Exponentials and Series

Contents

1.1 The Binomial Formula

This section is about expanding expressions like $(a+b)^8$ and $(1+x)^{100}$. First we need some notation.

The number $5!$ (read five factorial or factorial 5) means $5 \times 4 \times 3 \times 2 \times 1$. Note that $1! = 1$ and that by convention $0! = 1$. The reader should check by calculation that $4! = 24$ and use a calculator to check that $7! = 5040$ and $9! = 362880$.

We also need the binomial coefficient $\binom{n}{r}$ which is defined by

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Thus

$$\binom{8}{2} = \frac{8!}{2!\,6!} = \frac{8 \times 7}{2} = 28 \qquad \text{and} \qquad \binom{6}{3} = \frac{6 \times 5 \times 4}{6} = 20$$

You can also use a calculator to find the binomial coefficient; usually the nCr key. Note that we always have that

$$\binom{n}{0} = 1 \qquad \text{and} \qquad \binom{n}{n} = 1$$

The binomial formula is

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}\,b + \binom{n}{2} a^{n-2}\,b^2 + \cdots + \binom{n}{n-1} a\,b^{n-1} + \binom{n}{n} b^n$$

Note that the first term could be written as a^n and the last term as b^n. I have left the $\binom{n}{0}$ and $\binom{n}{n}$ terms in so the reader can see the pattern and hence make it easier to remember.

Example 1.1 *Use the binomial formula to expand* $(2+x)^3$.

Solution We use the binomial formula with $a = 2$, $b = x$ and $n = 3$. Then

$$\begin{aligned} (2+x)^3 &= \binom{3}{0} 2^3 + \binom{3}{1} 2^2\,x + \binom{3}{2} 2\,x^2 + \binom{3}{3} x^3 \\ &= 8 + 12x + 6x^2 + x^3 \end{aligned}$$

Example 1.2 *Use the binomial formula to expand* $(\theta + \frac{1}{\theta})^4$.

Solution

$$\begin{aligned} (\theta + \frac{1}{\theta})^4 &= \binom{4}{0}\theta^4 + \binom{4}{1}\theta^3\left(\frac{1}{\theta}\right) + \binom{4}{2}\theta^2\left(\frac{1}{\theta^2}\right) + \binom{4}{3}\theta\left(\frac{1}{\theta^3}\right) + \binom{4}{4}\left(\frac{1}{\theta^4}\right) \\ &= \theta^4 + 4\theta^3\frac{1}{\theta} + 6\theta^2\frac{1}{\theta^2} + 4\theta\frac{1}{\theta^3} + \frac{1}{\theta^4} \\ &= \theta^4 + 4\theta^2 + 6 + 4\theta^{-2} + \theta^{-4}. \end{aligned}$$

The binomial formula for the special case $(1+x)^n$ is worth remembering.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + x^n$$

Example 1.3 *Use the binomial formula to expand* $(1+x)^4$*.*

Solution Using the above formula with $n = 4$,

$$\begin{aligned}(1+x)^4 &= 1 + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + x^4 \\ &= 1 + 4x + 6x^2 + 4x^3 + x^4\end{aligned}$$

Example 1.4 *Expand* $(1+x)^{100}$ *up to the term in* x^2*.*

Solution

$$\begin{aligned}(1+x)^{100} &= 1 + \binom{100}{1}x + \binom{100}{2}x^2 + \text{terms in higher powers of } x \\ &= 1 + 100x + 4950x^2 + \text{terms in higher powers of } x\end{aligned}$$

Example 1.5 *Expand* $(1+x)^7$ *up to the term in* x^2*. Use your result to approximate* $(0.98)^7$.

Solution Using the same method as above we find that

$$(1+x)^7 = 1 + 7x + 21x^2 + \text{terms in higher powers of } x$$

Substituting $x = -0.02$ and ignoring higher powers ,

$$(0.98)^7 \approx 1 - 7(0.02) + 21(0.02)^2 = 0.8684$$

Example 1.6 *When a uniform beam is simply supported at its ends, the deflection* y *at the centre of the span is given by* $y = Cd^4$ *,where* d *is distance between the supports and* C *is a constant.*

Find the first two terms in the expansion of $(1+x)^4$ *and use this result to find the approximate change in* y *when* d *decreases by* 3% *.*

Solution Using the same method as above we find that $(1+x)^4 = 1 + 4x +$ terms in higher powers of x. For small x, $(1+x)^4 \approx 1 + 4x$. If d decreases by 3% then the new value of d is $d(1 - .03)$. The new value of y is $Cd^4(1 - .03)^4$.

Using $(1+x)^4 \approx 1 + 4x$ with $x = -.03$, gives the approximate new value of y as $Cd^4(1 - 0.12) = y(1 - 0.12)$. Hence y decreases by 12% approximately.

Exercises 1.1

1. State the binomial formula for $(a+b)^n$.

2. State the binomial formula for $(1+x)^n$.

3. Use the binomial formula to expand $(x+y)^3$.

4. Use the binomial formula to expand $(3+y)^4$.

5. Use the binomial formula to expand $(x+\frac{1}{x})^3$.

6. Use the binomial formula to expand $(1+x)^5$.

7. Expand $(1+x)^{20}$ up to the term in x^2.

8. Expand $(1+p^2)^{10}$ up to the term in p^4.

9. Expand $(1-x)^{10}$ up to the term in x^2.

10. Find the first two terms in the expansion of $(1+x)^6$ and use this result to find the approximate change in $y=p^6$ when p increases by 2%.

1.2 Geometric Sequences and Series

Situations in which things increase or decrease by some fixed ratio at each step are very common, e.g. interest on loans. A *geometric sequence* is some set of numbers of the form $\{a, ar, ar^2, ar^3, \ldots\}$ where a is the first term and r is the *common ratio.*

As an example, $\{3, 6, 12, 24, 48, \ldots\}$ is a geometric sequence with first term 3 and common ratio $r = 2$.

Geometric sequences arise in many areas including financial calculations. Suppose that you invest £500 at 10% interest compounded annually.

After 1 year the interest is 10% of 500, that is $500 \times (0.1) = 50$ so we now have £550. A more consise way to do the calculation is to note that at the end of year 1 we have £$500 \times (1.1)$.

At the end of the second year the interest is $550 \times (0.1) = 55$ so we now have £605. Again, the sum after two years is £$500 \times (1.1)^2$.

At the end of the third year the interest is $605 \times (0.1) = 60.5$ so we now have £665.5. Again, the sum after three years is £$500 \times (1.1)^3$.

The values in successive years, $500 \times (1.1)$, $500 \times (1.1)^2$, $500 \times (1.1)^3$ form a geometric sequence with common ratio $r = 1.1$. The value at the end of year n is $500 \times (1.1)^n$.

Example 1.7 *£1000 is borrowed for 2 years at a monthly compounded rate of interest of 2% . Explain why the amounts owed at the end of each month form a geometric sequence. Find the debt at the end of the period.*

Solution After 1 month $1000\times(1.02)$ is owed, after 2 months $1000\times(1.02)^2$ is owed and $1000\times(1.02)^n$ after n months, this is a geometric sequence with common ratio $r=1.02$. After 24 months the debt is $1000\times(1.02)^{24}=1604.44$ (Calculations like $(1.02)^{24}$ can be done on a calculator using the x^y button). Although the interest rate seems small, note how quickly the debt builds up.

If we sum up the members of the sequence it becomes a *geometric series.* The sum S_n of n terms of our geometric sequence $\{a, ar, ar^2, ar^3, \ldots\}$ is

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

As an example, for the geometric sequence $\{3, 6, 12, 24, 48, \ldots\}$ the sum of the first 6 terms is $S_6 = 3+6+12+24+48+96 = 189$.

The formula for getting the sum S_n of a geometric series is

$$S_n = \frac{a(r^n-1)}{r-1} \tag{1}$$

We show how to derive this formula at the end of the section. Examination questions about geometric series will always include the formula; you have to know how to use it.

Example 1.8 *Find the sum of 6 terms in the geometric sequence* $\{2, 6, 18, \ldots\}$

Solution I could work out the 6 terms and add them up, but we can use the formula above with $a=2, r=3, n=6$ to get

$$S_6 = \frac{2(3^6-1)}{3-1} = \frac{2(729-1)}{2} = 728$$

Example 1.9 *Mr Prudent saves £*100 *in a bank account at the beginning of each month. The bank offers a return of* 1% *compounded monthly. Find the total amount saved after* 12 *months.*

Solution Although a total of 12 payments of £100 are made, each payment is invested for a different period of time. The first payment is invested for 12 months so it is worth $100\times(1.01)^{12}$ at the end of the year. The second payment is invested for 11 months so it is worth $100\times(1.01)^{11}$ at the end. The final payment is invested for 1 month so it is worth 100×1.01 at the end. The total value at the end is

$$100(1.01)^{12} + 100(1.01)^{11} + 100(1.01)^{10} + \ldots + 100(1.01)$$

Writing this in reverse order the total is

$$100(1.01) + 100(1.01)^2 + \ldots + 100(1.01)^{11} + 100(1.01)^{12}$$

Thus we can apply the above formula with $a=100(1.01)$, $r=1.01$ and $n=12$:

$$\frac{100(1.01)\,[(1.01)^{12}-1]}{1.01-1} = 1280.93$$

Finally, we show the neat trick for getting the sum of a geometric series. Since

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

we see that

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$$

Subtracting the two gets rid of all but the first and last terms

$$rS_n - S_n = ar^n - a = a(r^n - 1)$$

which means that

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Exercises 1.2

1. For each of the following geometric sequences, find the common ratio and the next two terms: (a) $\{7, 21, 63, 189, \ldots\}$, (b) $\{17, 1.7, 0.17, 0.017, \ldots\}$.

2. A credit card company charges interest at 2% compounded monthly. Suppose that you owe £P to the company.

(a) Explain briefly why the amounts owed at the end of each month form a geometric sequence.

(b) Explain briefly why the interest rate charged by the company is not the same as an annual rate of 24%. What is the annual percentage rate ?

3. Find the sum of the first 8 terms in the geometric sequence $\{1, 3, 9, \ldots\}$ (Don't add them up by hand!)

4. A mathematically gifted child negotiates a pocket money deal with her parents : she receives 1p on the first day of the month, 2p on the second day, 4p on the third day , 8p on the fourth day and so on until the end of the month.

Find how much she would get (to the nearest million pounds) in a month of 30 days.

5. A person saves £1000 in a bank account at the beginning of each year. The bank offers a return of 8% compounded yearly. Find the total amount saved after 7 years.

6. A firm hires out computer equipment. In the first year the profit is £60000 but this diminishes by 5% on successive years. Explain briefly why the annual profits form a geometric sequence. Find the total of all profits during the first 6 years.

7. Find x if $x + 1$, $x + 11$, $x + 41$ are the first three terms in a geometric sequence.

1.3 Partial Fractions

The expression $\dfrac{2}{x-1} + \dfrac{5}{x+4}$ can be written as a single fraction by introducing the common denominator $(x-1)(x+4)$:

$$\frac{2}{x-1} + \frac{5}{x+4} = \frac{2(x+4)+5(x-1)}{(x-1)(x+4)} = \frac{7x+3}{(x-1)(x+4)}\,.$$

The method of partial fractions provides a means of performing the reverse operation.

We will study two basic types :

(i) The denominator of the fraction can be expressed as a product of linear factors and the numerator is linear; for example

$$\frac{7x+3}{(x-1)(x+4)}$$

(ii) The denominator of the fraction has a repeated factor and the numerator is linear; for example

$$\frac{11-3x}{(x-2)^2}$$

Example 1.10 *Express* $\dfrac{7x+3}{(x-1)(x+4)}$ *in terms of partial fractions.*

Solution Let

$$\frac{7x+3}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

We have to find the values of A and B so the above is true for all x.

Multiplying by $(x-1)(x+4)$:

$$\frac{(7x+3)(x-1)(x+4)}{(x-1)(x+4)} = \frac{A(x-1)(x+4)}{x-1} + \frac{B(x-1)(x+4)}{x+4}$$

Hence

$$7x+3 = A(x+4) + B(x-1)\,.$$

We choose values of x to make the term involving A or the term involving B disappear.

Put $x = 1$ then $7 \times 1 + 3 = A(1+4) + 0\,.$

Thus $10 = 5A$ so $A = 2\,.$

Put $x = -4$ then $7 \times (-4) + 3 = 0 + B(-4-1)\,.$

Hence $-25 = -5B$ so $B = 5\,.$

$$\frac{7x+3}{(x-1)(x+4)} = \frac{2}{x-1} + \frac{5}{x+4}$$

Example 1.11 *Express* $\dfrac{11-3x}{(x-2)^2}$ *in terms of partial fractions.*

Solution Let
$$\frac{11-3x}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

Note carefully the form that the partial fraction takes in this case.

Multiplying by $(x-2)^2$:
$$11-3x = A(x-2) + B\,.$$

Put $x=2$ then $11-6 = 0+B$ so that $B=5$.

To find A we equate the coefficents of x in the expression
$$11-3x = A(x-2)+B = Ax - 2A + B\,.$$

Hence $A=-3$ and
$$\frac{11-3x}{(x-2)^2} = \frac{5}{(x-2)^2} - \frac{3}{x-2}$$

It is important to start with the correct form of partial fractions. To express $\dfrac{2x-1}{(2x+1)(x-3)}$ in terms of partial fractions you start with letting
$$\frac{2x-1}{(2x+1)(x-3)} = \frac{A}{2x+1} + \frac{B}{x-3}$$
followed by multiplying by $(2x+1)(x-3)$.

To express $\dfrac{x+3}{(x+5)^2}$ in terms of partial fractions you would let
$$\frac{x+3}{(x+5)^2} = \frac{A}{x+5} + \frac{B}{(x+5)^2}$$
followed by multiplying by $(x+5)^2$.

Exercises 1.3

1. Express $\dfrac{2}{(x+7)(x+9)}$ in terms of partial fractions.

2. Express $\dfrac{2x+8}{(2x+5)(x+3)}$ in terms of partial fractions.

3. Express $\dfrac{5x+6}{(x-1)^2}$ in terms of partial fractions.

4. Express $\dfrac{2x-1}{(2x+1)(x-3)}$ in terms of partial fractions.

5. Express $\dfrac{10x+7}{(2x+3)^2}$ in terms of partial fractions.

1.4 Definition of Logarithm

If a and b are positive numbers we can always find x such that $a^x = b$. For example, if $a = 10$ and $b = 100$ then $x = 2$ since $10^2 = 100$. The number x is called the logarithm of b to base a, and we write $x = \log_a(b)$.

$$\boxed{\text{If } b = a^x \qquad \text{then} \qquad x = \log_a(b)}$$

As an example $1000 = 10^3$ so $\log_{10}(1000) = 3$.

If the base of the logarithm of b is 10 then we write $\log b$. On a calculator , logarithms to base 10 are usually denoted by $\log$. The reader should use a calculator to check that

$$\log(2) = 0.301 \qquad \log(45) = 1.653 \qquad \log(0.4) = -0.398$$

Most calculators have two different logarithm functions: base 10 logarithms and logarithms to base e The letter e stands for the number $2.718281\cdots$. On a calculator , logarithms to base e are usually denoted by $\ln$.

The reader should use the $\ln$ button on a calculator to check that

$$\ln(2) = 0.693 \qquad \ln(10) = 2.303 \qquad \ln(0.3) = -1.204$$

Note that $\ln A$ (and $\log A$) only make sense if $A > 0$. Hence, an expression such as $\ln(3x - 15)$ will only make sense if $3x - 15 > 0$, that is if if $x > 5$.

Example 1.12 *Show that* $\log_a 1 = 0$.

Solution If $x = \log_a 1$ then $a^x = 1$. Since $a^0 = 1$ we have that $x = 0$.

Exercises 1.4

1. Without using a calculator find : (a) $\log(10)$ (b) $\log(100)$ (c) $\log(0.001)$.

2. Evaluate $t = -\dfrac{\ln(x/y)}{10}$ when $x = 12$ and $y = 2$.

3. For what values of x does $\ln(7 - x)$ make sense.

4. For what values of x does $\ln(2x - 6)$ make sense. Find the value of x for which $\ln(2x - 6) = 0$.

5. The speed v of a signal in a submarine cable is related to the radius r of the cable's covering by the equation

$$v = \frac{25}{r^2} \ln\left(\frac{r}{5}\right)$$

Find v when $r = 10$.

1.5 Laws of Logarithms

Below we give the various laws of logarithms and show how to apply them.

Let B and C be positive numbers. Then for any base a:

1. $\log_a(BC) = \log_a(B) + \log_a(C)$
2. $\log_a\left(\frac{B}{C}\right) = \log_a(B) - \log_a(C)$
3. $\log_a(B^n) = n\log_a(B)$
4. $\log_a(1) = 0$

Example 1.13 *Reduce to a single log term :*
(a) $\log(x) + \log(x-3)$ *(b)* $4\log(2) + 2\log(3) - \log(12)$ *(c)* $\ln(x^2y) - \ln(y) + 3\ln(x)$.

Solution (a) Using rule 1, $\log(x) + \log(x-3) = \log(x\,(x-3)) = \log(x^2 - 3x)$.

(b) By rule 3, $4\log(2) = \log(2^4) = \log(16)$ and $2\log(3) = \log(9)$. Using rules 1 and 2,

$$4\log(2) + 2\log(3) - \log(12) = \log(16) + \log(9) - \log(12) = \log\left(\frac{16 \times 9}{12}\right) = \log(12)\,.$$

(c) $\ln(x^2y) - \ln(y) + 3\ln(x) = \ln(x^2) + \ln(y) - \ln(y) + \ln(x^3) = \ln(x^5)$.

Example 1.14 *Find the value of* x *which satisfies* $(1.06)^x = 2002$ *.*

Solution Taking logarithms to base 10 of each side of $(1.76)^x = 2002$ gives

$$\log((1.76)^x) = \log(2002)\,.$$

Using rule 3, $x\,\log(1.76) = \log(2002)$, so that

$$x = \frac{\log(2002)}{\log(1.76)} = \frac{3.301}{0.2455} = 13.45$$

Note that we could have taken logarithms to base e to obtain

$$x = \frac{\ln(2002)}{\ln(1.76)} = \frac{7.602}{0.5653} = 13.45$$

Exercises 1.5

1. If $\log 5 \sim 0.6989\ldots$, what is $\log 25$ (without taking a log on your calculator!!).

2. Reduce to a single log term :
(a) $\ln(xy^2) - 3\ln(y) + \ln(x)$ *(b)* $\ln x - 2\ln(3x+5)$.

3. Solve for x: $(1.1)^x = 1999$

4. The following equations occur in a problem on heat transfer:

$$-Q\ln r = mx \qquad \text{and} \qquad -Q\ln s = my$$

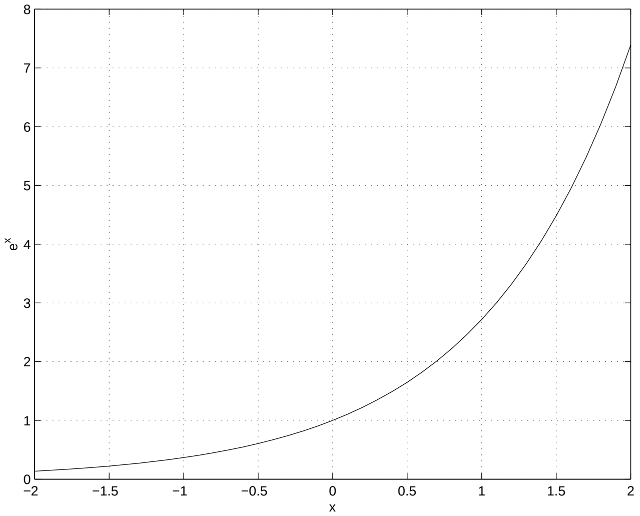
Show that

$$Q = \frac{m(x-y)}{\ln\left(\frac{s}{r}\right)}$$

1.6 The Exponential Function

We begin by looking at the graph of $y = e^x$

x	-2	-1	0	1	2	3	4
e^x	0.14	0.37	1	2.72	7.39	20.1	54.6



Graph of e^x

The function $y = e^x$ is called the exponential function. On a calculator , the exponential function is usually written as e^x . The reader should use the e^x button on a calculator to check that the above table. We can also write e^x as $\exp(x)$. The alternative notation $\exp(x)$ is is useful for complicated expressions like $\exp(-5x^2 + 3x)$.

The laws of indices apply to e^x so that

$$e^0 = 1\,, \qquad (e^x)^2 = e^{2x}\,, \qquad e^x\, e^y = e^{x+y}$$

If $y = e^x$ then taking taken logarithms to base e of each side gives

$\ln(y) = \ln(e^x) = x\ln(e) = x$ since $\ln(e) = 1$.

This result below is worth remembering:

$$\boxed{\text{if } y = e^x \quad \text{then} \quad x = \ln y}$$

Example 1.15 *Let* $f(t) = 3\exp(-5t^2)$ *. Find* $f(0)$ *and* $f(0.4)$ *.*

Solution $f(0) = 3e^0 = 3$.

When $t = 0.4$, $5t^2 = 0.8$. Using a calculator $f(0.4) = 3e^{-0.8} = 1.35$.

Example 1.16 *Simplify* $(e^{4x} - e^{-4x})^2$.

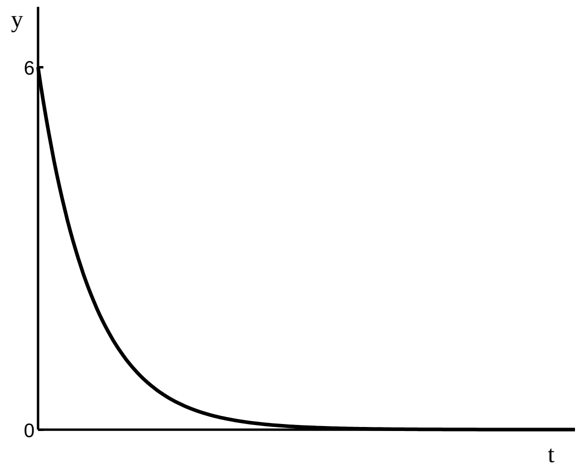
Solution $(e^{4x} - e^{-4x})^2 = (e^{4x})^2 - 2\,e^{4x}e^{-4x} + (e^{-4x})^2 = e^{8x} - 2 + e^{-8x}$.

Example 1.17 *Solve for* x : $100e^{-2x} = 26.2$.

Solution $e^{-2x} = 0.262$ so $-2x = \ln(0.262) = -1.34$. Hence $x = 0.67$.

Example 1.18 *Sketch the graph of* $y = 6e^{-2t}$ *for* $t \geq 0$.

Solution The main points are (a) $y = 6$ when $t = 0$, (b) $y \approx 0$ for large t , (c) y is rapidly decreasing. The graph is shown below.



$y = 6e^{-2t}$

Example 1.19 *Sketch the graph of* $y = 7 - 4e^{-2t}$ *for* $t \geq 0$. *Express* t *in terms of* y .

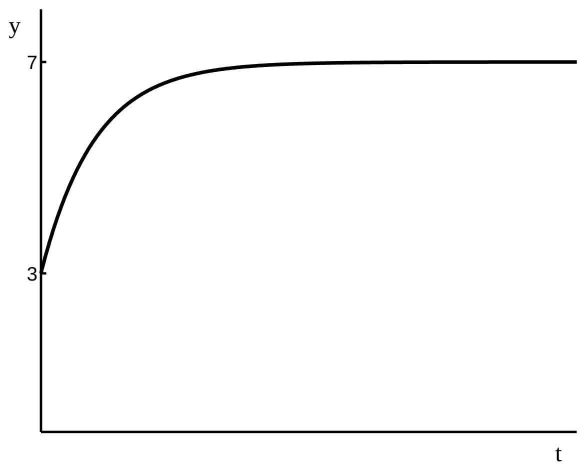
Solution The main points are (a) $y = 7 - 4 = 3$ when $t = 0$, (b) $y \approx 7$ for large t , (c) y is increasing. The graph is shown below.

$y = 7 - 4e^{-2t}$ so $4e^{-2t} = 7 - y$. Hence

$$e^{-2t} = \frac{7-y}{4} \qquad \text{and} \qquad -2t = \ln\left(\frac{7-y}{4}\right)$$

Hence

$$t = -\frac{1}{2}\ln\left(\frac{7-y}{4}\right)$$



$y = 7 - 4e^{-2t}$

Example 1.20 *A radioactive material decays according to* $m(t) = m_0 e^{-kt}$ *where* t *is the time (in years),* m_0 *is the initial mass (in grammes), and* $m(t)$ *is the mass at time* t *.*

A block of material with a mass of $500g$ *is reduced to* $50g$ *after* 40 *years. Find the time taken for half of its mass to decay.*

Solution We first find k. Since $m(40) = 50$ we have that $500e^{-40k} = 50$.

Then $e^{-40k} = 0.1$ so that $-40k = \ln(0.1) = -2.3025$ and $k = 0.0576$.

To find the time taken for half of its mass to decay we have to find t such that

$$500e^{-kt} = 250$$

with $k = 0.0576$. Then $e^{-kt} = 0.5$ so $-kt = \ln(0.5) = -0.693$ and $t = 12$ years.

Exercises 1.6

1. Let $f(t) = 2\exp(-t^2)$. Find $f(0)$ and $f(0.5)$.

2. Solve for x: $e^x = 666$

3. Simplify $(e^x - e^{-x})^2$.

4. On average the speed of computer processing hardware doubles every 18 months. This corresponds to an exponential growth for the speed, s, with $s = Ae^{kt}$ where A and k are constants. If t is measured in years, show that $k \approx 0.46$.

5. Sketch the graph of $y = 2e^{-4t}$ for $t \geq 0$.

6. Sketch the graph of $y = 3 - 2e^{-4t}$ for $t \geq 0$. Express t in terms of y.

7. Let $y = e^{3t}$. Express y in terms of x when

$$(a) \quad t = \ln x + \ln 2\,, \qquad (b) \quad t = -\ln x + 2$$

8. Let $z = \ln(x/y)$. If $x = me^{-at}$ and $y = me^{-bt}$, show that $z = (b - a)t$.

9. The intensity I_0 of a light source is reduced to I after passing through x meters of a fog where I is given by

$$I = I_0\, e^{-kx}$$

where k is a constant. Find k if $I = I_0$ is reduced to 0.01 of its original value when when $x = 33$.

10. BNFL at Sellafield have accidentally released some radioactive isotopes into the Irish sea. There are worries about whether the fish supper will be safe to eat in Dublin on Friday. The amount of radioactive substance N in a fish decays as $N = N_0\, e^{-kt}$ where N_0 is the initial amount the fish is exposed to, t is measured in days after exposure and $k = 1/3$. If N_0 is 100 units and consuming more than 20 units is dangerous, is it safe to eat the fish 6 days after exposure.

11. The atmospheric pressure p (mm of mercury) at height h (km) is given by

$$p = 760\, e^{-kh}$$

where k is a constant. The pressure at 1 kilometers is 670. Find k and hence find the pressure at height 2 km.

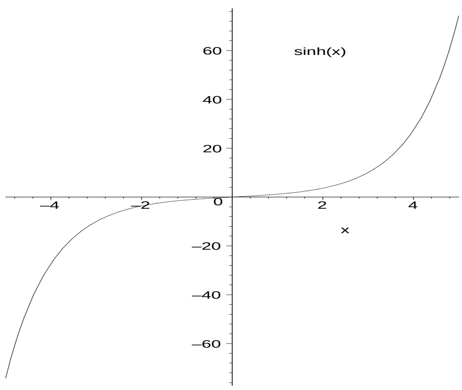
1.7 Hyperbolic Functions

We've seen in the last section how the exponential function e^x behaves. In this section we look at two close relatives of the exponential function, $\sinh x$ and $\cosh x$. Note the **h**'s! We write $\sinh x$ and $\cosh x$ as shorthand for the combination of exponentials defined below:

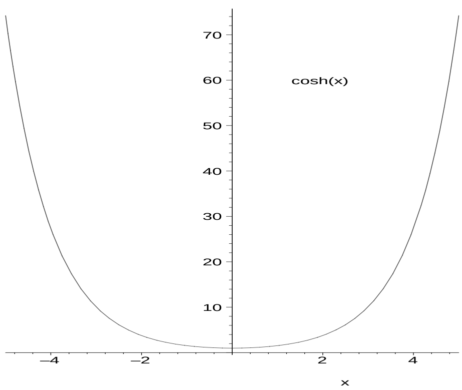
$$\boxed{\sinh x = \tfrac{1}{2}(e^x - e^{-x})}$$

$$\boxed{\cosh x = \tfrac{1}{2}(e^x + e^{-x})}$$

What do these look like?

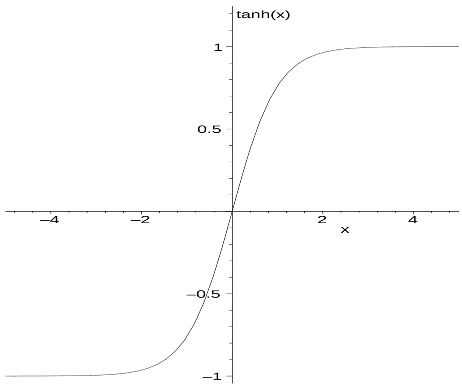


Graph of sinh x



Graph of cosh x

Just as for the trigonometric functions we can define $\tanh x = \sinh x / \cosh x$, which looks like a smooth step:



Graph of tanh x

How do these functions crop up? They appear, like sin and cos, as the solutions to many physical problems. For instance, although it was long thought that a hanging heavy chain or rope hung in the form of a parabola, it was eventually shown that it is in fact a section of the cosh curve.
As for the trigonometric functions the hyperbolic functions satisfy various identities, but these are easy to derive if we hark back to the definition above in terms of exponentials.

Example 1.21 *Express* $\sinh 8x$ *in terms of* e^{8x} *and* e^{-8x}.

Solution From the definition of $\sinh 8x$,

$$\sinh 8x = \frac{1}{2}(e^{8x} - e^{-8x})$$

Example 1.22 *Express* $5e^{4x} - 7e^{-4x}$ *in terms of* $\cosh 4x$ *and* $\sinh 4x$.

Solution From the definitions of $\cosh 4x$ and $\sinh 4x$,

$$\cosh 4x = \frac{1}{2}(e^{4x} + e^{-4x}) \qquad \text{and} \qquad \sinh 4x = \frac{1}{2}(e^{4x} - e^{-4x})$$

Hence,

$$\cosh 4x + \sinh 4x = \frac{1}{2}(e^{4x} + e^{-4x}) + \frac{1}{2}(e^{4x} - e^{-4x}) = e^{4x}$$

and

$$\cosh 4x - \sinh 4x = \frac{1}{2}(e^{4x} + e^{-4x}) - \frac{1}{2}(e^{4x} - e^{-4x}) = e^{-4x}$$

Then

$$5e^{4x} - 7e^{-4x} = 5(\cosh 4x + \sinh 4x) - 7(\cosh 4x - \sinh 4x) = -2\cosh 4x + 12\sinh 4x$$

Example 1.23 *Show that* $\cosh^2 x - \sinh^2 x = 1$

Solution Since

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

we can multiply these out, (remembering that $(e^x)^2 = e^{2x}$) :

$$\cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$$

$$\sinh^2 x = \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

so subtracting the two just gives one, as required.

Example 1.24 *Show that* $1 + 2\sinh^2 x = \cosh 2x$

Solution Again we go back to the definition in terms of exponentials:

$$\sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

so

$$1 + 2\sinh^2 x = 1 + \frac{1}{2}(e^{2x} - 2 + e^{-2x}) = \frac{1}{2}(e^{2x} + e^{-2x})$$

This is just the definition of $\cosh 2x$, which is what we wanted to show.

Example 1.25 *If* $y = \tanh x$ *show that* $2x = \ln\left(\frac{1+y}{1-y}\right)$

Solution

$$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Then

$$y(e^x + e^{-x}) = e^x - e^{-x}$$

and multiplying by e^x gives

$$y(e^{2x} + 1) = e^{2x} - 1$$

Hence

$$y + 1 = e^{2x}(1 - y)$$

so that

$$e^{2x} = \frac{1+y}{1-y}$$

Finally,

$$2x = \ln\left(\frac{1+y}{1-y}\right)$$

Exercises 1.7

1. Express $\cosh 7x$ in terms of e^{7x} and e^{-7x}.

2. Express $\sinh 6x$ in terms of e^{6x} and e^{-6x}.

3. Express $9e^{x} + 2e^{-x}$ in terms of $\cosh x$ and $\sinh x$.

4. Express $8e^{3x} - 5e^{-3x}$ in terms of $\cosh 3x$ and $\sinh 3x$.

5. Show that $2\cosh^2 x - 1 = \cosh 2x$.

6. In a semiconductor, the force F on an electron is given by

$$F = \frac{e^{-kx}}{(1+e^{-kx})^2}$$

where k is a constant and x is the distance into the semiconductor. Show that

$$F = \frac{1}{2 + 2\cosh kx}$$

1.8 Obtaining Formulae from Experimental Data

If two variables are believed to be connected , then this idea could be tested by collecting experimental data. As an example, we might want to investigate how the resistance R of a wire is related to the temperature of the wire. If we suppose that R and t are related by

$$R = at + b \tag{2}$$

we could test if (2) was true (allowing for experimental error). If (2) was true we could then attempt to find a and b.

Suppose that as a result of an experiment we collected the following data.

t °C	10	30	50	70	90
R Ω	4.9	5.2	5.8	6.2	6.8

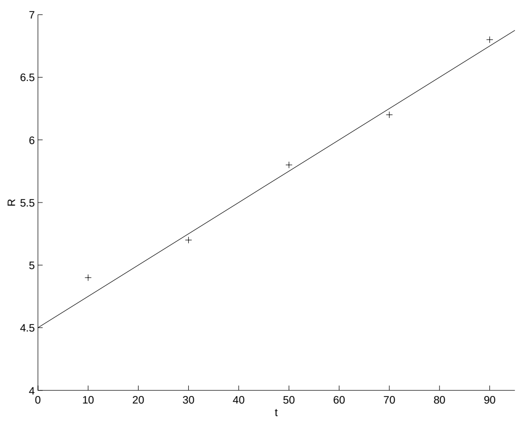


Figure 1

We plotted this data in Figure 1. From this we see that the points lie approximately on a straight line. Choosing the 'best' line to fit through will not be considered here. For the rest of the calculation we use points taken from the line (i.e. not from the data). To find a and b we first note that a is the gradient of the line. Taking for example, the points $(30, 5.25)$ and $(70, 6.25)$ which lie on the graph, we get that

$$a = \frac{6.25 - 5.25}{70 - 30} = \frac{1}{40} = 0.025$$

To obtain b we choose any point on the line, say $(10, 4.75)$, and substitute it into (2) with $a = 0.025$ to get $4.75 = 0.25 + b$ so that $b = 4.5$.

Suppose we want to test if t and y are related by

$$y = at^2 + b \tag{3}$$

and we have the following data:

t	0	1	2	3	4
y	3.1	4.9	11.2	20.8	34.9

It would be difficult to see from a plot of this data if (3) holds. Also it is not clear how to find a and b. If we let $x = t^2$ in (3) then

$$y = ax + b \tag{4}$$

which is the equation of a straight line. To test (4) we need a new data table.

$x = t^2$	0	1	4	9	16
y	3.1	4.9	11.2	20.8	34.9

From the data plot (see Figure 2) we see that to experimental error, (4) holds. The number a is the gradient of the line, so using two points on the line we get

$$a = \frac{23 - 3}{10 - 0} = 2$$

and since $y = 3$ when $x = 0$ we have that $b = 3$. Thus $y = 2t^2 + 3$.

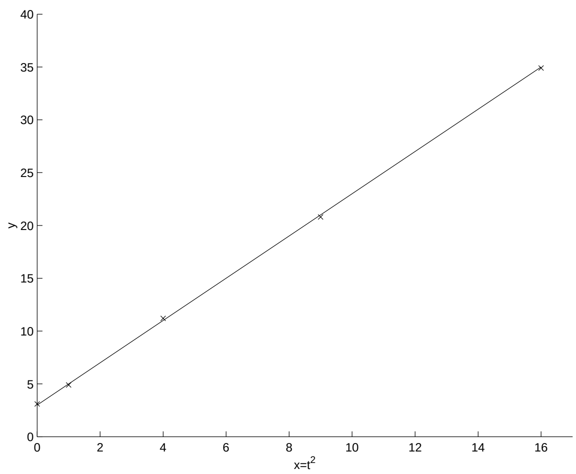


Figure 2

Example 1.26 *In each of the following cases explain how you would reduce each proposed law connecting x and y to straight line form.*

$$\text{(i) } y = ax^4 + b \qquad \text{(ii) } y = ax^2 + bx \qquad \text{(iii) } y = ax + \frac{b}{x^2}$$

Solution (i) If we let $t = x^4$ then $y = at + b$.

Hence if we plot y against x^4 we get a straight line.

(ii) Let $w = \dfrac{y}{x}$. Then $w = ax + b$.

If we plot $w = \dfrac{y}{x}$ against x we get a straight line.

(iii) From $y = ax + \frac{b}{x^2}$ we get $x^2 y = ax^3 + b$.

Then if $w = x^2 y$ and $t = x^3$ we get $w = at + b$.

If we plot $w = x^2 y$ against x^3 we get a straight line.

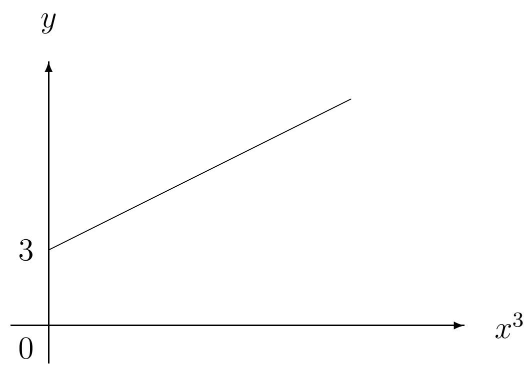
Example 1.27 *Explain how you would reduce the formula $y = ax^b$ to straight line form.*

Solution If $y = ax^b$ then

$$\ln y = \ln(ax^b) = \ln(a) + \ln(x^b) = \ln a + b \ln x$$

Let $w = \ln y$ and $t = \ln x$. Then $w = \ln a + bt$ and if we plot $\ln y$ against $\ln x$ we get a straight line.

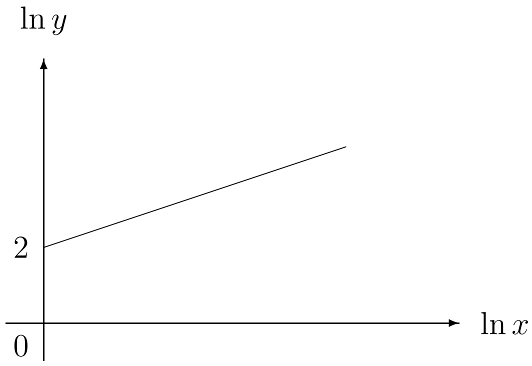
Example 1.28 *As shown in the figure below, a set of experimental results gives a straight line graph when y is plotted against x^3. The straight line passes through $(0, 3)$ and has a gradient of $1/2$. Express y in terms of x .*



Solution Put $t = x^3$ so that $y = at + b$. Then $b = 3$ and $a = 0.5$. Hence

$$y = (0.5)x^3 + 3$$

Example 1.29 *As shown below, a set of experimental results gives a straight line graph when* $\ln y$ *is plotted against* $\ln x$. *The straight line passes through* $(0,2)$ *and has a gradient of* $1/3$. *Express* y *in terms of* x.



Solution Let $w = \ln y$ and $t = \ln x$. Then

$$w = at + b = \frac{t}{3} + 2 = \frac{\ln x}{3} + 2 = \ln x^{\frac{1}{3}} + 2$$

Hence

$$y = \exp(\ln x^{\frac{1}{3}} + 2) = e^2\, x^{\frac{1}{3}} = (7.4)\, x^{\frac{1}{3}}$$

Exercises 1.8

1. The period T of oscillation of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum and g is gravity. Which (if any) of the following graphs would be a straight line ?

(a) T plotted against L **(b)** $\sqrt{T}$ plotted against L **(c)** $\sqrt{L}$ plotted against T

(d) T^2 plotted against L **(e)** L^2 plotted against T **(f)** T^{-2} plotted against L^{-1}

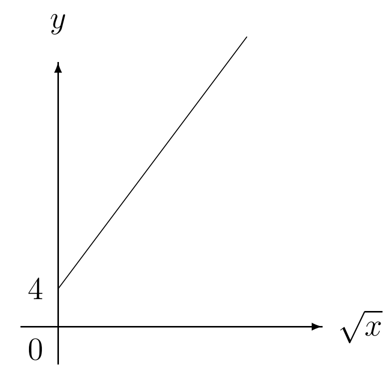
2. In each of the following cases explain how you would reduce each proposed law connecting x and y to straight line form.

$$\text{(i)}\ \ y = ax^2 + b \qquad\qquad \text{(ii)}\ \ y = \frac{a}{x} + b$$

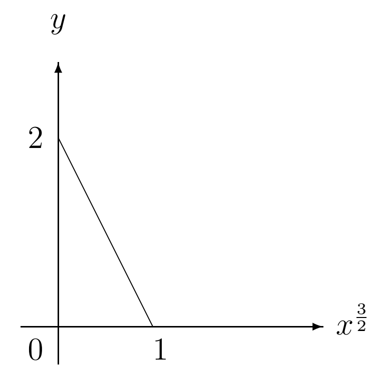
3. In each of the following cases explain how you would reduce each proposed law connecting x and y to straight line form.

$$\text{(i)}\ \ y = ax^3 + b \qquad\qquad \text{(ii)}\ \ y = \frac{1}{ax+b} \qquad\qquad \text{(iii)}\ \ y = ax^2 + \frac{b}{x}$$

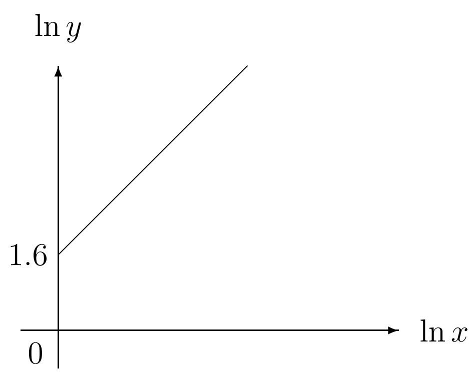
4. As shown in the figure below, a set of experimental results gives a straight line graph when y is plotted against $\sqrt{x}$. The straight line passes through $(0, 4)$ and has a gradient of 2. Express y in terms of x.



5. As shown below, a set of experimental results gives a straight line graph when y is plotted against $x^{\frac{3}{2}}$. The straight line passes through $(0, 2)$ and $(1, 0)$. Express y in terms of x.



6. As shown below, a set of experimental results gives a straight line graph when $\ln y$ is plotted against $\ln x$. The straight line has a gradient 2 and passes through $(0, 1.6)$. Express y in terms of x.



1.9 Answers to Exercises

Full solutions to the exercises are on the module website

Exercises 1.1

3. $x^3 + 3x^2y + 3xy^2 + y^3$ **4.** $81 + 108y + 54y^2 + 12y^3 + y^4$ **5.** $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

6. $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ **7.** $1 + 20x + 190x^2$ **8.** $1 + 10p^2 + 45p^4$

9. $1 - 10x + 45x^2$ **10.** $1 + 6x$, increase of 12%

Exercises 1.2

1(a) $r = 3$, next terms 567 and 1701 **1(b)** $r = 0.1$, next terms 0.0017 and 0.00017

2(a) Owe $(1.02)P, (1.02)^2P, (1.02)^3P, \ldots$ at the end of each month, a GS with $r = 1.02$

2(b) Because interest is paid on the interest. Annual rate is 26.8%

3. 3280 **4.** Eleven million pounds **5.** 9637

6. Profits are $60000, (.95) \times 60000, (.95)^2 \times 60000, \ldots$ Total 317900 **7.** $x = 4$

Exercises 1.3

1. $\dfrac{1}{x+7} - \dfrac{1}{x+9}$ **2.** $\dfrac{6}{2x+5} - \dfrac{2}{x+3}$ **3.** $\dfrac{5}{x-1} + \dfrac{11}{(x-1)^2}$

4. $\dfrac{4}{7(2x+1)} + \dfrac{5}{7(x-3)}$ **5.** $\dfrac{5}{2x+3} - \dfrac{8}{(2x+3)^2}$

Exercises 1.4

1(a) 1 **1(b)** 2 **1(c)** -3 **2.** -0.179 **3.** $x < 7$

4. $x > 3$, $x = \dfrac{7}{2}$ **5.** $v = 0.173$

Exercises 1.5

1. 1.3979 **2(a)** $\ln\left(\dfrac{x^2}{y}\right)$ **2(b)** $\ln\left(\dfrac{x}{(3x+5)^2}\right)$ **3.** $x = 79.743$

Exercises 1.6

1. $f(0) = 2$, $f(0.5) = 1.5576$ **2.** 6.501 **3.** $e^{2x} + e^{-2x} - 2$

Exercises 1.6

5. See Figure 3 **6.** $t = -\dfrac{1}{4}\ln\left(\frac{3-y}{2}\right)$ see Figure 4 for graph

7(a) $y = 8x^3$ **7(b)** $y = e^6x^{-3}$ **9.** $k = 0.14$

10. When $t = 6$, $N = 13.53$ so he's (fairly!) safe. **11.** $k = 0.126$ and $p = 590.7$

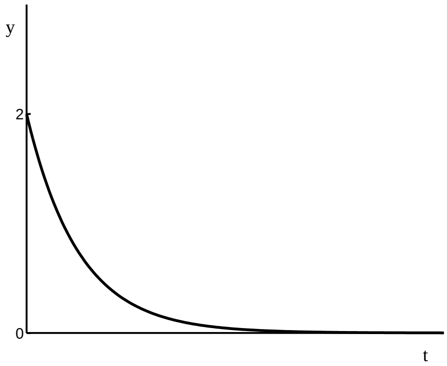


Figure 3

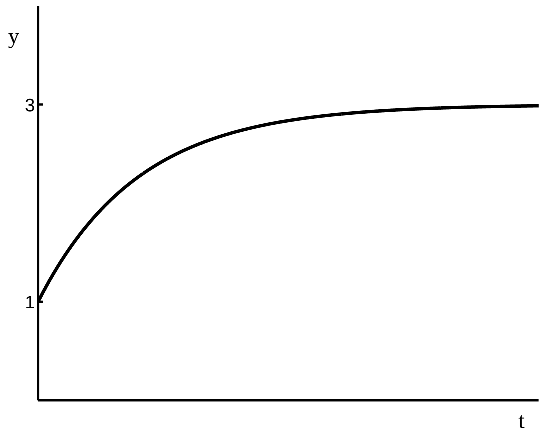


Figure 4

Exercises 1.7

1. $\dfrac{1}{2}(e^{7x} + e^{-7x})$ **2.** $\dfrac{1}{2}(e^{6x} - e^{-6x})$

3. $11\cosh x + 7\sinh x$ **4.** $3\cosh 3x + 13\sinh 3x$

Exercises 1.8

1. **(a)** , **(c)** and **(f)** gives straight line graphs, the rest do not.

2(i) Plot x^2 against y **2(ii)** Plot $\dfrac{1}{x}$ against y

3(i) Plot x^3 against y **3(ii)** Plot x against $\dfrac{1}{y}$ **3(iii)** Plot x^3 against xy

4. $y = 2\sqrt{x} + 4$ **5.** $y = -2x^{\frac{3}{2}} + 2$ **6.** $y = (4.95)\,x^2$