Differentiation

Contents

2.1 Introduction

Recall that the gradient of the straight line joining $A(x_1, y_1)$ to $B(x_2, y_2)$ is given by

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that the gradient of a straight line is the same at all points on the line. For a curve the gradient will depend on where we are on the line.
The gradient of a curve at a point A is the gradient of the tangent to the curve at A. In figure 1a, a tangent has been drawn at the point A. The gradient of the curve at A is equal to the gradient of the straight line AB.

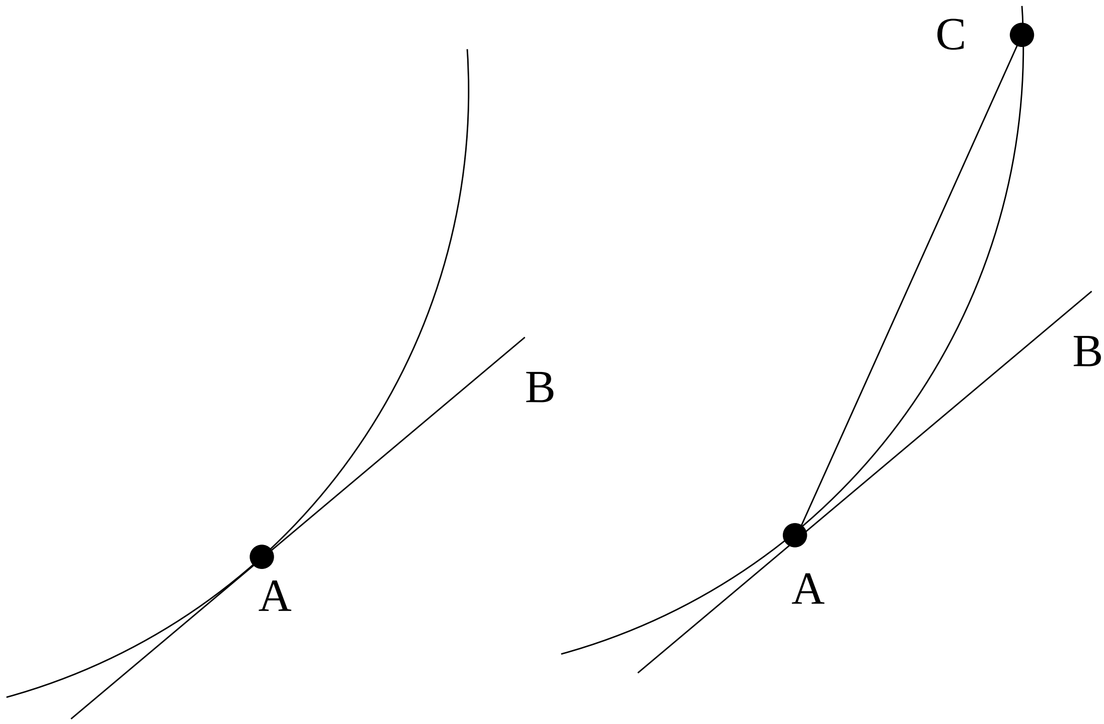

Figure 1

We can find approximations to the gradient of a curve by finding the gradient of the straight line AC (see figure 1b). Although this is clearly too steep here, we'll get a better approximation if we take C to lie closer to A. In fact, the closer C gets to A the better the approximation becomes.

For the curve $y = f(x) = x^2$, when $x = 1$, $y = f(x) = 1$ and when $x = 2$, $y = f(x) = 4$. Thus the points $A(1, 1)$ and $C(2, 4)$ are on the curve. We can calculate the gradient of the straight line AC :

$$\text{gradient of } AC = \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{2 - 1} = 3$$

To find a better approximation to the gradient of the curve $y = f(x) = x^2$, at $x = 1$, we consider a point C closer to the point A . Suppose C_1 is the point $(1.5, f(1.5)) = (1.5, 2.25)$.

$$\text{gradient of } AC_1 = \frac{f(1.5) - f(1)}{1.5 - 1} = \frac{2.25 - 1}{0.5} = 2.5$$

Now let C_2 be the point $(1.2, f(1.2))$.

$$\text{gradient of } AC_2 = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{1.44 - 1}{0.2} = 2.2$$

Similarly for $C_3\,(1.1, f(1.1))$ and $C_4\,(1.01, f(1.01))$:

$$\text{gradient of } AC_3 = \frac{f(1.1) - f(1)}{1.1 - 1} = 2.1$$

$$\text{gradient of } AC_4 = \frac{f(1.01) - f(1)}{1.01 - 1} = 2.01$$

As the point C gets closer and closer to the point A the gradients of the straight lines get nearer to 2. This is the gradient of the curve $y = f(x) = x^2$, at $x = 1$.

More generally, for any function $y = f(x)$, the gradient at x is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

The notation $\lim\limits_{h \to 0}$ means h approaches 0 .

- The gradient of the curve is written $f'(x)$ (pronounced f dash x).
- The process of finding $f'(x)$ from $f(x)$ is called differentiation or finding the derivative.

Using formula (1) to find the derivative is called differentiating from first principles. We use the above to find $f'(x)$ when $f(x) = x^2$. We write

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

We will develop rules to make the task of differentiating easier.

Exercises 2.1

1. If $f(x) = x^3$, then $f'(x) = 3x^2$ so $f'(1) = 3$. Geometrically this says that the gradient of the curve $f(x) = x^3$ at $x = 1$ is 3. Work out

$$\frac{f(1.3) - f(1)}{0.3}, \qquad \frac{f(1.2) - f(1)}{0.2}, \qquad \frac{f(1.1) - f(1)}{0.1}$$

to see if the definition

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

is approaching the correct value.

2.2 Derivatives of Algebraic Functions

The rule for differentiating $f(x) = x^n$ is

$$\text{if } f(x) = x^n \quad \text{then} \quad f'(x) = nx^{n-1} \tag{2}$$

The rule which tells us how to differentiate sums of functions such as $x^4 + x^2$ is :

$$\text{if } \; h(x) = f(x) + g(x) \qquad \text{then} \qquad h'(x) = f'(x) + g'(x) \tag{3}$$

In words : to differentiate a sum, differentiate each term seperately and add.

The next rule tells us how to differentiate constant multiples of a function such as $7x^3$. If c is a constant then

$$\text{if } \; g(x) = cf(x) \qquad \text{then} \qquad g'(x) = cf'(x) \tag{4}$$

Example 2.1 *Find* $f'(x)$ *if :* (a) $f(x) = x^7$ $\quad (b)$ $f(x) = \dfrac{1}{x}$ $\quad (c)$ $f(x) = \sqrt{x}$.

Solution (a) Using equation (2) with $n = 7$, if $f(x) = x^7$ then $f'(x) = 7x^6$.

(b) To use the rule (2) we must express $f(x)$ in the form $f(x) = x^n$.

Now $\dfrac{1}{x} = x^{-1}$ so $f(x) = x^{-1}$, and $n = -1$.
Then $n - 1 = -1 - 1 = -2$ and $f'(x) = -x^{-2}$.
(c) $\sqrt{x} = x^{\frac{1}{2}}$ so $f(x) = x^{\frac{1}{2}}$, and $n = \dfrac{1}{2}\cdot$
Then $n - 1 = \dfrac{1}{2} - 1 = -\dfrac{1}{2}\cdot$ Thus $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}\cdot$

Example 2.2 *Find* $f'(x)$ *if :* (a) $f(x) = x$, $\quad (b)$ $f(x) = 1$.

Solution (a) $f(x) = x$ is a straight line with gradient 1. Hence $f'(x) = 1$.
We could also obtain this result from (2) with $n = 1$: $f'(x) = x^0 = 1$.
(b) The graph of the function $f(x) = 1$ is a horizontal line so $f'(x) = 0$.
We could obtain this result from (2) using $f(x) = 1 = x^0$ so $n = 0$.

Example 2.3 *Find the derivative of* $f(x) = x^3 + x$.

Solution $f(x)$ is the sum of x^3 and x .
The derivative of x^3 is $3x^2$, and the derivative of x is 1 .
Using equation (3), $f'(x) = 3x^2 + 1$.

Example 2.4 *Find* $g'(x)$ *if :* (a) $g(x) = -2x^5$, (b) $g(x) = 4x(x-7)$.

Solution (a) To differentiate $g(x) = -2x^5$ first differentiate $f(x) = x^5$ to get $f'(x) = 5x^4$ and then multiply by -2 : $g'(x) = (-2)(5x^4) = -10x^4$.
(b) Multiplying out, $f(x) = 4x(x-7) = 4x^2 - 28x$. Hence $f'(x) = 8x - 28$.

Example 2.5 *Find the gradient of the tangent to* $f(x) = x^3 - 7x^2 + 4x - 9$ *at the point* $x = 2$.

Solution The gradient of the tangent at x is $f'(x)$.
Using the above rules, $f'(x) = 3x^2 - 14x + 4$. The gradient of the tangent at $x = 2$ is $f'(2) = 3(2)^2 - 14(2) + 4 = -12$.

An alternative notation for the derivative of $y = f(x)$ is $\dfrac{dy}{dx}$.
If $y = f(x) = 3x^2 - 7x + 2$ then $\dfrac{dy}{dx} = f'(x) = 6x - 7$.
If $y = f(x)$, then the following are all equivalent notations for the derivative:

$$f'(x) \qquad \frac{dy}{dx} \qquad y' \;\text{(read as ‘ y prime ’ or as ‘ y dash ’)}$$

We can differentiate functions where different symbols are used, for example :
if $y = f(u) = 2u^3 - 7u$ then $\dfrac{dy}{du} = f'(u) = 6u^2 - 7$.

Exercises 2.2

1. Find $f'(x)$ if : (a) $f(x) = x^3$, (b) $f(x) = x^5$, (c) $f(x) = \dfrac{1}{x}$, (d) $f(x) = \dfrac{1}{x^3}$.

2. Find the the gradient of the curve $f(x) = x^{3/2}$ at $x = 16$

3. Find the gradient of the curve $f(x) = \dfrac{1}{x}$ at $x = \frac{1}{3}$

4. Find $f'(x)$ if : (a) $f(x) = 3x^2$, (b) $f(x) = \dfrac{12}{x^3}$, (c) $f(x) = 3x^2 + 4x + 7$

5. Find $f'(x)$ if $f(x) = x^3 + (x+2)(x+3)$

6. Find $f'(t)$ if $f(t) = \dfrac{t^2 + 4t^3}{t^2} + \dfrac{3 + 4t}{t^2}$

7. The period T of oscillation of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum and g is gravity. Find $\dfrac{dT}{dL}$

2.3 Differentiation of Common Functions

There are formulae for finding the derivatives of many functions such as $\sin x$, $\cos x$, e^x and $\ln x$. Below we give a table with these formulae. Note that the rules for differentiating hyperbolic functions are in your formulae sheet.

The derivatives of $\sin x$ and $\cos x$ are simplest when x is measured in radians. For all subsequent work on differentiation of $\sin x$ and $\cos x$, x will be measured in radians.

In the following table a is a constant.

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
e^{ax}	ae^{ax}
$\ln x$	$\dfrac{1}{x}$

Note that your formulae sheet contains all the above results.

Example 2.6 *Find* $f'(x)$ *if :* (a) $f(x) = 5\sin x$ $\quad(b)$ $f(x) = 3\cos 2x$.

Solution (a) Using the table with $a = 1$, if $f(x) = 5\sin x$ then $f'(x) = 5\cos x$.

(b) Using the table with $a = 2$, if $f(x) = 3\cos 2x$ then $f'(x) = (-2)\times 3\sin 2x = -6\sin 2x$.

Example 2.7 *If* $f(x) = 5\sin 4x - 8\cos 4x$, *find* $f'(\frac{\pi}{2})$.

Solution $f'(x) = 20\cos 4x - 8(-4\sin 4x) = 20\cos 4x + 32\sin 4x$.

Then $f'(\frac{\pi}{2}) = 20\cos 2\pi + 32\sin 2\pi = 20$.

Example 2.8 *An alternating voltage is given by* $v = v(t) = 80\sin 10t$ *volts where* t *is the time in seconds. Find the rate of change of voltage when* $t = 0.2$.

Solution The rate of change of voltage at time t is $v'(t) = 800\cos 10t$.

When $t = 0.2$, $v'(t) = v'(0.2) = 800\cos 2 = 800(-0.4161) = -333$ (*Note: use radians*).

Example 2.9 *Find* y' *if* $y = 3e^{2x} + 1$ *and show that* $y' = 2(y-1)$.

Solution From the table above $y' = 3 \times 2e^{2x} = 6e^{2x}$.

Since

$$2(y-1) = 2(3e^{2x} + 1 - 1) = 6e^{2x}$$

we have that $y' = 2(y-1)$.

Example 2.10 *Find* $f'(x)$ *if* $f(x) = 2\ln x - x^2$.

Solution $f(x) = 2\ln x - x^2$ so from the table $f'(x) = \dfrac{2}{x} - 2x$.

Exercises 2.3

1. Find $f'(x)$ if : (a) $f(x) = 7\cos x$ (b) $f(x) = 2\sin 4x$.

2. Find $f'(x)$ when $f(x) = 6\sin 3x - \cos 3x$

3. If $f(x) = 5\cos 4x - 7\sin 2x$, find $f'(0)$.

4. Let $f(x) = a\sin 2x + b\cos 2x$ where a and b are constants. If $f(0) = 7$ and $f'(0) = -6$, find a and b.

5. Find $f'(x)$ if : (a) $f(x) = 5e^{3x}$ (b) $f(x) = 6e^{-5x}$.

6. Find $f'(x)$ if $f(x) = x - 3\ln x$.

7. Find $f(0)$ and $f'(0)$ if $f(x) = 3e^{-2x}$.

8. The number N of nasty bugs on a microbyte hamburger is given by $N(t) = 10e^{2t}$. Find $\dfrac{dN}{dt}$ and show that the number of bugs satisfies the equation $\dfrac{dN}{dt} - 2N = 0$.

9. If $y = 4e^{-5t} - 2$, show that $y' = -5(y+2)$.

10. Use your formulae sheet to find $f'(x)$ if $f(x) = 5\cosh 2x - 7\sinh 2x$.

2.4 Higher Derivatives

For $y = f(x)$ the derivative $f'(x)$ measures the gradient of the graph of y. The second derivative measures how the gradient changes and is calculated by differentiating $f'(x)$.

The following notations are used for the second derivative:

- $f''(x)$ read as ‘f double-dash’
- $\dfrac{d^2y}{dx^2}$ read as ‘ dee two y by dee x squared ’
- y'' read as ‘ y double-dash ’

Thus if $y = f(x) = x^4 - 5x$ then

$$f'(x) = \frac{dy}{dx} = y' = 4x^3 - 5 \qquad \text{and} \qquad f''(x) = \frac{d^2y}{dx^2} = y'' = 12x^2 .$$

Example 2.11 *Find* y'' *if* $y = 6\ln t - 4t$.

Solution $y' = \dfrac{6}{t} - 4$ and $y'' = -\dfrac{6}{t^2}$.

Example 2.12 *Find* $f''(x)$ *if :* (*a*) $f(x) = 9\sin 3x$, (*b*) $f(x) = 3e^{-5x}$.

Solution (a) If $f(x) = 9\sin 3x$ then $f'(x) = 27\cos 3x$ and $f''(x) = -81\sin 3x$.
(b) If $f(x) = 3e^{-5x}$ then $f'(x) = -15e^{-5x}$ and $f''(x) = 75e^{-5x}$.

Example 2.13 *If* $y = 5\cos 3x$ *show that* y *satisfies an equation of the form* $\dfrac{d^2y}{dx^2} + Ay = 0$ *where* A *is a constant and find* A.

Solution $\dfrac{dy}{dx} = -15\sin 3x$ and $\dfrac{d^2y}{dx^2} = -45\cos 3x$. Then

$$\frac{d^2y}{dx^2} + Ay = -45\cos 3x + 5A\cos 3x = (5A - 45)\cos 3x = 0$$

if $A = 9$.

Example 2.14 *If* $y = e^{5x}$ *, show that* y *satisfies an equation of the form* $y'' - 3y' + Ay = 0$ *where* A *is a constant and find* A.

Solution $y = e^{5x}$, so $y' = 5e^{5x}$ and $y'' = 25e^{5x}$. Then

$$y'' - 3y' + Ay = 25e^{5x} - 15\,e^{5x} + Ae^{5x} = e^{5x}\ (25 - 15 + A) = 0$$

if $A = -10$.

Exercises 2.4

1. Find $f''(x)$ if $f(x) = 3x^3 - 2x^2 + 4x$.

2. Find $f''(x)$ if $f(x) = 4\cos 5x$.

3. Find $f''(x)$ if $f(x) = 3e^{-2x} - 4\ln x$.

4. If $y = \sin 2x$ show that y satisfies an equation of the form $\dfrac{d^2y}{dx^2} + Ay = 0$ where A is a constant and find A.

5. If $y = e^{3x}$, show that y satisfies an equation of the form $y'' + Ay' + 3y = 0$ where A is a constant and find A.

6. If $y = e^{-2x}$, show that y satisfies an equation of the form $y'' - 3y' + Ay = 0$ where A is a constant and find A. Show that with this value of A , $y = e^{5x}$ also satisfies the equation.

2.5 The Chain Rule

In this section we differentiate functions such as

$$(x^2-3x+8)^5\,,\quad e^{-3x^2}\,,\quad \sin(5x-\pi)\,,\quad \cos^2 x\,,\quad \ln(9x+2)$$

Each of these functions can be regarded as the composition of simpler functions :

- if $y=f(x)=(x^2-3x+8)^5$ then $y=u^5$ where $u=x^2-3x+8$.
- if $y=f(x)=e^{-3x^2}$ then $y=e^{-u}$ where $u=-3x^2$.
- if $y=f(x)=\sin(5x-\pi)$ then $y=\sin u$ where $u=5x-\pi$.
- if $y=f(x)=\cos^2 x$ then $y=u^2$ where $u=\cos x$
- if $y=f(x)=\ln(9x+2)$ then $y=\ln u$ where $u=9x+2$.

In each case $y=f(x)$ is a function of u , where u is a function of x . We can find $\dfrac{dy}{dx}$ from

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

This is called the chain rule.

Example 2.15 *Find the derivative of* $(x^2-3x+8)^5$.

Solution $y=(x^2-3x+8)^5=u^5$ where $u=x^2-3x+8$. Now

$$\frac{dy}{du}=5u^4 \qquad \textit{and} \qquad \frac{du}{dx}=2x-3$$

Thus

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=5u^4(2x-3)=5(2x-3)(x^2-3x+8)^4\,.$$

Note : the final answer should be given in terms of x only.

With practice you will be able to use the chain rule without introducing much notation. For example, in the above $y=(x^2-3x+8)^5$. We can mentally use the substitution $u=x^2-3x+8$, giving

$$\frac{dy}{dx}=5u^4(2x-3)=5(2x-3)(x^2-3x+8)^4\,.$$

Example 2.16 *Find the derivative of* $y=e^{-3x^2}$.

Solution $y=e^u$ where $u=-3x^2$. Then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-6xe^u=-6xe^{-3x^2}\,.$$

Example 2.17 *Find the derivative of* $y = \sin(5x - \pi)$.

Solution $y = \sin u$ where $u = 5x - \pi$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5\cos(5x - \pi) .$$

Example 2.18 *Find the derivative of* $f(x) = \cos^2 x$.

Solution $y = f(x) = \cos^2 x = u^2$, where $u = \cos x$. Then

$$\frac{dy}{du} = 2u \qquad \textit{and} \qquad \frac{du}{dx} = -\sin x$$

Hence $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = -2\cos x \sin x$.

Example 2.19 *Find the derivative of* $y = \ln(9x + 2)$.

Solution $y = \ln u$ where $u = 9x + 2$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{9}{9x + 2} .$$

Example 2.20 *Find the derivative of* $y = \dfrac{1}{2x + 5}$.

Solution $y = \dfrac{1}{2x + 5} = (2x + 5)^{-1} = u^{-1}$ where $u = 2x + 5$. Then

$$\frac{dy}{du} = -u^{-2} \qquad \textit{and} \qquad \frac{du}{dx} = 2$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -2(2x + 5)^{-2} = \frac{-2}{(2x + 5)^2}$$

Exercises 2.5

1. Find $f'(x)$ if : (a) $f(x) = (7x-2)^3$ (b) $f(x) = (5x^2-1)^4$ (c) $f(x) = (5x-3)^{-2}$.

2. Find $f'(x)$ if : (a) $f(x) = 4\cos(2x-\pi/2)$ (b) $f(x) = \ln(3x-7)$ (c) $f(x) = e^{-5x^2}$.

3. Find the derivative of $y = \sin^2(7x)$.

4. Find $f'(x)$ if $f(x) = 3\ln(x) + 5\ln(4x - 2)$.

5. Find $f'(3)$ if $f(x) = (x^2 + 7)^{1/2}$.

6. Let $f(x) = \ln(x^2 + 8)$. Find $f'(x)$.

7. Express $f(x) = \dfrac{4x - 7}{(x - 1)^2}$ in terms of partial fractions. Hence find $f'(x)$.

2.6 Product Rule

The function $f(x) = x^2 e^{-4x}$ is the product of the functions $g(x) = x^2$ and $h(x) = e^{-4x}$, that is $f(x) = g(x)\,h(x)$. The product rule is

$$f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$$

The product rule is also written as : if $y = u\,v$ then $y' = u'\,v + u\,v'$.

Example 2.21 *Find the derivative of* $f(x) = x^2 e^{-4x}$.

Solution $f(x) = g(x)\,h(x)$ where $g(x) = x^2$ and $h(x) = e^{-4x}$.
By the product rule
$f'(x) = g'(x)\,h(x) + g(x)\,h'(x) = 2xe^{-4x} + x^2\,(-4e^{-4x}) = 2xe^{-4x} - 4x^2\,e^{-4x}$.

Example 2.22 *Find the derivative of* $f(x) = x^2 \cos x$.

Solution $f(x) = g(x)\,h(x)$ where $g(x) = x^2$ and $h(x) = \cos x$.
By the product rule
$f'(x) = g'(x)\,h(x) + g(x)\,h'(x) = 2x\cos x - x^2\,\sin x.$

Example 2.23 *Find the derivative of* $f(x) = x^3 \ln x$.

Solution $f(x) = g(x)\,h(x)$ where $g(x) = x^3$; and $h(x) = \ln x$. By the product rule
$f'(x) = g'(x)\,h(x) + g(x)\,h'(x) = 3x^2 \ln x + x^3 \left(\dfrac{1}{x}\right) = 3x^2 \ln x + x^2$

Example 2.24 *Find* $f''(x)$ *when* $f(x) = xe^{-3x}$.

Solution Using the product rule, $f'(x) = e^{-3x} - 3xe^{-3x}$.
Differentiating again and using the product rule for the second term gives
$f''(x) = -3e^{-3x} - 3e^{-3x} + 9xe^{-3x} = -6e^{-3x} + 9xe^{-3x}$.

Example 2.25 *If* $y = x\sin 4x$ *, show that* y *satisfies an equation of the form*

$$y'' + 16y = A\cos 4x$$

where A *is a constant and find* A.

Solution Using the product rule, $y' = \sin 4x + 4x\,\cos 4x$ and

$$y'' = 4\,\cos 4x + 4\,\cos 4x - 16x\,\sin 4x = 8\,\cos 4x - 16x\,\sin 4x$$

Then $y'' + 16y = 8\,\cos 4x$ so $A = 8$.

Exercises 2.6

1. Find $f'(x)$ if : (a) $f(x) = 2xe^{-7x}$, (b) $f(x) = 6x\sin 4x$, (c) $f(x) = x^2 \ln x$.

2. Find $f'(x)$ and $f''(x)$ when $f(x) = x\, e^{-2x}$.

3. Find $f'(x)$ and $f''(x)$ when $f(x) = x\ln x - x$.

4. If $y = x\cos 3x$, show that y satisfies an equation of the form

$$y'' + 9y = A\sin 3x$$

where A is a constant and find A.

5. If $y = e^{4x}$, show that y satisfies an equation of the form

$$y'' + Ay' + 16y = 0$$

where A is a constant and find A. Show that, with this value of A, $y = xe^{4x}$ also satisfies the equation.

6. A damped oscillator has displacement $y = e^{-kt}\cos t$ where k is a constant. Find the velocity $y'(t)$.

2.7 Quotient Rule

The quotient rule is :

$$if \quad f(x) = \frac{g(x)}{h(x)} \qquad \text{then} \qquad f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{h^2(x)}$$

The quotient rule rule is also written as : if $y = \dfrac{u}{v}$ then $y' = \dfrac{u'\,v - u\,v'}{v^2}$ ·

Example 2.26 *Find the derivative of* $f(x) = \dfrac{5x+1}{3x+2}$ ·

Solution Using the quotient rule with $g(x) = 5x + 1$ and $h(x) = 3x + 2$,

$$f'(x) = \frac{5(3x+2) - 3(5x+1)}{(3x+2)^2} = \frac{15x + 10 - 15x - 3}{(3x+2)^2} = \frac{7}{(3x+2)^2} = 7(3x+2)^{-2}.$$

Example 2.27 *Find the derivative of* $f(x) = \dfrac{x}{x^2+4}$ ·

Solution Using the quotient rule with $g(x) = x$ and $h(x) = x^2 + 4$,

$$f'(x) = \frac{x^2 + 4 - 2x^2}{(x^2+4)^2} = \frac{4 - x^2}{(x^2+4)^2}$$

Example 2.28 *Find* $f'(x)$ *when* $f(x) = \dfrac{e^{2x}}{1+3e^{2x}}$.

Solution Using the quotient rule with $g(x) = e^{2x}$ and $h(x) = 1 + 3e^{2x}$,

$$f'(x) = \frac{2e^{2x}(1+3e^{2x}) - e^{2x}(6e^{2x})}{(1+3e^{2x})^2}$$

$$f'(x) = \frac{2e^{2x} + 6e^{4x} - 6e^{4x}}{(1+3e^{2x})^2} = \frac{2e^{2x}}{(1+3e^{2x})^2}$$

Exercises 2.7

1. Find $f'(x)$ when $f(x) = \dfrac{4x+3}{2x+1}$

2. Find $f'(x)$ if : (a) $f(x) = \dfrac{2x}{5x+3}$ (b) $f(x) = \dfrac{x^2}{1+x^2}$ (c) $f(x) = \dfrac{2x^2}{x^3+1}$

3. Find $f'(x)$ if : (a) $f(x) = \dfrac{x}{1+\sin x}$ (b) $f(x) = \dfrac{1}{1+e^{4x}}$

4. Find $f'(x)$ if $f(x) = \dfrac{\sin x}{1+\cos x}$

5. The amount $N(t)$ of some substance in a chemical reaction is given by

$$N(t) = \frac{1}{2+3e^{-2t}}$$

Show that

$$\frac{dN}{dt} = \frac{6e^{-2t}}{(2+3e^{-2t})^2}$$

6. By writing $\tan x = \dfrac{\sin x}{\cos x}$ and using the quotient rule find the derivative of $\tan x$.

2.8 Parametric Differentiation

Often it is easier to get the equation of some curve in parametric form rather than as an explicit equation relating y and x.

The equation of a circle of radius r, for instance, is $x^2 + y^2 = r^2$, or in parametric form

$$x = r\cos t \qquad y = r\sin t$$

where t is the *parameter*. It can be thought of as a time co-ordinate for motion along the curve in question.

Suppose that a bicycle is moving in a straight line at constant speed. If we consider a particular point on a wheel, then this point will trace out a path. After t seconds, the position of the point is

$$x = t - \sin t \qquad\qquad y = 1 - \cos t$$

A picture of the path of the point is shown in Figure 1.

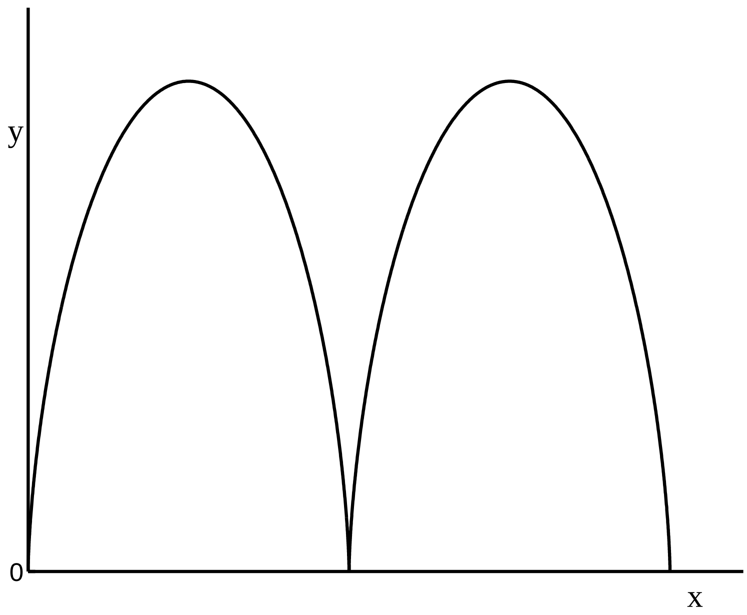

Figure 1 : $x = t - \sin t \qquad y = 1 - \cos t$

In this section we will be finding $\dfrac{dy}{dx}$ for a curve given by parametric equations. You should make sure you know what $\dfrac{dy}{dx}$ means in this case. For the above example shown in Figure 1, at a time t we will be at a certain point on the graph. The gradient $\dfrac{dy}{dx}$ is the slope at this point.

When a curve is given by parametric equations $x = x(t)$; $y = y(t)$, we use the chain rule to get

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times \frac{1}{\dfrac{dx}{dt}}$$

Example 2.29 *Let* $x = t^2 + 3$, $y = 4t^3$. *Find* $\dfrac{dy}{dx}$.

Solution Since $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 12t^2$,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\dfrac{dx}{dt}} = \frac{12t^2}{2t} = 6t$$

Example 2.30 *Find* $\dfrac{dy}{dx}$ *if* $x = t - \sin t$ *and* $y = 1 - \cos t$.

Solution In this case $\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$, so

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$$

Exercises 2.8

1. Let $x = t + 2t^2$ and $y = 3t$. Find $\dfrac{dy}{dx}$.

2. Let $x = 2\sin t$, $y = 6\cos t$. Find $\dfrac{dy}{dx}$.

3. Find $\dfrac{dy}{dx}$ for the curve defined parametrically by the equations

$$x = \frac{1}{t} \qquad\qquad y = 5t^2 + 1$$

4. Find $\dfrac{dy}{dx}$ for the curve defined parametrically by the equations

$$x = \frac{1}{t-1} \qquad\qquad y = \frac{t^2}{t-1}$$

2.9 Implicit Differentiation

If y is a *explicit* function of x then it is a clearly defined function, for example $y = \cos x$. However, we can also define functions *implicitly.* The equation

$$y^3 + 2xy - 5x^2 = 0$$

defines y as a function of x, but it is a bit awkward to disentangle y to give a explicit equation for y in terms of x.

Suppose we wanted to find $\dfrac{dy}{dx}$ for the y defined by this equation. We can still do this, without the effort of solving the equation for y.

Before working out an example we look at some of the techniques to be used. Suppose $y = y(x)$ and we want to find the derivative of $w = y^2$. By the chain rule,

$$\frac{dw}{dx} = \frac{dw}{dy} \times \frac{dy}{dx} = 2y\,\frac{dy}{dx}$$

To find the derivative of the expression $x^3\,y^2$ we would use the product rule and the above method to differentiate y^2 .

Example 2.31 *If* $y^3 + 2xy - 5x^2 = 0$, *find* $\dfrac{dy}{dx}$

Solution Although we don't know what y is explicitly, we do know that by the chain rule $\dfrac{dy^3}{dx} = 3y^2\dfrac{dy}{dx}$. Let's now differentiate the equation above

$$\frac{d}{dx}\left(y^3 + 2xy - 5x^2\right) = 3y^2\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} - 10x = 0$$

which can be solved for $\dfrac{dy}{dx}$ to give

$$\frac{dy}{dx} = \frac{10x - 2y}{3y^2 + 2x}$$

Example 2.32 *If* $x^2 - \sin y + xy = 0$ *find* $\dfrac{dy}{dx}$.

Solution Again we take the equation, differentiate using the chain rule and solve for $\dfrac{dy}{dx} \cdot$ We find

$$\frac{d}{dx}\left(x^2 - \sin y + xy\right) = 2x - \cos y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

which may be solved for $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2x + y}{\cos y - x}$$

.

Example 2.33 *Use implicit differentiation to show that* $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

Solution If $y = \sin^{-1} x$ then $x = \sin y$ and

$$\frac{d}{dx}(x - \sin y) = 1 - \cos y\frac{dy}{dx}$$

so

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We can put this into a more familiar form by remembering that $\cos^2 y + \sin^2 y = 1$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

and

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

.

Example 2.34 *$y(t)$ satisfies the equation $y''+ky=0$ where k is a constant. Find $g'(t)$ where*

$$g(t) = \left(y'(t)\right)^2 + k\left(y(t)\right)^2$$

Solution To find the derivative of $w = \left(y'(t)\right)^2$ write $w = u^2$ with $u = y'(t)$. Then

$$\frac{dw}{dt} = \frac{dw}{du}\frac{du}{dt} = 2uy'' = 2y''y'$$

Similarly, the derivative of $k\left(y(t)\right)^2$ is $2ky'y$.

Hence

$$g'(t) = 2y''y' + 2ky'y$$

Using $y'' = -ky$

$$g'(t) = 2y''y' + 2ky'y = 2y'\left(-ky\right) + 2ky'y = 0$$

In the above example, $g(t)$ is proportional to the energy of the system modelled by the differential equation $y'' + ky = 0$. Since $g'(t) = 0$, $g(t)$ is a constant which means that energy is conserved.

Exercises 2.9

1. Find $\dfrac{dy}{dx}$ in terms of x and y if $y^2 + x^2 = 18$.

2. Find $\dfrac{dy}{dx}$ if y is defined implicitly by $x^2 + 2xy + y^3 = 0$.

3. Find $\dfrac{dy}{dx}$ in terms of x and y if $5y^2 + 2y + xy = x^4$.

4. Find the gradient of the curve $x^2 + 5y^2 = 9$ at the point $(2, -1)$.

5. Let $3x^2 - xy - 2y^2 + 12 = 0$. Find $\dfrac{dy}{dx}$ at the point $(2, 3)$.

6. The equation of a circle of radius 1 is given by $x^2 + y^2 = 1$.

Work out $\dfrac{dy}{dx}$ on the circle at the point $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ by:

(i) solving for y ; (ii) using implicit differentiation. (they should be the same!)

7. $y(t)$ satisfies the equation $y'' + y + 2y^3 = 0$. Find $g'(t)$ where

$$g(t) = \left(y'(t)\right)^2 + \left(y(t)\right)^2 + \left(y(t)\right)^4$$

2.10 Answers to Exercises

Exercises 2.1

1. Approximate gradients : 3.99 , 3.64 , 3.31

Exercises 2.2

1(a) $3x^2$ **1(b)** $5x^4$ **1(c)** $-x^{-2}$ **1(d)** $-3x^{-4}$ **2.** 6 **3.** -9

4(a) $6x$ **4(b)** $-36x^{-4}$ **4(c)** $6x+4$ **5.** $3x^2+2x+5$

6. $4-6t^{-3}-4t^{-2}$ **7.** $\dfrac{dT}{dL}=\dfrac{\pi}{\sqrt{Lg}}$

Exercises 2.3

1(a) $-7\sin x$ **1(b)** $8\cos 4x$ **2.** $18\cos 3x+3\sin 3x$ **3.** -14

4. $a=-3\,,\ b=7$ **5(a)** $15e^{3x}$ **5(b)** $-30e^{-5x}$ **6.** $1-\dfrac{3}{x}$

7. $f(0)=3\,,\ f'(0)=-6$ **8.** $20e^{2t}$ **10.** $10\sinh 2x-14\cosh 2x$

Exercises 2.4

1. $18x-4$ **2.** $-100\cos 5x$ **3.** $12e^{-2x}+\dfrac{4}{x^2}$ **4.** $A=4$

5. $A=-4$ **6.** $A=-10$

Exercises 2.5

1(a) $21(7x-2)^2$ **1(b)** $40x(5x^2-1)^3$ **1(c)** $-10(5x-3)^{-3}$

2(a) $-8\sin(2x-\pi/2)$ **2(b)** $\dfrac{3}{3x-7}$ **2(c)** $-10xe^{-5x^2}$ **3.** $14\sin 7x\cos 7x$

4. $\dfrac{3}{x}+\dfrac{20}{4x-2}$ **5.** $\dfrac{3}{4}$ **6.** $\dfrac{2x}{x^2+8}$

7 $f(x)=\dfrac{4}{x-1}-\dfrac{3}{(x-1)^2}$ $f'(x)=\dfrac{6}{(x-1)^3}-\dfrac{4}{(x-1)^2}$

Exercises 2.6

1(a) $2e^{-7x} - 14xe^{-7x}$ **1(b)** $6\sin 4x + 24x\cos 4x$ **1(c)** $2x\ln x + x$

2. $f'(x) = e^{-2x} - 2xe^{-2x}\,,\quad f''(x) = 4xe^{-2x} - 4e^{-2x}$

3. $f'(x) = \ln x\,,\quad f''(x) = \dfrac{1}{x}$

4. $A = -6$ **5.** $A = -8$ **6.** $-e^{-kt}\sin t - ke^{-kt}\cos t$

Exercises 2.7

1. $\dfrac{-2}{(2x+1)^2}$

2(a) $\dfrac{6}{(5x+3)^2}$ **2(b)** $\dfrac{2x}{(1+x^2)^2}$ **2(c)** $\dfrac{4x-2x^4}{(1+x^3)^2}$

3(a) $\dfrac{1+\sin x - x\cos x}{(1+\sin x)^2}$ **3(b)** $-\dfrac{4e^{4x}}{(1+e^{4x})^2}$

4. $\dfrac{1}{1+\cos x}$ **6.** $\dfrac{1}{\cos^2 x} = \sec^2 x$

Exercises 2.8

1. $\dfrac{3}{1+4t}$ **2.** $-3\tan t$

3. $-10t^3$ **4.** $2t - t^2$

Exercises 2.9

1. $-\dfrac{x}{y}$ **2.** $\dfrac{-2x-2y}{2x+3y^2}$

3. $\dfrac{4x^3-y}{2+x+10y}$ **4.** $\dfrac{2}{5}$

5. $\dfrac{9}{14}$ **6.** -1 **7.** $g'(t) = 0$