Heriot-Watt University
BSc in Actuarial Science

Life Insurance Mathematics A (F70LA)
Solutions for Tutorial Problems

1.

$$
\begin{aligned}
Pr[T_x \le s | T_x \le 1] &= \frac{Pr[T_x \le s \text{ and } T_x \le 1]}{Pr[T_x \le 1]} \\
&= \frac{Pr[T_x \le s]}{Pr[T_x \le 1]} \\
&= \frac{{}_sq_x}{q_x} = \frac{sq_x}{q_x} \quad \text{by the UDD assumption} \\
&= s
\end{aligned}
$$

Hence T_x, given $T_x \le 1$ has a uniform distribution on the interval $(0, 1]$.

2. AM92 is based upon the mortality of male lives who have taken out life assurance policies during the years 1990–1992. We use the function $q_{[x]+r}$ to represent the mortality rate for policyholders, now aged $x + r$, who started their policies r years ago. The life office uses underwriting to select people who are in at least average health (preferrably in good health) and exclude applicants in poor health (from insurance at standard rates). This means that we must use a select mortality table to differentiate between new and existing policyholders.

   ELT15 (males/females) refers to mortality rates for the population of England and Wales during the years 1990–1992. There is no selection involved in the study. All individuals are assumed to have entered at the age of 0! All males the same age x are treated as indentical with the same mortality rates, q_x.

3. (a)

$$
\begin{aligned}
{}_2q_{[70]} &= 1 - {}_2p_{[70]} \\
&= 1 - p_{[70]}p_{[70]+1} = 1 - (1 - 0.04)(1 - 0.077) = 0.11392 \\
{}_{1|2}q_{[70]+1} &= Pr[1 < T_{[70]+1} \le 3] \\
&= Pr[1 < T_{[70]+1}] - Pr[3 < T_{[70]+1}] \\
&= p_{[70]+1} - {}_3p_{[70]+1} = (1 - 0.077) - (1 - 0.077)(1 - 0.108)(1 - 0.13) = 0.206715
\end{aligned}
$$

   (b)

$$
\begin{aligned}
l_{73} &= 10000 \\
l_{74} &= l_{73}p_{73} = 8700 \\
l_{[70]+2} &= \frac{l_{73}}{p_{[70]+2}} = \frac{l_{73}}{1 - q_{[70]+2}} = \frac{10000}{1 - 0.0108} = 11210.76 \\
l_{[70]+1} &= \frac{l_{[70]+2}}{p_{[70]+1}} = 12146.00 \\
l_{[70]} &= \frac{l_{[70]+1}}{p_{[70]}} = 12652.09
\end{aligned}
$$

Similarly,

$$\begin{aligned} l_{[71]+2} &= 9852.77 \\ l_{[71]+1} &= 10756.30 \\ l_{[71]} &= 11251.36. \end{aligned}$$

(c) From the life table in the previous part, we have $d_{[70]} = 506.09$, $d_{[70]+1} = 935.24$ and $d_{[70]+2} = 1210.76$.

4. $l_{72} = 62,562, \quad l_{[70]+1} = \frac{l_{72}}{1-q_{[70]+1}} = \frac{62,562}{1-0.6(0.04311)} = 64,223.20$

   $l_{[70]} = \frac{l_{[70]+1}}{1-q_{[70]}} = \frac{64,223.20}{1-0.3(0.03930)} = 64,989.42$

   Same question for ELT 12 Males:

   $l_{72} = 48625, \quad l_{[70]+1} = \frac{l_{72}}{1-q_{[70]+1}} = \frac{48625}{1-0.6(0.06047)} = 50455.63$

   $l_{[70]} = \frac{l_{[70]+1}}{1-q_{[70]}} = \frac{50455.63}{1-0.3(0.05566)} = 51312.45$

5. (a) An endowment is beneficial to seriously ill people. Therefore selection and/or underwriting by the life office takes place.

   (b) At an age of 20 deaths are mainly caused by accidents. At age 60, illnesses are a much more influential factor for the mortality.

   (c) We need $p_{[38]}$ and $p_{[38]+1}$.

$$\begin{aligned} p_{[38]+1} &= 1 - q_{[39-1]+1} = 1 - 0.8q_{39} = 1 - 0.8(1 - p_{39}) \\ &= 0.944 \end{aligned}$$

$$\begin{aligned} p_{[38]} &= 1 - q_{[38]} = 1 - 0.5q_{38} = 1 - 0.5(1 - p_{38}) \\ &= 0.97 \end{aligned}$$

$$l_{[38]} = \frac{l_{40}}{p_{[38]}p_{[38]+1}} = \frac{100,000}{0.97 \times 0.944} \approx 109,208.4571$$

6. (a) Let B_1 be the present value of the policy benefit (=pure endowment). We then have

$$B_1 = \begin{cases} 100v^{30} & \text{if } K_{30} \geq 30 \\ 0 & \text{if } K_{30} < 30 \end{cases}$$

Therefore

$$E[B_1] = 100v^{30}{}_{30}p_{30} = 100(0.23138)\left(\frac{78924}{95265}\right) = 19.16909$$

$$E[B_1^2] = \left(100v^{30}\right)^2{}_{30}p_{30} = (100(0.23138))^2\left(\frac{78924}{95265}\right) = 443.5344$$

$$\text{Var}[B_1] = 443.5344 - 19.168^2 = 76.0804$$

(b) We consider 1,000 identical such policies sold to lives which are independent. We denote the random variable denoting the present value of the benefits for the policies as B_i for $i = 1, 2, \cdots 1,000$. We define $B = \sum_{i=1}^{1,000} B_i$, as the present value of the benefits for the portfolio.

Since the lives are identical the $E[B] = 1,000E[B_1] = 19,169.1$. Since the lives are identical and also independent, $\text{Var}[B] = 1,000\text{Var}[B_1] = 76,080.4$. We assume that B is normally distributed (using the Central Limit Theorem) and if M is the sum of money that the company should set aside we actually want $P[B \leq M] = 0.95$. Now

$$P\left(\frac{B - E[B]}{\sqrt{\text{Var}[B]}} \leq \frac{M - E[B]}{\sqrt{\text{Var}[B]}}\right) = P\left(\underbrace{\frac{B - 19,169.1}{\sqrt{76,080.4}}}_{Z :=} \leq \underbrace{\frac{M - 19,169.1}{\sqrt{76,080.4}}}_{x :=}\right)$$
$$= P(Z \leq x) = 0.95$$

where approximately $Z \sim N(0,1)$, i.e. we must find x such that $\Phi(x) = 0.95$. From the Yellow Tables (p. 160) we obtain $\Phi(1.64) = 0.94950$ and $\Phi(1.65) = 0.95053$. A rough linear interpolation yields $\Phi(1.645) \approx 0.9500$. Therefore,

$$x = \frac{M - 19,169.1}{\sqrt{76,080.4}} = 1.645 \quad \text{giving} \quad M = 19,623.$$

7. (a) We can construct a table for this term assurance:

K_{45}	P.V. of benefit	Probability
0	$2,000v = 2,000(0.96154)$	${}_{0\|}q_{45} = \frac{d_{45}}{l_{45}} = \frac{369}{92433}$
1	$2,000v^2 = 2,000(0.92456)$	${}_{1\|}q_{45} = \frac{d_{46}}{l_{45}} = \frac{412}{92433}$
2	$2,000v^3 = 2,000(0.88900)$	${}_{2\|}q_{45} = \frac{d_{47}}{l_{45}} = \frac{463}{92433}$

The E.P.V. is

$$2,000A^{1}_{45:\overline{3}|} = 2,000\left[0.96154\left(\frac{369}{92433}\right) + 0.92456\left(\frac{412}{92433}\right) + 0.88900\left(\frac{463}{92433}\right)\right] = 24.83$$

(b) The E.P.V. of this pure endowment is

$$10,000A_{40:\overline{25}|}^{\ \ 1} = 10,000v^{25}{}_{25}p_{40} = 10,000(0.37512)\left(\frac{68490}{93790}\right) = 2,739.31$$

(c) We can construct a table.

K_{35}	P.V. of benefit	Probability
0	$v = 0.96154$	${}_{0}\vert q_{35} = \frac{d_{35}}{l_{35}} = \frac{147}{94652}$
1	$v^2 = 0.92456$	${}_{1}\vert q_{35} = \frac{d_{36}}{l_{35}} = \frac{158}{94652}$
2	$v^3 = 0.88900$	${}_{2}\vert q_{35} = \frac{d_{37}}{l_{35}} = \frac{171}{94652}$
3	$v^4 = 0.85480$	${}_{3}\vert q_{35} = \frac{d_{38}}{l_{35}} = \frac{185}{94652}$
4	$v^5 = 0.82193$	${}_{4}\vert q_{35} = \frac{d_{39}}{l_{35}} = \frac{201}{94652}$

The E.P.V is

$$A^{1}_{35:\overline{5}|} = 0.96154\left(\frac{147}{94652}\right) + 0.92456\left(\frac{158}{94652}\right) + \cdots + 0.82193\left(\frac{201}{94652}\right) = 0.00806.$$

(d)

$$A_{35:\overline{20}|}^{\ \ 1} = v^{20}{}_{20}p_{35} = (0.45639)\left(\frac{85916}{94652}\right) = 0.41427.$$

(e) We construct a table similar to the one in (c) but we use an interest rate of $0.04^2 + 2 \times 0.04 = 0.0816$.

K_{35}	P.V. of benefit	Probability
0	$v = 0.92456$	${}_{0}\vert q_{35} = \frac{d_{35}}{l_{35}} = \frac{147}{94652}$
1	$v^2 = 0.85481$	${}_{1}\vert q_{35} = \frac{d_{36}}{l_{35}} = \frac{158}{94652}$
2	$v^3 = 0.79032$	${}_{2}\vert q_{35} = \frac{d_{37}}{l_{35}} = \frac{171}{94652}$
3	$v^4 = 0.73070$	${}_{3}\vert q_{35} = \frac{d_{38}}{l_{35}} = \frac{185}{94652}$
4	$v^5 = 0.67558$	${}_{4}\vert q_{35} = \frac{d_{39}}{l_{35}} = \frac{201}{94652}$

Therefore

$${}^{2}A^{1}_{35:\overline{5}|} = 0.92456\left(\frac{147}{94652}\right) + 0.85481\left(\frac{158}{94652}\right) + \cdots + 0.67558\left(\frac{201}{94652}\right) = 0.007153.$$

(f) This is the same as in (d) but evaluated at interest rate of $0.04^2 + 2 \times 0.04 = 0.0816$.

$$^2A_{35:\overline{20}|}^{\;\;\;1} = v^{20}{}_{20}p_{35} = (0.20829)\left(\frac{85916}{94652}\right) = 0.18906.$$

8. (a) $A_{35:\overline{25}|} = 0.38359$ (Yellow Tables, p. 100).

(b) We have from (a), $A_{35:\overline{25}|} = 0.38359$. But $A_{35:\overline{25}|} = A^1_{35:\overline{25}|} + A_{35:\overline{25}|}^{\;\;\;1}$. Now

$$A_{35:\overline{25}|}^{\;\;\;1} = v^{25}{}_{25}p_{35} = v^{25}\left(\frac{l_{60}}{l_{35}}\right) = 0.37512\left(\frac{9287.2164}{9894.4299}\right) = 0.35210.$$

Therefore $A^1_{35:\overline{25}|} = 0.38359 - 0.35210 = 0.03149$.

9. The E.P.V. is $1,000A_{30:\overline{30}|}^{\;\;\;1} = 1,000v^{30}{}_{30}p_{30}$.

Now

$$\begin{aligned}
{}_{30}p_{30} &= \exp\left(-\int_{30}^{60} \mu_s ds\right) = \exp\left(-\int_{30}^{40} \mu_s ds\right) \times \exp\left(-\int_{40}^{60} \mu_s ds\right) \\
&= \exp\left(-\int_{30}^{40} 0.001 ds\right) \times \exp\left(-\int_{40}^{60} 0.001 + 0.01(s-40) ds\right) \\
&= \exp(-0.01)\exp(-0.02 - [0.01(s^2/2 - 40s)]_{40}^{60}) = e^{-2.03} = 0.131336
\end{aligned}$$

(note: untypical low value for ${}_{30}p_{30}$) Therefore

$$1,000A_{30:\overline{30}|}^{\;\;\;1} = 1,000\left(\frac{1}{1.065}\right)^{30}(0.131336) = 19.86.$$