A combination theorem for cubulation in small cancellation theory over free products

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Abstract

We prove that a group obtained as a quotient of the free product of finitely many cubulable groups by a finite set of relators satisfying the classical $C'(1/6)$–small cancellation condition is cubulable. This yields a new large class of relatively hyperbolic groups that can be cubulated, and constitutes the first instance of a cubulability theorem for relatively hyperbolic groups which does not require any geometric assumption on the peripheral subgroups besides their cubulability. We do this by constructing appropriate wallspace structures for such groups, by combining walls of the free factors with walls coming from the universal cover of an associated 2-complex of groups.

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1 Introduction

The geometry of non-positively curved cube complexes has attracted a lot of attention recently due to the spectacular progress in several related problems, most notably the solution to Thurston's remaining four questions on the structure of 3-dimensional manifolds, including the virtual Haken conjecture of Waldhausen [Ago13]. An important problem in this circle of ideas is to show that virtually special cubical complexes are *stable* under various geometric operations. A main geometric task is to *combine* the various wallspaces at hand to construct a wallspace structure for the group under study.

1.1 Combination problems

A general combination problem for wallspaces can be formulated as follows:

Combination Problem. *Let G be a group acting on a polyhedral complex X endowed with a wallspace structure, such that each non-trivial face stabiliser admits a wallspace structure.*

- *Under which conditions can we combine such structures into a wallspace structure for G?*

- *If each stabiliser is cubulable, under which conditions can we ensure that G is cubulable?*

This problem has been extensively studied to combine CAT(0) cube complexes under strong *(relative) hyperbolicity conditions on the group*:

Amalgams and HNN extensions. Haglund-Wise [HW12a] and [HW12b] prove that virtual specialness of groups is preserved under certain amalgamated products or HNN extensions. In a such setting, G is acting cocompactly on a tree X with vertex stabilisers that are CAT(0) cubulable. Theorem 1.2 of [HW12a] requires the vertex stabilisers and the whole groups to be *Gromov hyperbolic*, while Theorem A of [HW12b] requires that the group G is hyperbolic *relative to virtually abelian subgroups*.

Cubical small cancellation theory. The malnormal virtually special quotient theorem [Wis11] deals with cubulable *hyperbolic* groups and proves the cubulability of appropriate *hyperbolic* quotients, under strong *cubical* small cancellation conditions [Wis11, Def. 5.1].

Relatively hyperbolic groups. Hruska–Wise [HW14] prove that, for a group G that is hyperbolic relative to a finite set of parabolic subgroups (P_i), the G-action on the CAT(0) cube complex dual to a finite family of relatively quasiconvex subgroups is cocompact *relative to* P_i-invariant subcomplexes. If the parabolic subgroups are abelian and if the action on the dual cube complex is proper, they show that the action is cocompact on a *truncation* of that dual cube complex.

1.2 The main theorem

The main theorem of this article is a cubulation theorem for large classes of relatively hyperbolic groups, *without any* assumption on the peripheral subgroups besides their cubulability. These groups are realised as $C'(1/6)$ small cancellation groups over free products. Note that finitely presented $C'(1/6)$ small cancellation groups over a free product of groups are hyperbolic relative to their free factors [Pan99].

Theorem 1 (cf. Theorem 4.4 and Theorem 4.6)**.** *Let* F *be the free product of finitely many cubulable groups. If G is a quotient of* F *by a finite set of relators which satisfies the classical $C'(1/6)$–small cancellation condition over* F*, then G is cubulable.*

To the authors' knowledge, there are no prior results for relatively hyperbolic groups to provide a *cocompact* action on the dual CAT(0) cube complex *without* strong assumptions on the peripheral subgroups. In particular, the cocompactness of the action *does not* assume any condition on the free factors besides their cubulability. This contrasts with the aforementioned previous theorems where either stronger hyperbolic conditions on the group G, or stronger conditions on the peripheral subgroups are needed. In particular, we do not need the peripheral subgroups to be hyperbolic, nor do we need that they are virtually abelian.

Small cancellation over free products The class of small cancellation groups over free products, both classical and its graphical generalisation, provides a natural setting to study the cubulability of groups acting cocompactly but not properly on higher-dimensional complexes for two reasons. As we explain in this article, such groups act in a very controlled way on 2-dimensional $C'(1/6)$–polygonal complexes, and therefore provide a manageable framework to develop a good geometric intuition. Moreover, the small cancellation theory over free products allows for the construction of groups with a wide range of algebraic and geometric properties. It was fundamental in showing strong embedding properties of infinite groups [MS71, Sch76], in the solution of non-singular equations over groups [EJ11, EJ10], in the construction of torsion-free groups without the unique product property [RS87, Ste15, AS14, GMS15] and in the construction of acylindrically hyperbolic groups with unexpected properties [GS14, Theorem 1.7].

1.3 Comparison to previous work of Wise on cubical small cancellation

In the celebrated essay [Wis11], Wise outlines a far-reaching extension of his results on the action of finitely presented classical $C'(1/6)$–small cancellation quotients on CAT(0) cube complexes [Wis04]. We explain here how the small cancellation groups over free products considered in this paper can be considered examples of Wise's cubical small cancellation groups, and to what extent Wise's approach [Wis11, Th. 5.50, Cor. 5.53] is sufficient to recover some, but certainly not all, of the results obtained in this paper.

In *this* section, for the sake of a simplified comparison, we use the notations of [Wis11].

1. Cubical presentations. The general setting of Wise's cubical small cancellation theory deals with so-called *cubical presentations* $\langle X \mid Y \rangle$ [Wis11, Sec. 3.2], which consists of a non-positively curved cube complex X, and a local isometry $\varphi : Y \to X$ of non-positively curved cube complexes (Wise's theory deals with an arbitrary number of such maps, but for simplicity we will restrict ourselves to the case of a single local isometry). To such a data, one can associate its mapping cone X^*, whose fundamental group is the quotient of $\pi_1(X)$ by the normal subgroup generated by the image of φ.

We obtain a cubical presentation associated with a quotient over a free product of the form ${}^{A*B}/_{\ll w \gg}$, for some appropriate element w of $A * B$, as follows (Here we only treat the *torsion-free* case.): Let us assume that the word w is *not* a proper power, and that A and B are torsion-free. One first constructs a non-positively curved cube complex X with fundamental group $A * B$ by choosing two non-positively curved complexes X_A and X_B with fundamental groups A and B respectively, and by connecting them by an edge. One can then associate to the word w an immersed simplicial loop $\varphi : P \to X$. This yields a cubical presentation for the quotient $G := {}^{A*B}/_{\ll w \gg}$.

In Section 2, we explore such a construction in a technically precise way. We can then treat such groups in more generality than we could using only the methods of [Wis11].

2. Properness of the action and the generalised $B(6)$–condition. Wise gives conditions of a small cancellation nature so that the universal cover of X^* can be equipped with a wallspace structure that allows for the study of the cubulability and the specialness of $\pi_1(X^*)$. In particular, the generalised $B(6)$–condition [Wis11, Def. 5.1] is a key ingredient to construct an appropriate wallspace structure for the group in [Wis11, Th. 5.50], and to obtain the properness of the action on the dual CAT(0) cube complex. In presence of strong small cancellation conditions [Wis11, Th. 3.20, Cor. 3.32], Wise shows that the crucial non-positive curvature condition (2) in his generalised $B(6)$–condition holds.

In our previous construction, by choosing a sufficiently large length for the edge joining X_A and X_B, the generalised $B(6)$–condition can be verified for the cubical presentation of G. In particular, Wise's work can be adapted to our setting to show that the groups we consider in this paper act *properly* on a CAT(0) cube complex. Indeed, properness is treated in Theorem 5.50 of [Wis11]. (One can verify the assertions: The conditions (1),(3),(4),(5) and (6) of Wise generalised $B(6)$–condition are satisfied by construction; Condition (2) follows from Corollary 3.32 (1) in [Wis11]. The conditions (2),(3), and (4) of [Wis11, Th. 5.50] are as well satisfied by construction.)

With these general ideas of Wise in the background, the approach followed in this article, however, provides a shorter and more transparent explicit proof of the fact that such groups act properly on CAT(0) cube complexes, and does not require the full strength of Wise's machinery. In particular, we do neither use the generalised $B(6)$–condition, nor do we use Wise's detailed analysis of cubical van Kampen diagrams.

3. Cocompactness of the action. Our most important contribution lies in the cocompactness of the action. In Wise's Corollary 5.53 [Wis11] (and in other related results as mentioned above), cocompactness of the action follows from the hyperbolicity of the quotient group. It is therefore *not* possible to recover our cubulability results from Wise's argument in [Wis11, Th. 5.50, Cor. 5.53] when the free factors are not hyperbolic. In contrast to such strong conditions, in this article we control the geometry *of the universal cover* of the complex of groups associated with small cancellation groups over free products of groups, and not necessarily the geometry of the whole group, to understand the geometric structure of the wallspace.

1.4 Complexes of groups

In this article we adopt the point of view of *complexes of groups*, a high-dimensional generalisation of graph of groups, developed by Gersten–Stallings [Sta91], Corson [Cor92], and Haefliger [Hae91]. In particular, we associate to a small cancellation group G over a free product a 2-dimensional complex of groups with fundamental group G. Its universal cover is a $C'(1/6)$–small cancellation polygonal complex X on which G acts with vertex stabilisers being conjugates of the free factors. To obtain a space quasi-isometric to G, we then *blow up* vertices into CAT(0) cube complexes. As a result, we obtain a polyhedral complex with a proper and cocompact G-action. It is on such a polyhedral complex that we want to define a wallspace structure, by *combining* the walls in X and the walls of the various cube complexes present in the blown-up space.

This complex of groups approach is very natural: It allows us to work directly with the geometric structure of the small cancellation complex X. We can use it to explicitly combine walls of the free factors, to obtain a wallspace structure for the small cancellation quotient G.

It is this complex of groups approach that allows us to remove the strong (relative) hyperbolicity conditions required in aforementioned articles: The polygonal complex X itself is hyperbolic, but the blown-up space, which is quasi-isometric to G, can have a very different geometry. One of the key points in this complex of groups approach is to use the geometry of the polygonal complex X to study the walls constructed in the blown-up space.

Note that our main theorem then follows from the following, slightly more general, statement that can be extracted from our proof of Theorem 1.

Theorem 2. *Let X be $C'(1/6)$–small cancellation polygonal complex on which a group G acts cocompactly, with cubulable vertex stabilisers and trivial edge stabilisers. Then G is cubulable.*

1.5 Applications

The existence of a cubulation, or more generally of a proper action on a CAT(0) cube complex, has many interesting consequences. We list here several corollaries of our main theorem.

Baum-Connes conjecture. Recall that a group acting properly on a CAT(0) cube complex has the Haagerup property. In particular, such a group satisfies the strong Baum-Connes conjecture [HK01] and does not have Kazhdan's Property (T). By relaxing our assumptions on the free factors, we obtain a combination theorem for groups acting properly on *locally finite* CAT(0) cube complexes.

Theorem 3. *Let* F *be the free product of finitely many groups acting properly on a locally finite CAT(0) cube complex. If G is the quotient of* F *by a finite set of relators which satisfies the classical $C'(1/6)$–small cancellation condition over* F*, then G acts properly on a locally finite CAT(0) cube complex. In particular, G satisfies the Haagerup property and the strong Baum-Connes conjecture.*

Consequences of Agol's theorem. Let us mention two other significant applications of Theorem 1 in the particular case of (Gromov) hyperbolic groups. By a recent result of Agol [Ago13] building upon a work of Haglund–Wise [HW08, HW12a] among others, a hyperbolic group that acts properly and cocompactly on a CAT(0) cube complex is virtually a special subgroup of a right-angled Artin group. In particular, this implies that a cubulable hyperbolic group is residually finite, linear over the integers and has separable quasiconvex subgroups. We thus obtain the following:

Theorem 4. *Let* F *be the free product of finitely many hyperbolic cubulable groups. If G is a quotient of* F *by a finite set of relators which satisfies the* classical $C'(1/6)$*–small cancellation condition over* F*, then G is residually finite, linear over the integers and has separable quasiconvex subgroups.*

Another application of Agol's theorem, in the context of the Atiyah and Kaplansky zero-divisor conjectures, was provided by [Sch14]. The main result therein, based on the work of Linnell–Schick–Okun and collaborators, see for instance [LOS12], implies the Atiyah conjecture on ℓ^2-Betti numbers for a large class of groups having the Haagerup property, including cubulable hyperbolic groups. We thus obtain the following:

Theorem 5. *Let* F *be the free product of finitely many torsion-free hyperbolic cubulable groups. If* G *is a torsion-free quotient of* F *by a finite set of relators which satisfies the classical* $C'(1/6)$*–small cancellation condition over* F*, then* G *satisfies the strong Atiyah conjecture. In particular,* G *satisfies the Kaplansky zero-divisor conjecture over the complex numbers.*

The Kaplansky zero-divisor conjecture asserts that the group ring over the complex numbers of a torsion-free group contains no non-trivial zero-divisor. A usual method to show the Kaplansky conjecture is to prove the unique product property for the group. The question whether small cancellation groups have the unique product property is a difficult and long-standing open problem, cf. Problem N1140 of Ivanov in [MK14].

Open problem. Torsion-free groups without the unique product property were constructed as *graphical* small cancellation groups over free products [RS87, Ste15, AS14, GMS15]. It is unknown whether these so called generalised Rips-Segev groups satisfy the Kaplansky zero-divisor conjecture. It is therefore natural to ask, in light of Agol's theorem, whether our approach can be extended to cubulate some generalised Rips–Segev groups.

1.6 Methods

Let us detail the idea and structure of our proof.

Complexes of groups and spaces. In Section 2, we realise a $C'(1/6)$ small cancellation group G over the free product F as the fundamental group of a developable 2-dimensional complex of groups, the universal cover of which is a $C'(1/6)$–small cancellation polygonal complex. *From now on we denote this polygonal complex by* X*.* In order to prove that a group is cubulable, a useful approach—which goes back to ideas of Sageev [Sag95, HP98]—is to define an appropriate wallspace structure on it. Therefore, we first want a space with a proper and cocompact action of G. The polygonal complex X does not have this property in general. Indeed, vertex stabilisers are conjugates of (the image in G of) the possibly infinite free factors of the free product F.

The blow up space. To overcome this problem, we *blow up* vertices of X. More precisely, we construct a simply connected space $\mathcal{E}G$ with a proper and cocompact G-action as a *complex of spaces* (a high- dimensional generalisation of the notion of tree of spaces) over X. This complex has a polyhedral structure and is a union of CAT(0) cube complexes and polygons. The CAT(0) cube complexes are exactly the preimages of vertices of X and each one is endowed with a geometric action by the associate vertex stabiliser. The polygons of $\mathcal{E}G$ are in one-to-one correspondence with the polygons of X; some of their edges map homeomorphically to edges of X, while portions of their boundary are geodesics in some of the CAT(0) cube complexes contained in $\mathcal{E}G$ (see Figure 3). This construction can be thought as a generalisation of the action of a classical $C'(1/6)$–small cancellation quotient over the *free group* on the universal cover of its presentation complex.

Walls on the building blocks of $\mathcal{E}G$. The space $\mathcal{E}G$ is built up from X and the fibre CAT(0) cube complexes.

In Section 3, we put a wallspace structure on (the set of vertices of) $\mathcal{E}G$. First notice that the walls of the small cancellation complex X, the so-called *hypergraphs* introduced by Wise [Wis04], naturally lift to walls of $\mathcal{E}G$. In the case where one of the free factors in the free product F is infinite however, this collection of walls is not enough to separate elements of G in a conjugate of the image of that factor. This corresponds to the problem of separating vertices of $\mathcal{E}G$ in one of the CAT(0) cube complexes which is the preimage of a vertex of X with an infinite stabiliser. Nonetheless, vertices of a CAT(0) cube complex are separated by so-called hyperplanes. We therefore want to "extend" hyperplanes in a given CAT(0) cube complex to walls of the whole space $\mathcal{E}G$. In order to do that, we extend Wise's approach [Wis04, Wis11] to this more general setting.

Walls on complexes of CAT(0) cube complexes. Namely, every time a polygon $\widetilde{R}$ of $\mathcal{E}G$ crosses a hyperplane in some vertex fibre along an edge e, we want to combine this hyperplane with the diameter of $\widetilde{R}$ (seen as a wall) starting at the midpoint of e. Such a procedure should have the feature that the resulting walls should be realised as trees of hyperplanes over *generalised hypergraphs* of X. However, since polygons of $\mathcal{E}G$ have part of their boundary contained in the vertex fibres, the overlaps between polygons of $\mathcal{E}G$ can be quite different from the well controlled overlaps between polygons of the small cancellation complex X. In order to overcome this problem, we first perform an appropriate subdivision, called "balancing", of the complex (see Definition 3.20 for a precise definition). This procedure, as well as the construction of walls, is detailed in Sections 2.2, 2.3 and 2.4. The aforementioned generalised hypergraphs of X, together with the associated generalised *hypercarriers*, are introduced in Section 2.1. They enjoy the same properties as the usual notions introduced in [Wis04], and Wise's argument extends to this more general setting in a straightforward way; we give the full proofs of these results in an Appendix.

Properness and cocompactness. Finally, we study in Section 4 the set of walls of $\mathcal{E}G$. Namely, we prove that this set of walls satisfies criteria which, as shown by Chatterji–Niblo [CN05], imply that the action of G on the CAT(0) cube complex associated with the wallspace structure is proper and cocompact. This concludes the proof of Theorem 1.

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2 Complexes of groups and small cancellation over free products of groups

Suppose G is a finitely presented group, viewed as a *quotient of the free group* $\mathbb{F}_n$ on n generators. That is, G is given by generators $g_1, \ldots, g_n$ and finitely many relators $r_1, \ldots, r_m \in \mathbb{F}_n$ such that G is the quotient of the free group by the normal closure of the subgroup generated by the relators. We now recall the constructions of the *presentation complex* and the *Cayley complex* associated with such presentations. We start from the bouquet of n oriented cycles $c_1, \ldots c_n$. We label each cycle c_i by the generator g_i. For each relator r_j we take a polygonal 2-cell R_j, whose boundary edges are oriented and labelled by the generators such that the label of a boundary path of R_j equals r_j. Then glue R_j to the bouquet along its boundary word. The complex so obtained is the *presentation complex of* G. Its universal cover is the *Cayley complex of* G. Note that the fundamental group of the presentation complex is G, and G has a free and cocompact action on the associated Cayley complex.

In this paper, we are interested in properties of groups G that are quotients of the (non-trivial) *free product* F of finitely many groups. In this section, we associate to a small cancellation quotient G of the free product of two groups a developable 2-dimensional complex of groups with fundamental group G, the universal cover of which is a small cancellation polygonal complex, see Definition 2.9. We shall think about this complex of groups as of an analogue for the presentation complex in the case of quotients of *free product of groups*. The action of G on the universal cover is no longer proper as soon as one of the free factors is infinite. More precisely, stabilisers of vertices correspond to conjugates of the free factors in G. However, we can construct another polyhedral complex with a proper and cocompact G-action, by *blowing up* vertices of the universal cover. This polyhedral complex is the analogue of the Cayley complex for quotients of free products of groups, and is obtained as a *complex of spaces* over the universal cover.

2.1 Small cancellation groups over free products of groups

We summarize some aspects of the small cancellation theory of the free product of two groups. A more complete treatment can be found in [LS77, Chapter V.9] or [Ol′91, Ch. 11]. We let $\mathrm{F} = A * B$ be the free product of two groups A and B. The groups A and B are called the *free factors*. Every non-trivial element of F can be represented in a unique way as a product $w = h_1 \cdots h_n$, called the *normal form*, where h_i is a non-trivial element in either A or B and no two consecutive h_i, h_{i+1} belong to the same free factor. Then the *free product length* of w is given by $|w| := n$.

The normal form of w is *weakly cyclically reduced* if $|w| \leqslant 1$ or $h_1 \neq h_n^{-1}$. If $u, v \in \mathrm{F}$, $u = h_1 \cdots h_n$, $v = k_1 \cdots k_m$, and $h_n = k_1^{-1}$, then h_n and k_1 *cancel* in the product uv. Otherwise, we say that the product uv is *weakly reduced*.

Let $\mathcal{R} \subset \mathrm{F}$ be a subset of F, each element of which is represented by a weakly cyclically reduced normal form. Let G be the group defined as

$$G := {}^{\mathrm{F}}/\!\ll \mathcal{R} \gg,$$

where $\ll \mathcal{R} \gg$ denotes the normal closure of $\mathcal{R}$ in F. We say that $\mathcal{R}$ is *symmetrised* if it is stable by taking cyclic conjugates and inverses. Up to adding all cyclic conjugates of elements of $\mathcal{R}$ and their inverses, we can always assume that $\mathcal{R}$ is symmetrised.

An element p in F is a *piece* if there are distinct relators $r_1, r_2 \in \mathcal{R}$ such that the products $r_1 = pu_1$ and $r_2 = pu_2$ are weakly reduced.

The set $\mathcal{R}$ satisfies the $C'(1/6)$*–condition (over* F*)* if it is symmetrised and if for every piece p and every relator $r \in \mathcal{R}$ such that the product $r = pu$ is weakly reduced, we have that

$$|p| < \frac{1}{6}|r|.$$

In this case we say that G is a $C'(1/6)$*–group (over* F*)*.

Theorem 2.1 (Corollary V.9.4 of [LS77])**.** *Let G be a $C'(1/6)$–group over the free product* F*. Then the projection map* $\mathrm{F} \to G$ *embeds each free factor of* F. $\square$

Theorem 2.2 (cf. [Pan99])**.** *Let G be a $C'(1/6)$–group over the free product* F*. Then G is hyperbolic relative to the free factors. If all free factors are hyperbolic, then so is G.*

Example 2.3 (Fuchsian groups)**.** Fuchsian groups are the fundamental groups of surfaces of genus g with r cone-points of order $m_1, \dots m_r$, and s points or closed discs removed. They are generated by $a_1, \dots, a_g, b_1, \dots, b_g, x_1, \dots, x_r, y_1, \dots, y_s$, with the relators

$$\Pi_{i=1}^{g}[a_i, b_i]\Pi_{j=1}^{r}x_j\Pi_{k=1}^{s}y_k, x_1^{m_1}, \dots, x_r^{m_r}.$$

If $4g + r + s + t \geq 6$, then the set of relators obtained from symmetrising the word

$$\Pi_{i=1}^{g}[a_i, b_i]\Pi_{j=1}^{r}x_j\Pi_{k=1}^{s}y_k$$

satisfies the $C'(1/6)$–condition over the free product

$$\langle a_1 \rangle * \langle b_1 \rangle * \cdots * \langle a_g \rangle * \langle b_g \rangle * \langle x_1 \mid x_1^{m_1} \rangle * \cdots * \langle x_r \mid x_r^{m_r} \rangle * \langle y_1 \rangle * \cdots \langle y_s \rangle.$$

2.2 Complex of groups associated with $C'(1/6)$–groups over a free product of groups

Let $w \in \mathrm{F}$ be an element satisfying the $C'(1/6)$–condition over $\mathrm{F} = A * B$ and define the group

$$G := {}^{A*B}/_{\ll w \gg}.$$

Observe that w acts hyperbolically on the Bass-Serre tree associated with $A * B$ by the small cancellation condition, and thus we can write

$$w = (a_0 b_0 \dots a_{N-1} b_{N-1})^d,$$

where $d \geq 1$ and $a_0 b_0 \dots a_{N-1} b_{N-1}$ is not a proper power in $A * B$. The theory that we develop in this paper can readily be extended to the free product of finitely many groups and to quotients with respect to finitely many relators.

We now construct a complex of groups whose fundamental group is G. We start by defining several complexes, see Figure 1.

- Let L be the simplicial complex consisting of a single edge with vertices u_A and u_B, and let L' be its barycentric subdivision with c being the barycentre of L. The space L' consists of two edges e_A (containing u_A) and e_B (containing u_B).
- Let R_0 be the *model polygon* on $2N$ sides, that is, a polygonal complex consisting of a single 2-cell whose boundary consists of $2N$ edges. We choose an orientation of R_0, a vertex v_0 in ∂R_0, and then denote by $(v_i)_{i \in \mathbb{Z}/2N\mathbb{Z}}$ the remaining vertices, so that, seen from v_i, the vertex v_{i+1} is the next vertex in the positive direction on ∂R_0.
- Let $R_{0,\mathrm{simpl}}$ be the simplicial complex obtained from R_0 by adding a vertex, called *apex*, in the centre of the 2-cell, and, for each vertex v_i an edge, called *radius*, joining the apex to v_i. In particular, $R_{0,simpl}$ is the simplicial cone over a loop on $2N$ edges.
- Let $R'_{0,\mathrm{simpl}}$ be the the barycentric subdivision of $R_{0,\mathrm{simpl}}$.

Let us *orient the edges* in the 1-skeleton of L' and $R'_{0,simpl}$ as follows. If $\sigma \subsetneq \sigma'$ are two simplices of L or R (i.e. vertices, edges, or faces) with barycentres c and c' respectively, then the edge between c and c' is oriented from c' to c; the barycentre c' is called the *initial vertex* of that edge, the barycentre c is called the *terminal vertex* of that edge. The two edges of L' are, in particular, oriented towards the vertices u_A and u_B respectively. If $\sigma \subsetneq \sigma' \subsetneq \sigma''$ with barycentres c, c' and c'' respectively, then the edges a from c'' to c' and b from c' to c are said to be *composable*, and their composition is defined to be the edge from c'' to c, which we denote ba.

Starting from these complexes, we now define the CW-complexes

$$K := {}^{(L \sqcup R_0)}/_{\simeq}, \quad K_{\mathrm{simpl}} := {}^{(L \sqcup R_{0,\mathrm{simpl}})}/_{\simeq} \text{ and } K'_{\mathrm{simpl}} := {}^{(L' \sqcup R'_{0,\mathrm{simpl}})}/_{\simeq}.$$

Let us first describe K'_{simpl}. Here we identify oriented edges in the boundary of $R'_{0,\mathrm{simpl}}$ pointing towards a vertex v_{2i} with the oriented edge e_A of L', while oriented edges in the boundary of $R'_{0,\mathrm{simpl}}$

pointing towards a vertex v_{2i+1} are identified with the oriented edge e_B. The resulting simplicial complex is K'_{simpl}. The construction is illustrated in Figure 1. Now, let

$$q : L' \sqcup R'_{0,\mathrm{simpl}} \to K'_{simpl}$$

denote the projection, seen as the map between the underlying topological spaces. The map q restricts to a homeomorphism on the interior of each cell of $L \sqcup R_0$ and $L \sqcup R_{0,\mathrm{simpl}}$. We can therefore push forward the CW-structures of $L \sqcup R_0$ and $L \sqcup R_{0,\mathrm{simpl}}$ using the map q, and we denote by

$$K_{simpl} := q(L \sqcup R_{0,\mathrm{simpl}}) \text{ and } K := q(L \sqcup R_0)$$

the associated CW-complexes. In other words, K_{simpl} and K are obtained from K'_{simpl} by forgetting, in each case from left to right, the additional structure we have put on $R'_{0,simpl}$ and $R_{0,simpl}$ respectively. In all three cases, we use apex, radii, and v_i to refer to their respective images in $R'_{0,simpl}$ and K'_{simpl} respectively.

A *small category without loop*, or *scwol* in short, is an oriented graph without loop with a notion of *composability* of edges, see [BH99, Chapter III.$\mathcal{C}$ Definitions 1.1]. The oriented 1-skeleton of the first barycentric subdivision of a simplicial complex can be endowed with a structure of scwol. In particular, we described a structure of scwol on the oriented 1-skeleton of L', which we denote $\mathcal{L}'$, and on the oriented 1-skeleton of $R'_{0,simpl}$. These scwols can be glued together along the map q, yielding a structure of scwol on the 1-skeleton of K'_{simpl}, which we denote $\mathcal{K}'_{simpl}$.

Observe that pairs of composable edges of $\mathcal{K}'_{\mathrm{simpl}}$ are in 1-to-1 correspondence with triangles of K'_{simpl}. The simplicial complex K'_{simpl} is said to be a *geometric realisation* of the scwol $\mathcal{K}'_{\mathrm{simpl}}$.

A *complex of groups* over a scwol $\mathcal{Y}$ consists of the data $(G_\sigma, \psi_\sigma, g_{b,a})$ of *local groups* G_σ, *local maps* ψ_σ, and *twisting elements* $g_{b,a}$ for every pair (b, a) of composable edges of $\mathcal{Y}$ subject to additional compatibility conditions, see [BH99, Chapter III.$\mathcal{C}$, Definition 2.1]. To follow our construction details of such kind are not a prerequisite. However, we refer the interested reader to Bridson–Haefliger [BH99, Chapter III.$\mathcal{C}$] for more terminology and background on complexes of groups.

Definition 2.4. We define a complex of groups $G(\mathcal{K}'_{\mathrm{simpl}})$ over $\mathcal{K}'_{\mathrm{simpl}}$ as follows:

- the local groups at u_A and u_B are respectively A and B, the local group at the apex is $\mathbb{Z}/d\mathbb{Z}$, and all the other local groups are trivial;
- all the local maps are trivial;
- the twisting element associated with a pair of composable edges (b, a), or equivalently to the associated triangle of K'_{simpl}, is represented in Figure 1.

We now define a morphism $F = (F_\sigma, F(a))$ of complexes of groups from $G(\mathcal{K}'_{\text{simpl}})$ to G. (A general definition of morphism of complexes of groups can be found in [BH99, Chapter III.$\mathcal{C}$, Definition 2.5].) We first fix some notation.

For $i = 0, \dots, 2N-1$, let c_i be the barycentre of the radius at the vertex v_i. For $i = 0, \dots, N-1$, let e_i be the oriented edge of $\mathcal{K}'_{\text{simpl}}$ from c_{2i} to v_{2i}, and let f_i be the oriented edge of $\mathcal{K}'_{\text{simpl}}$ from c_{2i+1} to v_{2i+1}. Let also e'_0 be the oriented edge of $\mathcal{K}'_{\text{simpl}}$ from c_0 to the apex of $R_{0,\text{simpl}}$.

Now let us set the local maps of F as follows.

- The local morphisms $F_{u_A} : A \to G$ and $F_{u_B} : B \to G$ are the natural projections, the map $\mathbb{Z}/d\mathbb{Z} \to G$ sends the generator $\overline{1}$ of $\mathbb{Z}/d\mathbb{Z}$ to the image of $a_0 b_0 \dots a_{N-1} b_{N-1}$ in G, and the other maps are trivial;
- For $i = 0, \dots, N-1$, we set $F(e_i) := a_i$, $F(f_i) := b_i$, $F(e'_0) := \overline{1}$, and F is trivial on all the other edges of $\mathcal{K}'_{\text{simpl}}$.

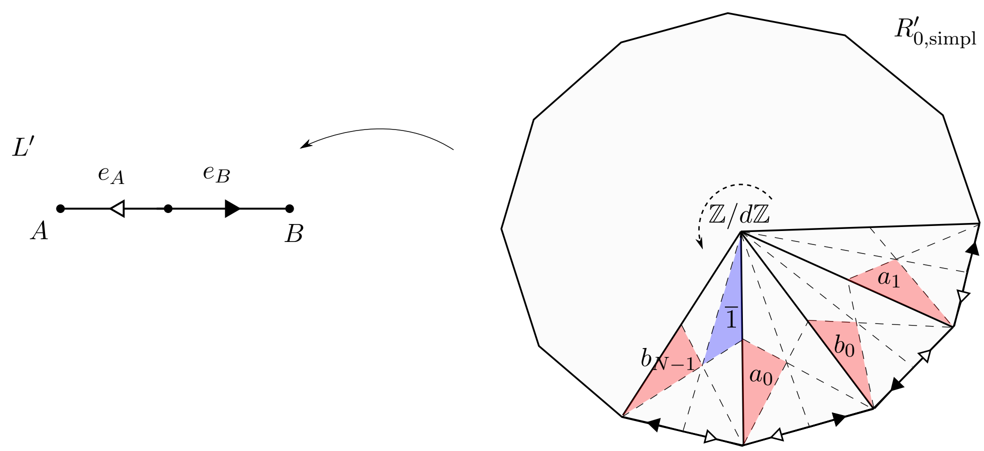

Figure 1: Part of the complex of groups $G(\mathcal{K}'_{\text{simpl}})$. Twisting elements corresponding to white triangles of $R'_{0,\text{simpl}}$ are trivial. (The element $\overline{1}$ denotes the generator of $\mathbb{Z}/d\mathbb{Z}$.)

Let $\pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A)$ be the fundamental group of $G(\mathcal{K}'_{\text{simpl}})$ at the vertex u_A, seen as the group of homotopy classes of $G(\mathcal{K}'_{\text{simpl}})$-loops, see [BH99, Chapter III.$\mathcal{C}$, Definition 3.5]. Let $\pi_1 F : \pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A) \to G$ be the associated morphism of fundamental groups, see [BH99, Chapter III.$\mathcal{C}$, Proposition 3.6]. The following result is not surprising when viewed against the aforementioned construction of the presentation complex. However, as complexes of groups are in some technical points surprisingly different to the standard situation, we give an elementary proof using the language of [BH99, Chapter III.$\mathcal{C}$ Section 3].

Proposition 2.5. *The map*

$$\pi_1 F : \pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A) \to G$$

is an isomorphism.

Proof. Since A and B generate $A * B$, and thus G, the map $\pi_1 F$ is surjective. Let g be an element of $\ker \pi_1 F \subseteq \pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A)$, and let γ be a $G(\mathcal{K}'_{\text{simpl}})$-loop based at u_A in the homotopy class g. Note that it is possible to homotop γ to a loop the support of which is contained in the image of L' in K'_{simpl}.

In other words, if we denote by $i : G(\mathcal{L}') \to G(\mathcal{K}'_{\text{simpl}})$ the natural embedding of complexes of groups (that is, the pullback of $G(\mathcal{K}'_{\text{simpl}})$ under the inclusion of scwols $\mathcal{L}' \hookrightarrow \mathcal{K}'_{\text{simpl}}$), then the induced morphism of fundamental groups $\pi_1 i : \pi_1(G(\mathcal{L}'), u_A) \to \pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A)$ is surjective. Let h be an element of $\pi_1(G(\mathcal{L}'), u_A)$ such that $g = \pi_1 i(h)$. We thus have $\pi_1 F(\pi_1 i(h)) = 0$. But since $\pi_1 F \circ \pi_1 i : \pi_1(G(\mathcal{L}'), u_A) \to G$ is the natural projection $A * B \to {}^{A * B}/_{\ll w \gg}$, it follows that h is in the normal subgroup generated by the $G(\mathcal{L}')$-loop $(a_0, e_A^{-1}, e_B, b_0, e_B^{-1}, e_A, a_1, \ldots)^d$. Thus, g is in the normal closure of the $G(\mathcal{K}'_{\text{simpl}})$-loop $(a_0, e_A^{-1}, e_B, b_0, e_B^{-1}, e_A, a_1, \ldots)^d$. It is now enough to prove that such a $G(\mathcal{K}'_{\text{simpl}})$-loop is homotopically trivial. But the definition of $\pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A)$ implies that this loop is homotopic to the following edge-path (seen as a $\pi_1(G(\mathcal{K}'_{\text{simpl}}), u_A)$-loop):

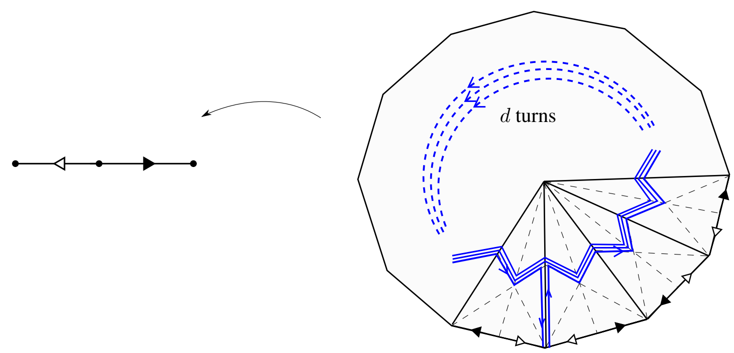

Figure 2: A homotopically trivial $G(\mathcal{K}'_{\text{simpl}})$-loop.

which is homotopically trivial since the local group at the apex is $\mathbb{Z}/d\mathbb{Z}$, hence the result. $\square$

Let $\mathfrak{A}^{(0)}(\mathcal{K}'_{\text{simpl}})$ be the set of vertices of $\mathcal{K}'_{\text{simpl}}$, let $\mathfrak{A}^{(1)}(\mathcal{K}'_{\text{simpl}})$ be the set of edges of $\mathcal{K}'_{\text{simpl}}$, and let $\mathfrak{A}^{(2)}(\mathcal{K}'_{\text{simpl}})$ denote the set of pairs $\mathfrak{a} = (a_2, a_1)$ of composable edges of $\mathcal{K}'_{\text{simpl}}$. For every (oriented) edge a define $i(a)$ to be the initial vertex, and $t(a)$ to be the terminal vertex. For $\mathfrak{a} = (a_2, a_1) \in A^{(2)}(\mathcal{Y})$, we set $i(\mathfrak{a}) := i(a_1)$ and $t(\mathfrak{a}) := t(a_2)$. We define maps

$$\partial_0, \partial_1 : \mathfrak{A}^{(1)} \to \mathfrak{A}^{(0)}$$

by setting $\partial_0(a) := i(a)$ and $\partial_1(a) := t(a)$. For $0 \leq i \leq 2$, we define maps

$$\partial_i : \mathfrak{A}^{(2)}(\mathcal{K}'_{\text{simpl}}) \to \mathfrak{A}^{(1)}(\mathcal{K}'_{\text{simpl}})$$

by setting $\partial_0(a_2, a_1) := a_2$, $\partial_1(a_2, a_1) := a_2 a_1$, and $\partial_2(a_2, a_1) := a_1$.

Let Δ^k be the standard Euclidean k-simplex, that is, the set of elements $(t_0, \ldots, t_k)$ with $t_i \geq 0$ and $\sum_i t_i = 1$. For $k \geq 1$ and $0 \leq i \leq k$, we denote the embeddings of the sides of Δ^k by

$$d_i : \Delta^{k-1} \to \Delta^k,$$

defined by sending $(t_0, \ldots, t_{k-1})$ to $(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1})$.

Since the morphism F is injective on the local groups, we can define the following complex.

Definition 2.6. Let X'_{simpl} be the simplicial complex obtained from the disjoint union

$$\coprod_{0 \leq k \leq 2} \coprod_{\mathfrak{a} \in \mathfrak{A}^{(k)}(\mathcal{K}'_{simpl})} \Big(F_{i(\mathfrak{a})}(G_{i(\mathfrak{a})}) \backslash G \times \{\mathfrak{a}\} \times \Delta^k \Big)$$

by identifying pairs of the form

$$([gF(a)^{-1}], \partial_i \mathfrak{a}, x) \text{ and } \big([g], \mathfrak{a}, d_i(x)\big) \text{ for } 0 \leq i \leq k \ ,$$

where a denotes the edge with initial vertex $i(\mathfrak{a})$ and terminal vertex $i(\partial_i \mathfrak{a})$.

Note that there is a natural projection

$$\pi : X'_{simpl} \to K'_{simpl}$$

obtained by forgetting the first coordinate. The CW-structure on K_{simpl} can be pulled-back along π, yielding a simplicial complex X_{simpl} with barycentric subdivision X'_{simpl}. For simplicity reasons, we still denote by π the projection map $X_{\mathrm{simpl}} \to K_{\mathrm{simpl}}$.

We now construct our *polygonal complex* X as the pull back of the CW-structure on K along π. We can obtain X from X_{simpl} as follows. We denote by $s \in K_{\mathrm{simpl}}$ the apex of K_{simpl} and by $S \subset X_{\mathrm{simpl}}$ the preimage of s under the projection $\pi : X_{\mathrm{simpl}} \to K_{\mathrm{simpl}}$, called the *set of apices* of X_{simpl}. A *simplicial polygon* of X_{simpl} is the star in X_{simpl} of an apex of S, that is, the subcomplex consisting of all simplices containing that apex as a vertex. Two distinct simplicial polygons of X_{simpl} are either disjoint or meet along a subset of $\pi^{-1}(L)$. Let us delete all the apices of X_{simpl} and all the edges containing them to obtain a polygonal complex denoted X, that is, a CW-complex such that 2-cells are modelled after a *model polygon* $\widetilde{R}_0$ on $d \cdot 2N$ sides (which is an orbifold cover of the model polygon R_0 on $2N$ sides), and such that the various gluing maps $\partial \widetilde{R}_0 \to \pi^{-1}(L)$ are simplicial. Furthermore, we identify $\widetilde{R}_0$ with the polygon of X whose apex in X_{simpl} corresponds to the point $\{1\} \times \{s\} \times \Delta^0$.

By definition, X'_{simpl} is the geometric realisation of the development $D(\mathcal{K}'_{simpl}, F)$, see [BH99, Chapter III.$\mathcal{C}$, Theorem 2.13]. Note that the following result on complexes of groups follows directly from [BH99, Chapter III.$\mathcal{C}$, Proposition 3.14].

Proposition 2.7. *Let $G(\mathcal{Y})$ be a complex of groups over a scwol $\mathcal{Y}$ whose geometric realisation is a simplicial complex, v be a vertex of $\mathcal{Y}$ and $F : G(\mathcal{Y}) \to G$ a morphism from $G(\mathcal{Y})$ to some group G that is injective on the local groups.*

The geometric realisation of the development $D(\mathcal{Y}, F)$ is a universal cover of the complex of groups $G(\mathcal{Y})$ if and only if the induced morphism $\pi_1 F : \pi_1(G(\mathcal{Y}), v) \to G$ is an isomorphism. □

We thus obtain the following.

Proposition 2.8. *The simplicial complex X'_{simpl} is (equivariantly isomorphic to) a universal cover of $G(\mathcal{K}'_{simpl})$. In particular, the small cancellation group G acts on X'_{simpl} with quotient K'_{simpl}, with vertex stabilisers A, B, or $\mathbb{Z}/d\mathbb{Z}$ at vertices mapped under π on the vertices u_A, u_B, or the apex respectively, and with trivial edge stabilisers.*

Proof. It is enough to prove that the conditions of Proposition 2.7 are satisfied. The geometric realisation of $\mathcal{K}'_{simpl}$ is the simplicial complex K'_{simpl}. The morphism $F : G(\mathcal{K}'_{simpl}) \to G$ is injective on the local groups as G is a $C'(1/6)$–small cancellation group, see Theorem 2.1. The result thus follows from Proposition 2.5. □

Definition 2.9 (piece, $C'(1/6)$ polygonal complex)**.** Let Y be a polygonal complex. A *path of* Y is an injective path in the 1-skeleton of Y. For a path P of Y, we denote by $|P|$ the number of edges of P, called its *length*.

A *piece* of a polygonal complex Y is a path P of Y such that there exist polygons R_1 and R_2 such that the map $P \to Y$ factors as $P \to R_1 \to Y$ and $P \to R_2 \to Y$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \partial R_2 \arrow[d] \\ \partial R_1 \arrow[r] \arrow[ur] & Y. \end{tikzcd}$$

By convention, we also consider edges of Y as pieces.

The polygonal complex Y is said to be a $C'(\lambda)$ *polygonal complex*, $\lambda > 0$, if for every piece P of Y and every polygon R of Y containing P in its boundary, we have $|P| < \lambda \cdot |\partial R|$.

Proposition 2.10. *Let G be a $C'(1/6)$–small cancellation group over the free product* F. *Then, the polygonal complex X defined above is a $C'(1/6)$ polygonal complex.*

Proof. Consider two polygons of X sharing an edge. Up to the action of G, we can assume that such an edge contains the vertex v_A. The two chosen polygons then correspond to two cyclic conjugates of w. By construction of $G(\mathcal{K}'_{\mathrm{simpl}})$, these cyclic conjugates must be distinct. The result thus follows from the $C'(1/6)$–condition satisfied by G. □

The Greendlinger Lemma [LS77] immediately implies the following, see for instance [MW02, Lemma 13.2].

Corollary 2.11. *The polygons of X are embedded.* □

2.3 Complex of spaces with proper and cocompact action

A group is *cubulable* if it acts geometrically, i.e. properly discontinuously and cocompactly, on a CAT(0) cube complex. From now on, we assume that A and B are cubulable groups, and denote CAT(0) cube complexes with a geometric action of A and B respectively by EA and EB respectively.

Let Y be a CW-complex. We consider the *vertex set* of Y as a metric space, equipped with the *graph- or edge metric* on the 1-skeleton of Y. We abuse notation and refer to this metric space again as Y.

We now apply a useful theory for classifying spaces of complexes of groups [Mar14b, Section 2]. This theory provides us with an explicit construction of a simply connected polyhedral complex with a geometric action of G. The construction can be thought of as of *blowing up* the vertices of the polygonal complex X.

Definition 2.12 (Definition 2.2 of [Mar14b])**.** Let $G(\mathcal{Y})$ be a complex of groups over a scwol $\mathcal{Y}$. A *complex of classifying spaces $EG(\mathcal{Y})$ compatible with the complex of groups* $G(\mathcal{Y})$ consists of the following:

- For every vertex σ of $\mathcal{Y}$, a space EG_σ, called a *fibre*, which is a cocompact model for the classifying space for proper actions of the local group G_σ,
- For every edge a of $\mathcal{Y}$ with initial vertex $i(a)$ and terminal vertex $t(a)$, a $G_{i(a)}$-equivariant map $\phi_a : EG_{i(a)} \to EG_{t(a)}$, that is, for every $g \in G_{i(a)}$ and every $x \in EG_{i(a)}$, we have
$$\phi_a(g.x) = \psi_a(g).\phi_a(x),$$
and such that for every pair (b, a) of composable edges of $\mathcal{Y}$, we have
$$g_{b,a} \circ \phi_{ba} = \phi_b\phi_a.$$

Complexes of classifying spaces compatible with a given complex of groups were shown to exist in full generality in [Mar14a]. However, we define here an explicit complex of classifying spaces compatible with $G(\mathcal{K}'_{\mathrm{simpl}})$. We use this space to define a wallspace structure in Section 3.3. Let us denote by c_i the barycentre of the radius of $\mathcal{K}'_{simpl}$ from the apex to the vertex v_i. Recall from Definition 2.4 that e_i is the edge of $\mathcal{K}'_{\mathrm{simpl}}$ starting at $s_i := c_{2i}$ and terminating at v_{2i}. Furthermore, recall that f_i is the edge of $\mathcal{K}'_{\mathrm{simpl}}$ starting at $t_i := c_{2i+1}$ and terminating at the vertex v_{2i+1}.

- The fibre $EG_{u_A} := EA$ and $EG_{u_B} := EB$ are the given CAT(0) cube complexes. We fix base vertices $x_A \in EG_{u_A}$ and $x_B \in EG_{u_B}$ respectively.
- For each $i = 0, 1, \ldots, N-1$, we choose an oriented *geodesic*
$$\gamma_{A,i} \text{ from } x_A \text{ to } a_i \cdot x_A$$
in EG_{u_A}, and denote by $|\gamma_{A,i}|$ its edge length. Let EG_{s_i} be the oriented simplicial segment of $|\gamma_{A,i}|$ edges, and let $\phi_{e_i} : EG_{s_i} \to EG_{u_A}$ be a parametrisation of $\gamma_{A,i}$.

- For each $i = 0 \dots, N-1$, we choose an oriented *geodesic*

$$\gamma_{B,i} \text{ from } x_B \text{ to } b_i \cdot x_B$$

in EG_{u_B}, and denote by $|\gamma_{B,i}|$ its edge length. Let EG_{t_i} be the oriented simplicial segment of $|\gamma_{B,i}|$ edges, and let $\phi_{f_i} : EG_{t_i} \to EG_{u_B}$ be a parametrisation of $\gamma_{B,i}$.

- All the other fibres are reduced to a single point and all the other maps are the trivial ones.

It is straightforward to check that this indeed defines a complex of classifying spaces compatible with $G(\mathcal{K}'_{\text{simpl}})$.

Definition 2.13 (The space $\mathcal{E}G$)**.** We construct a space $\mathcal{E}G$, obtained from the disjoint union

$$\coprod_{0 \leq k \leq 2} \coprod_{\mathfrak{a} \in \mathfrak{A}^{(k)}(\mathcal{K}'_{simpl})} \left(F_{i(\mathfrak{a})}(G_{i(\mathfrak{a})}) \backslash G \times \{\mathfrak{a}\} \times \Delta^k \times EG_{i(\mathfrak{a})} \right)$$

by identifying pairs of the form

$$([gF(a)^{-1}], \partial_i \mathfrak{a}, x, \phi_a(\xi)) \text{ and } \big([g], \mathfrak{a}, d_i(x), \xi\big), \text{ for } 0 \leq i \leq k,$$

where a is the edge with initial vertex $i(\mathfrak{a})$ and terminal vertex $i(\partial_i \mathfrak{a})$. The various maps

$$F_{i(\mathfrak{a})}(G_{i(\mathfrak{a})}) \backslash G \times \{\mathfrak{a}\} \times \Delta^k \times EG_{i(\mathfrak{a})} \to F_{i(\mathfrak{a})}(G_{i(\mathfrak{a})}) \backslash G \times \{\mathfrak{a}\} \times \Delta^k$$

obtained by forgetting the last coordinate yield a projection

$$p : \mathcal{E}G \to X.$$

The preimage of a vertex v of X under p is called the *fibre* over v and denoted EG_v, as it is a cocompact model for the classifying space for proper actions of the stabiliser G_v of v.

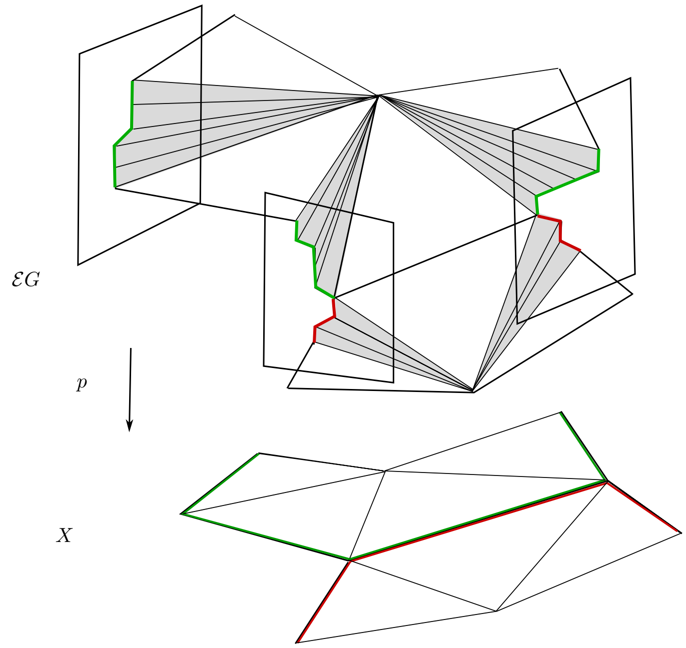

Figure 3: The polyhedral structure of $\mathcal{E}G$. The figure presents portions of two simplicial polygons of X (one green, one red) and its preimage in $\mathcal{E}G$. Vertical triangles are shaded. Attaching paths in the various fibres are coloured with respect to the associated polygon of X.

The following proposition is an application of Theorem 2.4 of [Mar14b].

Proposition 2.14. *The space $\mathcal{E}G$ is simply connected, and the G-action on it is proper and cocompact.* $\square$

Remark 2.15. This result was proven in [Mar14b, Theorem 2.4] in the case of a complex of groups over a simplicial complex. Here, while $G(\mathcal{K}'_{\text{simpl}})$ is not a complex of groups over a simplicial complex, the geometric realisation of $\mathcal{K}'_{\text{simpl}}$ is nonetheless a simplicial complex, and the proof of [Mar14b] carries over to this case without any change.

Definition 2.16 (polyhedral structure on $\mathcal{E}G$, Figure 3)**.** The space $\mathcal{E}G$ can be endowed with a polyhedral structure as follows. First note that we have, in particular, a projection $\mathcal{E}G \to K'_{simpl}$.

- Each fibre is isomorphic to a locally finite CAT(0) cube complex, more specifically EG_v is isomorphic to EA if v is a vertex in the preimage of u_A, and EG_v is isomorphic to EB if v is a vertex in the preimage of u_B.
- Let R be a polygon of X, and denote by $\mathring{R}$ its interior. The boundary of $p^{-1}(\mathring{R})$ in $\mathcal{E}G$ is a path of $\mathcal{E}G$ which is the concatenation of geodesics in the fibres (which are translates of the

chosen geodesics $\gamma_{A,i}, \gamma_{B,i}$) and paths which map homeomorphically onto edges of X_{simpl}. Thus, such a boundary comes equipped with a simplicial structure, and we identify the closure of $p^{-1}(\mathring{R})$ with the simplicial cone over such a boundary path. The preimage of the closure $p^{-1}(\mathring{R})$ with this simplicial structure is called a *simplicial polygon* of $\mathcal{E}G$.

This endows $\mathcal{E}G$ with a polyhedral structure, and the projection map $p : \mathcal{E}G \to X_{\mathrm{simpl}}$ is a polyhedral map. For a polygon R of X, we denote by

$$\widetilde{R} := \text{closure of } p^{-1}(\mathring{R})$$

the associated simplicial polygon of $\mathcal{E}G$.

Definition 2.17 (horizontal, vertical polyhedrons)**.** We say that a polyhedron of $\mathcal{E}G$ is *horizontal* if p restricts to a homeomorphism on it, and *vertical* otherwise. For an edge e of X, we denote by $\widetilde{e}$ the unique horizontal edge of $\mathcal{E}G$ which maps onto e under p.

Definition 2.18 (attaching paths)**.** Let R be a polygon of X and v be a vertex of R. We define the *attaching path of $\widetilde{R}$ along EG_v*:

$$p_{v,R} := EG_v \cap \widetilde{R}.$$

In this section we have constructed a polygonal complex $\mathcal{E}G$ which is a realisation of an analogue for quotients of free products of the Cayley complex of the group G . In particular, the group G acts properly and cocompactly on $\mathcal{E}G$. The space $\mathcal{E}G$ was realised as a complex of spaces over the $C'(1/6)$–small cancellation polygonal complex X constructed in Section 2.2. $\mathcal{E}G$ is equipped with a polyhedral structure consisting of the following two *building blocks*: *polygons* of $\mathcal{E}G$, which are mapped to polygons of X (the latter being modelled after the polygon $\widetilde{R_0}$), and *CAT(0) cube complexes*, which are fibre of vertices of X, and are isomorphic to the chosen complexes E_A or E_B.

Our next aim is to describe a wall structure on $\mathcal{E}G$. One family of walls on $\mathcal{E}G$ is obtained by lifting the walls of X. A second family of walls is obtained by combining natural wall structures on the polygons of $\mathcal{E}G$ and the various CAT(0) cube complexes. There is however, a priori, no canonical way to combine these walls, see our explanation in Section 3.3. The geometric structure of the corresponding wallspace associated with $\mathcal{E}G$ is controlled using the properties of the $C'(1/6)$–small cancellation polygonal complex X in combination with the properties of the fibre CAT(0) cube complexes. The properties of X are discussed in Section 3.1 and the Appendix.

3 The wallspace

Spaces with walls were introduced by Haglund–Paulin [HP98] and generalise essential properties of CAT(0) cube complexes.

Definition 3.1 (wallspace)**.** A *wallspace* is a pair $(Y, \mathcal{H})$ consisting of a set Y together with a collection $\mathcal{H}$ of non-empty subsets of Y, called *half-spaces*, such that:

- for every half-space H in $\mathcal{H}$, its complement $Y \setminus H$ is also in $\mathcal{H}$,
- for every x, y of Y, there are only finitely many half-spaces H such that $x \in H$ and $y \notin H$.

A partition of Y into two half-spaces is called a *wall*, and we denote the set of walls of $(Y, \mathcal{H})$ (short: Y) by $\mathcal{W}(Y)$.

We say that a wall *separates* a pair of points of Y if each half-space associated with that wall contains exactly one point of the pair. We say that two walls $W = \{H, Y \setminus H\}$ and $W' = \{H', Y \setminus H'\}$ *cross* if all the intersections $H \cap H', H \cap (Y \setminus H'), (Y \setminus H) \cap H', (Y \setminus H) \cap (Y \setminus H')$ are non-empty. We define the *wall-pseudometric* $d_{\mathcal{W}(Y)}(x, y)$ between two points x, y of Y to be the number of walls separating them. We say that a group *acts on a wallspace* if it acts on the underlying set and preserves the set of half-spaces.

Definition 3.2 (wallspace on a polyhedral complex)**.** A structure of wallspace *on a polyhedral complex* is a structure of wallspace on its vertex set.

If a wall of a polyhedral complex is defined by means of the complement of a separating subset containing no vertex, we will *abuse notation* and not distinguish the associated wall and the separating subset.

Whenever a group acts on a space with walls, one can associate an action of the group on a CAT(0) cube complex by isomorphisms. The CAT(0) cube complex can explicitly be described using the walls, see [CN05, Nic04] for the explicit construction. Let G be a small cancellation group over the free product of two groups. The aim of this section is to define a set of walls $\mathcal{W}$ on the polyhedral complex $\mathcal{E}G$, turning $\mathcal{E}G$ into a wallspace. The above mentioned general procedure then yields the cube complex $C_{\mathcal{W}}$ associated with the action of G on the wallspace $(\mathcal{E}G, \mathcal{W})$.

Again, G denotes the small cancellation quotient ${}^{A * B}/_{\ll w \gg}$, and X the $C'(1/6)$–polygonal complex constructed in Section 2.

3.1 Galleries, hypercarriers and hypergraphs

In this section we introduce fundamental notions and theory that we use later to define walls and then to study their geometric structure. In what follows, while results are stated for the polygonal complex X, the results hold for an arbitrary $C'(1/6)$–polygonal complex.

Definition 3.3 (far apart)**.** Let R be a polygon of a $C'(1/6)$–polygonal complex and τ_1, τ_2 two simplices of its boundary ∂R. We say that τ_1 and τ_2 are *far apart* in R if no path P in ∂R containing both τ_1 and τ_2 is a concatenation of strictly less than four pieces.

Example 3.4. In a $C'(1/6)$ polygonal complex, opposite edges of a given polygon are far apart.

If two cells of a given polygon R of a $C'(1/6)$ polygonal complex are far apart in R, then the polygon R is unique by the small cancellation condition. We thus simply say that these cells are *far apart*, the reference to R being implicit.

Definition 3.5 (polygon with doors, system of doors)**.** A *polygon with doors* is a polygon R of X, referred as the *underlying cell*, together with a choice of simplices τ_1, τ_2 of ∂R called *doors*. We will denote such a data $R_{\{\tau_1,\tau_2\}}$. (We often write $R_{\{\tau_1,\tau_2\}}$ indistinctly for a polygon with doors and for its underlying cell.)

A *system of doors* is a collection $\mathcal{C}$ of polygons with doors. We will simply speak of a *polygon* of $\mathcal{C}$ when speaking of a polygon with doors of $\mathcal{C}$. A *door* of $\mathcal{C}$ is a door of a polygon of $\mathcal{C}$.

Note that a door can be an edge as well as a vertex in the boundary of a polygon.

Definition 3.6 (Gallery)**.** A *gallery* is a system of doors $\mathcal{C}$ satisfying the following conditions.

- (coherence condition) For every pair of polygons $R_{\{\tau_1,\tau_2\}}, R_{\{\tau_1',\tau_2'\}}$ of $\mathcal{C}$ with the same underlying cell and such that $\tau_1 = \tau_1'$, we also have $\tau_2 = \tau_2'$.
- (far apart condition) For every polygon $R_{\{\tau_1,\tau_2\}}$ of $\mathcal{C}$, the doors τ_1 and τ_2 are far apart in the sense of Definition 3.3.
- (connectedness condition) For every pair of doors τ, τ' of $\mathcal{C}$, there exists a sequence

$$R_{\{\tau_1,\tau_2\}}, R_{\{\tau_2,\tau_3\}}, \ldots, R_{\{\tau_{n-1},\tau_n\}}$$

  of polygons of $\mathcal{C}$ such that $\tau = \tau_1$ and $\tau' = \tau_n$.

Definition 3.7 (hypercarrier and hypergraph associated with a gallery)**.** Given a gallery $\mathcal{C}$, we associate a polygonal complex to it as follows. Take the disjoint union of all polygons $R_{\{\tau_1,\tau_2\}}$ of $\mathcal{C}$. Whenever P is a path embedded in $\partial R_{\{\tau_1,\tau_2\}}$ and $\partial R_{\{\tau_2,\tau_3\}}$, and if P embeds in X such that P is contained in the intersection of $\partial R_{\{\tau_1,\tau_2\}}$ and $\partial R_{\{\tau_2,\tau_3\}}$ in X, then we identify $\partial R_{\{\tau_1,\tau_2\}}$ and $\partial R_{\{\tau_2,\tau_3\}}$ along P. The resulting polygonal complex is denoted by $Y_{\mathcal{C}}$ and called the *hypercarrier* associated with $\mathcal{C}$.

For each polygon $R_{\{\tau_1,\tau_2\}}$ of $\mathcal{C}$, we denote by $L_{\{\tau_1,\tau_2\}}$ the path of $R_{\{\tau_1,\tau_2\}}$ which is the union of the radii of $R_{\{\tau_1,\tau_2\}}$ joining the apex of $R_{\{\tau_1,\tau_2\}}$ to the barycentres of τ_1 and τ_2. Let

$$\Lambda_{\mathcal{C}} := \bigcup L_{\{\tau_1,\tau_2\}} \subset Y_{\mathcal{C}}.$$

We call $\Lambda_{\mathcal{C}}$ the *hypergraph* associated with $\mathcal{C}$.

The hypercarrier $Y_{\mathcal{C}}$ comes endowed with a map $i_{\mathcal{C}} : Y_{\mathcal{C}} \to X$, by mapping every polygon in $Y_{\mathcal{C}}$ to the corresponding polygon in X. This map is by construction an immersion on the 1-skeletons.

We note that our hypercarriers and hypergraphs extend the corresponding notions of Wise [Wis04, Definition 3.2 and 3.3]. In particular, Wise's hypercarriers and hypergraphs are defined by means of opposite edges, see Section 3.2.1. Our far apart condition allows, in contrast, the study of hypergraphs and hypercarriers that are *not* associated with opposite edges. Our definition moreover includes hypergraphs going through the vertices of X. The hypercarriers we consider are therefore allowed to have cutpoints at such vertices, cf. Figure 4. Such configurations do not appear in [Wis04].

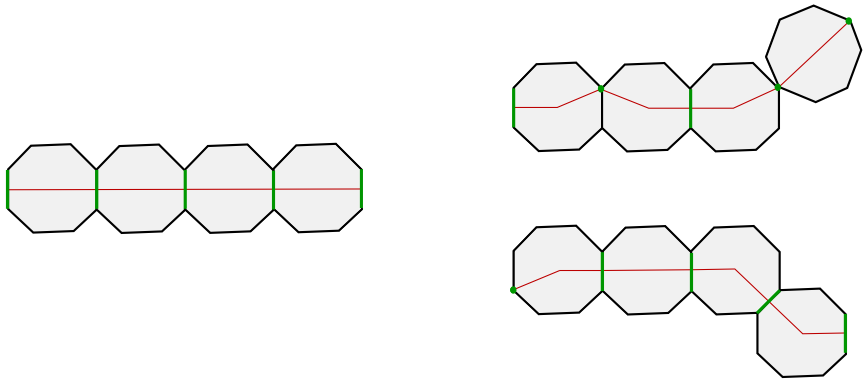

Figure 4: Examples of hypercarriers with their associated hypergraphs (in red) and doors (in green). Configurations on the left are studied in detail in [Wis04]. The configurations on the right are studied in detail in the appendix.

Definition 3.8 (convex)**.** A subcomplex Y is called *convex* if every geodesic between two vertices of Y is contained in Y.

The following results extend Lemma 3.11 and Theorem 3.18 of [Wis04], cf. Proposition 3.12.

Theorem 3.9 (cf. Proposition A.9, Corollary A.14 and Proposition A.16)**.** *Let $\mathcal{C}$ be a gallery in X. Then:*

- *Its hypercarrier $Y_{\mathcal{C}}$ is connected and simply connected and the map $i_{\mathcal{C}} : Y(\mathcal{C}) \to X$ is an embedding.*
- *The associated hypergraph $\Lambda_{\mathcal{C}}$ is a tree which embeds in X.*
- *The subcomplex $Y_{\mathcal{C}}$ of X is convex.*

Corollary 3.10. *Polygons of X are convex.* □

The proofs are by - now standard - small cancellation arguments, and extend the original arguments of Wise in a straightforward way, using our far apart condition. We give a complete account of the arguments in Appendix A.

We study several examples of galleries, hypergraphs and hypercarriers below. In Section 3.2.1 we review Wise's hypergraphs and hypercarriers associated with diametrically opposed edges in the $C'(1/6)$–small cancellation complex X. In Section 3.3.2 we lift such hypergraphs and hypercarriers to $\mathcal{E}G$. Finally, in Section 3.3.3 we modify $\mathcal{E}G$ to extend the hyperplanes in the fibres of $\mathcal{E}G$. We obtain graphs of spaces whose projection to X are hypergraphs associated with a gallery of X. In all three situations, we show that the complement of a hypergraph defines a wall. Here Theorem 3.9 is essential.

3.2 Walls on the building blocks

Recall that the space $\mathcal{E}G$ has two building blocks, the polygons of $\mathcal{E}G$ and the CAT(0) cube complexes which are fibres of vertices of X. Its geometric structure and the combination of these building blocks is controlled using the properties of the underlying $C'(1/6)$–small cancellation polygonal complex X. For these three types of spaces, the $C'(1/6)$–small cancellation polygonal complex, the polygons of $\mathcal{E}G$, and the fibre CAT(0) cube complexes, we describe the associated wallspace structures.

3.2.1 Walls of diametrically opposed edges

As usual, X denotes the $C'(1/6)$–polygonal complex constructed in Section 2.2. Note that what follows can be applied to an arbitrary $C'(1/6)$–polygonal complex.

We first put a wall structure on X. Then we discuss hypergraphs and walls on polygons of $\mathcal{E}G$. We define an equivalence class on the set of edges of X as follows. Two edges e and e' are said to be *diametrically opposed* or *opposite* if there exists a polygon R containing them and such that e and e' are diametrically opposed in R. We denote by $R_{\{e,e'\}}$ the associated polygon with doors.

Definition 3.11 (Equivalence class of opposite edges)**.** Two edges e and e' are *equivalent* if there is a sequence $e = e_1, \dots, e_n = e'$ of edges such that any two consecutive ones are diametrically opposed.

For an edge e of X, we define the complex with doors $\mathcal{C}_e^X$ to be the disjoint union of all the polygons with doors $R_{\{e_1,e_2\}}$ where e_1, e_2 are diametrically opposed and in the equivalence class of e. Observe that $\mathcal{C}_e^X$ is a gallery by definition. The far apart condition follows immediately from the fact that X is a $C'(1/6)$–polygonal complex. We denote the associated hypergraph by Λ_e^X, and the associated hypercarrier by Y_e^X. This coincides with Wise's hypergraphs and hypercarriers [Wis04, Definition 3.2, 3.3]. Theorem 3.9 implies:

Proposition 3.12 ([Wis04, Lemma 3.11, Theorem 3.18])**.** *Every hypergraph Λ_e^X embeds in X, is contractible and separates X into two connected components.* □

Definition 3.13 (Walls on X)**.** For every edge e of X, the associated hypergraph Λ_e^X separates X in two components. Let W_e^X be the wall of X associated with this decomposition. We say that W_e^X is the wall *associated with* e.

Let $\mathcal{W}^X$ be the set of all these walls.

Proposition 3.14 ([Wis04])**.** *The space X with the walls $\mathcal{W}^X$ is a wallspace. The wall pseudometric on X is a metric.* □

In Section 3.3.2, we lift the walls $\mathcal{W}^X$ to $\mathcal{E}G$.

Remark 3.15 (Hypergraphs and Walls on polygons)**.** Consider a single polygonal cell R on an even number of edges as a $C'(1/6)$–small cancellation polygonal complex. It then comes with the above defined hypergraphs and walls of diametrically opposed edges. We denote the hypergraph of R associated with e by Λ_e^R. The corresponding wall on R is denoted by W_e^R.

Up to taking a subdivision of $\mathcal{E}G$, this, in particular, endows each polygon of $\mathcal{E}G$ with a wallspace structure.

3.2.2 Hyperplanes in CAT(0) cube complexes

We recall some facts on hyperplanes in CAT(0) cube complexes. Let C be a CAT(0) cube complex. The building blocks of C are cubes, each k-cubing isomorphic to $[-1,1]^k$ for some integer $k \geq 0$. A cube hyperplane associated with a cube I is obtained by setting exactly one coordinate to zero, and is therefore of the form $[-1,1]^i \times \{0\} \times [-1,1]^j$ with $i+j=k-1$. A hyperplane on C is a connected nonempty subspace whose intersection with each cube I of C is either empty or a cube hyperplane associated with I. Every edge of C has a unique hyperplane intersecting it.

Proposition 3.16. *[Sag95, Th. 4.10, Th. 4.13] Let H be a hyperplane of C.*

- *The hyperplane H is contractible and separates C into two connected components.*
- *The neighbourhood of a hyperplane H is convex.* □

In particular, given two vertices of C there is a hyperplane separating them, and every hyperplane defines a wall of C. The following follows from the work of Sageev [Sag95].

Proposition 3.17. *A CAT(0) cube complex C with the collection of the complements of its hyperplanes as walls is a wallspace. The wall pseudometric on C is a metric.* □

In Section 3.3.3 we extend the walls in the fibres EG_v, using the walls on the polygons of $\mathcal{E}G$. We therefore need the following observations.

Lemma 3.18. *Let v be a vertex in X, and let EG_v be the corresponding fibre in $\mathcal{E}G$. Let $p_{v,R}$ be the attaching path where R is a polygon R of X. Suppose H is a hyperplane that crosses an edge e of $p_{v,R}$. Then,*

- *(Fibre separation) The hyperplane H separates the vertices of e* in *EG_v. In particular, the hyperplane intersects every path in EG_v that connects the starting and endpoint of the attaching path $p_{v,R}$.*
- *(No turns) The hyperplane H does not intersect $p_{v,R}$ more than once.* □

The first fact is immediate from the above properties of CAT(0) cube complexes. For the second fact recall that $p_{v,R}$ is geodesic in EG_v. Hence, a turn would contradict the convexity of hyperplanes in CAT(0) cube complexes.

3.3 Construction of the new walls

In this section we lift the walls of X to $\mathcal{E}G$, and explain how to combine the walls on the building blocks of $\mathcal{E}G$. The space $\mathcal{E}G$ is build up from the various CAT(0) cube complexes EG_v, modelled after the CAT(0) cube complexes E_A and E_B, and the various polygons of $\mathcal{E}G$. We just saw that these building blocks of $\mathcal{E}G$ are equipped with natural wallspace structures. The idea is to combine walls defined by the hyperplanes on the fibre CAT(0) cube complexes with the walls of opposite edges for polygons of $\mathcal{E}G$. We now observe that there is a priori no canonical way to do this. In particular, it is not possible to employ the viewpoint of Wise's seminal paper [Wis11, Section 5]: To adapt to the viewpoint of Wise, view the boundary path of a polygon $\widetilde{R}$ of $\mathcal{E}G$ as a cube complex, and $\widetilde{R}$ as a cone over this boundary path. It comes with the wall structure associated with opposite edges. Combining the walls of E_A, E_B and $\widetilde{R}$ as in [Wis11, Section 5.f], cf. Definition 3.26 below, does not yield walls; in particular, conditions (1), (2) and (3) of Lemma 5.13 in [Wis11] fail. Indeed, the subspaces we obtain with such a procedure no longer embed. More precisely, as the small cancellation condition over the free product of two groups does not control the length of the attaching paths, a hypergraph of diametrically opposed edges of $\widetilde{R}$ is likely to intersect two distinct edges of the same attaching path of the same fibre. The corresponding new hyperplane then consists of the two distinct hyperplanes associated with the aforementioned edges of that fibre and the hypergraph of diametrically opposed edges intersecting them. Note that we have no control of the position of these two hyperplanes of the fibre cube complex, meaning that they can intersect, osculate, or just not intersect any other attaching path, hence the claim.

3.3.1 Balancing

We now modify the complex $\mathcal{E}G$. This then allows us to combine the hyperplanes in the various CAT(0) cube complexes with the walls associated with opposed edges in polygons of $\mathcal{E}G$.

Definition 3.19 (the subdivided complexes X_k and $(\mathcal{E}G)_k$)**.** Let $k \geq 0$ be an even integer. We define a new polygonal structure from X by subdividing each edge of X exactly k times. We denote by X_k the resulting polygonal complex.

Similarly, we define a new polyhedral structure from $\mathcal{E}G$ by subdividing each horizontal edge, see Definition 2.17, exactly k times. We denote by $(\mathcal{E}G)_k$ this new polyhedral structure, and by

$$p : (\mathcal{E}G)_k \to X_k$$

the induced projection map.

Note that this procedure does *not* modify the CAT(0) cubical structures of the various fibres of $\mathcal{E}G$, and it does *not* modify the attaching paths. Moreover, each complex X_k does again satisfy the $C'(1/6)$–condition, and pieces of X_k are subdivisions of pieces of X.

Definition 3.20. (balanced) We say that $(\mathcal{E}G)_k$ is *balanced* if for every polygon $\widetilde{R}$ of $(\mathcal{E}G)_k$ and every edge e of $\widetilde{R}$ with opposite edge e', the projections $p(e)$ and $p(e')$ are far apart (see Definition 3.3) in X_k.

Lemma 3.21. *There exists an even integer $k \geq 0$ such that $(\mathcal{E}G)_k$ is balanced.*

Proof. Since the number of edges in the various attaching paths $p_{v,R}$ is uniformly bounded above by the maximum of the edge lengths of the geodesics $\gamma_{A,i}, \gamma_{B,i}$, the subdivided complex $(\mathcal{E}G)_k$ becomes balanced for k large enough by the $C'(1/6)$–condition. □

Definition 3.22. Let $k \geq 0$ be the smallest even number such that $(\mathcal{E}G)_k$ is balanced. We denote by $\mathcal{E}G_{bal}$ and X_{bal} the complexes $(\mathcal{E}G)_k$ and X_k respectively.

In the next section *the properties of X*, in combination with the properties of the fibre CAT(0) cube complexes, will be used to control the geometric structure of $\mathcal{E}G$. We first endow $\mathcal{E}G$ with a wall structure.

3.3.2 Lifted hypergraphs

The polygonal complexes X and X_{bal} satisfy the small cancellation condition $C'(1/6)$, hence the hypergraphs of diametrically opposed edges of Section 3.2.1 define a wallspace on X_{bal} We now lift the corresponding family of walls on X_{bal} to define a first family of walls on $\mathcal{E}G_{bal}$.

Definition 3.23 (hypergraph associated with an edge of X_{bal})**.** Let e be an edge of X_{bal} and Λ_e^X the hypergraph of diametrically opposed edges in X_{bal} defined in Section 3.2.1. We call Λ_e^X the hypergraph *associated with the edge* e of X_{bal}.

We define the subset $\widetilde{\Lambda_e^X}$ of $\mathcal{E}G_{bal}$ as the preimage of Λ_e^X under $p : \mathcal{E}G_{bal} \to X_{bal}$. We call $\widetilde{\Lambda_e^X}$ the *lifted hypergraph* (of $\mathcal{E}G_{bal}$) *associated with the edge* e of X_{bal}.

Lemma 3.24. *Each lifted hypergraph $\widetilde{\Lambda_e^X}$ of $\mathcal{E}G$ associated with an edge of X_{bal} is contractible and separates $\mathcal{E}G_{bal}$ into two connected components.*

Proof. We use Proposition 3.12. Note that p restricts to a homeomorphism $\widetilde{\Lambda_e^X} \to \Lambda_e^X$. Hence, $\widetilde{\Lambda_e^X}$ is contractible. The fact that $\widetilde{\Lambda_e^X}$ disconnects $\mathcal{E}G_{bal}$ follows from the fact that Λ_e^X disconnects X_{bal} into two components. The fact that $\mathcal{E}G_{bal} - \widetilde{\Lambda_e^X}$ has exactly two connected components follows from the fact that the preimage of a connected set under p is again connected. □

Definition 3.25 (wall of $\mathcal{E}G_{bal}$ associated with an edge of X_{bal})**.** We define the *wall of $\mathcal{E}G_{bal}$ associated with the edge* e of X_{bal} as $W_e^X := \widetilde{\Lambda_e^X}$.

Note that this family of walls is not large enough to define a wallspace structure on $\mathcal{E}G$ whose associated CAT(0) cube complex is endowed with a proper action, as this family of walls does not separate vertices in a given fibre.

3.3.3 Combining the walls on the building blocks

In this section, we combine walls on the building blocks of $\mathcal{E}G$ to a wall on the whole space $\mathcal{E}G_{bal}$. Let e be an edge of $\mathcal{E}G_{bal}$. If e is a vertical edge (that is, contained in one of the fibre CAT(0) cube complexes), we denote by H_e the hyperplane in that fibre associated with e. If e is a horizontal edge (that is, projects to an edge of X_{bal}), we denote by H_e the midpoint of e. In both cases we call H_e the *hyperplane associated with* e.

Definition 3.26. We define an elementary equivalence relation on the set of edges of $\mathcal{E}G_{bal}$ as follows. Two edges e, e' of $\mathcal{E}G_{bal}$ are said to be *elementarily equivalent*, and we denote it $e \sim_1 e'$, if one of the following situations occurs:

- e and e' are opposite edges in some polygon of $\mathcal{E}G_{bal}$,
- e, e' are vertical edges in the same fibre and the hyperplanes H_e and H_e' coincide.

The transitive closure defines an equivalence relation on the set of edges of $\mathcal{E}G_{bal}$.

Definition 3.27 (systems of doors associated with an edge of $\mathcal{E}G_{bal}$)**.** Let e be an edge of $\mathcal{E}G_{bal}$. We associate to e a system of doors $\mathcal{C}_e^{\mathcal{E}G}$ of X as follows. To every polygon $\widetilde{R}$ of $\mathcal{E}G_{bal}$ together with a pair of diametrically opposed edges $e_1, e_2 \in \widetilde{R}$ in the equivalence class of e, we associate a polygon with doors of $\mathcal{C}_e^{\mathcal{E}G}$ with underlying cell $p(\widetilde{R})$ and with doors being the projections $p(e_1)$ and $p(e_2)$.

Proposition 3.28. *The system of doors $\mathcal{C}_e^{\mathcal{E}G}$ is a gallery.* $\square$

Proof. We have to verify the conditions listed in Definition 3.6. The connectedness condition follows immediately from the definition. The doors of a given polygon of $\mathcal{C}_e^{\mathcal{E}G}$ are far apart because $\mathcal{E}G_{bal}$ is balanced. Suppose by contradiction that there exists a pair of polygons of $\mathcal{C}_e^{\mathcal{E}G}$ violating the coherence condition, i.e. a pair of polygons $R_{\{\tau_1,\tau_2\}}, R_{\{\tau_1',\tau_2'\}}$ of $\mathcal{C}_e^{\mathcal{E}G}$ with the same underlying cell R, such that $\tau_1 = \tau_1'$ and $\tau_2 \neq \tau_2'$. By the connectedness condition, let

$$R_{\{\tau_1,\tau_2\}}, R_{\{\tau_2,\tau_3\}}, \ldots, R_{\{\tau_{n-1},\tau_n\}}$$

be a sequence of polygons of $\mathcal{C}_e^{\mathcal{E}G}$ with $\tau_{n-1} = \tau_2', \tau_n = \tau_1$, and

$$R_1, R_2, \ldots, R_{n-1}$$

the associated sequence of underlying cells, with $R_1 = R_{n-1} = R$. We can assume that such a sequence of polygons is minimal.

We claim that, for $1 < i < j \leq n-1$, we have $R_i \neq R_j$. Indeed, if this was not the case, then either the set of doors $\{\tau_i, \tau_{i+1}\}$ of R_i is disjoint from the set of doors $\{\tau_j, \tau_{j+1}\}$ of R_j, or those polygons share a door. In the former case, $R_{\{\tau_i,\tau_{i+1}\}}, \ldots, R_{\{\tau_j,\tau_{j+1}\}}$ defines a gallery (the coherence condition now being trivially verified), the hypercarrier of which does not embed, contradicting

Theorem 3.9. In the latter case, this contradicts the minimality of the initial sequence of polygons. This proves our claim.

It now follows that $R_{\{\tau_2,\tau_3\}}, \dots, R_{\{\tau_{n-1},\tau_n\}}$ defines a gallery (the coherence condition being trivially verified), and we have $R_2 \cap R_{n-1} \neq \varnothing$ by hypothesis. Since $n \geq 4$ by the far apart condition, it follows that the hypercarrier of that gallery does not embed in X, contradicting Theorem 3.9. $\square$

Let $\Lambda_e^{\mathcal{E}G}$ be the hypergraph in X associated with $\mathcal{C}_e^{\mathcal{E}G}$. It follows from Proposition 3.28 and Proposition 3.9 that $\Lambda_e^{\mathcal{E}G}$ is a tree.

Lemma 3.29. *The hypergraph $\Lambda_e^{\mathcal{E}G}$ is a tree embedded in X.* $\square$

Definition 3.30 (wall of $\mathcal{E}G$ associated with an edge of $\mathcal{E}G_{bal}$)**.** Let e be an edge of $\mathcal{E}G_{bal}$. We define the *wall associated with* e as a tree of spaces over the hypergraph $\Lambda_e^{\mathcal{E}G}$ as follows. Let $\widetilde{R}$ be a polygon of $\mathcal{E}G$ and let e_1 and e_2 be opposite edges of a polygon $\widetilde{R}$ of $\mathcal{E}G_{bal}$, which are in the equivalence class of e, see Definition 3.26. Note that the polygon $p(\widetilde{R})$ of X_{bal}, together with the doors $p(e_1)$ and $p(e_2)$, defines a polygon of $\mathcal{C}_e^{\mathcal{E}G}$. We define

$$W_e^{\mathcal{E}G} := \bigcup_{\substack{e_1 \sim_1 e_2 \\ \text{in the equivalence class of } e}} \left(H_{e_1} \cup \Lambda_{e_1}^{\widetilde{R}} \cup H_{e_2}\right),$$

where $\Lambda_{e_1}^{\widetilde{R}}$ is the hypergraph of the polygon $\widetilde{R}$ defined in Remark 3.15. $W_e^{\mathcal{E}G}$ is called the *wall of $\mathcal{E}G$ associated with e.*

We readily observe that the above defined wall $W_e^{\mathcal{E}G}$ is a combination of hyperplanes of the various CAT(0) cube complexes of $\mathcal{E}G$ and hypergraphs of the various polygons of $\mathcal{E}G$.

Note that the projection of the wall $W_e^{\mathcal{E}G}$ under $p : \mathcal{E}G_{bal} \to X_{bal}$ is the hypergraph $\Lambda_e^{\mathcal{E}G}$ associated with the gallery $C_e^{\mathcal{E}G}$. Let us distinguish two types of walls $W_e^{\mathcal{E}G}$ associated with an edge of $\mathcal{E}G_{bal}$ according to their projections $\Lambda_e^{\mathcal{E}G}$ in X.

- The wall $W_e^{\mathcal{E}G}$ and its associated hypergraph $\Lambda_e^{\mathcal{E}G}$ are said to be *of first type* if $\Lambda_e^{\mathcal{E}G}$ consists of a single vertex.
- Otherwise, $W_e^{\mathcal{E}G}$ and $\Lambda_e^{\mathcal{E}G}$ are said to be *of second type*.

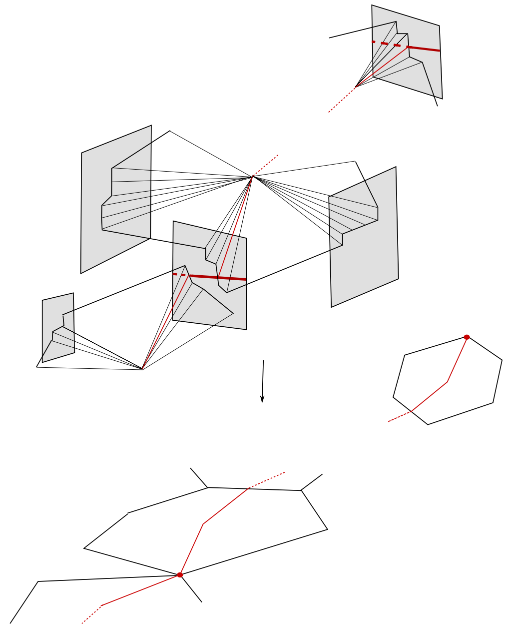

Figure 5: A portion of a wall associated with an edge of $\mathcal{E}G_{bal}$ of second type, together with its hypergraph. To avoid drawing too many edges, we assume here that $\mathcal{E}G$ is balanced, that is, $\mathcal{E}G = \mathcal{E}G_{bal}$.

Note that a wall $W_e^{\mathcal{E}G}$ associated with a vertical edge e of $\mathcal{E}G_{bal}$ is of first type if and only if e is contained in a fibre CAT(0) cube complex and the associated hyperplane crosses none of the attaching paths defined in Definition 2.18. An example where all occurring types of hypergraphs $\Lambda_e^{\mathcal{E}G}$and $\widetilde{\Lambda_e^{\mathcal{E}G}}$ are displayed is shown in Figure 7.

We now show that the walls of $\mathcal{E}G$ associated with edges of $\mathcal{E}G_{bal}$ are walls in the sense of Definition 3.1, that is, they separate $\mathcal{E}G$ into exactly two connected components. As noted in the introduction, the results and methods of [Wis11, Section 5] cannot be applied to conclude in our situation. Instead, we use, as already mentioned, the properties of hypercarriers in the $C'(1/6)$–polygonal complex X and the properties of the fibre CAT(0) cube complexes. Hence, we give a more direct approach to the cubulation problem.

Lemma 3.31. *A wall associated with an edge of $\mathcal{E}G_{bal}$ of first type is contractible and separates $\mathcal{E}G$ into two connected components.*

Proof. Let e be an edge of $\mathcal{E}G_{bal}$ whose associated hypergraph is of first type. The hypergraph is then completely contained in a CAT(0) cube complex of the form EG_v, that is, it coincides with one of the hyperplanes of EG_v, and such a hyperplane does not cross any attaching path. Thus, the wall is contractible and separates $\mathcal{E}G$ locally into two connected components by Proposition 3.16. Since $\mathcal{E}G$ is simply connected, the wall separates $\mathcal{E}G$ globally into two connected components. □

Lemma 3.32. *A wall associated with an edge of $\mathcal{E}G_{bal}$ of second type is contractible and separates $\mathcal{E}G$ into two connected components.*

Proof. Let e be an edge of $\mathcal{E}G_{bal}$ whose associated hypergraph is of second type. We first use properties of hypergraphs in X. Using Lemma 3.29, we observe that the wall associated with e has a structure of tree of spaces over $\Lambda_e^{\mathcal{E}G}$ with fibres being (contractible) hyperplanes. The contractibility of such a wall thus follows.

Since $\mathcal{E}G_{bal}$ is simply connected, it is enough to prove that the associated hypergraph separates locally $\mathcal{E}G_{bal}$ into two connected components. Therefore, we now use geometric properties of X to reduce the problem to the hyperplanes in the CAT(0) cube complexes.

The only non-trivial case to consider is the preimage of a neighbourhood of a vertex of X_{bal} contained in $\Lambda_e^{\mathcal{E}G}$, that is, a point of $\Lambda_e^{\mathcal{E}G}$ whose preimage in $\mathcal{E}G_{bal}$ is a hyperplane in the associate fibre. Let v be such a vertex of X_{bal} and H the hyperplane associated with an edge e on the attaching path $p_{v,R}$ in EG_v corresponding to a polygon of X.

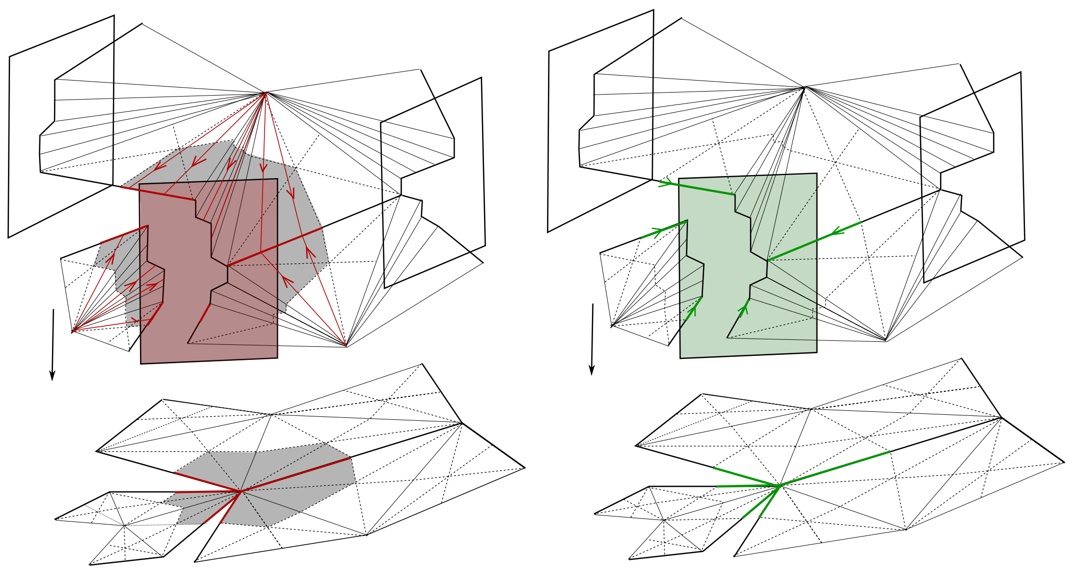

Figure 6: The construction of the projection map q_v. The star of v and its preimage in $\mathcal{E}G_{bal}$ are represented in shaded. On the left, the various radial projections $\widetilde{R} \cap S(v) \to \partial\widetilde{R} \cap S(v)$. On the right, the various projections $\partial\widetilde{R} \cap S(v) \to EG_v$.

We now work with a finer polyhedral structure on $\mathcal{E}G_{bal}$ obtained as follows. First consider the simplicial polygon associated with each polygon of $\mathcal{E}G_{bal}$ (as explained in Section 2.2), then take its first barycentric subdivision. For this new polyhedral structure, consider the star st(v) of v, that is, the union of all the simplices containing v. Denote by $S(v)$ the preimage of st(v) under the projection map $p : \mathcal{E}G_{bal} \to X_{bal}$. We define a projection map $q_v : S(v) \to EG_v$ in two steps.

Let R be a polygon of X_{bal} containing v and $\widetilde{R}$ its lift to $\mathcal{E}G_{bal}$. First retract radially $\widetilde{R} \cap S(v)$ onto $\partial\widetilde{R} \cap S(v)$, then retract $\partial\widetilde{R} \cap S(v)$ onto $\partial\widetilde{R} \cap EG_v$ (see Figure 6). It is straightforward to check that these projections are compatible and define a map from $S(v)$ to EG_v. Furthermore, by definition of $W_e^{\mathcal{E}G}$, q_v restricts to a surjective map from $S(v) \setminus W_e^{\mathcal{E}G}$ onto $EG_v \setminus H$.

Finally, we use the properties of CAT(0) cube complexes to conclude. As EG_v is a CAT(0) cube complex, by Proposition 3.28(1) the latter space is disconnected into exactly two components, so, using Proposition 3.28(2), is $S(v) \setminus W_e^{\mathcal{E}G}$. As the preimage under q_v of a path of EG_v is a connected subset of $S(v)$ and $EG_v \setminus H$ has exactly two connected components, $S(v) \setminus W_e^{\mathcal{E}G}$ has at most two connected components, hence it has exactly two connected components. □

We now have defined many walls on $\mathcal{E}G$: lifts of walls of X, and extension of hyperplanes of the fibre CAT(0) cube complexes to the whole space $\mathcal{E}G$. In the next section we use all these walls to define a wallspace structure on $\mathcal{E}G$ that makes $\mathcal{E}G$ a wallspace. We then associate a CAT(0) cube complex to such a structure.

3.4 The wallspace and its associated CAT(0) cube complex

In this section we combine the walls associated with edges of X_{bal}, Definition 3.25, and the walls associated with edges of $\mathcal{E}G_{bal}$, Definition 3.30. This yields a wallspace structure on $\mathcal{E}G_{bal}$. Figure 7 shows an example of $\mathcal{E}G$ with all three types of walls, together with their corresponding hypergraphs.

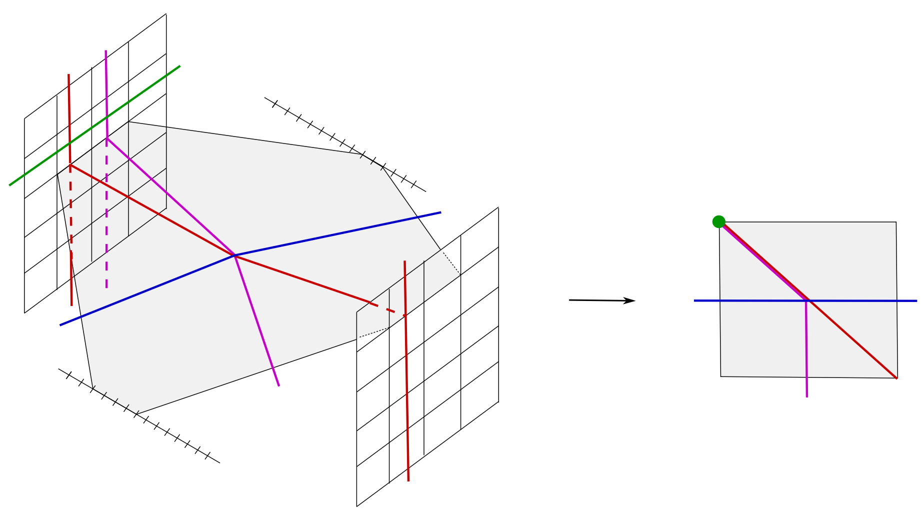

Figure 7: Examples of the three types of walls of $\mathcal{E}G$ (left) and their associated hypergraphs in X (right), in the case $A = \mathbb{Z}^2$, $B = \mathbb{Z}$. To avoid a busy picture, we only represent the case of a polygon of X with 4 sides. Blue: Walls/Hypergraphs associated with edges of X. Green: Walls/Hypergraphs associated with edges of $\mathcal{E}G_{\mathrm{bal}}$ of first type. Red and pink: Walls/Hypergraphs associated with edges of $\mathcal{E}G_{\mathrm{bal}}$ of second type.

Definition 3.33. We denote by $\mathcal{W}$ the family of walls of $\mathcal{E}G_{bal}$ consisting of:

- the walls associated with an edge of X_{bal},
- the walls of first type associated with an edge of $\mathcal{E}G_{bal}$,
- the walls of second type associated with an edge of $\mathcal{E}G_{bal}$.

We call an element of $\mathcal{W}$ a *wall* of $\mathcal{E}G_{bal}$.

The next result follows from combining the properties of the three types of walls that we have discussed above.

Proposition 3.34. *The complex $\mathcal{E}G$ with the previous family of walls $\mathcal{W}$ is a wallspace.* $\square$

The wallspace $(\mathcal{E}G, \mathcal{W})$ comes with an action of G, by setting $g \cdot W_e^X := W_{g\cdot e}^X$ and $g \cdot W_e^{\mathcal{E}G} := W_{g\cdot e}^{\mathcal{E}G}$ respectively.

Proof of Proposition 3.34. By definition and Lemma 3.24 every edge in X_{bal} defines a unique wall of $\mathcal{E}G$. By definition and Lemmas 3.31 and Lemmas 3.32 an edge of $\mathcal{E}G_{bal}$ defines a unique wall of $\mathcal{E}G$. Two vertices of $\mathcal{E}G_{bal}$ can be joined by a finite path in the 1-skeleton of $\mathcal{E}G_{bal}$. As a wall separating two vertices must cross every path connecting them, the result follows. $\square$

The proof immediately implies the following useful remark.

Corollary 3.35. *There is an upper bound on the number of walls of $\mathcal{E}G$, the hypercarriers of which contain a given polygon of X.* $\square$

Remark 3.36. We note that the wall-pseudometric on $\mathcal{E}G$ is a metric. Indeed, every pair of vertices of $\mathcal{E}G$ is separated by a wall. To see this first consider two vertices in the same fibre. By assumption the fibre is a CAT(0) cube complex. Then, the proof of Lemma 3.32 implies in particular that two such points are separated by at least one wall associated with a vertical edge. For vertices in two different fibres, as X is a $C'(1/6)$–complex it follows from the fact that the family of hypergraphs $\mathcal{W}^X$ separates any two vertices of X; this last statement follows directly from [Wis04, Lemma 4.3].

Let us now associate a CAT(0) cube complex to the wallspace $(\mathcal{E}G, \mathcal{W})$, and to the wallspace X. A vertex of this complex is a map $\sigma : \mathcal{W} \to \mathcal{H}$ sending each wall to one of the two half-spaces it defines, with some additional conditions, see [CN05]. Two vertices σ_1 and σ_2 are connected by an edge if σ_1 and σ_2 differ on exactly one wall.

Definition 3.37. Let $C_{\mathcal{W}}$ denote the CAT(0) cube complex associated with $(\mathcal{E}G, \mathcal{W})$.

Remark 3.38. The action of G on $\mathcal{W}$ induces an action of G on $C_{\mathcal{W}}$.

Remark 3.39. Using the definition of the CAT(0) cube complexes associated with a wallspace [CN05], one can show that the embedding of wallspaces associated with the embedding $EG_v \hookrightarrow \mathcal{E}G$ yields an embedding $EG_v \hookrightarrow C_{\mathcal{W}}$ of CAT(0) cube complexes which is equivariant with respect to the map $G_v \hookrightarrow G$.

Note however that there is a priori no link between the CAT(0) cube complex associated with the wallspace $(X, \mathcal{W}^X)$ and $C_{\mathcal{W}}$. Therefore, the results of Wise [Wis04] that are valid for C_X cannot directly be used to conclude anything about $C_{\mathcal{W}}$. It would technically be possible to reason solely with walls associated with the edges of $\mathcal{E}G_{bal}$ and the associated cube complex. However, adding walls associated with edges of X_{bal} only increases the dimension of the cube complex acted upon. We have decided to follow this approach as it seemed to us more natural from the viewpoint of the combination argument.

In the next section, we will combine results of Wise on the geometric positions of walls of $\mathcal{W}^X$ [Wis04, Lemma 6.4, Theorem 6.9, Theorem 11.1] with new results on the combination of such walls with the walls associated with edges of $\mathcal{E}G_{bal}$. This will be used to prove that the wallspace structure $\mathcal{W}$ on $\mathcal{E}G$ is such that the induced action on the associated CAT(0) cube complex $C_{\mathcal{W}}$ is proper and cocompact.

4 Cubulation theorem

The aim of this section is to prove our main result.

Theorem 4.1. *The action of G on the CAT(0) cube complex $C_{\mathcal{W}}$ is proper and cocompact.*

The following two criteria provide information about the group action on a cube complex from the properties of the action on the wallspace used to define this cube complex.

Proposition 4.2 (Theorem 3 in [CN05])**.** *Let H be a group acting by isometries on a space with walls $(Y, \mathcal{W}(Y))$, where Y is a metric space. The H-action on the associated CAT(0) cube complex is proper if for some $y \in Y$, we have $d_{\mathcal{W}(Y)}(y, h \cdot y) \to \infty$ when $h \to \infty$.* □

Proposition 4.3. *Let H be a group acting on a space with walls $(Y, \mathcal{W}(Y))$. The H-action on the associated cube complex is cocompact if and only if there exist only finitely many configurations of pairwise crossing walls of Y, up to the action of H.* □

Therefore, we continue to study the combination of the various type of walls underlying $C_{\mathcal{W}}$.

4.1 Properness

Theorem 4.4. *The action of G on $C_{\mathcal{W}}$ is proper.*

Let us mention, once again, that we do not follow a more general approach of Wise [Wis11, Th. 5.50]. This has an advantage of a more elementary proof. Again, we combine in an appropriate way properties of the fibre CAT(0) cube complexes and properties of the $C'(1/6)$–polygonal complex X.

Proof. We first prove that the wall distance $d_{\mathcal{W}}$ is proper, that is, for every vertex x of $\mathcal{E}G_{bal}$ and every integer $M \geq 0$, the set of vertices separated by at most M walls from x is compact. We give an inductive procedure to describe the set of vertices separated of x by at most M walls.

Let v be a vertex of X, x_v be a vertex of EG_v and $M \geq 0$ an integer. Let

$$K_0 := \{x \in \mathcal{E}G \mid x \text{ is a vertex of } EG_v \text{ and } d_{\mathcal{W}(EG_v)}(x_v, x) \leq M\},$$

be the ball in EG_v of radius M around x_v. As EG_v is a locally finite CAT(0) cube complex we see that K_0 is finite.

We now *orient* edges e of X by setting one vertex of e the initial and the other vertex the terminal vertex, denoted by $i(e)$ and $t(e)$ respectively. Given an oriented edge e we denote by $x_{i(e)}$ and $x_{t(e)}$ the respective attaching points of the lift $\widetilde{e}$ in $\mathcal{E}G$. Let us orient each edge e of X at v such that $i(e) = v$. Let E_0 be the set of those such edges with $x_{i(e)} \in K_0$.

Suppose we have inductively defined sets $K_0 \subseteq \ldots \subseteq K_k$ of vertices of $\mathcal{E}G$ and finite sets $E_0 \subseteq \ldots \subseteq E_k$ of oriented edges of X such that for every such edge $e \in E_i$ we have that $x_{i(e)} \in K_i, 0 \leq i \leq k$ and $x_{t(e)} \in K_{i-1}, 0 < i \leq k$. For every edge $e \in E_k - E_{k-1}$ denote by K_e the ball in $EG_{t(e)}$ of radius $M + d_{\mathcal{W}}(x_v, x_{t(e)})$ around x_v. Denote by E_e the set of edges e' of X at $t(e)$ such that $i(e') \in K_e$. Set

$$K_{k+1} := K_k \cup \bigcup_{e \in E_k} K_e,$$

and let

$$E_{k+1} := E_k \cup \bigcup_{e \in E_k} E_e.$$

Again, as the various spaces EG_v are CAT(0) cube complexes and $\mathcal{E}G$ is locally finite, the sets E_k and K_k are finite.

Since X is a $C'(1/6)$ polygonal complex, there exists a constant k_M such that a vertex of X at distance at least k_M from v is separated from v by at least M walls of X. Therefore and by construction, the set of vertices of $\mathcal{E}G$ which are separated from x_v by at most M walls of $\mathcal{E}G$ is contained in the set K_{k_M}. This set was shown to be finite, hence the claim.

Finally, let (g_n) be an injective sequence of elements of G. Since G acts properly discontinuously on $\mathcal{E}G$, there are for any integer $m \geq 0$ only finitely many $n \geq 0$ such that $g_n x_v \in K_m$. Thus, $d_{\mathcal{W}}(x_v, g_n x_v) \to \infty$, and the result now follows from Proposition 4.2. $\square$

Note that the proof of Theorem 4.4 uses only the fact that the various fibres are locally finite CAT(0) cube complexes, and that $\mathcal{E}G$ is a locally finite polyhedral complex, which follows from the fact that the fibres are locally finite and that G is obtained by considering only *finitely* many relators in $A * B$. In particular, redoing the whole construction in this more general framework, we obtain a proof of Theorem 3.

Corollary 4.5. *If A and B are only assumed to act properly on locally finite CAT(0) cube complexes EA and EB respectively, then G acts properly on $C_{\mathcal{W}}$.* $\square$

4.2 Cocompactness

Here we prove the cocompactness of the action on $C_{\mathcal{W}}$.

Theorem 4.6. *The action of G on $C_{\mathcal{W}}$ is cocompact.*

This follows once we have shown that $\mathcal{W}$ satisfies the assumptions of Proposition 4.3. In order to do that, we combine, again, the properties of the fibre CAT(0) cube complexes E_A and E_B with the properties of hypergraphs in the $C'(1/6)$–small cancellation polygonal complex X.

In particular, we use the following properties of CAT(0) cube complexes, cf. [Sag95, Ger97].

Theorem 4.7. *Let Y be a CAT(0) cube complex.*

- *Given a convex subcomplex of Y, its neighbourhood, that is, the union of all the cubes meeting it, is again convex.*
- *neighbourhoods of hyperplanes of Y are convex.*
- *(Helly's theorem) Let (Y_i) be a family of pairwise convex subcomplexes of Y such that any two such subcomplexes have a non-empty intersection. Then $\cap_i Y_i$ is non-empty.* □

We use the following result on the hypercarriers in X of pairwise crossing walls of $\mathcal{E}G_{bal}$.

Proposition 4.8. *Let $W_1, W_2, \ldots$ be a set of pairwise crossing walls of $\mathcal{E}G$, and let $Y_1, \ldots, Y_k$, $k \geq 3$, be the set of corresponding hypercarriers of X. Then the intersection $\bigcap Y_i$ is non-trivial.*

This result extends the following result of Wise.

Lemma 4.9 (cf. Theorem 6.9 of [Wis04])**.** *Let $\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$ be a set of pairwise crossing hypercarriers of X defined by equivalence of diametrically opposed edges, see Section 3.2.1. If $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ pairwise cross, then their common intersection contains a vertex.*

Let us emphasise once again, that Lemma 4.9 cannot directly be applied because our hypercarriers have cutpoints, and our far apart condition allows hypercarriers that differ significantly from those defined by equivalence classes of opposite edges.

Proof of Proposition 4.8. We consider three cases. If all walls $W_1, W_2, \ldots$ are associated with vertical edges in $\mathcal{E}G_v$, then v is contained in the intersection of their hypercarriers. This is the only configuration where a wall of first type can occur. If all walls $W_1, W_2, \ldots$ are walls coming from X, associated with edges of X_{bal}, then Wise's Lemma 4.9 immediately implies the claim. All other configurations contain no wall of first type, and at least one wall of second type. In this case, the proof of Wise's Lemma 4.9 can be extended in a straightforward way, using our generalised notions of hypergraphs and hypercarriers. Our far apart condition is again essential. We give a full account of the arguments in Appendix A.4, see Lemma A.24 □

Theorem 4.10. *There is only finitely many configurations of pairwise crossing walls of $\mathcal{E}G$, up to the action of G.*

Proof. Let l be the maximal length of an attaching path in the fibres of $\mathcal{E}G$. Let (W_i) be a system of pairwise crossing walls of $\mathcal{E}G$ and denote by (Y_i) the associated system of hypercarriers in X. By Lemma 4.8, let v be a vertex in the intersection of these hypercarriers. For each i, let K_i be the union of all the attaching paths $p_{v,R} \subset EG_v$, where R is a polygon of Y_i containing v. We now describe the sets K_i, depending on the relative position of the hypergraph and the vertex v, as illustrated in Figure 8.

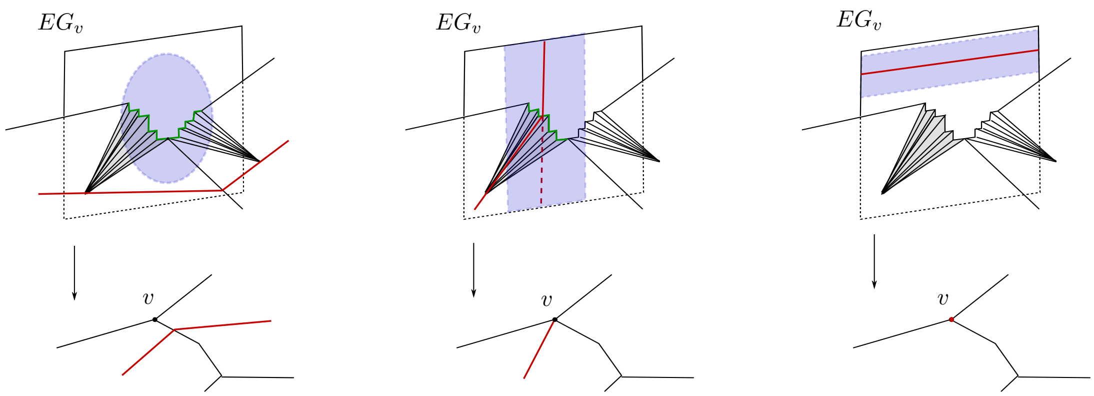

Figure 8: The three possible configurations, depending on the relative position of the hypergraph associated with W_i and the vertex v. In red: walls and hypergraphs. In green: K_i. In blue: C_i.

If $W_i \cap EG_v$ is empty, then the vertex v either belongs to an exterior arc of a polygon of Y_i, or v belongs to a door-tree of Y_i. In the former case, K_i consists of the single attaching path $p_{v,R}$. We then denote by u the starting vertex of $p_{v,R}$ in EG_v. In the latter case, all the polygons R_j of Y_i containing v share a common edge containing v. Then K_i consists of the union of all the attaching path p_{v,R_j}. These paths intersect in one vertex in EG_v, that we denote by u. In both cases, let C_i be the $2l$-ball around u. It follows that K_i is contained in C_i.

If $W_i \cap EG_v$ is nonempty, then W_i is a wall associated with a vertical edge of $\mathcal{E}G_{\mathrm{bal}}$ of first or of second type. If W_i is a wall of first type associated with an edge of $\mathcal{E}G_v$, then $W_i \cap EG_v = W_i$, and K_i is empty. Then let C_i be the $2l$-neighbourhood of the hyperplane corresponding to W_i.

If W_i is a wall of second type associated with an edge of of EG_v, then let C_i be the $2l$-neighbourhood of the hyperplane $W_i \cap EG_v$. By definition the attaching path of any polygon of Y_i containing v must cross the hyperplane $W_i \cap EG_v$. Thus the subset K_i is contained in C_i.

For two given indices i and j, we have that $C_i \cap C_j \neq \varnothing$. Indeed, if W_i and W_j cross in EG_v this is immediate. If W_i and W_j do not cross in EG_v, then choose a cell R of X_{bal} whose preimage in $\mathcal{E}G_{bal}$ contains a point of $W_i \cap W_j$. Choose a vertex w of R other than v, and consider a geodesic between v and w. By Proposition 3.9, such a geodesic is contained in $Y_i \cap Y_j$. In particular, the unique edge of that geodesic containing v is in $Y_i \cap Y_j$, which implies that $C_i \cap C_j \neq \varnothing$. The various subcomplexes C_i are convex by Theorem 4.7. Thus, Helly's theorem implies that the intersection $\cap_i C_i$ is non-empty. Let w be a vertex in this intersection, and let C be the $4l$-ball around w. Note that, as EG_v is a locally finite CAT(0) cube complex, the set C is finite.

Let us now consider two cases. First suppose all hypergraphs W_i intersect in EG_v. Then for each hypergraph W_i there is an edge e_i contained in C such that the hyperplane associated with e_i equals $\widehat{\Lambda_i} \cap EG_v$. Therefore the information that is necessary to reconstruct such a situation is contained in the finite subset C of EG_v.

Now suppose that at least one hypergraph W_j does not contain v, that is $W_j \cap EG_v = \varnothing$. First note that there is no wall of first type in this situation. Then observe that C contains K_j. Hence C contains the attaching paths $p_{v,R}$ contained in $K_i \cap K_j$ for all i. Thus, C contains an attaching path associated with W_i for all i. Again, as C is finite, there are only finitely many attaching paths contained in C, and therefore the information that is necessary to reconstruct such a situation is contained in the finite subset C of EG_v.

Since the action of A on EA (resp. of B on EB) is cocompact, choose a compact subcomplex K_A (resp. K_B) of EG_{v_A} (resp. EG_{v_B}) which contains an A-translate (resp. a B-translate) of every $4l$-ball of EG_{v_A} (resp. EG_{v_B}).

Let g be an element of G which sends C to a subcomplex of $K_A \cup K_B$. In the first case above, as the fibres are locally finite there are only finitely many possibilities for the walls (gWi). In the second above case, let $\mathcal{P}$ be the set of polygons of $\mathcal{E}G$ such that one of their attaching paths meet K_A or K_B. This set is finite since the action of A on EA (resp. of B on EB) is properly discontinuous. As $\mathcal{P}$ is finite, and by Corollary 3.35, there are only finitely many possibilities for the walls (gW_i). Hence, in total there are only finitely many possibilities for the walls (gW_i). □

Theorem 4.6 now follows from Proposition 4.3 and Theorem 4.10. □

A Appendix: Small cancellation polygonal complexes

Let us denote by X a $C'(1/6)$ polygonal complex. Here, we study the geometry of X. The results can then be applied to the $C'(1/6)$ polygonal complex defined in Section 2.2.

A.1 Classification of disc diagrams

Definition A.1 (disc diagram over X, reduced disc diagrams, arcs)**.** A disc diagram D *over the* $C'(1/6)$ *polygonal complex* X is a contractible planar polygonal complex endowed with a map $D \to X$ which is an embedding on each polygon. A disc diagram D over X is called *reduced* if no two distinct polygons of D that share an edge are sent to the same polygon of X.

For a disc diagram D, we denote by ∂D its boundary and $\mathring{D}$ its interior. The *area* of a diagram D, denoted Area(D), is the number of polygons of D. For a polygon R of D, the intersection $\partial R \cap \partial D$ is called the *outer component* of R (and the *outer path* if such an intersection is connected), the closure of $\partial R \cap \mathring{D}$ is called the *inner component* of R (and the *inner path* if such an intersection is connected).

A diagram is called *non-degenerate* if its boundary is homeomorphic to a circle, *degenerate* otherwise. An *arc* of D is a path of D whose interior vertices have valence 2 and whose boundary vertices have valence at least 3. Such an arc is called *internal* if its interior is contained in $\mathring{D}$, *external* if the arc is fully contained in ∂D.

We have the following fundamental result:

Theorem A.2 (Lyndon–van Kampen)**.** *Every loop of X is the boundary of a reduced disc diagram.* $\square$

All disc diagrams considered in this Appendix will be reduced without further notice. We now present a classification theorem for reduced disc diagrams.

Definition A.3 (ladder)**.** A reduced disc diagram D of X is a *ladder* if it can be written as a union $D = c_1 \cup \ldots \cup c_n$, where the c_i are edges or polygons of X and such that:

- $D \setminus c_1$ and $D \setminus c_n$ are connected,
- $D \setminus c_i$ has exactly two connected components for $1 < i < n$.

Definition A.4 (shell, spur)**.** Let D be a reduced disc diagram of X. A *shell* of D is a polygon of D such that $\partial R \cap \partial D$ is connected and whose inner path is the concatenation of at most 3 internal arcs of D. A *spur* of D is an edge of D with a vertex of valence 1.

Remark A.5. Note that the internal arcs involved in the previous definition are automatically sent to pieces of X by the properties of a reduced disc diagram.

The following is the fundamental result of small cancellation theory (a version of the well-known Greendlinger Lemma, see Theorem 4.5 in [LS77, Chapter V.4]). This version follows directly from Theorem 9.4 of [MW02].

Theorem A.6 (Classification Theorem for disc diagrams)**.** *Let D be a reduced disc diagram of X. Then either:*

- *D consists of a single vertex, edge or polygon,*
- *D is a ladder,*
- *D contains at least three shells or spurs.* $\square$

The proof of this theorem is based on a negative curvature phenomenon described via a version of Gauß-Bonnet's Theorem. We now explain this theorem as it is used later.

Definition A.7 (corner, disc diagram with angles)**.** A *corner* of a (reduced) disc diagram D of X is a pair (v, R) where v is a vertex of D and R a polygon containing it. We denote by Corner(v) (resp. Corner(R)) the set of corners of the form (v, R') (resp. (v', R)).

We say that D is a disc diagram *with angles* if each corner c is assigned an *angle* $\angle(c) \geq 0$.

For a vertex v of D, we define its *curvature*:

$$\kappa(v) = 2\pi - \pi \cdot \chi(\text{link}(v)) - \sum_{c \in \text{Corner}(v)} \angle(c).$$

For a polygon R of X, we define its *curvature*:

$$\kappa(R) = \sum_{c \in \text{Corner}(R)} \angle(c) - \pi \cdot |\partial R| + 2\pi.$$

Theorem A.8 (Gauß-Bonnet Theorem)**.** *For a (reduced) disc diagram of X with angles, we have:*

$$\sum_{v \text{ vertex of } D} \kappa(v) + \sum_{R \text{ polygon of } D} \kappa(R) = 2\pi.$$

□

A.2 Hypercarriers embed

Galleries were defined in Definition 3.6. We prove the following result, which generalises a result of Wise [Wis04, Lemma 3.11]:

Proposition A.9. *Let $\mathcal{C}$ be a gallery. Then its hypercarrier $Y_{\mathcal{C}}$ is connected and simply connected and the map $i_{\mathcal{C}} : Y(\mathcal{C}) \to X$ is en embedding.*

The proof of this proposition is using all three properties of a gallery, in particular the far apart condition. Extending the arguments of [Wis04] in a straight-forward way, we give the detailed proof below.

Lemma A.10. *Let $\mathcal{C}$ be a gallery and let $R_{\{\tau_1,\tau_2\}}$ be a polygon of $\mathcal{C}$. Let $P_1, P_2 \subset \partial R_{\{\tau_1,\tau_2\}}$ be distinct paths such that the concatenations $\tau_1 P_1$ and $\tau_2 P_2$ are pieces of X. Then* no *connected component of $\partial R \setminus (\tau_1 P_1 \cup \tau_2 P_2)$ is covered by a single piece, and P_1 and P_2 are disjoint.*

Proof. If a connected component C of $\partial R \setminus (\tau_1 P_1 \cup \tau_2 P_2)$ is covered by a single piece, the path from τ_1 to τ_2 covering C consists of at most three pieces. This contradicts the far apart condition. If P_1 and P_2 intersect, the path covering $\tau_1 P_1$ and $\tau_2 P_2$ consists of at most two pieces, again contradicting the far apart condition. □

Definition A.11 (canonical decomposition of a 2-cell, exterior arcs, door-trees)**.** Let $\mathcal{C}$ be a gallery with hypercarrier $Y_{\mathcal{C}}$ and let $R_{\{\tau_1,\tau_2\}}$ be a polygon of $\mathcal{C}$. Let $P_1, P_1', P_2, P_2' \subset \partial R_{\{\tau_1,\tau_2\}}$ be maximal paths such that the concatenations $\tau_1 P_1, \tau_1 P_1', \tau_2 P_2, \tau_2, P_2'$ are pieces.

Let $A, A' \subset \partial R_{\{\tau_1,\tau_2\}}$ be the paths joining the extremities of P_1, P_2 and P_1', P_2', called the *exterior arcs* of $R_{\{\tau_1,\tau_2\}}$.

The union of all the paths of the form $\tau_1 P_1$ and $\tau_1 P_1'$, where R runs over the polygons of $\mathcal{C}$ containing τ_1 as door, is a tree, called the *door-tree* associated with the door τ_1.

By definition of $Y_{\mathcal{C}}$, no edge of A or A' is identified to the edge of a distinct polygon of $Y_{\mathcal{C}}$ which is glued to $R_{\{\tau_1,\tau_2\}}$ along either τ_1 or τ_2. This implies in particular that two distinct polygons of $Y_{\mathcal{C}}$ sharing a door of $\mathcal{C}$ are sent to different polygons of X. As the map $i_{\mathcal{C}} : Y_{\mathcal{C}} \to X$ is already an immersion at the level of the 1-skeleton, the following follows:

Corollary A.12. *Let $\mathcal{C}$ be a gallery of X. Then the map $i_{\mathcal{C}} : Y_{\mathcal{C}} \to X$ is an immersion.* □

Lemma A.13. *Let $\mathcal{C}$ be a gallery of X. Let R be a polygon of X meeting $i_{\mathcal{C}}(Y_{\mathcal{C}})$ which does not contain a door of $\mathcal{C}$. Let P be a path of $\partial R \cap i_{\mathcal{C}}(Y_{\mathcal{C}})$ which admits a lift to $Y_{\mathcal{C}}$ under $i_{\mathcal{C}}$. Then P is covered by the concatenation of at most two pieces.*

Proof. Lemma A.10 implies that P cannot cover a complete exterior arc A. Thus, either P is a proper subpath of A, or P intersects exactly two polygons of $\mathcal{C}$. In the former case, P is covered by one piece, in the latter case P is covered by two pieces. □

Note that we have, so far, not used the connectedness nor the coherence condition in the definition of a gallery, see Definition 3.6.

Proof of Proposition A.9. The fact that $Y_{\mathcal{C}}$ is connected is a direct consequence of the connectedness condition.

We say that a path P of $Y_{\mathcal{C}}$ is *essential* if it is a loop representing a non-trivial element of the fundamental group of $Y_{\mathcal{C}}$, or if it is a path with distinct extremities which are sent to the same vertex of X. In the latter case, we call such a vertex of X the *unique singular vertex of* $i_{\mathcal{C}}(P)$. The proposition amounts to proving that there exists no essential path in $Y_{\mathcal{C}}$.

We reason by contradiction. Let P be such an essential path of $Y_{\mathcal{C}}$. Since X is simply-connected, the loop $i_{\mathcal{C}}(P)$ is the boundary of a disc diagram D. Notice first that D cannot be a single vertex or edge. Without loss of generality, we can assume that the number of polygons of D is minimal among such diagrams. In particular, D is non-degenerate and each path of its boundary $i_{\mathcal{C}}(P)$ that does not contain the singular vertex of $i_{\mathcal{C}}(P)$ lifts to a path of $P \subset Y_{\mathcal{C}}$.

First suppose that D is a single polygon. By hypothesis on P, D cannot be contained in $i_{\mathcal{C}}(Y_{\mathcal{C}})$. Let us decompose the boundary of D as the union of two paths P_1 and P_2 neither of which contains the singular vertex of $i_{\mathcal{C}}(Y_{\mathcal{C}})$ in their interior. Both paths P_1 and P_2 thus lift to paths of $Y_{\mathcal{C}}$. By Lemma A.10, this implies that P_1 and P_2 can be covered by the concatenation of two pieces, and so the boundary of D is covered by fours pieces, contradicting the condition $C'(1/6)$.

By the classification theorem A.6, this implies that the disc diagram D contains at least two shells, and we can choose one of these shells, say R, so that its outer path does not contain the singular vertex of $i_{\mathcal{C}}(Y_{\mathcal{C}})$ in its interior. Such a shell must be contained in $i_{\mathcal{C}}(Y_{\mathcal{C}})$, for otherwise Lemma A.13 would imply that $R \cap \partial D$ is covered by two pieces, making the boundary of R covered by five pieces, a contradiction with condition $C'(1/6)$. Thus $R \subset i_{\mathcal{C}}(Y_{\mathcal{C}})$ and we can push the path P through the lift of R in $Y_{\mathcal{C}}$ to obtain a new essential path, the image of which in X is the image in X of the boundary of the disc diagram $D \setminus R$. As such a diagram contains strictly fewer polygons than D, we get a contradiction. □

Corollary A.14. *For every gallery $\mathcal{C}$, the associated hypergraph $\Lambda_{\mathcal{C}}$ is a tree which embeds in X.*

Proof. It is enough by Proposition A.9 to see that the associated hypercarrier $Y_{\mathcal{C}}$ retracts by deformation onto $\Lambda_{\mathcal{C}}$. Such a deformation is easily defined using the canonical decomposition of a polygon of $\mathcal{C}$ introduced in Definition A.11. $\square$

Remark A.15 (minimal ladder between two simplices of a hypercarrier)**.** Let $\mathcal{C}$ be a gallery and τ and τ' be two simplices of $Y_{\mathcal{C}}$ that are not contained in the same door-tree of $Y_{\mathcal{C}}$. There exists a unique non-degenerate ladder of minimal area containing τ and τ', which we call the *(minimal) ladder of $Y_{\mathcal{C}}$ between τ and τ'.*

A.3 Convexity of hypercarriers

Here we prove the following:

Proposition A.16. *Let $\mathcal{C}$ be a gallery. Then the subcomplex $Y_{\mathcal{C}}$ of X is convex, that is, a geodesic between two vertices of $Y_{\mathcal{C}}$ is contained in $Y_{\mathcal{C}}$.*

We will prove that proposition by contradiction. Let us assume that there exists a geodesic P between two vertices of $Y_{\mathcal{C}}$ and such that P is not contained in $Y_{\mathcal{C}}$. Let Q be a path of $Y_{\mathcal{C}}$ joining the two extremities of P. The union of P and Q yields a loop of X, and thus there exists a disc diagram with such a loop as boundary. We choose P, Q and D in such a way that $(|P|, \text{Area}(D))$ is minimal for the lexicographic order. In particular, P does not cross the hypergraph $\Lambda_{\mathcal{C}}$. We now study separately three cases.

Lemma A.17. *The diagram D cannot consist of a single polygon.*

Proof. By contradiction, suppose that D consists of a single polygon R of X. Since R is not contained in $Y_{\mathcal{C}}$ by assumption, the path $Q = R \cap Y_{\mathcal{C}}$ is covered by at most two pieces by Lemma A.13. Thus, condition $C'(1/6)$ implies that $|Q| < \frac{1}{2}|\partial D|$, hence $|P| > \frac{1}{2}|\partial D| > |Q|$, contradicting the fact that P is a geodesic. $\square$

Lemma A.18. *The diagram D cannot contain three shells.*

Proof. By contradiction, suppose that D contains three shells. We can thus choose one of them, say R, whose outer boundary is contained either in P or in Q.

First assume that such an outer path is contained in P. We can thus push P through R to get a new path P' such that the union $P' \cup Q$ is the boundary of the disc diagram $D \setminus R$. Let L be the concatenation of the inner arcs of R. Since R is a shell, the $C'(1/6)$–condition implies $|L| < \frac{1}{2}|\partial R|$, hence $|P'| < |P|$, a contradiction.

Assume now that this outer path of R is contained in Q. First notice that R has to be contained in $Y_{\mathcal{C}}$, for otherwise such an arc would be covered by two pieces by Lemma A.13 and since R is a shell the whole of ∂R would be covered by five pieces, contradicting the $C'(1/6)$–condition. Thus $R \subset Y_{\mathcal{C}}$ and we can push Q through R to obtain a new path Q' of $Y_{\mathcal{C}}$ such that $P \cup Q'$ is the boundary of the disc diagram $D \setminus R$, contradicting the minimality of D. $\square$

Lemma A.19. *The disc diagram D cannot be a ladder.*

Proof. By contradiction, suppose that D is a (non-trivial) ladder. The minimality assumption implies that D is non-degenerate. Let us write $D = R_1 \cup R_2 \cup \dots$ and let P_1 be the portion of P contained in R_1, and P_2 its complement in R_1.

We can push P through R_1 to obtain a new path P_1'. Since P does not cross $\Lambda_{\mathcal{C}}$, R_1 is not contained in $Y_{\mathcal{C}}$ and thus $R_1 \cap Y_{\mathcal{C}}$ is covered by two pieces by Lemma A.13. As $R_1 \cap R_2$ is also a piece, it follows that P_2 is covered by three pieces, and condition $C'(1/6)$ now implies $|P_2| < \frac{1}{2}|\partial R_1| < |P_1|$, a contradiction. □

Proof of Proposition A.16. This follows from Lemmas A.17, A.18, A.19, together with the classification theorem for disc diagrams A.6. □

Corollary A.20. *Polygons of X are convex.* □

A.4 Intersections of hypercarriers

In this section, we extend the following results of Wise.

Lemma A.21 (cf. Lemma 6.4 of [Wis04])**.** *Let Y_1, Y_2 and Y_3 be hypercarriers of X defined by equivalence of diametrically opposed edges, see Section 3.2.1. If Y_1, Y_2 and Y_3 pairwise cross, then their common intersection is non-trivial.*

Lemma A.22 (cf. Theorem 6.9 of [Wis04])**.** *Let $\{Y_1, Y_2, Y_3, \dots\}$ be a set of pairwise crossing hypercarriers of X defined by equivalence of diametrically opposed edges, see Section 3.2.1. If Y_1, Y_2 and Y_3 pairwise cross, then their common intersection contains a vertex.*

Again, the proofs are extensions of Wise's original proofs, the small difference being related to cut-points in hypercarriers. The generalised hypercarriers coming from the far apart condition play no particular role here, as we treat them with the results of the previous sections. However, the corresponding results of [Wis04] are not sufficient.

Lemma A.23. *Let Y_1, Y_2, Y_3 be three pairwise crossing hypercarriers of X_{bal}. Then the intersection $Y_1 \cap Y_2 \cap Y_3$ contains a vertex.*

Proof. We can restrict to the case where $Y_1 \cap Y_2 \cap Y_3$ does not contain a polygon. First choose cells $\sigma_{1,2} \subset Y_1 \cap Y_2$, $\sigma_{2,3} \subset Y_2 \cap Y_3$ and $\sigma_{3,1} \subset Y_3 \cap Y_1$ of maximal dimension such that the preimage of $\sigma_{i,j}$ in $\mathcal{E}G_{bal}$ contains a point of $W_i \cap W_j$. The cell $\sigma_{i,j}$ is either a polygon $R_{i,j}$ or a vertex $v_{i,j}$. In the former case, the hypergraphs of Y_i and Y_j intersect in the apex of $R_{i,j}$, in the latter, the fibre over $v_{i,j}$ contains both, a hyperplane of Y_i, and a hyperplane of Y_j.

If two of these cells $\sigma_{i,j}$ coincide, then it defines a cell in $Y_1 \cap Y_2 \cap Y_3$. Suppose this is not the case. For pairwise distinct $i, j, k \in \{1, 2, 3\}$, consider the minimal ladder L_i in Y_i between $\sigma_{i,j}$ and $\sigma_{i,k}$. We choose such a configuration in such a way that the number of polygons in $L_1 \cup L_2 \cup L_3$ is minimal. Denote by $\lambda_1 \subset L_1$ the portion of the *hypergraph* Λ_1 *associated with* Y_1 which is the geodesic of Λ_1 joining the barycentres of $\sigma_{1,2}$ and $\sigma_{3,1}$, and define similarly $\lambda_2 \subset L_2$ and $\lambda_3 \subset L_3$. Subdivide the polygons of $L_1 \cup L_2 \cup L_2$ in a minimal way such that $\lambda_1 \cup \lambda_2 \cup \lambda_3$ defines a triangle of the 1-skeleton of X. Denote by $v_{i,j}$ the vertex associated with the cell $\sigma_{i,j}$. Consider now a reduced disc diagram whose boundary path is $\lambda_1 \cup \lambda_2 \cup \lambda_3$. We now endow D with a structure of disc diagram with angles:

- If $\sigma_{i,j}$ is a polygon $R_{i,j}$, the corner at the vertex corresponding to $R_{i,j}$ is given the angle $\frac{(n_{i,j}-3)\pi}{3}$, where $n_{i,j}$ is the number of sides of the polygon of D containing that vertex. Note that by minimality of the number of polygons in $L_1 \cup L_2 \cup L_3$, we necessarily have $n_{i,j} \geq 4$. If $\sigma_{i,j}$ is a vertex $v_{i,j}$, then by minimality of the number of polygons of $L_1 \cup L_2 \cup L_3$, there are at least two distinct polygons of D containing $v_{i,j}$.
- Each other corner of D relying on an edge of ∂D is given an angle $\frac{\pi}{2}$.
- All remaining corners are given an angle $\frac{2\pi}{3}$.

It is straightforward to check that with such a choice of angles, every polygon and every vertex of D has non-positive curvature by the $C'(1/6)$–condition, apart maybe from the the vertices corresponding to the various $R_{i,j}$. The curvature at each such vertex being at most $\frac{2\pi}{3}$, it must be exactly $\frac{2\pi}{3}$ by the Gauss Bonnet Theorem A.8 (in particular, each $\sigma_{i,j}$ is a polygon $R_{i,j}$). Thus, there is no vertex or polygon with negative curvature. In particular, since an internal polygon of D would have at least 7 sides by the $C'(1/6)$–condition, and since such a cell would have negative curvature, D contains no internal polygon. Thus the image of D is contained in $L_1 \cup L_2 \cup L_3$ and $L_1 \cap L_2 \cap L_3$, hence $Y_1 \cap Y_2 \cap Y_3$, must be non-empty. ☐

Lemma A.24. *Let $Y_1, \ldots, Y_k$, $k \geq 3$, be a set of pairwise crossing hypercarriers of X_{bal}. Then the intersection $\bigcap Y_i$ contains a vertex.*

Proof. We again use the methods we have developed in Section 3.3.3 and this Appendix to extend the original arguments of Wise's proof of Lemma A.22. We prove the result by induction on $k \geq 3$, the case $k = 3$ being Lemma A.23. For a subset S of $I := \{1, \ldots, k\}$, we denote by Y_S the intersection of the hypergraphs Y_i for $i \in S$.

By the induction hypothesis, the intersections $Y_{I-\{1\}}$, $Y_{I-\{2\}}$ and $Y_{I-\{3\}}$ contain a vertex, denoted respectively v_1, v_2 and v_3. Choose a geodesic between v_i and v_j for $1 \leq i \neq j \leq 3$, which we denote $P_{i,j}$. By Proposition A.16, we have that $P_{i,j} \subset Y_{I-\{i,j\}} \subset Y_k$.

If Y_i is a hypercarrier defined by equivalence of diametrically opposed edges, see Section 3.2.1, its boundary ∂Y_i is the disjoint union of two trees, $\partial_+ Y_i$ and $\partial_- Y_i$ and Y_i retracts by deformation on each of these trees. The situation is slightly different here since vertices can be local cut-points of Y_i. However, by reasoning separately on the closure of each component of Y_i with its cut-points removed, we can write ∂Y_i as the union of two trees $\partial_+ Y_i$ and $\partial_- Y_i$ whose intersection is contained in the set of cut-points of Y_i and such that Y_i retracts by deformation on each of these two trees.

We now consider two cases, depending on the relative position of v_1, v_2 and v_3 inside Y_k. First assume that v_1, v_2 and v_3 are contained in the same boundary component of Y_k, say $\partial_+ Y_k$. We can thus replace the paths $P_{i,j}$ by immersed paths $P'_{i,j}$ between v_i and v_j, and which is contained in the tree $\partial_+ Y_k$. In particular, the intersection $P'_{1,2} \cap P'_{2,3} \cap P'_{3,1}$ contains a vertex, which is thus contained in $Y_{I-\{1,2\}} \cap Y_{I-\{2,3\}} \cap Y_{I-\{3,1\}} = Y_I$.

Let us now assume that v_1 and v_2 are contained in the same component $\partial_+ Y_k$ and v_3 is contained in $\partial_- Y_k$. For $i = 1, 2$, consider the minimal ladder $L_{i,3} \subset Y_k$ between v_i and v_3 and define the path $P'_{i,3} := L_{i,3} \cap \partial_+ Y_k$. Consider the sequence of doors between v_3 and v_1, and between v_3 and v_2. If these sequences do not share the same initial door, then v_3 belongs to one of the exterior arcs of some polygon R of Y_k. Since both doors of R also belong to $Y_{I-\{3\}}$, this subcomplex contains a subpath of ∂R of length $\frac{|\partial R|}{2}$ by Corollary A.20. This implies that $R \subset Y_{I-\{3\}}$ by Lemma A.13, and thus the other exterior arc of R is contained $P'_{1,2} \cap Y_{I-\{3\}} \subset Y_I$. Otherwise consider the last door in this initial common subsequence. Then one of the vertices of this door is contained in $P'_{1,2} \cap Y_{I-\{3\}} \subset Y_I$. $\square$

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