I\ddot{a})_{\overline{n}|} = (\ddot{a}_{\overline{n}|} - nv^n)/d$ to prove that $(I\ddot{a})_x = (\ddot{a}_x - (IA)_x)/d$.

    Verify that this identity holds in the case of Question 29 (a) and (d).

31. A 23-year endowment assurance has just been issued to a life aged 42 exact. The sum assured is £50,000 and the single premium was £20,000.

    Calculate, approximately, the internal rate of return on this policy assuming mortality is A1967-70 Ultimate and no expenses and using each of the following methods:

    (a) Use the tables of values for $A_{x:\overline{n}|}$ for various rates of interest and interpolate.

    (b) Calculate the expected present value of the profit at 4%. Then calculate the derivative of the expected present value of the profit with respect to δ, the force of interest. Finally, use the Newton-Raphson method to find, approximately, the force of interest (and hence the internal rate of return) at which the expected present value of profit is zero.

32. Consider an annuity payable for life to an ultimate life aged x which pays:

    - 1 at times 0 and 0.5
    - 2 at times 1 and 1.5
    - 3 at times 2 and 2.5
    - .....
    - $k+1$ at times k and $k+0.5$

    Show that the expected present value of this annuity is approximately:

$$2(I\ddot{a})_x - \frac{1}{2}\ddot{a}_x$$

33. A life office issues a whole-life with-profits assurance to a life aged 40. The basic sum assured is £30,000 and is payable immediately on death. Level premiums are payable annually in advance for at most 25 years. Calculate the annual premium.

    Premium basis:

Mortality:	AM92	Ultimate
Interest:	4%	
Expenses:	Initial:	£200 plus 25% of the annual premium
	Renewal:	£15 plus 2% of the annual premium at the time of the second and each subsequent premium
	Claims:	£100 at the time of payment of the sum assured
Bonus:	simple reversionary bonus of 4% per annum vesting at the start of each policy year	

    Profit criterion: $E[PV_\pi] = 50\%$ of the annual premium.

34. (a) A life office wishes to calculate the monthly premium on a 20-year with-profits endowment assurance contract to be issued to a woman aged 45. The initial sum assured is £35,000 and is payable at the end of the year of death or on survival to age 65. Level premiums are payable monthly in advance for at most 15 years.

    What is the monthly premium?

    Premium basis:

Mortality:	A1967-70	Select
Interest:	5%	
Expenses:	Initial:	£175
	Renewal:	£5 at the time of payment of each premium including the first
Bonus:	compound reversionary bonus of 1.9417% per annum vesting at the start of each policy year	

Profit criterion: $E[PV \text{ profit}] = 1\%$ of the initial sum assured.

(b) Suppose instead that the sum assured is payable immediately on death within the next 20 years and that the following additional claim expenses will be payable: £150 on death within the first 15 years; £125 on death during the final 5 years of the policy; or £50 on survival to age 65.

i. Explain why the claim expenses might vary in this way.

ii. Calculate the revised monthly premium.

35. A life office is to sell a 25-year with-profits endowment assurance to a man aged 35. The basic sum assured is $S = £30,000$ payable at the end of the year of death or on survival to age 60. Level premiums are payable monthly in advance.

Premium basis:

Mortality:	A1967-70	Ultimate
Interest:	6%	
Expenses:	Initial:	£150
	Renewal:	£4 at the time of payment of each premium including the first
	Claims:	£80 at the time of payment of the sum assured
Bonus:	super-compound reversionary bonus vesting at the start of each policy year	
	$b_1 = 3\%$ = rate on bonus on the basic sum assured	
	$b_2 = 6\%$ = rate of bonus on bonus already declared	

Profit criterion: $E[PV_\pi] = 20\%$ of the total annual premium.

Calculate the monthly premium.

36. Given

$$(\bar{I}\bar{a})_{\overline{m}|} = \int_0^m te^{-\delta t}dt = \frac{\bar{a}_{\overline{m}|} - me^{-\delta m}}{\delta}$$

show that

$$(\bar{I}\bar{a})_{x:\overline{n}|} = E\left[(\bar{I}\bar{a})_{\overline{\min(T_x,n)}|}\right] = \frac{1}{\delta}\left(\bar{a}_{x:\overline{n}|} - (\bar{I}\bar{A})_{x:\overline{n}|}\right)$$

37. A life office wishes to issue a whole life with profits assurance to a man aged 35. The basic sum assured is £50000 payable immediately on death. Simple reversionary bonus accrues continuously at the rate of 3% per annum of the basic sum assured (that is, if the man dies at age 50.6 exact then the death benefit will be $50000(1 + 15.6 \times 0.03) = 73400$). Premiums (increasing continuously) are payable throughout the term of the policy. The premium rate at time t will be $(1 + 0.05t)$ times the initial premium rate for all $t \geq 0$.

Calculate the initial premium rate.

Premium basis:

Mortality: ELT-12 males
Interest: 4%
Expenses: none

38. Consider an annuity policy which pays £1000 per annum annually in arrears, issued to a woman aged 60.

    (a) Calculate the smallest single premium which satisfies the profit criterion: $Pr(\text{PV profit } \geq 0) \geq 0.8$.

    (b) What is the expected present value of profit with this single premium?

    Premium basis:
    Mortality: a(55) Ultimate
    Interest: 4%
    Expenses: None

39. A life office is to sell identical 5-year endowment assurances to a group of 1600 lives all aged 70 exact. For each life the mortality rates are as follows:

x	70	71	72	73	74
q_x	0.05	0.06	0.07	0.08	0.09

    The sum assured is £10,000 payable at the end of the year of death or on survival to age 75. Level premiums are payable annually in advance.

    The rate of interest is 10% per annum and there are no expenses.

    (a) What additional assumption do you need to make about mortality?

    (b) i. Show that the present value of the profit on policy i is:

$$PV_{\pi i} = 11P - (11P + 10000)v^{X_i}$$

    where $X_i = \min\{K_i + 1, 5\}$, K_i is the curtate future lifetime of policyholder i and P is the annual premium.

    ii. Estimate the annual premium with the profit criterion:

$$Pr[\text{portfolio profit } \geq 0] = 0.975$$

40. With a rate of interest of 4% per annum calculate the values of $\ddot{a}_{40:\overline{20}|}$ on the following mortality bases:

    (a) AM92 Select

    (b) AM92 Ultimate

    (c) AM92 Ultimate with an age rating of 5 years

    (d) AM92 Ultimate plus 10 years

    (e) AM92 Ultimate with a constant addition to the force of mortality of 0.019048.

41. Recall Question 30 (show that $(I\ddot{a})_x = (\ddot{a}_x - (IA)_x)/d$).

    Now prove the more general result (using the same method):

$$(I\ddot{a})_{x:\overline{n}|} = (\ddot{a}_{x:\overline{n}|} - (IA)_{x:\overline{n}|})/d$$

42. A life office is to issue a special with profits endowment assurance with a term of 15 years to an impaired life aged 45. The sum assured is payable at the end of the year of death or on survival to age 60. The basic sum assured is £50,000 and level premiums are payable annually in advance.

Calculate the annual premium.

Premium basis:

Mortality:	A1967-70 Ultimate with a constant addition to the force of mortality of 0.0096619
Interest:	3%
Expenses:	none
Bonus:	Simple reversionary bonus of 3% of the basic sum assured per annum, vesting at the start of each policy year

Profit criterion: $E[PV \text{ profit}] = £500$.

[*Hint: Calculate* $\ddot{a}'_{45:\overline{15}|}$ *and* $(I\ddot{a})'_{45:\overline{15}|}$ *first and then use Question 41 to calculate* $(IA)'_{45:\overline{15}|}$.]

43. A heavy smoker aged 40 has applied for a deferred annuity contract. The life office estimates that he should have an age rating of 5 years relative to standard mortality.

The annuity will pay £12,000 per annum monthly in advance from age 65.

Level premiums are payable monthly in advance until age 65.

In addition to the deferred annuity benefit there is a death benefit of S which is payable immediately on death before age 65.

Premium basis:

Standard mortality:	AM92	Ultimate
Interest:	4%	
Expenses:	Initial:	£100
	Renewal:	1% of all premiums
	Claims:	£100 immediately on death before age 65
		£50 immediately on death after age 65
		£75 on survival to age 65

Profit criterion: $E[PV \text{ profit}] = 0$.

Let the total annual premiums be P for a standard life and P' for the smoker.

(a) Show that P and P' can be expressed in the form $P = \alpha S + \beta$ and $P' = \alpha' S + \beta'$ for constants α, β, α' and β'.
Give the values for α, β, α' and β'.

(b) What are the monthly premium rates for a standard life and for the smoker when $S = 100,000$, $S = 200,000$ and $S = 300,000$?

(c) For what value of S will we have $P = P'$?

44. Given the appropriate annual or single gross premium P, derive formulae for the gross premium prospective reserve for the following policies t years after they were issued.

Assume there are no expenses.

(a) An $n-$year non-profit endowment assurance with sum assured S payable at the end of the year of death.