ACYLINDRICAL ACTIONS FOR TWO-DIMENSIONAL ARTIN GROUPS OF HYPERBOLIC TYPE

ALEXANDRE MARTIN†∗ AND PIOTR PRZYTYCKI‡∗

Abstract. For a two-dimensional Artin group $A$ whose associated Coxeter group is hyperbolic, we prove that the action of $A$ on the hyperbolic space obtained by coning off certain subcomplexes of its modified Deligne complex is acylindrical and universal. As a consequence, we obtain the Tits alternative for $A$, and we classify its subgroups that virtually split as a direct product. We also extend the classification of maximal virtually abelian subgroups to arbitrary two-dimensional Artin groups. A key ingredient in our approach is a simple criterion to show the acylindricity of an action on a two-dimensional $\mathrm{CAT}(-1)$ complex.

1. Introduction

Artin groups form a class of intensively studied groups generalising braid groups and closely related to Coxeter groups. Let us recall their definition. Let $S$ be a finite set and for all $s \neq t \in S$ let $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$. We encode this data in the defining graph with vertex set $S$ and edges between $s$ and $t$ with label $m_{st}$ whenever $m_{st} < \infty$. The associated Artin group $A_S$ is given by generators and relations:

$$A_S = \langle S \mid s t s \cdots = t s t \cdots \rangle.$$

The associated Coxeter group $W_S$ is obtained by adding the relations $s^2 = 1$ for every $s \in S$.

Coxeter groups are well understood in many respects. In particular, they are known to be $\mathrm{CAT}(0)$ [Mou88]. Artin groups on the other hand form a class of groups with a more mysterious structure and geometry, and many problems remain open in general: whether they are torsion-free, linear, or whether they satisfy the celebrated $K(\pi, 1)$ conjecture. (See [Cha07] for a survey of many open problems on Artin groups, as well as partial results.) While little is known about general Artin groups, they are expected to be as well-behaved as Coxeter groups. In particular, braid groups are conjectured to be $\mathrm{CAT}(0)$, which was verified in low dimensions [BM10, HKS16]. Recently, Artin groups of large-type (i.e. such that $m_{st} \geq 3$ for every $s \neq t \in S$) were shown to be systolic [HO17a], a simplicial analogue of $\mathrm{CAT}(0)$.

A unifying theme for many groups in geometric group theory has been to find interesting actions on hyperbolic spaces, and in particular acylindrical actions on hyperbolic spaces. Recall that an action of a group $G$ on a metric space $X$ is acylindrical [Bow08] if for every $r \geq 0$, there exist constants $L, N \geq 0$ such that for every $x, y \in X$ at distance at least $L$, there are at most $N$ elements $g \in G$...
such that $d(x, gx) \leq r$, $d(y, gy) \leq r$. The prime example of this phenomenon is the mapping class group of a closed hyperbolic surface acting acylindrically on its curve complex [MM99, Bow08]. Since then, many groups have been shown to admit acylindrical actions on hyperbolic spaces [Osi16], including large classes of Artin groups and related groups [CM16, CW17, CMW18]. While a group may have many acylindrical actions on hyperbolic spaces, there has been a lot of interest recently in understanding which groups, like mapping class groups, admit a ‘largest’ acylindrical action on a hyperbolic space. Let $G$ be a group acting on a hyperbolic space $X$. Recall that an element $g \in G$ is loxodromic for this action if for some (hence any) point $x \in X$, its orbit map $\mathbb{Z} \ni n \to g^n x \in X$ is a quasi-isometric embedding. An acylindrical action of a group $G$ on a hyperbolic metric space $X$ is called universal [Osi16] if the elements of $G$ that are loxodromic for this action are exactly the elements of $G$ that are loxodromic for some acylindrical action of $G$ on a hyperbolic space (such elements are called generalised loxodromic). Universal actions often offer much deeper insight into the structure of the underlying group, and are known to exist for instance for right-angled Artin groups [ABD17]. The goal of this article is to show the existence of such a universal acylindrical action for a large class of Artin groups, and to use the dynamics of the action to understand the structure of certain of its subgroups.

Statement of results. An Artin group is two-dimensional if for every $s, t, r \in S$, we have

$$\frac{1}{m_{st}} + \frac{1}{m_{tr}} + \frac{1}{m_{sr}} \leq 1.$$  

This class contains in particular all large-type Artin groups. We say that an Artin group is of hyperbolic type if the associated Coxeter group is hyperbolic. For two-dimensional Artin groups, by [Mou88] this is equivalent to requiring that for any $s, t, r \in S$, we have

$$\frac{1}{m_{st}} + \frac{1}{m_{tr}} + \frac{1}{m_{sr}} < 1.$$  

This class contains in particular all Artin groups of extra-large type, that is, with all $m_{st} \geq 4$.

An important complex associated to an Artin group is its modified Deligne complex (see Definition 3.6) introduced by Charney–Davis [CD95], and generalising a construction of Deligne for Artin groups of spherical type [Del72]. While the action of an Artin group on its modified Deligne complex is almost never acylindrical, we are able to construct an acylindrical action for a two-dimensional Artin group of hyperbolic type by coning off an appropriate family of subcomplexes of its modified Deligne complex called standard trees (see Definition 4.1). Our main result is the following:

**Theorem A.** The action of a two-dimensional Artin group $A_S$ of hyperbolic type on its coned off Deligne complex is acylindrical. If for each $s \in S$ there is $t \in S$ with $m_{st} < \infty$, then this action is universal.

This result is the first example of a universal acylindrical action for Artin groups that are not right-angled. The dynamics of such an action can be used to understand the structure of certain subgroups of these Artin groups, as we now explain.

**Tits alternative.** A group satisfies the Tits alternative if every finitely generated subgroup is either virtually soluble or contains a non-abelian free subgroup. This dichotomy has been shown for many groups of geometric interest, such as linear groups [Tit72], mapping class groups of hyperbolic surfaces [Iva84, McC85], outer automorphism groups of free groups [BFH00, BFH05], groups of birational transformations of surfaces [Can11], etc. A general heuristic is that a group that
is non-positively curved in a very broad sense should satisfy the Tits alternative. In particular, it is conjectured that all CAT(0) groups satisfy the Tits alternative. Several classes of CAT(0) groups have been shown to satisfy this alternative, in particular cocompactly cubulated groups [SW05], and groups acting geometrically on two-dimensional systolic complexes or buildings [OP19], but the problem remains open in general.

Following the heuristic that Artin groups should be non-positively curved in an appropriate sense, it is natural to ask whether Artin groups satisfy the Tits alternative, as already noted in [Bes99]. It should be mentioned that Coxeter groups are linear, which implies that they do satisfy the Tits alternative. So far, the following classes of Artin groups have been shown to satisfy the Tits alternative:

- Artin groups that can be cocompactly cubulated. An important class of such Artin groups is the class of right-angled Artin groups, namely Artin groups such that \( m_{st} = 2 \) or \( \infty \) for every \( s \neq t \in S \). Beyond them, a few classes of Artin groups have been shown to be cocompactly cubulated [HJP16, Hae17], but the conjectural picture states that the class of cocompactly cubulated Artin groups is extremely constrained [Hae17].
- Artin groups of finite type (also known as spherical Artin groups), i.e. Artin groups whose associated Coxeter group is finite, since they were shown to be linear [CW02].
- Artin groups acting geometrically on certain two-dimensional complexes, including Artin groups of extra-large type [OP19, Thm B] and Artin groups acting geometrically on a two-dimensional systolic complex [OP19, Thm A] provided by [BM00].

We obtain a strengthening of the Tits alternative for two-dimensional Artin groups of hyperbolic type.

**Corollary B (Tits alternative).** Let \( A_S \) be a two-dimensional Artin group of hyperbolic type. Then \( A_S \) satisfies the Tits alternative. More precisely, every subgroup of \( A_S \) that is not virtually cyclic either contains a nonabelian free group or is virtually \( \mathbb{Z}^2 \).

In a forthcoming article, we prove the Tits alternative for Artin groups of type FC, using different techniques [MP19].

**Virtual abelian and virtual product subgroups.** We complete Corollary B by obtaining a classification of the virtually \( \mathbb{Z}^2 \) subgroups of two-dimensional Artin groups. Before stating our result, let us first recall some of the known \( \mathbb{Z}^2 \) of Artin groups (see Section 2 for references).

A dihedral Artin group, that is, an Artin group on two generators, contains a finite index subgroup isomorphic to \( \mathbb{Z} \times F \), where \( F \) is a free group of rank \( \geq 1 \). In particular, dihedral parabolic subgroups of a given Artin group contain many \( \mathbb{Z}^2 \) subgroups.

Another source of \( \mathbb{Z}^2 \) subgroups are the central elements of dihedral parabolic subgroups. The centre of a dihedral Artin group on two generators \( s, t \) with \( m_{st} \geq 3 \) is infinite cyclic, generated by an element \( z_{st} \). In particular, for a general Artin group and \( s \in S \), \( s \) commutes with any element in the subgroup generated by \( z_{t_1 \ldots t_m} \), where \( t_1, \ldots, t_m \) denote the neighbours of \( s \) in the defining graph.

For two-dimensional Artin groups of hyperbolic type, our result states that these \( \mathbb{Z}^2 \) subgroups are close to being the only ones, and relies crucially on the dynamics of the action of these Artin groups on their coned off Deligne complex.

**Corollary C (Classification of virtually \( \mathbb{Z}^2 \) subgroups).** Let \( A_S \) be a two-dimensional Artin group of hyperbolic type, and let \( H \) be a subgroup that is virtually \( \mathbb{Z}^2 \). Up to conjugation, one of the following occurs:
H is contained in the stabiliser of a vertex of the modified Deligne complex (i.e. is conjugate to a subgroup of a dihedral parabolic subgroup), or

- $H$ is contained in the stabiliser of a standard tree of the modified Deligne complex. In particular, $H$ contains a conjugate of a non-trivial power of some $s \in S$. We refer to Remark 4.6 for an explicit description of these subgroups.

By contrast, for more general two-dimensional Artin groups where $A_S$ contains a Euclidean parabolic subgroup, there exist ‘exotic’ virtually $\mathbb{Z}^2$ subgroups coming from periodic flats of the modified Deligne complex. In order to complete the picture given by Corollary C and to highlight the particular behaviours of Artin groups of hyperbolic type, we give a complete classification of all virtually $\mathbb{Z}^2$ subgroups for general two-dimensional Artin groups. The classification of these exotic $\mathbb{Z}^2$ subgroups is however more technical and we refer to Proposition 6.2 for the precise statement.

In a similar direction, we also obtain a complete classification of the subgroups of $A_S$ that decompose as a non-trivial product in a two-dimensional Artin group of hyperbolic type.

**Corollary D (Classification of virtual products).** Let $A_S$ be a two-dimensional Artin group of hyperbolic type, and let $H$ be a subgroup that virtually splits as a (non-trivial) direct product. Then $H$ is virtually of the form $\mathbb{Z} \times F$, where $F$ is a free group.

**Strategy of the proof.** The key to all the theorems is to find a convenient hyperbolic space on which $A_S$ acts acylindrically. The first space we study is the modified Deligne complex $\Phi$ associated to $A_S$, where it was shown that the Mousong metric on $\Phi$ is CAT(0) for two-dimensional Artin groups [CD95]. In the case of two-dimensional Artin groups of hyperbolic type, this metric can be modified to be CAT$(-1)$. However, since dihedral Artin groups have non-trivial centres, the action on $\Phi$ is not acylindrical. More precisely, $\Phi$ contains unbounded trees with infinite pointwise stabilisers (see Definition 4.1). To circumvent this, we construct a new space $\Phi^*$ by coning off these trees, and we show that it is still possible to endow this new space $\Phi^*$ with an equivariant CAT$(-1)$ metric. More crucially, removing these obvious obstructions to acylindricity turns out to be enough, as we prove that the action of $A_S$ on the coned off Deligne complex $\Phi^*$ is acylindrical and universal.

Theorem A is proved by means of a general result on acylindrical actions on two-dimensional CAT$(-1)$ spaces. Recall that an action of a group $G$ on a metric space $X$ is weakly acylindrical [Mar15] if there exist constants $L, N \geq 0$ such that two points of $X$ at distance $\geq L$ are fixed by at most $N$ elements of $G$. Weak acylindricity is a dynamical condition that is weaker and much easier to deal with than acylindricity, especially for actions on non-locally compact spaces. Weak acylindricity was already known to be equivalent to acylindricity for actions on trees, and more generally for actions on finite-dimensional CAT(0) cube complexes [Gen16]. The following theorem, which is the central result of this article, is thus a powerful tool to study the dynamics of actions on two-dimensional CAT$(-1)$ spaces (the result extends in a straightforward manner to CAT$(\kappa)$ spaces, $\kappa < 0$).

**Theorem E.** Let $G$ be a group acting by simplicial isometries on a two-dimensional piecewise hyperbolic CAT$(-1)$ simplicial complex $Y$ with finitely many isometry types of simplices. If the action of $G$ on $Y$ is weakly acylindrical, then it is acylindrical.

**Organisation of the article.** In Section 2, we recall a few basic facts about dihedral Artin groups, whose properties are used to understand the links of vertices
of the modified Deligne complex $\Phi$ in a two-dimensional Artin group. In Section 3, we recall the definition of $\Phi$ and endow it with a particular CAT($-1$) metric. In Section 4, we introduce the standard trees, which are the main obstruction to the acylindricity of the action of $A_S$ on $\Phi$. We describe their geometry and use it to construct the coned off space $\Phi^*$, which we show to admit a CAT($-1$) metric. Section 5 is devoted to the proof of acylindricity Theorems E and A, and relies on a fine control of geodesics in a two-dimensional CAT($-1$) simplicial complex. With the acylindricity of the action of $A_S$ on $\Phi^*$, we are then able to prove Corollaries B–D. Finally, Section 6 classifies the virtually $\mathbb{Z}$ subgroups for general two-dimensional Artin groups, where the heart of the matter is to classify the virtually $\mathbb{Z}^2$ coming from periodic flats in $\Phi$.

2. Preliminaries on dihedral Artin groups

Let $S = \{s, t\}$ with $m = m_{st} < \infty$, and let us consider the dihedral Artin group $A_S$. In this section, we recall a few facts about $A_S$. The following is well known, see for example [HJP16, Lem 4.3(1)] for a proof.

**Lemma 2.1.** $A_S$ has a finite index subgroup isomorphic to $\mathbb{Z} \times F$, where $F$ is a free group (non-abelian if $m \geq 3$).

We denote $\Delta_{st} = \Delta \in A_S$.

**Lemma 2.2** ([Del72, Thm 4.21]). Let $m \geq 3$. The centre of $A_S$ is generated by $\Delta_{st}$ for $m$ even and by $\Delta_{st}^2$ for $m$ odd.

We therefore denote $z_{st} = \Delta_{st}$ for $m$ even (including the case $m = 2$) and $z_{st} = \Delta_{st}^2$ for $m$ odd.

**Lemma 2.3** ([Cri05, Lem 7(ii)]). Let $m \geq 3$. For any $k \neq 0$ the centralizer in $A_S$ of $s^k$ is the rank 2 abelian group generated by $s$ and $z_{st}$.

**Lemma 2.4** ([AS83, Lem 6]). Let $n < m$. A word with $2n$ syllables (i.e. of form $s^{i_1}t^{j_1} \cdots s^{i_n}t^{j_n}$ with all $i_k, j_k \in \mathbb{Z} - \{0\}$) is non-trivial in $A_S$.

The following result will be only used in Section 6, so we recommend to skip it at a first reading.

**Lemma 2.5.** Let $m \geq 3$. A word with $2m$ syllables is trivial in $A_S$ if and only if up to interchanging $s$ with $t$, and a cyclic permutation, it is of the form:

- $s^k t \cdots s t^{-k} s^{-1} \cdots t^{-1}$ for $m$ odd,
- $s^k t s \cdots t s^{-k} t^{-1} s^{-1} \cdots t^{-1}$ for $m$ even,

where $k \in \mathbb{Z} - \{0\}$.

**Proof.** The ‘if’ part follows immediately from Figure 1. We prove the ‘only if’ part by induction on the size of any reduced (van Kampen) diagram $M$ of the word $w$ in question, where we prove the stronger assertion that, up to interchanging $s$ with $t$, $M$ is as in Figure 1.
We use the vocabulary from [AS83], where the 2-cells of $M$ are called \textit{regions} and the \textit{interior degree} $i(D)$ of a region is the number of interior edges of $\partial D$ (after forgetting vertices of valence 2). For example the two extreme regions in Figure 1 have interior degree 1. A region $D$ is a \textit{simple boundary region} if $\partial D \cap \partial M$ is nonempty, and $M - \overline{D}$ is connected. For example, the two extreme regions in Figure 1 are simple boundary regions, but the remaining ones are not. A \textit{singleton strip} is a simple boundary region with $i(D) \leq 1$. A \textit{compound strip} is a subdiagram $R$ of $M$ consisting of regions $D_1, \ldots, D_n$, with $n \geq 2$, with $D_{k-1} \cap D_k$ a single interior edge of $R$ (after forgetting vertices of valence 2), satisfying $i(D_1) = i(D_n) = 2, i(D_k) = 3$ for $1 < k < n$ and $M - R$ connected.

Let $R$ be a strip of $M$ with boundary labelled by $rb$, where $r$ labels $\partial R \cap \partial M$ and so $w = ra'$. Assume also that $R$ shares no regions with some other strip (such a pair of strips exists by [AS83, Lem 2]). By [AS83, Lem 5], we have that the syllable lengths satisfy $||r|| \geq ||b|| + 2$ and so by Lemma 2.4 $||r|| \geq m + 1$, hence $||w'|| \leq m + 1$. In fact, since the outside boundary of the other strip has also syllable length $\geq m + 1$, we have $||r|| = m + 1$, and hence $||b|| = m - 1$. Let $M'$ be the diagram with boundary labelled by $b^{-1}w'$ obtained from $M$ by removing $R$. By the induction hypothesis, $M'$ is as in Figure 1. If $R$ is a singleton strip, then there is only one way of gluing $R$ to $M'$ to obtain $||w|| = 2m$ and it is as in Figure 1. If $R$ is a compound strip, then by the induction hypothesis $R$ is also as in Figure 1. Moreover, since all regions of $R$ share exactly one edge with $M'$, up to interchanging $s$ with $t$, and/or $b$ with $b^{-1}$, we have $b = s^k ts \cdots s^{-2}$ or $b = \cdots st^k$, where $k \geq 2$.

Since $m > 2$, we have that $b$ cannot be a subword of the boundary word of $M'$, unless $M'$ is a mirror copy of $R$, contradiction. 

\textbf{Remark 2.6.} Let $m = 2$. A word with 4 syllables is trivial in $A_2 = \mathbb{Z}^2$ if and only if up to interchanging $s$ with $t$ it is of the form $s^k l^s s^{-k} t^{-1}$, where $k, l \in \mathbb{Z} - \{0\}$.

\textbf{3. The modified Deligne complex and its geometry}

Let $A_2$ be a two-dimensional Artin group with $|S| = n$. Let $\Delta$ be an $(n - 1)$-simplex with faces labelled by $S$. In the barycentric subdivision of $\Delta$, a vertex is of type $T \subset S$, if it corresponds to a simplex that is the intersection of faces labelled by $s$, over $s \in T$. Let $K$ be the subcomplex of the barycentric subdivision of $\Delta$ spanned on the vertices of type $\emptyset$, types $\{s\}$ for all $s \in S$ and types $\{s, t\}$ for all $m_{st} < \infty$.

We give $K$ the following structure of a simple complex of groups $\mathcal{K}$ (see [BH99, §II.12] for background). The vertex groups are trivial, $\mathbb{Z} \simeq A_s$, or $A_{st}$, when the vertex is of type $\emptyset, \{s\}, \{s, t\}$, respectively. For an edge joining a vertex of type $\{s\}$ to a vertex of type $\{s, t\}$, its edge group is $A_s$; all other edge groups and all triangle
groups are trivial. All inclusion maps are the obvious ones. It follows directly from the definitions that $A_S$ is the fundamental group of $K$.

### 3.1. Modified Deligne complex

Assume now that $A_S$ is of hyperbolic type. We will equip $K$ with a CAT($-1$) metric, which is inspired by the construction of Moussong for Coxeter groups [Mou88, §13]. Let $\varepsilon > 0$ be small enough so that for all $m \geq 2$ we have $(\pi/2 + \varepsilon) + (\pi/2m) \leq \pi - \varepsilon$. For any $m \geq 2$, let $\theta(m, \varepsilon)$ be the third angle of any Euclidean triangle with angles $\pi/2 + \varepsilon, \pi/2m + \varepsilon$. In other words, $\theta(m, \varepsilon) = (m-1)\pi/2m - 2\varepsilon \geq \varepsilon$.

**Lemma 3.1.** There exists $\varepsilon$ such that for any cycle $(s_i)$ in the defining graph of any Artin group of hyperbolic type, for $m_i = m_{s_is_{i+1}}$, we have $\sum_i (\theta(m_i, \varepsilon) - \varepsilon) \geq \pi + \varepsilon$.

**Proof.** Step 1. There is $\varepsilon$ such that for each hyperbolic triangle group with exponents $(m, m', m'')$ we have $\theta(m, \varepsilon) + \theta(m', \varepsilon) + \theta(m'', \varepsilon) \geq \pi + 4\varepsilon$.

To this end, let $\varepsilon$ be such that $\frac{1}{m} + \frac{1}{m'} + \frac{1}{m''} \leq 1 - 24\varepsilon$ for $(2, 3, 7), (2, 4, 5)$ and $(3, 3, 4)$ triangle groups. Since the exponents $(m, m', m'')$ of any hyperbolic triangle group dominate the exponents of one of these three, the same inequality holds for all hyperbolic triangle groups. Consequently,

$$\theta(m, \varepsilon) + \theta(m', \varepsilon) + \theta(m'', \varepsilon) = \frac{(m-1)\pi}{2m} + \frac{(m'-1)\pi}{2m'} + \frac{(m''-1)\pi}{2m''} - 6\varepsilon = \frac{3\pi}{2} - \left( \frac{\pi}{2m} + \frac{\pi}{2m'} + \frac{\pi}{2m''} \right) - 6\varepsilon \geq \frac{3\pi}{2} - 12\varepsilon \geq \pi + 6\varepsilon.$$

**Step 2.** There is $\varepsilon$ such that for any 4-cycle we have $\sum_i (\theta(m_i, \varepsilon) - \varepsilon) \geq \pi + \varepsilon$.

Indeed, since $A_S$ is of hyperbolic type, at least one $m_i$ is $\geq 3$, and consequently $\sum \theta(m_i, \varepsilon) \geq 3\frac{\pi}{2} + \frac{\pi}{2} - 8\varepsilon$. Hence it suffices to take $(8 + 4 + 1)\varepsilon \leq \frac{\pi}{2}$.

**Step 3.** There is $\varepsilon$ such that for any $k$-cycle with $k \geq 5$ we have $\sum_i (\theta(m_i, \varepsilon) - \varepsilon) \geq \varepsilon$.

Indeed, we have $\sum_i \theta(m_i, \varepsilon) \geq 5(\frac{\pi}{2} - 2\varepsilon)$. Hence it suffices to take $(10 + 5 + 1)\varepsilon \leq \frac{\pi}{4}$. □

From now on, we fix any $\varepsilon$ satisfying Lemma 3.1.

**Lemma 3.2.** Given a finite set $M \subset \{2, 3, \ldots\}$, for sufficiently small $r > 0$ we have that for any $m \in M$ there exists a hyperbolic triangle $vv'v''$ with angles $\angle v = \frac{\pi}{2m} + \varepsilon, \angle v' = \frac{\pi}{2} + \varepsilon, \angle v'' = \theta(\varepsilon, m) - \varepsilon$, and $|v'v''| = r$.

**Proof.** We claim that for any value of $|v'v''| = r$ there is a hyperbolic triangle $vv'v''$ with $\angle v = \frac{\pi}{2m} + \varepsilon$ and $\angle v' = \frac{\pi}{2} + \varepsilon$. Indeed, fix $v', v''$ at distance $r$ and a half-line at angle $\frac{\pi}{2} + \varepsilon$ to $v'v''$ at $v'$. Varying $v$ along that half-line we can achieve any angle at $v$ between $0$ and $\frac{\pi}{2} - \varepsilon$. Since we assumed at the beginning of the section $\left( \frac{\pi}{2} + \varepsilon \right) + \left( \frac{\pi}{2m} + \varepsilon \right) \leq \pi - \varepsilon$, we have $\frac{\pi}{2m} + \varepsilon < \frac{\pi}{2} - \varepsilon$, justifying the claim.

It is also easy to see that for fixed $m \in M$ and $r \to 0$, we have $v \to v'$. Thus for sufficiently small $r$, the area of the triangle $vv'v''$ is arbitrarily small, hence by Gauss–Bonnet Theorem so is its defect and thus $\angle v'' \geq \theta(\varepsilon, m) - \varepsilon$. Since $M$ is finite, for sufficiently small $r$ this holds for all $m \in M$ simultaneously. □

To choose $r$ appropriately we need the following, the role of which will become clear in Section 4.

**Remark 3.3.** In a right-angled hyperbolic triangle with legs of lengths $1$ and $d$, for $d$ sufficiently small, the other angle at length $d$ leg is $\geq \frac{\pi}{2} - \varepsilon$. 


We fix arbitrary $r > 0$ satisfying Lemma 3.2 and with $d = |vw'|$ satisfying Remark 3.3, for all exponents $m = m_{st}$ of $A_S$. We now equip $K$ with a piecewise hyperbolic metric. Let $\tau$ be a triangle of $K$ with vertices $v, v', v''$ of types $\emptyset, \{s\}, \{s, t\}$ with $m = m_{st}$, respectively. We equip $\tau$ with the metric of the unique hyperbolic triangle from Lemma 3.2. Note that this choice is consistent on the edges, since two triangles sharing an edge either share a vertex of type $\{s, t\}$ and are thus congruent, or they share an edge with vertices of types $\emptyset, \{s\}$, which is of common length $r$.

**Lemma 3.4.** For each vertex $v$ of $K$, its link in the local development at $v$ of $K$ has girth $\geq 2\pi + \varepsilon$.

**Proof.** At $v$ of type $\emptyset$, its local development coincides with $K$. The link at $v$ coincides with barycentric subdivision of the defining graph with the length of the edge $(s, t)$ being $\geq 2(\theta(\varepsilon, m) - \varepsilon)$ (Lemma 3.2). Hence the lemma follows from Lemma 3.1.

At $v$ of type $\{s\}$, the link of the local development is a bipartite graph of edge length $\frac{\pi}{2} + \varepsilon$ and the lemma follows as well.

At $v$ of type $\{s, t\}$ with $m = m_{st}$, the link of the local development is the barycentric subdivision of the development $D$ of the edge of groups $Z + Z$ associated to the obvious morphism into $A_{st}$, with edge length $2(\frac{\pi}{m} + \varepsilon)$. Thus it suffices to show that $D$ has girth $2m$. This is exactly Lemma 2.4. □

By cite [BH99, Thm II.12.28] we obtain the following (which is also a consequence of [vdL83, Thm 4.13]).

**Corollary 3.5.** $K$ is strictly developable.

**Definition 3.6.** The development of $K$ is called the modified Deligne complex $[CD95]$ and is denoted $\Phi$. Note that it is a CAT($-1$) triangle complex with a cocompact action of $A_S$. Its vertex stabilisers are trivial or conjugates of $A_s$ and $A_{st}$, depending on their type, and its edge stabilisers are trivial or conjugates of $A_s$. Furthermore, $\Phi$ has finitely many isometry types of simplices and thus it is complete by [BH99, Thm I.7.19].

### 3.2. Non-hyperbolic case.

Here we drop the hypothesis that $A_S$ is of hyperbolic type. Let $K$ be the same complex of groups as before with the metric on each triangle being Euclidean with angles $\frac{\pi}{2}, \frac{\pi}{2m_{st}}, \frac{(m_{st}-1)\pi}{2m_{st}} = \theta(m_{st}, 0)$. Setting $\varepsilon = 0$, the same arguments as before give that the local developments of $K$ are CAT(0) and hence $K$ is strictly developable and its development $\Phi$ exists and is CAT(0). See [CD95] for detailed proof and the description of this Moussong metric in general.

### 4. Standard trees and the coned off space $\Phi^*$

Let $A_S$ be a two-dimensional Artin group, possibly not of a hyperbolic type. Let $\Phi_T \subset \Phi$ be the subcomplex that is the union of all the edges of $\Phi$ joining vertices of type $\{s, t\}$ and $\{s\}$ for all $s, t \in S$. Let $r \in S$ and let $T$ be the fixed-point set in $\Phi$ of $r$. Note that since $A_S$ acts on $\Phi$ without inversions, $T$ is a subcomplex of $\Phi$. Since the stabilisers of the simplices of $\Phi$ outside $\Phi_T$ are trivial, we have that $T \subset \Phi_T$. In particular $T$ is a graph. Since $\Phi$ with the Moussong metric is CAT(0), $T$ is convex and thus it is a tree.

**Definition 4.1.** A standard tree is the fixed-point set in $\Phi$ of a conjugate of a generator $r \in S$ of $A_S$.

From the convexity of $T$ we also have immediately:
Corollary 4.2. Let $T$ be a standard tree, and let $v$ be a vertex of $T$ incident to edges $e, e'$ of $T$. Then the combinatorial distance between their corresponding vertices in the link of $v$ is at least $2m_{st}$ for $v$ of type $\{s, t\}$, or exactly $2$ for $v$ of type $\{s\}$. Consequently, in the case where $A_S$ is of hyperbolic type, their distance in the angular metric induced from the piecewise hyperbolic metric is at least $\pi + 2m_{st} \varepsilon$ for $v$ of type $\{s, t\}$ or exactly $\pi + 2 \varepsilon$ for $v$ of type $\{s\}$.

The following lemma will allow us to describe the structure of the stabiliser of a standard tree. For a vertex $v$ of type $\{s, t\}$, with $v = gv_K$ for $g \in A_S$ and $v_K$ the unique vertex of type $\{s, t\}$ in $K$, we define $z_v = gz_v g^{-1}$. Note that $z_v$ does not depend on $g$, since if $g'v_K = v_K$, then $g^{-1} g' \in A_{st}$ and hence $g^{-1} g'$ commutes with $z_v$ implying $gz_v g^{-1} = g(g^{-1} g') z_v (g^{-1} g')^{-1} = g' z_v (g')^{-1}$.

Lemma 4.3. Let $v, v'$ be edges in $\Phi$ with a common vertex $v$ of type $\{s, t\}$. Then either $\text{Stab}(v) \cap \text{Stab}(v') = \{ \text{Id} \}$ or $\text{Stab}(v) = \text{Stab}(v')$. Moreover in the latter case, if $e, e'$ are of the same type, then there is $g \in \langle z_v \rangle$ with $e' = ge$, and if $e$ corresponds to the coset $A_s$ and $e'$ to the coset $h'A_t$, then $m_{st}$ is odd and there is $hA_t = h'A_t$ with $h \in \Delta_{st}(z_v)$.

Proof. Assume without loss of generality that $e$ corresponds to the trivial coset $A_s$. Then $\text{Stab}(v) = A_{st}$. Assume first that $e'$ corresponds to a coset $h'A_t$. Note that whenever we will establish $e' = ge$ for some $g \in \langle z_v \rangle$, we will have $\text{Stab}(v) = \text{Stab}(e')$. If $m = 2$, then $\langle z_v \rangle A_s = A_{st}$. Thus there is $g \in \langle z_v \rangle$ with $gA_s = h'A_s$, and so $e' = ge$, as desired. Suppose now $m \geq 3$. If $\text{Stab}(v) \cap \text{Stab}(e') \neq \{ \text{Id} \}$, then we have $h^s k (h')^{-1} e = e$ for some $k > 0$. This means that $h^t k (h')^{-1} \in A_s$, and using the homomorphism $A_{st} \to \mathbb{Z}$ mapping both generators to $1$ we obtain $h^s k (h')^{-1} = s^k$. By Lemma 2.3, there is $g \in \langle z_v \rangle$ with $h'A_t = gA_s$, as desired.

If $e'$ corresponds to $h'A_t$ and $\text{Stab}(v) \cap \text{Stab}(e') \neq \{ \text{Id} \}$, then we have $h^t k (h')^{-1} e = e$ for some $k > 0$. This means that $h^t k (h')^{-1} \in A_s$, and using the same homomorphism $A_{st} \to \mathbb{Z}$ we obtain $h^t k (h')^{-1} = s^k$. If $m_{st}$ is even, then $s$ and $t$ are not conjugate (use a homomorphism to $\mathbb{Z}$ killing $s$ but not $t$), contradiction. When $m_{st}$ is odd, we have $s \Delta_{st} = \Delta_{st} t$ and consequently $s^k = \Delta_{st} t^k \Delta_{st}^{-1}$, so that $(h')^{-1} \Delta_{st}$ commutes with $t^k$. By Lemma 2.3 there is $h$ with $hA_t = h'A_t$ and $h^{-1} \Delta_{st} \in \langle z_v \rangle$. Thus $\text{Stab}(e') = hA_t h^{-1} = A_s = \text{Stab}(e)$.

Remark 4.4. By Lemma 4.3, the stabilisers of all edges in a standard tree coincide. Consequently each edge of $\Phi_T$ belongs to exactly one standard tree and each vertex of type $\{s\}$ belongs to exactly one standard tree.

Lemma 4.5. The stabiliser of a standard tree is of the form $\mathbb{Z} \times F$ for some free group $F$.

Proof. Let $T$ be the standard tree that is the fixed-point set of $r \in S$. Any element $g \in \text{Stab}(T)$ conjugates $r$ to some $r^k$, and in fact we obtain $k = 1$ using the homomorphism $A_S \to \mathbb{Z}$ mapping all the generators to $1$. Hence $\mathbb{Z} = A_r$ is in the centre of $\text{Stab}(T)$. The quotient $F = \text{Stab}(T)/\mathbb{Z}$ acts on $T$ with trivial edge stabilisers. Thus $F$ is the fundamental group of the quotient graph of groups $T/F$, whose edge groups are trivial and whose vertex groups are $\mathbb{Z}$ by Lemma 4.3. Consequently $F$ is free. Taking any splitting $F \to \text{Stab}(T)$ gives us $\text{Stab}(T) = \mathbb{Z} \times F$.

Remark 4.6. We have the following explicit description of the stabiliser of the standard tree $T$ in Lemma 4.5. Let $\overline{T}$ be the graph obtained from $K_T = K \cap \Phi_T$ by cutting it along all the vertices of type $\{s, t\}$ with $m_{st}$ even. Vertices of $\overline{T}$ inherit types from the vertices of $K_T$. By Lemma 4.3, we have $T/F = \overline{T}$, with vertex groups $\mathbb{Z}$ at all the vertices of type $\{s, t\}$. Introduce an order $s_1, s_2, \ldots$ on
the elements of $S$. Label each directed edge $e$ of $T$ connecting the vertex of type $\{s_i\}$ to the vertex of type $\{s_i, s_j\}$ with $i < j$ and $m_{s_is_j}$ odd with the element $\phi(e) = \Delta_{s_is_j} \in A_S$. Label all other edges of $T$ by the trivial element. From Lemma 4.3 one can deduce that we can take $F$ freely generated by

- the words labelling a set of closed paths in $T$ based at the vertex of type $\{r\}$ forming a free basis of $\pi_1 T$, and
- the conjugates of $z_{s,t}$ by the words labelling some paths in $T$ joining the vertex of type $\{r\}$ with each of the vertices of type $\{s,t\}$.

**Definition 4.7.** Suppose now that $A_S$ is of hyperbolic type, and equip $\Phi$ with the CAT($-1$) metric of Section 3.1. Let $\Phi^*$ be the 2-complex obtained by coning off simplicially each of the standard trees. In $\Phi$ consider an edge of a standard tree with vertices $v, v'$ of type $\{s, t\}, \{s\}$ respectively and let $e$ be its cone vertex. We put on the triangle $cvv'$ the metric of a right-angled hyperbolic triangle with the right angle at $v'$, $|cv| = 1$ and the length $d = |vv'|$ (depending on $m_{st}$) as in $\Phi$ so that by Remark 3.3 the angle at $v$ is $\geq \frac{\pi}{2} - \varepsilon$.

Since $\Phi$ has finitely many isometry types of simplices, $\Phi^*$ also has finitely many isometry types of simplices. In particular $\Phi^*$ is complete by [BH99, Thm I.7.19].

**Proposition 4.8.** For $\varepsilon$ sufficiently small, $\Phi^*$ is CAT($-1$).

**Proof.** Since $\Phi$ is simply connected and standard trees are simply connected, $\Phi^*$ is simply connected. $\Phi^*$ is piecewise hyperbolic, so by [BH99, Thms II.4.1(2) and II.5.24] it suffices to show that the link of each vertex $v$ is of girth $\geq 2\pi + \varepsilon$. If $v$ is a cone point, its link is a tree and there is nothing to prove. The vertex links $L(v)$ in $\Phi$ are of girth $\geq 2\pi + \varepsilon$ by Lemma 3.4. Hence if $v$ is of type $\emptyset$, we are done as well.

If $v$ if of type $\{s\}$, and $\alpha$ is a cycle in its link not contained in $L(v)$, then $\alpha$ passes through a vertex corresponding to an edge joining $v$ to a cone point, which is unique by Remark 4.4. Hence $\alpha$ travels through two adjacent edges of length $\frac{\pi}{2}$, and through at least two edges in $L(v)$, which by Corollary 4.2 have length $\geq \frac{\pi}{2} + \varepsilon$, as desired.

If $v$ if of type $\{s,t\}$, and $\alpha$ is a cycle in its link not contained in $L(v)$, then analogously $\alpha$ passes through a vertex corresponding to an edge joining $v$ to a cone point. Hence $\alpha$ travels through two adjacent edges of length $\geq \frac{\pi}{2} - \varepsilon$. If the remaining part of $\alpha$ is contained in $L(v)$, then it suffices to use Corollary 4.2. Finally, by Remark 4.4, if $\alpha$ passes through exactly one other vertex (respectively, at least two other vertices) corresponding to a cone point, it is of length $\geq (2\pi - 4\varepsilon) + 2\left(\frac{\pi}{m_{st}} + 2\varepsilon\right)$ (respectively, $\geq 3\pi - 6\varepsilon$), which is $\geq 2\pi + \varepsilon$ for $\varepsilon$ small enough with respect to $m_{st}$. \(\square\)

**Convention 4.9.** From now on, we will fix a value of $\varepsilon$ such that the coned off space $\Phi^*$ is CAT($-1$).

5. **Dynamics of the action**

The goal of this section is to prove the acylindricity of the action of $G$ on $\Phi^*$ for two-dimensional Artin groups of hyperbolic type (Theorem A) and its Corollaries B–D. In order to do this, we prove the more general Theorem E whose proof we postpone to the next subsection.

5.1. **Weak acylindricity of the action of $A_S$ on $\Phi^*$**

**Lemma 5.1.** Let $Y$ be a piecewise hyperbolic simplicial complex with finitely many isometry types of simplices. Then for any non-empty subcomplexes $T, T' \subset Y$ there is a pair of points $(x, x') \in T \times T'$ realising the distance between $T$ and $T'$.
Note that if $Y$ is CAT($-1$) and $T, T'$ are convex and disjoint, then such $x, x'$ must be unique.

Proof. Let $d_0 = d(x_0, x'_0)$ for some $(x_0, x'_0) \in T \times T'$. By [BH99, Thm I.7.28] there is a constant $c$ such that for any pair of simplices $\sigma \times \sigma' \subset T \times T'$ with $d(\sigma, \sigma') \leq d_0$, the distance realising path from $\sigma$ to $\sigma'$ passes through at most $c$ simplices. Consequently, there are only finitely many possible $d(\sigma, \sigma')$ and thus the distance $d(T, T')$ is realised.$\square$

Proposition 5.2. The action of $A_S$ on $\Phi^*$ is weakly acylindrical.

Proof. Suppose $g \in A_S$ is a non-trivial element fixing points $y, y' \in \Phi^*$. To prove weak acylindricity it suffices to bound the distance $d(y, y')$ from above. Let $v$ (resp. $v'$) be a vertex of the simplex containing $y$ (resp. $y'$) in its interior. We will bound from above the combinatorial distance between $v$ and $v'$.

Since $A_S$ acts without inversions, $g$ fixes $v, v'$. We now define the following subcomplexes $T, T'$ of $\Phi$. If $v$ is a vertex of $\Phi$, set $T = v$. If $v$ is a cone vertex, set $T$ to be the standard tree corresponding to $v$. Define $T'$ analogously. We can assume that $T, T'$ are disjoint, since otherwise $v, v'$ are at combinatorial distance $\leq 2$ in $\Phi^*$, as desired. Let $\gamma$ be the geodesic in $\Phi$ between $x$ and $x'$ provided by Lemma 5.1 applied to $Y = \Phi$. Note that $g$ fixes $\gamma$.

If $\gamma$ is not contained in $\Phi_T$, then it has a point with trivial stabiliser, contradicting the assumption that $g$ is non-trivial. Assume now that $\gamma$ is contained in $\Phi_T$. Suppose that $\gamma$ has two consecutive edges $e, e'$ that belong to distinct standard trees. By Lemma 4.3, we have that $\text{Stab}(e) \cap \text{Stab}(e')$ is trivial, contradicting again the assumption that $g$ is non-trivial. In the remaining case, entire $\gamma$ is contained in one standard tree. Consequently $v, v'$ are at combinatorial distance $\leq 4$ in $\Phi^*$, as desired.$\square$

We say that an action is elliptic if it has bounded orbits. A non-cyclic group is acylindrically hyperbolic if it has an acylindrical action on a hyperbolic space that is not elliptic. (For equivalent definitions, see [Osi16, Thm 1.2].)

Remark 5.3. If $H \subset A_S$ acts elliptically on $\Phi^*$, then since $\Phi^*$ is CAT($-1$) and complete, by [BH99, Thm II.2.8(1)] $H$ fixes a point $v$ of $\Phi^*$. Since $H$ acts without inversions, we can take $v$ a vertex, which is a vertex of $\Phi$ or a cone vertex. Thus by Lemmas 2.1 and 4.5 $H$ has a finite index subgroup contained in $\mathbb{Z} \times F$ for a free group $F$.

Proof of Theorem A. Since the action of $A_S$ on $\Phi^*$ is weakly acylindrical by Proposition 5.2, it follows from Theorem E that the action is acylindrical.

Let us now show the universality of this action. By a theorem of Bridson [Bri99, Thm A], a simplicial isometry $g$ of a piecewise hyperbolic complex with finitely many isometry types of simplices is either loxodromic or elliptic. In particular, this applies to the action of each $g \in A_S$ on $\Phi^*$. Thus if $g$ is not loxodromic, by Remark 5.3 there is $k > 0$ with $g^k \in \mathbb{Z} \times F \subset A_S$ for a free group $F$. The group $F$ has rank $\geq 1$ since in the case where $g$ stabilises a standard tree that is a translate of the fixed-point set of $s \in S$, we assumed that there is $t \in S$ with $m_{st} < \infty$. Thus $g^k$ generates an infinite cyclic subgroup with infinite index in its centraliser. It follows from [Osi16, Cor 6.9] that $g$ cannot be generalised loxodromic.$\square$

Note that the condition that for each $s \in S$ there is $t \in S$ with $m_{st} < \infty$ is necessary for the action to be universal. Indeed, otherwise $A_S = A_s * A_{S-s}$ and $\Phi^*$ is a tree of spaces with vertex spaces of two types. The first type are the coned off modified Deligne complexes for $A_{S-s}$. The second type are edges joining the fixed points of the conjugates of $s$ with their cone points. These vertex spaces are joined.
by edges with vertices of type \{s\} and \emptyset. If we replace the vertex spaces of the second type with real lines containing a \(Z\)’s worth of vertices of type \{s\}, the action stays acylindrical but \(s\) becomes loxodromic. Note also that after performing these replacements for all free factors \(A_s\) of \(A_S\) we obtain a universal acylindrical action.

**Proof of Corollary B.** Let \(H\) be a subgroup of \(A_S\) that is not virtually cyclic. Since the action of \(A_S\) on the hyperbolic space \(\Phi^*\) is acylindrical by Theorem A, \(H\) is elliptic or acylindrically hyperbolic. If \(H\) is elliptic, then by Remark 5.3 \(H\) is virtually contained in \(Z \times F\) for a free group \(F\), and thus it is virtually \(Z^2\) or contains a non-abelian free group. If \(H\) is acylindrically hyperbolic, then it contains a non-abelian free group by [DGO17, Thm 6.14].

**Proof of Corollary C.** Let \(H\) be a subgroup of \(A_S\) that is virtually \(Z^2\). Since \(H\) is not acylindrically hyperbolic by [Osi16, Cor 7.2(b)], it is elliptic by Theorem A. By Remark 5.3, \(H\) stabilises a vertex of \(\Phi\) or a cone vertex of \(\Phi^*\), and hence a standard tree of \(\Phi\). In the latter case by Lemma 4.5 and Remark 4.6, \(H\) is contained in a \(Z \times F\) with \(F\) a free group, and \(Z\) conjugate to some \(A_s\), as desired.

**Proof of Corollary D.** Let \(H\) be a subgroup of \(A_S\) that is virtually a non-trivial direct product. Since \(A_S\) is torsion-free, by [Osi16, Cor 7.2(b)] \(H\) is not acylindrically hyperbolic, and thus it is elliptic by Theorem A. By Remark 5.3, \(H\) is virtually of form \(Z \times F\), as desired.

5.2. **The general acylindricity theorem.** We now turn to the proof of Theorem E. In this section, \(Y\) denotes a two-dimensional piecewise hyperbolic simplicial complex, with finitely many isometry types of simplices, and that is CAT\((-1)\).

**Simplifications.** We first explain how to alter the metric \(d\) on \(Y\) so that girths of vertex links become uniformly greater than \(2\pi\). Replace every hyperbolic triangle of \(Y\) with side lengths \(a, b, c\) by a hyperbolic triangle of side lengths \(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\) respectively, and call this new metric \(d_2\). With respect to \(d_2\) all the angles are strictly larger than with respect to \(d\). Since \((Y, d_2)\) still has finitely many isometry types of simplices, the girths of vertex links are now uniformly greater than \(2\pi\). Therefore, without loss of generality we assume from now on that the same was true with respect to \(d\). Note that in the case where \(Y\) is the coned off space \(\Phi^*\), it follows from the proof of Proposition 4.8 that vertex links already satisfy this condition to start with.

Secondly, we explain how to subdivide the complex so that all the triangles become acute. Namely, any piecewise euclidean two-dimensional simplicial complex with finitely many isometry types of simplices admits an equivariant subdivision with all the triangles acute [BZ60]. While the piecewise hyperbolic counterpart of that result does not seem to appear in the literature, we can make use of the piecewise euclidean statement in the following way. For any \(\epsilon\) there is \(n\) such that for all triangles of \(Y\) with side lengths \(a, b, c\) there are \(C^1\) maps from the hyperbolic triangles with side lengths \(\frac{a}{n}, \frac{b}{n}, \frac{c}{n}\) to euclidean triangles with the same side lengths that are isometries on the sides and with first differentials at distance \(\leq \epsilon\) from isometries. Consider an acute triangulation \(\Delta\) of the piecewise euclidean simplicial complex obtained from \(Y\) by replacing each hyperbolic triangle by the Euclidean triangle with the same side lengths. Pulling back the vertices of dilated \(\Delta\) via the \(C^1\) maps yields an acute triangulation of \((Y, d_n)\) (defined analogously to \(d_2\)) for sufficiently large \(n\). Therefore without loss of generality we assume that all the triangles of \((Y, d)\) are acute. In particular, stars are convex, and the union of two triangles sharing an edge is convex.
Notation. Let \( v \) be a vertex of \( Y \). We denote by \( \pi_v : Y - \{v\} \to \text{lk}(v) \) the map assigning to each \( y \in Y - \{v\} \) the direction of the geodesic from \( v \) to \( y \). The angular distance between two points \( x, x' \in \text{lk}(v) \) will be denoted \( \angle_{\pi_v}(x, x') \). For \( y, y' \in Y - \{v\} \) we extend this notation so that \( \angle_{\pi_v}(y, y') = \angle_{\pi_v}(\pi_v(y), \pi_v(y')) \). For \( k > 0 \), we define the metric \( k \)-neighbourhood of a subset \( Y' \) of \( Y \) as the set
\[
\mathcal{N}_k(Y') = \{ y \in Y \mid d(y, Y') < k \}.
\]
For a point \( y \in Y \), by \( \tau_y \) we denote the simplex of \( Y \) containing \( y \) in its interior. For a simplex \( \tau \) of \( Y \), its open star \( st(\tau) \) is the union of the interiors of the simplices containing \( \tau \).

Definition 5.4. We fix a constant \( \alpha > 0 \) such that:
- the link of every vertex of \( Y \) has girth \( \geq 2\pi + 4\alpha \),
- every triangle of \( Y \) has all angles \( \geq 12\alpha \).

Choosing an open cover. A key tool in controlling geodesics of \( Y \) will be to subdivide them in pieces that are easier to understand locally. This will be done by means of an appropriate cover of \( Y \), which will take us some time to define. The cover will consist of an open set \( U_\tau \) for each simplex \( \tau \) of \( Y \). We will also define a constant \( \epsilon \) (distinct from \( \epsilon \) in Convention 4.9) that depends only on \( Y \). Their main properties will be:

\((*)\): Each \( \mathcal{N}_\epsilon(U_\tau) \) is contained in \( st(\tau) \).

\((\cap)\): If \( U_\tau \) intersects \( U_{\tau'} \), then \( \tau \) contains \( \tau' \) or vice versa.

To start with, since \( Y \) has only finitely many isometry types of simplices, we can fix a constant \( \epsilon_0 > 0 \) such that the balls
\[
U_v = \mathcal{N}_{6\epsilon_0}(v)
\]
around the vertices \( v \) of \( Y \) satisfy \( \mathcal{N}_{4\epsilon_0}(U_v) \subset st(v) \) and property \((\cap)\) for \( \tau, \tau' \) vertices. Before we proceed with the construction of the remaining elements of the cover, we need the following.

Definition 5.5. We fix a constant \( 0 < \epsilon \leq \epsilon_0 \) such that for each vertex \( v \) of \( Y \) and a pair of points \( x, y \in Y - \mathcal{N}_{\epsilon_0}(v) \) with \( d(x, y) \leq \epsilon \), we have
\[
\angle_{\pi_v}(x, y) \leq \alpha.
\]

Now, for an edge \( e = vw \) of \( Y \), we define
\[
U_e = \mathcal{N}_{4\epsilon}(e - U_v \cup U_w).
\]
Note that for any point \( x \in U_e \), we have \( d(v, x) \geq d(v, Y - U_v) - 4\epsilon \geq 6\epsilon_0 - 4\epsilon \geq 2\epsilon_0 \). Then by Definitions 5.4 and 5.5, for any edge \( f = vu \) with \( u \neq w \), we have \( \angle_{\pi_v}(x, u) \geq 12\alpha - 4\alpha > 0 \). Consequently, \( U_e \) is disjoint from \( f \), and so we have property \((*)\) for \( \tau \) an edge. Similarly we have property \((\cap)\) for \( \tau, \tau' \) edges. Property \((\cap)\) for \( \tau \) an edge and \( \tau' \) a vertex follows from \( \mathcal{N}_{4\epsilon_0}(U_v) \subset st(v) \).

Note also that for points \( v', w' \in e \) at distance \( 6\epsilon_0 - 2\epsilon \) from \( v, w \), we have \( \mathcal{N}_{2\epsilon}(v'v') \subset U_v \) and \( \mathcal{N}_{2\epsilon}(v'w') \subset U_w \), and consequently \( \mathcal{N}_{2\epsilon}(e) \subset U_v \cup U_e \cup U_w \). This is why when for \( \sigma \) a triangle of \( Y \), we define
\[
U_\sigma = \mathcal{N}_\epsilon(\sigma - \bigcup_{\tau \subset \partial \sigma} U_\tau),
\]
we have property \((*)\) and consequently property \((\cap)\) for \( \tau \) a triangle and \( \tau' \) arbitrary. Furthermore, again by Definitions 5.4 and 5.5 we have the following:

Corollary 5.6. Let \( v \) be a vertex of \( Y \), and let \( \tau, \tau' \neq v \) be simplices of \( Y \) containing \( v \) and such that neither of them is contained in the other. Then for every \( x \in U_\tau \) and \( y \in U_{\tau'} \), we have
\[
\angle_{\pi_v}(x, y) \geq 4\alpha.
\]
Galleries and extended galleries.

**Definition 5.7** (Gallery). Let \( \gamma \) be a geodesic segment in \( Y \). We denote by \( \text{Gal}(\gamma) \) the minimal subcomplex of \( Y \) that contains \( \gamma \), and we call it the *gallery* of \( \gamma \). It is the union of all the simplices \( \tau_y \) over \( y \in \gamma \).

We now want to slightly enlarge \( \text{Gal}(\gamma) \).

**Definition 5.8** (Extended gallery). Let \( \gamma \) be a geodesic segment in \( Y \), let \( \text{Gal}(\gamma) \) be its gallery, and let \( V(\gamma) \) be the (possibly empty) set of vertices of \( Y \) contained in \( \gamma \) and which are not an endpoint of \( \gamma \). For each \( v \in V(\gamma) \), we perform the following construction.

Since the geodesic \( \gamma \) passes through \( v \), \( \gamma \) defines two points \( x_1, x_2 \) in the link \( \text{lk}(v) \). Since vertex links have girth \( \geq 2\pi + 4\alpha \), there exists at most one geodesic \( \ell_v \) of \( \text{lk}(v) \) (for the angular metric) of length \( < \pi + 2\alpha \) between \( x_1 \) and \( x_2 \). If no such geodesic \( \ell_v \) exists, we set \( E_v = \emptyset \). Otherwise, let \( E_v \) be the set of edges of the minimal subgraph of \( \text{lk}(v) \) containing \( \ell_v \). Each edge \( e \in E_v \) corresponds to a triangle \( \sigma_e \) of \( Y \) containing \( v \). We set

\[
\text{Gal}^*(\gamma) = \text{Gal}(\gamma) \cup \bigcup_{v \in V(\gamma)} \bigcup_{e \in E_v} \sigma_e,
\]

which we call the *extended gallery* of \( \gamma \).

**Remark 5.9.** By [BH99, Thm I.7.28], and since angles of triangles in \( Y \) are bounded from below, there is a constant \( C \) such that for each geodesic segment \( \gamma \) of length \( |\gamma| \), the extended gallery \( \text{Gal}^*(\gamma) \) contains at most \( C|\gamma| \) vertices.

Appropriate subdivision of a geodesic.

**Definition 5.10** (Decomposition). Let \( \gamma = [x, y] \) be a geodesic in \( Y \), oriented from \( x \) to \( y \). By property \((\cap)\), we can choose a shortest sequence of simplices \( (\Sigma_i)_{i=0}^{k+1} \) such that there are points \( x_i \in U_{\Sigma_i} \) lying on \( \gamma \) in that order, with \( x_0 = x, x_{k+1} = y \) and \( \sigma_i = \Sigma_{i-1} \cap \Sigma_i \neq \emptyset \). We call the data of all \( \Sigma_i \) and \( x_i \) (and \( \sigma_i, \gamma_i \) determined by them) a decomposition of \( \gamma \).

**Remark 5.11.** If in each pair \( \Sigma_i, \Sigma_{i+1} \) none of the simplices is contained in the other, we say that the decomposition is *anchored*. Note that by the minimality of \( k \), \( \Sigma_i \) cannot be contained in \( \Sigma_{i+1} \) unless \( i = 0 \), and \( \Sigma_i \) cannot contain \( \Sigma_{i+1} \) unless \( i = k \). Thus the geodesic \([x_1, x_k]\) has an anchored decomposition obtained by discarding \( \Sigma_0, \Sigma_{k+1} \).

**Technical lemma.** The following lemma is the key technical result allowing us to control the simplices met by a geodesic sufficiently close to another one. We advise the reader to skip its proof during a first reading.

**Lemma 5.12.** Let \( \gamma = [x, y], \gamma' = [x', y'] \) be two geodesics in \( Y \) with \( d(x, y), d(x', y') < \epsilon \). Suppose that \( \gamma \) has an anchored decomposition with \( k \geq 0 \). Then

\[
\text{Gal}(\gamma') - \tau_{x'} \cup \tau_{y'} \subset \text{Gal}^*(\gamma).
\]

We first prove a local version of Lemma 5.12:

**Lemma 5.13.** Lemma 5.12 holds under the additional assumption that the anchored decomposition of \( \gamma \) has \( k = 0 \).

**Proof.** Let \( \Sigma_0, \Sigma_1 \) be the simplices of the anchored decomposition of \( \gamma \). First notice that if \( \sigma_1 \) is an edge, then \( \Sigma_0, \Sigma_1 \) are triangles and so by property \((\ast)\) we have \( \tau_{x'} \cup \tau_{y'} = \Sigma_0 \cup \Sigma_1 \), which is is convex. Thus we have \( \gamma' \subset \tau_{x'} \cup \tau_{y'} \) and consequently \( \text{Gal}(\gamma') - \tau_{x'} \cup \tau_{y'} = \emptyset \) so the lemma follows. We can thus assume that \( \sigma_1 \) is a vertex
We decompose Proof of Lemma 5.12.

Case 1. \( \mathcal{L}_v(x', y') \geq \pi \).

Then \( \gamma' \) passes through \( v \). Thus \( \text{Gal}(\gamma') - \tau_{x'} \cup \tau_{y'} = \emptyset \) and the lemma follows.

Case 2. \( \mathcal{L}_v(x', y') < \pi \).

Since the decomposition was anchored, Corollary 5.6 implies \( \mathcal{L}_v(x, y) \geq 4\epsilon \). Moreover, Definition 5.5 and the fact that \( \mathcal{L}_v(x, y) < \pi + 2\epsilon \).

Since \( \text{lk}(v) \) has girth \( \geq 2\pi + \alpha \), the convex hull of the points \( \pi_v(x), \pi_v(y), \pi_v(x'), \pi_v(y') \) in \( \text{lk}(v) \) is a tree indicated in Figure 2. Since \( \text{set}(v) \) is convex and contains \( x', y' \) by property \((*)\), we have that \( \text{Gal}(\gamma') \) consists of the triangles \( \sigma_e \) with \( e \) in the geodesic \( [\pi_v(x'), \pi_v(y')] \) in \( \text{lk}(v) \). Since triangle angles are \( > \alpha \), we have that among these \( \sigma_e \) only \( \tau_{x'}, \tau_{y'} \) might not lie in \( \text{Gal}^*(\gamma) \), as desired. \( \square \)

![Figure 2. Convex tree in the link of v.](image-url)

**Proof of Lemma 5.12.** We decompose \( \gamma' \) as a concatenation \( \gamma' = \gamma'_1 \cup \cdots \cup \gamma'_k \) with \( \gamma'_i = [x'_{i-1}, x'_i] \) for a sequence of points \( x'_0 = x', x'_1, \ldots, x'_{k+1} = y' \in \gamma' \) such that \( d(x_i, x'_i) < \epsilon \) for all \( i \). By Lemma 5.13, for each \( 1 \leq i \leq k + 1 \), we have \( \text{Gal}(\gamma'_i) - \tau_{x'_{i-1}} \cup \tau_{x'_i} \subseteq \text{Gal}^*(\gamma_i) \). To conclude that \( \text{Gal}(\gamma') - \tau_{x'_0} \cup \tau_{x'_{k+1}} \subseteq \text{Gal}^*(\gamma) \), it suffices to prove the following.

**Claim.** For every \( 1 \leq i \leq k \), we have \( \tau_{x'_i} \subseteq \text{Gal}^*(\gamma_i) \cup \text{Gal}^*(\gamma_{i+1}) \).

To justify the claim, let us assume by contradiction that for some \( 1 \leq i \leq k \), the simplex \( \tau_{x'_i} \) is neither contained in \( \text{Gal}^*(\gamma_i) \) nor in \( \text{Gal}^*(\gamma_{i+1}) \). Then in particular \( \tau_{x'_i} \neq \tau_{x_i} \), so by property \((*)\) \( \Sigma_i \) is not a triangle and thus it is an edge. Furthermore, \( \tau_{x_i} \) contains \( \Sigma_i, \) so \( \Sigma_i \subseteq \text{Gal}^*(\gamma_i) \), and thus \( x'_i \notin \Sigma_i \). For simplicity, let us denote the vertices \( \sigma_i, \sigma_{i+1} \) of \( \Sigma_i \) by \( v_i \) and \( v_{i+1} \) respectively. We will prove that each of \( \gamma'_i, \gamma'_{i+1} \) intersects \( \Sigma_i \), which contradicts the convexity of \( \Sigma_i \) and \( x'_i \notin \Sigma_i \). To show, say, \( \gamma'_i \cap \Sigma_i \neq \emptyset \), we consider the following cases.

**Case 1.** \( \mathcal{L}_{v_i}(x'_{i-1}, x'_i) \geq \pi \).

Then \( \gamma'_i \) passes through \( v_i \in \Sigma_i \), as required.

**Case 2.** \( \mathcal{L}_{v_i}(x'_{i-1}, x'_i) < \pi \).
Since the decomposition was anchored, Corollary 5.6 implies \( \angle_{v_0}(x_{i-1}, x_i) \geq 4\alpha \).
Moreover, as before Definition 5.5 implies \( \angle_{v_0}(x_{i-1}, x_{i-1}') \leq \alpha \) and \( \angle_{v_0}(x_i, x_i') \leq \alpha \).
In particular, \( x_{v_0}(x_{i-1}, x_i) < \pi + 2\alpha \). Since \( lk(v_i) \) has girth \( \geq 2\pi + 4\alpha \), the convex hull of the points \( \pi_{v_0}(x_{i-1}), \pi_{v_0}(x_i), \pi_{v_0}(x_{i-1}'), \pi_{v_0}(x_i') \) in \( lk(v_i) \) is a tree indicated in Figure 3. Note that since \( \tau_{x_i'} \) is not contained in \( \text{Gal}^*(\gamma_i) \), the point \( \pi_{v_0}(x_i) \) does not lie on the geodesic \( [\pi_{v_0}(x_{i-1}), \pi_{v_0}(x_i)] \) in \( lk(v_i) \). Since \( x_i \in U_{\Sigma_i} \), the branching point indicated in the figure must be \( \pi_{v_0}(v_{i+1}) \) (all other branching points are at distance \( \geq 12\alpha - 4\alpha \) from \( \pi_{v_0}(x_i) \) by Definition 5.5), which justifies \( \gamma_i' \cap \Sigma_i \neq \emptyset \).

\[
\begin{array}{ccc}
\pi_{v_0}(x_{i-1}) & \pi_{v_0}(x_{i-1}') & \pi_{v_0}(x_i') \\
\tau_{x_i'} & \ell_i & \pi_{v_0}(x_i) \\
\end{array}
\]

**Figure 3**

\[
\square
\]

**Proof of Theorem E.** We are finally ready to prove Theorem E. We will use the following immediate consequence of [Kap09, Lem 3.10], convexity of the distance function, and the triangle inequality.

**Lemma 5.14.** For each \( r \geq \epsilon > 0 \) there is \( l > 0 \) satisfying the following. Let \( \gamma = [x, y] \) be a geodesic in a CAT\((-1)\) space \( Y \), and let \( g \) be an isometry of \( Y \) with \( d(x, gx) \leq r \) and \( d(y, gy) \leq r \). For each subsegment \( \gamma' \) of \( \gamma \) with endpoints at distance \( \geq l \) from \( x \) and \( y \), we have that the \( 2r \)-neighbourhood \( \gamma'_+ \) of \( \gamma' \) in \( \gamma \) satisfies \( g\gamma' \subset N_r(\gamma'_+) \).

**Proof of Theorem E.** Consider \( r \geq \epsilon \) as in Definition 5.5, and let \( L_0, N_0 > 0 \) be such that two points of \( Y \) at distance at least \( L_0 \) are stabilised by at most \( N_0 \) elements of \( G \). Let \( l = l(r, \epsilon) \) be as in Lemma 5.14. Since \( Y \) has only finitely many isometry types of simplices, there is an upper bound \( B \) on the length of a geodesic contained in the star of a vertex of \( Y \) [BH99, Lem I.7.23 and Thm I.7.28]. Let \( L = L_0 + 4B + 2l \). For a point \( x \in Y \), we define the \( r \)-stabiliser of \( x \) as

\[\text{Stab}_r(x) = \{ g \in G \mid d(x, gx) \leq r \}\]

To prove acylindrical hyperbolicity, consider \( x, y \in Y \) with \( d(x, y) \geq L \). We will bound the size of \( \text{Stab}_r(x) \cap \text{Stab}_r(y) \).

Let \( \gamma \) be the geodesic between \( x \) and \( y \). By Lemma 5.14, there is a subsegment \( \gamma' = [x', y'] \subset \gamma \) of length \( L_0 + 4B \) such that for all \( g \in \text{Stab}_r(x) \cap \text{Stab}_r(y) \) we have \( d(gx', x'_g), d(gy', y'_g) < \epsilon \) for some \( x'_g, y'_g \in \gamma'_+ \). By Remark 5.11, each \( [x'_g, y'_g] \) has a subsegment with an anchored decomposition obtained by removing from \( [x'_g, y'_g] \) a subsegment of length at most \( B \) at each of \( x'_g, y'_g \). Consequently, by Lemma 5.12 there is a subsegment \( \gamma'' = [x''_g, y''_g] \subset [x', y'] \), obtained by removing from \( \gamma' \) a subsegment of length at most \( B \) at each of \( x', y' \), with \( g^0(\gamma'' - \tau_{x''_g} \cup \tau_{y''_g}) \subset \text{Gal}^*(\gamma'_+) \). Thus the subsegment \( \gamma''' \) of \( \gamma' \) of length \( L_0 \) and centred at the midpoint of \( \gamma' \) satisfies \( g\gamma''' \subset \text{Gal}^*(\gamma'_+) \). By Remark 5.9, there is a constant \( C \) such that \( \text{Gal}^*(\gamma'_+) \) contains at most \( C^0 = (L_0 + 4B + 4r)C \) vertices. Consequently \( \text{Stab}_r(x) \cap \text{Stab}_r(y) \subset N_0C^0 \), since otherwise there would be \( g_0, g_1, \ldots, g_{N_0} \) with each \( g_0^{-1}g_i \) fixing \( \text{Gal}(\gamma''') \). \( \square \)
6. Abelian subgroups in general two-dimensional Artin groups

6.1. Abelian subgroups containing elliptic elements. In this section, we prove the following:

**Theorem 6.1.** Let $A_S$ be a two-dimensional Artin group, and let $H$ be a subgroup of $A_S$ that is virtually $\mathbb{Z}^2$. Up to conjugation, one of the following occurs:

(i) $H$ is contained in the stabiliser of a vertex of $\Phi$, or
(ii) $H$ is contained in the stabiliser of a standard tree of $\Phi$, or
(iii) $H$ acts properly on a Euclidean plane isometrically embedded in $\Phi$.

In Proposition 6.2 we will list all $H \cong \mathbb{Z}^2$ satisfying (iii).

**Proof.** Let $\Gamma \subset H$ be a finite index normal subgroup isomorphic to $\mathbb{Z}^2$. By [Bri99], $\Gamma$ acts on $\Phi$ by semi-simple isometries. Let $\text{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Min}(\gamma)$, where $\text{Min}(\gamma)$ is the Minset of $\gamma$ in $\Phi$. By a variant of the Flat Torus Theorem not requiring properness [BH99, Thm II.7.20(1)], $\text{Min}(\Gamma)$ is nonempty. By [BH99, Thm II.7.20(4)] we have that $H$ stabilises $\text{Min}(\Gamma)$.

Suppose first that each element of $\Gamma$ fixes a point of $\Phi$. Then $\Gamma$ acts trivially on $\text{Min}(\Gamma)$. By the fixed-point theorem [BH99, Thm II.2.8(1)] the finite group $H/\Gamma$ fixes a point of $\text{Min}(\Gamma)$, and since the action is without inversions, we can take this point to be a vertex as required in (i).

Secondly, suppose that $\Gamma$ has both an element $\gamma$ that fixes a point of $\Phi$ and an element that is loxodromic. Then $\text{Min}(\Gamma)$ is not a single point. Since $\text{Min}(\Gamma) \subset \text{Fix}(\gamma)$, we have that $\gamma$ is a conjugate of an element of $S$ and thus $\text{Min}(\Gamma)$ is contained in a standard tree $T$. For any $h \in H$ we have that the intersection $h(T) \cap T$ contains $\text{Min}(\Gamma)$ which is not a single point and thus by Remark 4.4 we have $h \in \text{Stab}(T)$, as required in (ii).

Finally, suppose that all elements of $\Gamma$ are loxodromic. By [BH99, Thm II.7.20(1,4)] we have $\text{Min}(\Gamma) = Y \times \mathbb{R}^a$ with $H$ preserving the product structure and $\Gamma$ acting trivially on $Y$. As before $H/\Gamma$ fixes a point of $Y$ and so $H$ stabilises $\mathbb{R}^a$ isometrically embedded in $\Phi$. By [BH99, Thm II.7.20(2)], we have $n \leq 2$, but since $H$ acts by simplicial isometries, we have $n = 2$ and the action is proper, as required in (iii).

6.2. Purely loxodromic abelian subgroups. In this section we finish classification of $\mathbb{Z}^2$ subgroups of two-dimensional Artin groups. We use the following notation. Let $S$ be an alphabet. If $s \in S$, then $s^* \in S^*$ denotes the language (i.e. set of words) of form $s^n$ for $n \in \mathbb{Z} - \{0\}$. We treat a letter $s \in S^*$ as a language consisting of a single word. If $\mathcal{L}, \mathcal{L}'$ are languages, then $\mathcal{L}\mathcal{L}'$ denotes the language of words of the form $ww'$ where $w \in \mathcal{L}, w' \in \mathcal{L}'$. If $\mathcal{L}$ is a language, then $\mathcal{L}^*$ denotes the union of the languages $\mathcal{L}^n$ for $n \geq 1$.

**Proposition 6.2.** Suppose $\mathbb{Z}^2 \subset A_S$ acts properly on a Euclidean plane isometrically embedded in $\Phi$. Then $\mathbb{Z}^2$ is conjugate into one of the following, where $s,t,r \in S$.

(a) $\langle w, w' \rangle$, where $w \in A_T, w' \in A_{T'}$ and $m_{tt'} = 2$ for all $t \in T, t' \in T'$.
(b) $\langle s^{str} t^{str} \rangle$, where $w \in (s^{str})^* \land m_{st} = m_{st'} = m_{sr} = 3$.
(c) $\langle s^{str} t^{str} \rangle$, where $w \in (s^{t^{-1} s^{*}} t^{*})^* \land m_{st} = m_{st'} = 4, m_{sr} = 2$.
(d) $\langle s^{str} t^{str} \rangle$, where $w \in (s^{str})^* \land m_{st} = m_{st'} = 4, m_{sr} = 2$.
(e) $\langle s^{str} t^{str} r^{str} \rangle$, where $w \in (s^{str})^* \land m_{st} = 6, m_{st'} = 3, m_{sr} = 2$.
(f) $\langle s^{str} t^{str} \rangle$, where $w \in (s^{* t^{str}} r^{*})^* \land m_{st} = 6, m_{st'} = 3, m_{sr} = 2$.

It is easy to check directly that the above groups are indeed abelian. Since $A_S$ is torsion-free, the only other subgroups of $A_S$ that are virtually $\mathbb{Z}^2$ are isomorphic
to the fundamental group of the Klein bottle. They can be also classified, see Remark 6.16.

In the proof of Proposition 6.2 we will describe in detail the Euclidean planes in \( \Phi \) stabilised by \( \mathbb{Z}^2 \subset A_5 \). Huang and Osajda established properties of arbitrary quasiflats in the Cayley complex of \( A_5 \), and one can find similarities between our results and [HO17b, §5.1-5.2 and Prop 8.3].

6.2.1. Polarisation. We equip \( \Phi \) with the CAT(0) Moussong metric described in Subsection 3.2.

**Definition 6.3.** Let \( F \) be a Euclidean plane isometrically embedded in \( \Phi \). Then for each vertex \( v \) in \( F \) of type \( \{s, t\} \) there are exactly \( 4m_{st} \) triangles in \( F \) incident to \( v \). We assemble them into regular \( 2m_{st} \)-gons, and call this complex the tiling of \( F \). We say that a cell of this tiling has type \( T \) if its barycentre in \( \Phi \) has type \( T \).

For a Coxeter group \( W \), let \( \Sigma \) denote its Davis complex, i.e. the complex obtained from the standard Cayley graph by adding \( k \)-cells corresponding to cosets of finite \( (T) \) for \( T \subset S \) of size \( k \). For example for \( W \) the triangle Coxeter group with exponents \( \{3, 3, 3\} \), the complex \( \Sigma \) is the tiling of the Euclidean plane by regular hexagons.

**Lemma 6.4.** Let \( F \) be a Euclidean plane isometrically embedded in \( \Phi \). Then the tiling of \( F \) is either the standard square tiling, or the one of the Davis complex \( \Sigma \) for \( W \) the triangle Coxeter group with exponents \( \{3, 3, 3\}, \{2, 4, 4\} \) or \( \{2, 3, 6\} \).

**Proof.** The 2-cells of the tiling are regular polygons with even numbers of sides, hence their angles lie in \([\pi/2, \pi] \). If there is a vertex \( v \) of \( F \) incident to four 2-cells, then all these 2-cells are squares. Consequently, any vertex of \( F \) adjacent to \( v \) is incident to at least two squares, and thus to exactly four squares. Then, since the 1-skeleton of \( F \) is connected, the tiling of \( F \) is the standard square tiling.

If \( v \) is incident to three 2-cells, which are \( 2m, 2m', 2m'' \)-gons, then since \( \frac{1}{m} + \frac{1}{m'} + \frac{1}{m''} = 1 \), we have \( \{m, m', m''\} = \{3, 3, 3\}, \{2, 4, 4\} \) or \( \{2, 3, 6\} \). Moreover, a vertex \( u \) of \( F \) adjacent to \( v \) is incident to two of these three 2-cells, and this implies that the third 2-cell incident to \( u \) has the same size as the one incident to \( v \). This determines uniquely the tiling of \( F \) as the one of \( \Sigma \). \( \square \)

Henceforth, Let \( \Sigma \) be the Davis complex for \( W \) the triangle Coxeter group with exponents \( \{3, 3, 3\}, \{2, 4, 4\} \) or \( \{2, 3, 6\} \).

**Lemma 6.5.** Suppose \( \Sigma \) is the tiling of a Euclidean plane isometrically embedded in \( \Phi \). Then the natural action of \( W \) on \( \Sigma \) preserves the edge types coming from \( \Phi \).

In particular, \( W = W_T \) for some \( T \subset S \) with \( |T| = 3 \).

**Proof.** Chose a vertex \( v \) of \( \Sigma \), and let \( \{s\}, \{t\}, \{r\} \) be the types of edges incident to \( v \). Let \( u \) be a vertex of \( \Sigma \) adjacent to \( v \), say along an edge \( e \) of type \( \{r\} \). Hence the 2-cells in \( \Sigma \) incident to \( e \) have types \( \{s, r\} \) and \( \{t, r\} \). Consequently, the types of remaining two edges incident to \( u \) are also \( \{s\} \) and \( \{t\} \), and in such a way that the reflection of \( \Sigma \) interchanging the endpoints of \( e \) preserves the types of these edges. This determines uniquely the types of the edges of \( \Sigma \), and guarantees that they are preserved by \( W \). \( \square \)

**Definition 6.6.** A polarisation of \( \Sigma \) is a choice of a longest diagonal \( l(\sigma) \) in each 2-cell \( \sigma \) of \( \Sigma \). A polarisation is admissible if every vertex of \( \Sigma \) belongs to exactly one \( l(\sigma) \).
Definition 6.7. Suppose $\mathbb{Z}^2 \subset A_S$ acts properly and cocompactly on $\Sigma \subset \Phi$. For an edge $e$ of type $\{s\}$ in $\Sigma$, its vertices correspond to elements $g, gs^k \in A_S$ for $k > 0$. We direct $e$ from $g$ to $gs^k$. By Lemma 2.5, the boundary of each 2-cell $\sigma$ is subdivided into two directed paths joining two opposite vertices. The induced polarisation of $\Sigma$ assigns to each $\sigma$ the longest diagonal $l(\sigma)$ joining these two vertices.

Lemma 6.8. An induced polarisation is admissible.

Proof. Step 1. For each vertex $v$ of $\Sigma$, there is at most one $l(\sigma)$ containing $v$.

Indeed, suppose that we have $v \in l(\sigma), l(\tau)$. Without loss of generality suppose that the edge $e = \sigma \cap \tau$ is directed from $v$. Then the other two edges incident to $v$ are also directed from $v$. We will now prove by induction on the distance from $v$ that each edge of $\Sigma$ is oriented from its vertex closer to $v$ to its vertex farther from $v$ in the 1-skeleton $\Sigma^1$ (they cannot be at equal distance since $\Sigma^1$ is bipartite).

For the induction step, suppose we have already proved the induction hypothesis for all edges closer to $v$ than an edge $uu'$, where $u'$ is closer to $v$ than $u$. Let $u''$ be the first vertex on a geodesic from $u'$ to $v$ in $\Sigma^1$. Let $\sigma$ be the 2-cell containing the path $uu''$. By [Ron09, Thms 2.10 and 2.16], $\sigma$ has two opposite vertices $u_0$ closest to $v$ and $u_{\text{max}}$ farthest from $v$. By the induction hypothesis, the edge $u''u''$ is oriented from $u''$ to $u'$, and both edges of $\sigma$ incident to $u_0$ are oriented from $u_0$. Thus if the edge $uu''$ was oriented to $u'$ we would have that $u'$ is opposite to $u_0$, so $u'' = u_{\text{max}}$, contradiction. This finishes the induction step.

As a consequence, $v$ is the unique vertex of $\Sigma$ with all edges incident to $v$ oriented from $v$. This contradicts the cocompactness of the action of $\mathbb{Z}^2$ on $\Sigma$ and proves Step 1.

Step 2. For each $v$ there is at least one $l(\sigma)$ containing $v$.

Among the edges incident to $v$ there are at least two edges directed from $v$ or at least two edges directed to $v$. The 2-cell $\sigma$ containing such two edges satisfies $l(\sigma) \ni v$. □

6.2.2. Classification. We are now ready to classify $\mathbb{Z}^2$-subgroups of $A_S$.

Proposition 6.9. Let $\Sigma$ be the Davis complex for $W_T$ the triangle Coxeter group with exponents $\{3,3,3\}$ with an admissible polarisation $l$. Then there is an edge $e$ such that each hexagon $\gamma$ of $\Sigma$ satisfies

$\clubsuit$: the diagonal $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$.

Note that if the conclusion of Proposition 6.9 holds, then the translation $\rho$ mapping one hexagon containing $e$ to the other preserves $l$.

Remark 6.10. It is easy to prove the converse, i.e. that if each $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$, and if $l$ is $\rho$-invariant, then $l$ is admissible. This can be used to classify all admissible polarisations, but we will not need it.

To prove Proposition 6.9 we need the following reduction.

Lemma 6.11. Let $e$ be an edge and $\rho$ a translation mapping one hexagon containing $e$ to the other. If $\clubsuit$ holds for all hexagons $\gamma$ in one $\rho$-orbit, then it holds for all $\gamma$.

Proof. Suppose that $\clubsuit$ holds for all hexagons $\gamma$ in the $\rho$-orbit of a hexagon $\sigma$. Let $\tau$ be a hexagon adjacent to two of them, say to $\sigma$ and $\rho(\sigma)$. Let $v = \sigma \cap \rho(\sigma) \cap \tau$. Since $\clubsuit$ holds for $\gamma = \sigma$ and $\gamma = \rho(\sigma)$, by the admissibility of $l$, $v$ belongs to one of $l(\sigma), l(\rho(\sigma))$. Thus $v \notin l(\tau)$ and hence $\clubsuit$ holds for $\gamma = \tau$. Proceeding inductively, by the connectivity of $\Sigma$, we obtain $\clubsuit$ for all $\gamma$. □
Proof of Proposition 6.9. Case 1. There are adjacent hexagons \( \sigma, \tau \) with non-parallel \( l(\sigma), l(\tau) \).

Let \( f = \sigma \cap \tau \). Without loss of generality \( l(\sigma) \cap f = \emptyset, l(\tau) \cap f \neq \emptyset \). Let \( v \) be the vertex of \( f \) outside \( l(\tau) \). By the admissibility of \( l \), \( v \) is contained in \( l(\sigma') \) for the third hexagon \( \sigma' \) incident to \( v \). Hence \( \spadesuit \) holds for \( e = \sigma \cap \sigma' \) and \( \gamma = \sigma, \sigma', \tau \) (see Figure 4).

Let \( \rho \) be the translation mapping \( \sigma \) to \( \sigma' \). Replacing the pair \( \sigma, \tau \) with \( \tau, \sigma' \) and repeating inductively the argument shows that \( \spadesuit \) holds for \( \gamma = \rho^n(\sigma), \rho^n(\tau) \) for all \( n > 0 \) (note that \( e \) gets replaced by parallel edges in this procedure).

![Figure 4](image_url)

Furthermore, by the admissibility of \( l \), since \( l(\rho^{-1}(\tau)) \) is disjoint from \( l(\sigma) \) and \( l(\tau) \), it leaves us only one choice for \( l(\rho^{-1}(\tau)) \), and it satisfies \( \spadesuit \) for \( \gamma = \rho^{-n}(\sigma), \rho^{-n}(\tau) \) for all \( n > 0 \). It remains to apply Lemma 6.11.

Case 2. All the \( l(\sigma) \) are parallel.

In this case it suffices to take any edge \( e \) intersecting some \( l(\sigma) \).

Proposition 6.12. Let \( \Sigma \) be the Davis complex for \( W_T \) the triangle Coxeter group with exponents \( \{2, 4, 4\} \) with an admissible polarisation \( l \). Then there is an edge \( e \) such that each octagon \( \gamma \) of \( \Sigma \) satisfies

\[ \diamondsuit: \text{the diagonal } l(\gamma) \text{ has endpoints on edges of } \gamma \text{ parallel to } e. \]

An edge \( e \) of \( \Sigma \) lies either in two octagons \( \sigma, \sigma' \) or there is a square with two parallel edges \( e, e' \) in octagons \( \sigma, \sigma' \). The translation of \( \Sigma \) mapping \( \sigma \) to \( \sigma' \) is called an \( e \)-translation. Note that if the conclusion of Proposition 6.12 holds, then an \( e \)-translation preserves \( l \).

Lemma 6.13. Let \( e \) be an edge and \( \rho \) an \( e \)-translation. If \( \diamondsuit \) holds for all octagons \( \gamma \) in one \( \rho \)-orbit, then it holds for all \( \gamma \).

Proof. Suppose that \( \diamondsuit \) holds for all octagons \( \gamma \) in the \( \rho \)-orbit of an octagon \( \sigma \). We can assume \( e \subset \sigma \). Suppose first that \( e \) lies in another octagon \( \sigma' \). Then let \( \tau \) be an octagon outside the \( \rho \)-orbit of \( \sigma \) adjacent to some \( \rho^k(\sigma) \), say \( \sigma \). Let \( \Box, \rho(\Box) \) be the two squares adjacent to both \( \sigma \) and \( \tau \) (see Figure 5). By the admissibility of \( l \), we have that \( l(\Box), l(\rho(\Box)) \) contain the two vertices of \( \sigma \cap \tau \). Consequently, \( l(\tau) \) intersects the edge \( \tau \cap \rho(\tau) \), and so \( \gamma = \tau \) satisfies \( \diamondsuit \). It is easy to extend this to all the octagons \( \gamma \).
It remains to consider the case where there is a square with two parallel edges $e, e'$ in octagons $\sigma, \sigma'$. Let $\tau$ be an octagon adjacent to two of them, say to $\sigma$ and $\rho(\sigma)$. Let $v = e \cap \tau, x = e' \cap \tau$. Since $\diamond$ holds for $\gamma = \sigma, \sigma'$, each of $v, x$ lies in one of $l(\sigma), l(\square), l(\sigma')$. Thus by the admissibility of $l$, we have $v, x \notin l(\tau)$. Any of the two remaining choices for $l(\tau)$ satisfy $\diamond$ for $\gamma = \tau$. It is again easy to extend this to all the octagons $\gamma$.

Proof of Proposition 6.12. Note that we fall in one of the following two cases.

**Case 1.** There is an edge $f$ in octagons $\sigma, \tau$ with $l(\sigma) \cap f = \emptyset, l(\tau) \cap f \neq \emptyset$.

Let $v$ be the vertex of $f$ distinct from $u = l(\tau) \cap f$. By the admissibility of $l$, the vertex $v$ is contained in $l(\square)$ for the square $\square$ incident to $v$. Let $x$ be the vertex in $\tau \cap \square$ distinct from $v$, and let $\sigma'$ be the octagon incident to $x$ distinct from $\tau$. By the admissibility of $l$, the vertex $x$ is contained in $l(\sigma')$. Hence $\diamond$ holds for $e = \sigma \cap \square$ and $\gamma = \sigma', \tau$ (see Figure 6). Let $\rho$ be the translation mapping $\sigma$ to $\sigma'$.

Now let $\square$ be the square incident to $u$, and let $z$ be the vertex in $\sigma \cap \square$ distinct from $u$. Note that by the admissibility of $l$, we have $z \in l(\square)$, and consequently $\rho^{-1}(x) \in l(\sigma)$ and $\rho^{-1}(v) \in l(\rho^{-1}(\square))$. Hence $l(\rho^{-1}(\tau))$ cannot contain neither $z$, nor $\rho^{-1}(v)$, nor $\rho^{-1}(x) \in l(\sigma)$. There is only one remaining choice for $l(\rho^{-1}(\tau))$, and it satisfies $\diamond$ for $\gamma = \rho^{-1}(\tau)$.

We can now argue exactly as in the proof of Proposition 6.9 that that $\diamond$ holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n \in \mathbb{Z}$. It then remains to apply Lemma 6.13.

**Case 2.** For each edge $e$ in octagons $\sigma, \sigma'$ with $l(\sigma) \cap e \neq \emptyset$ we have $l(\sigma') \cap e \neq \emptyset$.

Let $\sigma$ be any octagon and $e$ an edge contained in another octagon $\sigma'$ and intersecting $l(\sigma)$. Let $\rho$ be the translation mapping $\sigma$ to $\sigma'$. One can show inductively
that ◊ holds for octagons $\gamma = \rho^n(\sigma)$ for all $n \in \mathbb{Z}$. It then remains to apply Lemma 6.13.

Note that for $l$ satisfying ◊ for all octagons $\gamma$, the values of $l$ on octagons determine its values on squares.

**Proposition 6.14.** Let $\Sigma$ be the Davis complex for $W_T$ the triangle Coxeter group with exponents $\{2,3,6\}$ with an admissible polarisation $l$. Then there is an edge $e$ such that such that each 12-gon $\gamma$ of $\Sigma$ satisfies

◊: the diagonal $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$.

Let $e$ be an edge. An $e$-translation is the translation of $\Sigma$ mapping $\sigma$ to $\sigma'$ in of the two following configurations. In the first configuration we have a square with two parallel edges $e,e'$ in 12-gons $\sigma,\sigma'$. In the second configuration we have four parallel edges $e,e',e'',e'''$ such that $e',e''$ lie in a square, $e,e'$ in a hexagon $\phi$ and $e'',e'''$ in another hexagon, and we consider 12-gons $\sigma \supset e,\sigma' \supset e''$. Again, if the conclusion of Proposition 6.14 holds, then an $e$-translation preserves $l$. To see this in the configuration with hexagons it suffices to observe that $l(\phi)$ (and similarly for the other hexagon) is not parallel to $e$: otherwise $l(\phi)$ would intersect $l(\square)$ for $\square$ the square containing $e \cap l(\sigma)$.

**Lemma 6.15.** Let $e$ be an edge and $\rho$ an $e$-translation. If ◊ holds for all 12-gons $\gamma$ in one $\rho$-orbit, then it holds for all $\gamma$.

The proof is easy, it goes along the same lines as the proofs of Lemmas 6.11 and 6.13 and we omit it.

**Proof of Proposition 6.14.** We adopt the convention that if we label the vertices of an edge in a 12-gon $\sigma$ by $v_0v_1$, then all the other vertices of $\sigma$ get cyclically labelled by $v_2 \cdots v_{11}$.

Let $\tau$ be a 12-gon and suppose that $l(\tau)$ contains a vertex $v_1$ of an edge $v_0v_1 \subset \tau$ for a square $\square = v_0v_1v_2v_0$. Let $\sigma$ be the 12-gon containing $u_0u_1$. Then $l(\square) = u_1u_0$ and furthermore $l$ assigns to the hexagon and square containing $u_1u_2, u_2u_3$, respectively, the longest diagonal containing $u_2, u_3$, respectively. Thus the only three remaining options for $l(\sigma)$ are the diagonals $u_0u_6, u_5u_11$, and $u_4u_{10}$. Thus we fall in one of the following three cases.

**Case 0.** There is such a $\tau$ with $l(\sigma) = u_4u_{10}$.

Let $\phi$ be the hexagon containing $u_3u_4$. Then $l(\phi)$ is parallel to $u_3u_4$ and there is no admissible choice for $l$ in the square containing $u_4u_5$ (see Figure 7). This is a contradiction.

![Figure 7](image-url)
**Case 1.** There is such a $\tau$ with $l(\sigma) = u_5u_{11}$.

It is easy to see that $l$ agrees with Figure 8 on the hexagon containing $v_0v_{11}$ and the square containing $v_{11}v_{10}$. Thus the only 2-cell $\phi$ with $l(\phi)$ containing $v_{10}$ may be (and is) the hexagon containing $v_{10}v_9$. Consequently the only 2-cell $\square$ with $l(\square)$ containing $v_9$ may be (and is) the square containing $v_9v_8$. Denote by $\sigma'$ the 12-gon adjacent to both $\phi$ and $\square$ at the vertex $x \neq v_9$. Then $x$ may lie (and lies) only in $l(\sigma')$. Denote by $\rho$ the translation mapping $\sigma$ to $\sigma'$.

![Figure 8](image)

It is also easy to see that the 2-cells surrounding $\sigma$ and $\rho^{-1}(\tau)$ depicted in Figure 9 have $l(\cdot)$ as indicated. This leaves only two choices for $l(\rho^{-1}(\tau))$, where one of them leads to Case 0, and the other satisfies $\heartsuit$.

![Figure 9](image)

Replacing repeatedly $\sigma$ and $\tau$ in the above argument by $\tau$ and $\sigma'$ or by $\rho^{-1}(\tau)$ and $\sigma$ gives that $\heartsuit$ holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n \in \mathbb{Z}$. It remains to apply Lemma 6.15.

**Case 2.** There is no such $\tau$ as in Case 0 or 1.

It is then easy to see that $e = v_0v_1$ and $\rho$ mapping $\sigma$ to $\tau$ satisfy the hypothesis of Lemma 6.15.

Note that for $l$ satisfying $\heartsuit$ for all 12-gons $\gamma$, the values of $l$ on 12-gons determine its values on squares and hexagons.

We are now ready for the following.
Proof of Proposition 6.2. Let \( F \) be a Euclidean plane isometrically embedded in \( \Phi \) with a proper (and thus cocompact) action of \( \mathbb{Z}^2 \). By Lemma 6.4, the tiling of \( F \) is either the standard square tiling, or the one of the Davis complex \( \Sigma \) for \( W \) the triangle Coxeter group with exponents \( \{3,3,3\}, \{2,4,4\} \) or \( \{2,3,6\} \).

First consider the case where the tiling of \( F \) is the standard square tiling. We can partition the set of edges into two classes horizontal and vertical of parallel edges. Let \( T \) be the set of types of horizontal edges and \( T' \) be the set of types of vertical edges. Since for each square the type of its two horizontal (respectively, vertical) edges is the same, we have \( m_{tt'} = 2 \) for all \( t \in T, t' \in T' \). Moreover, by Remark 2.6, if one of the edges is of form \( g, gt^k \), then the other is of form \( h, ht^k \).

Thus, up to a conjugation, the stabiliser of \( F \) in \( A_S \) is generated by a horizontal translation \( w \in A_T \) and a vertical translation \( w' \in A_T' \). This brings us to Case (a) in Proposition 6.2.

It remains to consider the case where the tiling of \( F \) is the one of \( \Sigma \). Consider its induced polarisation \( l \) from Definition 6.7. By Lemma 6.8, \( l \) is admissible. By Propositions 6.9, 6.12, and 6.14, there is an edge \( e \) such that each for each \( \gamma \) a maximal size 2-cell, the diagonal \( l(\gamma) \) has endpoints on edges of \( \gamma \) parallel to \( e \), and there is a particular translation \( p \) in the direction perpendicular to \( e \) preserving \( l \).

For an edge \( f \) of type \( \{s\} \) in \( \Sigma \), its vertices are of form \( g, gs^k \) for \( k > 0 \), directed from \( g \) to \( gs^k \). If \( k > 1 \), then we call \( f \) \( k \)-long. By Lemma 2.5 and Remark 2.6, if \( f \) is \( k \)-long, then so is its opposite edge in both of the 2-cells that contain \( f \). Consequently all the edges crossing the bisector of \( f \) are \( k \)-long. Moreover, all such bisectors are parallel, since otherwise the 2-cell \( \square \) where they crossed would have four long edges, so \( \square \) would be a square by Lemma 2.5. Analysing \( l \) in the 2-cells adjacent to \( \square \) leaves then no admissible choice for \( l(\square) \).

Furthermore, by Lemmas 2.5, 6.11, 6.13 and 6.15, if \( f \) is a long edge, then we can assume that \( f \) is parallel to \( e \).

Suppose first that \( W_T \) is the triangle Coxeter group with exponents \( \{3,3,3\} \) and \( T = \{s,t,r\} \). Let \( \omega \) be a combinatorial axis for the action of \( \rho \) on \( \Sigma \). Since none of the edges of \( \omega \) are parallel to \( e \), by the definition of the induced polarisation we see that they are all directed consistently (see Figure 10).

Thus, up to replacing \( F \) by its translate and interchanging \( t \) with \( r \), the element \( strtr \) preserves \( \omega \), and coincides on it with \( \rho^3 \). In fact, \( \rho^3 \) not only preserves the types of edges, but also by Lemma 2.5 their direction and \( k \)-longness. Thus \( strtr \) preserves \( F \). The second generator of the type preserving translation group of \( \Sigma \) in \( W_T \) is \( tstr \). Note that the path representing it in \( \Sigma \subset \Phi \) corresponds to a word in \( t^*str \subset A_T \). That word depends on whether the second edge of the path is long and on the polarisation. Since \( \mathbb{Z}^2 \) acts cocompactly, there is a power of \( tsstr \) such that its corresponding path in \( \Sigma \subset \Phi \) reads off a word in \( (t^*str)^* \subset A_T \) that is the other generator of the orientation preserving stabiliser of \( F \) in \( A_S \). This brings us to Case (b) in Proposition 6.2. One similarly obtains the characterisations of orientation preserving stabilisers of \( F \) for the two other \( W \) (see Figures 11 and 12).
Figure 11

Figure 12
Remark 6.16. Analysing the full stabilisers of $F$ in $A_S$ one can easily classify also the subgroups of $A_S$ acting properly on $\Phi$ isomorphic to the fundamental group of the Klein bottle. For example, suppose that the second generator of the $\mathbb{Z}^2$ in Case (b) of Proposition 6.2 has the form $g = (t^k s t^{-k}) s t$. Then our $\mathbb{Z}^2$ is generated by $s t s r$ and $g' = g(s t s r)^{-1} = t^k s r^{-k} s^{-1}$. Note that $s t$ normalises our $\mathbb{Z}^2$ with $(s t)^{-1} g(s t) = (g')^{-1}$. Thus $(s t, g')$ is isomorphic to the fundamental group of the Klein bottle. We do not include a full classification, since it is not particularly illuminating.

REFERENCES


ACYLINDRICAL ACTIONS FOR TWO-DIMENSIONAL ARTIN GROUPS


DEPARTMENT OF MATHEMATICS AND THE MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, HERIOT-WATT UNIVERSITY, RECCARTON, EH14 4AS EDINBURGH, UNITED KINGDOM
E-mail address: alexandre.martin@hw.ac.uk

DEPARTMENT OF MATHEMATICS AND STATISTICS, McGILL UNIVERSITY, BURNSIDE HALL, 805 SHERBROOKE STREET WEST, MONTREAL, QC, H3A 0B9, CANADA
E-mail address: piotr.przytycki@mcgill.ca