Applications of Differentiation

Contents

3.1 Tangent Lines

We have already seen that differentiating a function $f(x)$ at some point x_0 can be thought of as getting the slope of the tangent to $f(x)$ at x_0. In the diagram below the tangent AB touches the curve at the point $(x_0, f(x_0))$.

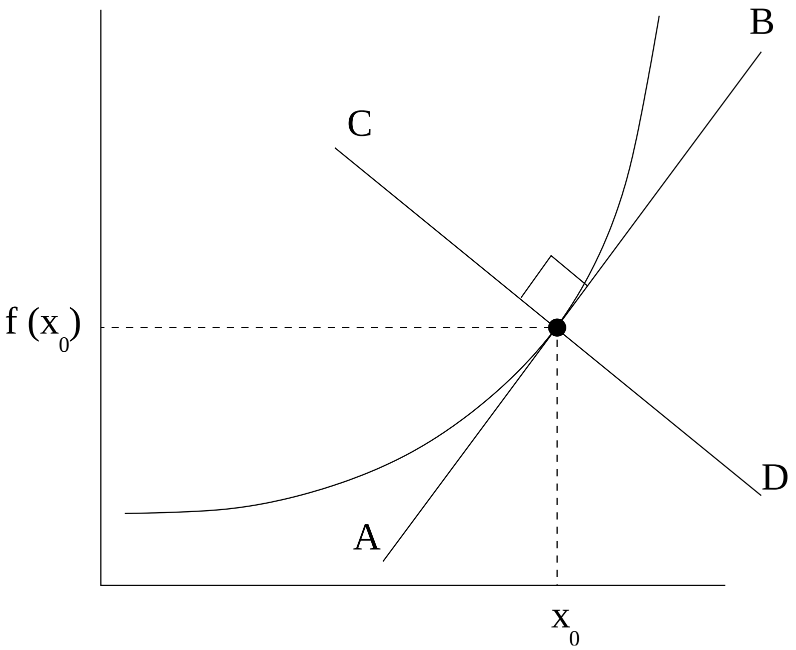


Figure 1

We can use what we already know, namely, the slope of AB and the fact that AB goes through the point of contact $(x_0, f(x_0))$ to determine the equation of both the tangent to the curve and the normal, which is shown as CD in the diagram.

The general formula for a straight line is $y = mx + c$, where m is the gradient. To find the equation of the tangent line to a curve $f(x)$ at x_0 we need to calculate m and c. The gradient to the curve $f(x)$ at x_0 is $f'(x_0)$ so $m = f'(x_0)$. We can find c by using the fact that the tangent line passes through the point $(x_0, f(x_0))$.

The equation of the normal can be found in a similar way using the fact that the gradient of the normal is equal to $-\dfrac{1}{m}$ where m is the gradient of the tangent. We will not find (or examine) equations of the normal in this module.

Example 3.1 *Find the equation of the tangent line to the curve* $f(x) = x^2 - 3x + 7$ *at* $x = 4$.

Solution Equation of the tangent line is $y = mx + c$. We first find m.

$f'(x) = 2x - 3$ and $f'(4) = 2(4) - 3 = 5$. Hence $m = 5$ and $y = 5x + c$.

$f(4) = 4^2 - 3(4) + 7 = 11$. Putting this in $y = 5x + c$ gives $11 = 5(4) + c = 20 + c$ so $c = -9$ and the equation of the tangent is

$$y = 5x - 9$$

Finally, we note that the tangent line at a point x_0 gives a good approximation to $f(x)$ for x close to x_0. For example, a calculation shows that the equation of the tangent line to the curve $f(x) = \sin x$ at the point $x = 0$ is given by $y = x$. Thus for small x, $\sin x \approx x$. For $x = 0.1$ (note: x is in radians) , $\sin(0.1) = 0.998334 \approx 0.1$.

Exercises 3.1

1. Find the equation of the tangent line to the curve $f(x) = x^2 - 9x + 8$ at the point $x = 2$.

2. Find the equation of the tangent line to the curve $f(x) = x^3 + 4x^2 - 2x - 1$ at the point $x = 1$.

3. Find the equation of the tangent line to the curve $f(x) = e^x$ at the point $x = 0$.

4. Find the equation of the tangent line to the curve $f(x) = \ln x$ at the point $x = 1$.

5. Show that the equation of the tangent line to the curve $f(x) = \dfrac{1}{x}$ at the point $x = a > 0$ is given by

$$a^2 y = 2a - x$$

The tangent line cuts the x-axes at A and the y-axes at B. Find the area of the triangle ABO where O is the origin . The interesting point is that the area does not depend on a .

3.2 Maclaurin Series

A calculation shows that the equation of the tangent line to the curve $f(x) = e^x$ at the point $x = 0$ is given by $y = x + 1$ (see the exercises in the previous section). This tangent line is a good approximation to $f(x) = e^x$ for x close to 0. However, at $x = 1$ the tangent line is equal to 2 while the curve $e^x = e^1 = e = 2.718\ldots$. We could improve the approximation $1 + x$ by adding terms in x^2, x^3 etc. This section is about doing this for a general function $f(x)$.

Suppose

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \tag{1}$$

we want to find the numbers $a_0. a_1$ etc. Putting $x = 0$ in (1) gives

$$f(0) = a_0 + a_1.0 + a_2.0^2 + \ldots$$

so we find $a_0 = f(0)$. Differentiating (1) gives

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots \tag{2}$$

Putting $x = 0$ in (2) gives

$$f'(0) = a_1 + 2a_2.0 + 2a_3.0^2 + \ldots$$

so we find $a_1 = f'(0)$. Differentiating (2) gives

$$f''(x) = 2a_2 + (3 \times 2)a_3 x + (4 \times 3)a_4 x^2 + (5 \times 4)a_5 x^3 + \ldots \tag{3}$$

Putting $x = 0$ in (3) gives $2a_2 = f''(0)$. Continuing this process we obtain

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \ldots \tag{4}$$

which is called the Maclaurin series for $f(x)$. Recall the notation $4! = 4 \times 3 \times 2 \times 1 = 24$ (see the section on the binomial formula).

Example 3.2 *Find the Maclaurin series for e^x up to and including the x^3 term.*

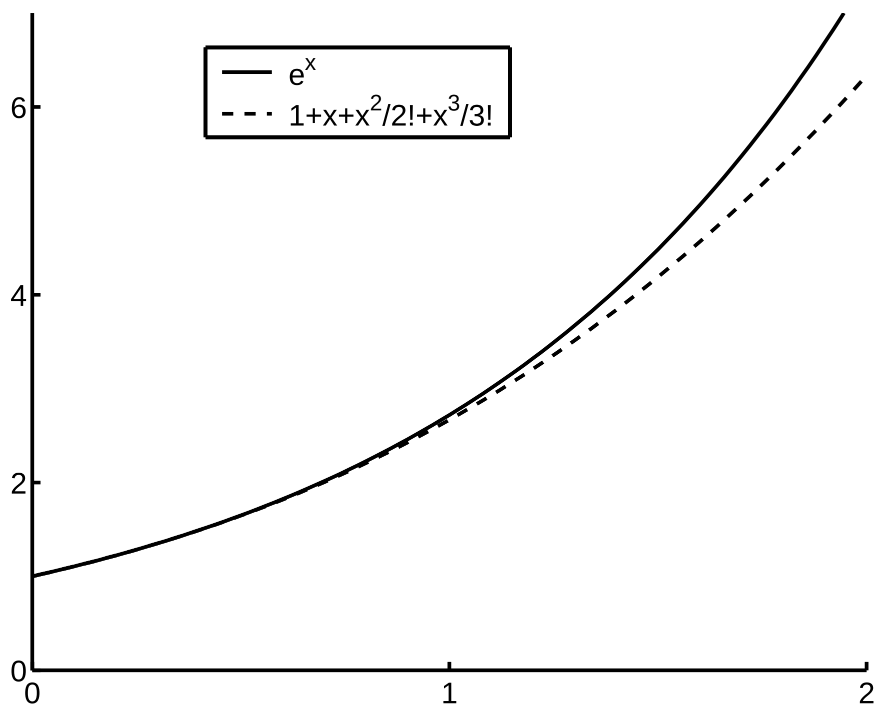
Solution $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$
$f(x) = e^x$ so $f(0) = e^0 = 1$. Also, $f'(x) = e^x$, $f''(x) = e^x$ and $f'''(x) = e^x$.
Hence $f'(0) = f''(0) = f'''(0) = 1$ and the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

In the figure below we have plotted the graph of e^x and the the first four terms of the Maclaurin series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Taking more terms in the Maclaurin series would give a better approximation.



Example 3.3 *Find the Maclaurin series for* $\sin x$ *up to the* x^5 *term.*

Solution $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots$

$f(x) = \sin x$ so

$f'(x) = \cos x \qquad f''(x) = -\sin x \qquad f'''(x) = -\cos x \qquad f^{(4)}(x) = \sin x \qquad f^{(5)}(x) = \cos x$

Substituting $x = 0$ and using $\sin 0 = 0$, $\cos 0 = 1$ we have

$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = -1 \qquad f^{(4)}(0) = 0 \qquad f^{(5)}(0) = 1$

The Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots$$

Exercises 3.2

1. Find the Maclaurin series for $\sqrt{1+x}$ up to the x^2 term.

2. Find the Maclaurin series for $\ln(1+x)$ up to the x^3 term.

3. Find the Maclaurin series for $\cos x$ up to the x^4 term.

4. Find the Maclaurin series for $\ln(1+e^x)$ up to the x^2 term.

3.3 Approximate Solutions of Equations

In general it is impossible to find an exact solution to the equation $f(x) = 0$. In this section we study the Newton-Raphson method for finding approximate solutions. The method is based on the following idea:

If $x = a$ is an estimate of a solution to $f(x) = 0$ then $x = b$ is a closer estimate when :

$$b = a - \frac{f(a)}{f'(a)} \tag{5}$$

The rational behind this will be explained in the lectures.

Starting with the approximation $x = a$ of a solution to $f(x) = 0$ we get a better estimate b calculated from (5). Replacing b by c , and a by b in (5), we get an even better approximation $x = c$ of a solution to $f(x) = 0$:

$$c = b - \frac{f(b)}{f'(b)}$$

In general, starting with the estimate x_0 for the solution to $f(x) = 0$ we generate a sequence of better estimates $x_1, x_2, x_3 \dots$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{6}$$

which is called the Newton-Raphson method.

Example 3.4 *Let* $f(x) = x^2 - 3$ *. Find the Newton-Raphson recurrence relation for the solution of* $f(x) = 0$ *. Do three iterations starting with* $x_0 = 1$ *to find an approximation to* $\sqrt{3}$*.*

Solution $f(x) = x^2 - 3$ so $f'(x) = 2x$ and

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3}{2x} = \frac{2x^2 - x^2 + 3}{2x} = \frac{x^2 + 3}{2x}$$

Hence, using (6), the Newton-Raphson recurrence relation is

$$x_{n+1} = \frac{x_n^2 + 3}{2x_n}$$

Taking $x_0 = 1$,

$$x_1 = \frac{1+3}{2} = 2\,, \qquad x_2 = \frac{4+3}{4} = 1.75\,, \qquad x_3 = \frac{(1.75)^2 + 3}{2(1.75)} = 1.73$$

Exercises 3.3

1. Let $f(x) = 1 - 10x^{-2}$. Use the Newton-Raphson method to obtain the recurrence relation

$$x_{n+1} = (1.5)x_n - (0.05)x_n^3$$

for the solution of $f(x) = 0$.

2. By considering

$$f(x) = 1 - \frac{1}{cx} = 0$$

use the Newton-Raphson method to obtain the recurrence relation

$$x_{n+1} = 2x_n - cx_n^2$$

for finding the value of $1/c$. Use this to approximate the reciprocal of 7 by doing two iterations starting with $x_0 = 0.1$.

3. By considering the equation $f(x) = x^3 - c = 0$, use the Newton-Raphson method to obtain the recurrence relation

$$x_{n+1} = \frac{1}{3}(2x_n + \frac{c}{x_n^2})$$

for finding the cube root of c. Find an approximation to the cube root of 10 starting with $x_0 = 2$ and doing three iterations.

4. $f(x) = e^x - 4x$. Show that the equation $f(x) = 0$ has a solution in the interval $[2, 3]$. Use the Newton-Raphson method to estimate the solution of $f(x) = 0$ by doing two iterations starting with $x_0 = 2$.

3.4 Stationary Points

The derivative $f'(x)$ measures the rate at which $f(x)$ increases with respect to x.

If $f'(x) > 0$ on an interval then $f(x)$ is an increasing function on that interval.

If $f'(x) < 0$ on an interval then $f(x)$ is a decreasing function on that interval.

Example 3.5 *If a fixed resistor of r ohms is connected in parallel with a variable resistor of x ohms, the resistance $R(x)$ of the combination is given by*

$$R(x) = \frac{rx}{x+r}$$

Show that $R(x)$ is an increasing function of x.

Solution We have to show that $R'(x) > 0$ (note that $x > 0$). Using the quotient rule

$$R'(x) = \frac{r(x+r) - rx}{(x+r)^2} = \frac{r^2}{(x+r)^2} > 0$$

Hence $R(x)$ is an increasing function of x.

In Figure 2, f is increasing if $x < a$ and $x > b$, while for $a < x < b$, it is decreasing. At the points $x = a$ and $x = b$ the gradient is zero, that is $f'(x) = 0$.

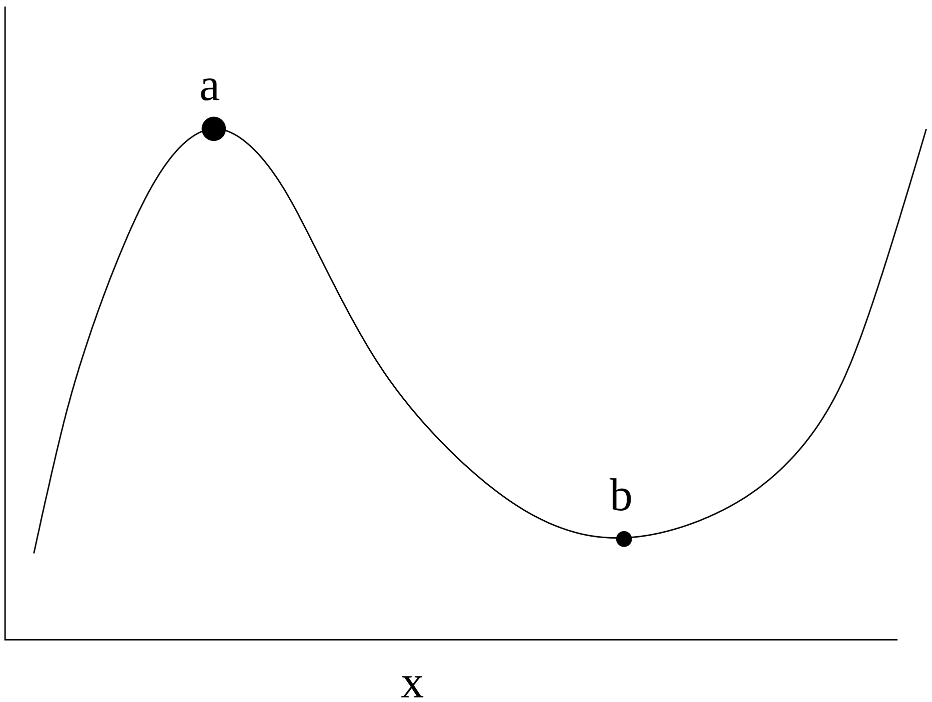


Figure 2: Graph of f(x)

In general, if $f'(c) = 0$ then c is a stationary point.

Example 3.6 *Find the stationary points of* $f(x) = x^3 - 3x^2$.

Solution At stationary points $f'(x) = 0$. Since $f'(x) = 3x^2 - 6x$, we solve

$$3x^2 - 6x = 3x(x-2) = 0$$

Thus $x = 0$ and $x = 2$ are the stationary points.

The value of $f(x)$ at a stationary point is called the stationary value. A stationary value may be a local maximum, a local minimum or a point of inflexion. In Figure 2, the stationary point at $x = a$ is a local maximum while $x = b$ is a local minimum.

There are two methods for finding the nature of a stationary point at $x = c$.

The first method compares the sign of $f'(x)$ for $x < c$ and $x > c$ (with x close to c). For a maximum, $f'(x)$ is positive for $x < c$ and negative for $x > c$. For a minimum, $f'(x)$ is negative for $x < c$ and positive for $x > c$.

In this module, we focus on the method called the second derivative test. Since $f''(x)$ measures the rate at which $f'(x)$ increases, if $f''(c) < 0$ then $f'(x)$ is decreasing at c. If $f''(c) > 0$ then $f'(x)$ is increasing at c. Hence

- if $f'(c) = 0$ and $f''(c) < 0$ then $x = c$ is a local maximum.
- if $f'(c) = 0$ and $f''(c) > 0$ then $x = c$ is a local minimum.

Example 3.7 *For* $f(x) = 6x - 2x^3$, *find the stationary points of* f *and the nature of each stationary point.*

Solution $f'(x) = 6 - 6x^2$ and $f''(x) = -12x$.

If x is a stationary point then $f'(x) = 0$, that is $x^2 = 1$ so $x = \pm 1$.

$f''(-1) = -12(-1) = 12 > 0$ and $f''(1) = -12 < 0$. Hence $x = -1$ is a local minimum and $x = 1$ is a local maximum.

Example 3.8 *Find* $f'(x)$ *and* $f''(x)$ *when* $f(x) = xe^{-x}$. *Show that* $x = 1$ *is a stationary point and determine its nature.*

Solution Using the product rule,

$$f'(x) = e^{-x} - xe^{-x}, \qquad f''(x) = -e^{-x} - e^{-x} + xe^{-x} = -2e^{-x} + xe^{-x}$$

Then $f'(1) = e^{-1} - e^{-1} = 0$ so $x = 1$ is a stationary point.

$f''(1) = -2e^{-1} + e^{-1} = -e^{-1} < 0$ so $x = 1$ is a local maximum.

Exercises 3.4

1. Find the stationary point of $f(x) = x + \dfrac{9}{x}$, $x > 0$ and determine its nature.

2. For $f(x) = x^3 - 12x$, find the stationary points of f and the nature of each stationary point.

3. For $f(x) = x^3 + 3x^2 - 45x$, find the stationary points of f and the nature of each stationary point.

4. Find $f'(x)$ and $f''(x)$ when $f(x) = e^{3x} - 3e^x$. Show that $x = 0$ is a stationary point and determine its nature.

5. Find $f'(x)$ and $f''(x)$ when $f(x) = \ln x - x$. Show that $x = 1$ is a stationary point and determine its nature.

6. Find $f'(x)$ and $f''(x)$ when $f(x) = 3x^2 + \cos 4x$. Show that $x = 0$ is a stationary point and determine its nature.

7. Show that $f(x) = x^2 + \dfrac{54}{x}$ is increasing for $x > 3$.

8. Express $f(x) = \dfrac{3x+2}{(x+1)(x+2)}$ in terms of partial fractions. Hence find $f'(x)$ and $f''(x)$. Show that $x = 0$ is a stationary point and determine its nature.

3.5 Optimization Problems

In many real world problems we want to make something as large or as small as possible by changing some of the parameters in the problem ("optimization").

Example 3.9 *A variable rectangle has a constant perimeter of* 16*. Find the lengths of the sides when the area is a maximum.*

Solution There are many rectangles with a perimeter 16, for example sides 7 and 1 (area 7) or 5 and 3 (area 15). To do problems like this we have to express the given information about the area as a function of one variable.

If the sides are x and y then the area is $A = xy$.

Since $2x + 2y = 16$ we have that $A(x) = x(8 - x) = 8x - x^2$.

At stationary points $A'(x) = 0$. Then $8 - 2x = 0$ and $x = 4$ is the stationary point.

Since $A''(x) = -2$, $A''(4) < 0$ so $x = 4$ is a maximum. Hence the area is a maximum when it is a square of side 4.

Example 3.10 *An open rectangular tank with a square base is to be constructed so that it has a volume of* 500 *cubic metres. Find the dimensions of the tank if the total surface area is a minimum.*

Solution Let h be the height and x the length of the side of the base. The surface area S is the area of the four walls and base, that is $S = x^2 + 4xh$. To proceed we need to use the information about the volume to get S as a function of one variable.

Since the volume is 500 we have $x^2h = 500$ so

$$h = \frac{500}{x^2}$$

and

$$S(x) = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + \frac{2000}{x}$$

Then

$$S'(x) = 2x - \frac{2000}{x^2} = 0$$

if $x = 10$. Hence $x = 10$ is the stationary point.

Since $S''(x) = 2 + 4000x^{-3}$, $S''(10) > 0$ and $x = 10$ is a minimum.

Using $x^2h = 500$, $h = 5$ and the required dimensions are 10 by 10 by 5.

Exercises 3.5

1. A variable rectangle has a constant area of 49. Find the lengths of the sides when the perimeter is a minimum.

2. A rectangular block has a square base. Its total surface area is 150.

(a) If the base length is x, show that the volume V of the block is

$$V = \frac{75x - x^3}{2}$$

(b) Find the dimensions of the block if the volume is a maximum.

3. An open box is to be made with a square base. The volume of the box is 32 cm^3. Find the dimensions of the box if the surface area of the box is a minimum.

4. A rectangular box-shaped house is to have a square floor. Four times as much heat per square metre is lost through the roof as through the walls: no heat is lost through the floor. The house has to enclose 2000 cubic metres. Find the dimensions of the house so as to minimise heat loss.

5. A rectangular field is to be fenced off along a road. The fence along the road costs 3 pounds per metre while on the other sides it costs 2 pounds per metre. Calculate the maximum area that can be fenced off for 400 pounds.

3.6 Velocity and Acceleration

Suppose that a car is travelling in a straight line and that after t it is at a distance $s(t)$ from its starting point. If the velocity $v(t)$ is constant then the graph of $s(t)$ against t is a straight line and the acceleration of the car is zero.

If the velocity $v(t)$ is not constant then the graph of $s(t)$ against t will not be a straight line. The average velocity of the car in the time interval t to $t+h$ is given by

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{s(t+h)-s(t)}{h}$$

The velocity of the car at the exact time t can be approximated by making h small so that the average velocity is calculated over a small time interval. Taking the limit, the velocity at time t is given by $v(t) = s'(t)$.

Similarly, the acceleration at time t is given by $a(t) = v'(t)$.

Example 3.11 *A particle moves along a straight line such that its distance $s(t)$ metres from a fixed point P after t seconds is given by $s(t) = t^3 - 3t^2 + 9t$.*

(a) Find the distance of the particle from P after 2 seconds.

(b) Find the velocity and acceleration after t seconds.

(c) Find the velocity when the acceleration of the particle is zero.

Solution (a) $s(t)$ is the distance after t seconds so $s(2) = 7$ metres is the distance from P after 2 second.

(b) $v(t) = s'(t) = 3t^2 - 6t + 9$ and $a(t) = v'(t) = 6t - 6$.

(c) If the acceleration of the particle is zero then $a(t) = 0$.

$6t - 6 = 0$ so $t = 1$. The velocity at this time is $v(1) = 3 - 6 + 9 = 6$ m/s .

Exercises 3.6

1. A particle moves along a straight line such that its distance $s(t)$ from a fixed point **P** after t seconds is given by $s(t) = t^3 - 6t^2 + 12$. Find

(a) its distance from **P** after 1 second,

(b) its acceleration at the time $t > 0$ when its velocity is zero.

2. A ball is thrown vertically upwards and after t seconds its height $s(t)$ above the ground is given by $s(t) = 8 + 10t - 5t^2$. Find

(a) the height from which the ball is thrown

(b) the initial velocity of the ball

(c) the maximum height reached by the ball.

3. A particle moves along a straight line such that its distance $s(t)$ metres from a fixed point at time t is given by $s(t) = te^{-2t}$. Find the velocity and acceleration.

3.7 First Order Differential Equations

Many scientific laws are formulated in terms of differential equations. In later courses you will learn how to solve such equations. In this module we will be studying their formulation and checking that given expressions satisfy particular differential equations. First order differential equations involve one derivative y'.
Consider the differential equation

$$\frac{dy}{dx} = 3$$

Since the gradient is constant (equal to 3) the solution is the straight line $y = 3x + C$ where C is a constant. If we have extra information such as $y = 7$ when $x = 1$ we can determine the value of C. In general, solutions of first order differential equations involve one arbitary constant.

Example 3.12 *The velocity $y(t)$ of a particle moving in a resisting medium is decreasing at a rate proportional to $y^2(t)$. If $k > 0$ is the constant of proportionality, obtain the differential equation satisfied by $y(t)$.*

Solution The rate of decrease of $y(t)$ is $-\frac{dy}{dt}$ so $-\frac{dy}{dt} = ky^2$. Hence $\frac{dy}{dt} = -ky^2$.

Example 3.13 *In a model to estimate the depreciation of the value of a computer, the value $y(t)$ at age t months, decreases at a rate proportional to $y(t)$.*

(a) If $k > 0$ is the constant of proportionality, obtain the differential equation satisfied by $y(t)$.

(b) The initial value of the computer is £1000. Verify that the solution is $y(t) = 1000e^{-kt}$.

Solution (a) The rate of decrease of $y(t)$ is $-\frac{dy}{dt}$ so $-\frac{dy}{dt} = ky$. Hence $\frac{dy}{dt} = -ky$.

(b) We have to

- Show that $y(t) = 1000e^{-kt}$ satisfies $\frac{dy}{dt} = -ky$.
- Check that for $y(t) = 1000e^{-kt}$, $y(0) = 1000$.

For $y(t) = 1000e^{-kt}$, $\frac{dy}{dt} = -1000ke^{-kt}$ and $-ky(t) = -1000ke^{-kt}$. Hence $\frac{dy}{dt} = -ky$.
Putting $t = 0$ in $y(t) = 1000e^{-kt}$ gives $y(0) = 1000e^0 = 1000$ as required.

Example 3.14 *Verify that $y = 1 + Ce^{3t}$ is a solution of*

$$\frac{dy}{dt} = 3(y - 1)$$

where C is a constant. Hence find the solution with $y = 5$ when $t = 0$.

Solution $\dfrac{dy}{dt} = 3Ce^{3t}$. Also, $3(y-1) = 3(1 + Ce^{3t} - 1) = 3Ce^{3t}$. Hence

$$\frac{dy}{dt} = 3(y-1)$$

as required.

Putting $t = 0$ in $y = 1 + Ce^{3t}$ gives

$$5 = 1 + Ce^0 = 1 + C$$

so $C = 4$ and $y = 1 + 4e^{3t}$.

Example 3.15 *In a certain chemical reaction, the concentration* $y(t)$ *of a substance is decreasing at a rate proportional to the cube of its value at that instant. If* $k > 0$ *is the constant of proportionality, obtain the differential equation satisfied by* $y(t)$ *. If the initial concentration is* $y(0) = 1$ *, verify that the solution is*

$$y(t) = (2kt + 1)^{-1/2}$$

Find T *such that* $y(T) = 1/2$ *.*

Solution The rate of decrease of $y(t)$ is $-\dfrac{dy}{dt}$ so $-\dfrac{dy}{dt} = ky^3$. Hence $\dfrac{dy}{dt} = -ky^3$.

For $y(t) = (2kt + 1)^{-1/2}$

$$\frac{dy}{dt} = (2k)(-1/2)(2kt + 1)^{-3/2} = -k(2kt + 1)^{-3/2}$$

and $-ky^3 = -k(2kt + 1)^{-3/2}$ so $\dfrac{dy}{dt} = -ky^3$.

Putting $t = 0$ in $y(t) = (2kt + 1)^{-1/2}$ gives $y(0) = (1)^{-1/2} = 1$ as required.

If $y(T) = 1/2$

$$\frac{1}{(2kT + 1)^{1/2}} = \frac{1}{2}$$

Then $(2kT + 1)^{1/2} = 2$ and $2kT + 1 = 4$ so that $T = \dfrac{3}{2k}$ ·

Exercises 3.7

1. Let $y(t)$ be the population of a certain species at time t. The rate of increase of $y(t)$ is proportional to the size of the population at any time. If $k > 0$ is the constant of proportionality, obtain the differential equation satisfied by $y(t)$.

2. In a certain chemical reaction, the concentration $y(t)$ of a substance is deceasing at a rate proportional to the square of its value at that instant. If $k > 0$ is the constant of proportionality, obtain the differential equation satisfied by $y(t)$.

3. Verify that $y = 2 + \dfrac{C}{t}$ is a solution of $t\dfrac{dy}{dt} + y = 2$ where C is a constant. Hence find the solution with $y = 7$ when $t = 3$.

4. The velocity $y(t)$ of a particle moving in a resisting medium satisfies the differential equation

$$\frac{dy}{dt} = -t - y$$

Verify that the solution with $y(0) = 5$ is $y(t) = 1 - t + 4e^{-t}$.

5. The height $y(t)$ of a tank of water which is being drained satisfies the differential equation $y'(t) = -16\sqrt{y(t)}$. Verify that the solution with $y(0) = 1$ is $y(t) = (1 - 8t)^2$.

6. In an electric circuit, the current $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = E - y$$

where the applied voltage E is a constant. Verify that the solution with $y(0) = 0$ is $y(t) = E(1 - e^{-t})$. What is the value of the current for large t ?

7. In a model to estimate the spread of a disease , the proportion $y(t)$ of the population infected at time t satisfies the differential equation $y'(t) = y\,(1 - y)$. Verify that the solution with $y(0) = 1/2$ is

$$y(t) = \frac{1}{1 + e^{-t}}$$

What proportion of the population is infected in the long term ?

8. A hard-boiled egg is put in a basin of water whose temperature is $18°C$. The temperature of the egg at time t is $y(t)$, and $y(t)$ is decreasing at a rate proportional to $y(t) - 18$.

If $k > 0$ is the constant of proportionality, obtain the differential equation satisfied by $y(t)$. Verify that $y(t) = 18 + Ce^{-kt}$ is a solution where C is a constant. Hence find $y(t)$ if the initial temperature of the egg is $98°C$.

9. Verify that $y = 2 + Ce^{4t}$ is a solution of

$$\frac{dy}{dt} = 4(y - 2)$$

where C is a constant. Hence find the solution with $y = 9$ when $t = 0$.

10. Verify that $y = t\ln t + Ct$ is a solution of

$$t\frac{dy}{dt} = t + y$$

where C is a constant. Hence find the solution with $y = 5$ when $t = 1$.

11. Verify that $y = (2t + C)^{-1/2}$ is a solution of $y' = -y^3$ where C is a constant. Hence find the solution with $y = 3$ when $t = 0$.

3.8 Second Order Differential Equations

Second order differential equations are equations involving the derivatives y'' and y'. Such equations occur in many areas of science and engineering. The equation

$$\frac{d^2y}{dt^2} + k^2 y = 0$$

where k is a constant arises in vibrational problems. Equations such as

$$y'' + ay' + by = 0$$

where a and b are constants arise in the analysis of electric circuits and in mechanics problems.

The differential equation $y'' = 0$ has the solution $y(t) = A + Bt$ where A and B are constants. In general, for second order differential equations we need two pieces of information to find the two arbitary constants in a solution.

Example 3.16 *Show that* $y(t) = A\sin 3t + B\cos 3t$ *satisfies the differential equation*

$$\frac{d^2y}{dt^2} + 9y = 0$$

where A *and* B *are constants. Hence find the solution with* $y(0) = -5$ *and* $y'(0) = 6$.

Solution $y' = 3A\cos 3t - 3B\sin 3t$ and $y'' = -9A\sin 3t - 9B\cos 3t$. Then

$$\frac{d^2y}{dt^2} + 9y = -9A\sin 3t - 9B\cos 3t + 9A\sin 3t + 9B\cos 3t = 0$$

as required.

Putting $t = 0$ in $y(t) = A\sin 3t + B\cos 3t$ gives $-5 = A\sin 0 + B\cos 0 = B$ so $B = -5$.

Putting $t = 0$ in $y'(t) = 3A\cos 3t - 3B\sin 3t$ gives $6 = 3A$ so $A = 2$.

Hence $y(t) = 2\sin 3t - 5\cos 3t$.

Example 3.17 *Show that* $y(t) = (A + Bt)e^t$ *satisfies the differential equation*

$$y'' - 2y' + y = 0$$

where A *and* B *are constants. Hence find the solution with* $y(0) = 3$ *and* $y'(0) = 1$.

Solution Using the product rule, $y' = Be^t + (A + Bt)e^t = (A + B + Bt)e^t$

$$y'' = Be^t + (A + B + Bt)e^t = (A + 2B + Bt)e^t$$

$$y'' - 2y' + y = (A + 2B + Bt)e^t - 2(A + B + Bt)e^t + (A + Bt)e^t = 0$$

as required.

Putting $t = 0$ in $y(t) = (A + Bt)e^t$ gives $A = 3$.

Putting $t = 0$ in $y' = (A + B + Bt)e^t$ gives $1 = A + B = 3 + B$.

Hence $B = -2$ and $y(t) = 3e^t - 2te^t$.

Example 3.18 *Find the value of* A *if* $y(t) = Ae^{-2t}$ *is a solution of* $y'' + 5y' + 4y = 6e^{-2t}$.

Solution $y' = -2Ae^{-2t}$ and $y'' = 4Ae^{-2t}$. Then

$$y'' + 5y' + 4y = 4Ae^{-2t} - 10Ae^{-2t} + 4Ae^{-2t} = -2Ae^{-2t} = 6e^{-2t}$$

if $A = -3$.

Example 3.19 *Verify that* $y(t) = e^{mt}$ *is a solution of*

$$y'' + 6y' + 5y = 0$$

if $m^2 + 6m + 5 = 0$. *Hence obtain two solutions of the above differential equation.*

Solution $y(t) = e^{mt}$ so $y' = me^{mt}$ and $y'' = m^2e^{mt}$. Then

$$y'' + 6y' + 5y = m^2e^{mt} + 6me^{mt} + 5e^{mt} = (m^2 + 6m + 5)e^{mt} = 0$$

if $m^2 + 6m + 5 = 0$. To find two solutions we solve

$$m^2 + 6m + 5 = (m + 5)(m + 1) = 0$$

to get $m = -5$ and $m = -1$. The solutions of the differential equation are $y(t) = e^{-5t}$ and $y(t) = e^{-t}$.

Exercises 3.8

1. Find the value of A if $y(t) = Ae^{3t}$ is a solution of $y'' - 6y' + 3y = 12e^{3t}$.

2. Show that $y(t) = Ae^{4t} + Be^{-4t}$ satisfies the differential equation

$$\frac{d^2y}{dt^2} - 16y = 0$$

where A and B are constants. Hence find the solution with $y(0) = 0$ and $y'(0) = 24$.

3. Show that $y(t) = A\sin 2t + B\cos 2t - \sin 4t$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + 4y = 12\sin 4t$$

where A and B are constants. Hence find the solution with $y(0) = 5$ and $y'(0) = 8$.

4. Verify that $y(t) = e^{mt}$ is a solution of

$$y'' - 36y = 0$$

if $m^2 - 36 = 0$. Hence obtain two solutions of the above differential equation.

5. Verify that $y(t) = e^{mt}$ is a solution of

$$y'' + y' - 6y = 0$$

if $m^2 + m - 6 = 0$. Hence obtain two solutions of the above differential equation.

6. Find the value of A if $y(t) = A\sin 2t$ is a solution of

$$y'' + y = 21\sin 2t$$

3.9 Answers to Exercises

Exercises 3.1

1. $y = -5x + 4$ **2.** $y = 9x - 7$ **3.** $y = x + 1$

4. $y = x - 1$ **5.** Area $= 2$

Exercises 3.2

1. $1 + \dfrac{x}{2} - \dfrac{x^2}{8}$ **2.** $x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

3. $1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ **4.** $\ln 2 + \dfrac{x}{2} + \dfrac{x^2}{8}$

Exercises 3.3

2. $x_1 = 0.13\,,\ x_2 = 0.142$

3. $x_1 = 2.16667\,,\ x_2 = 2.15450\,,\ x_3 = 2.15443$

4. $x_1 = 2.18\,,\ x_2 = 2.15$

Exercises 3.4

1. $x = 3$ a local minimum.

2. $x = -2$ a local maximum. $x = 2$ a local minimum.

3. $x = -5$ a local maximum. $x = 3$ a local minimum.

4. $x = 0$ is a local minimum.

5. $x = 1$ a local maximum. **6.** $x = 0$ a local maximum.

8. $f(x) = \dfrac{4}{x+2} - \dfrac{1}{x+1}$ $f'(x) = (x+1)^{-2} - 4(x+2)^{-2}$

8. $f''(x) = 8(x+2)^{-3} - 2(x+1)^{-3}$ $x = 0$ a local maximum.

Exercises 3.5

1. 7 by 7 **2.** 5 by 5 by 5

3. 4 by 4 by 2

4. 10 by 10 by 20 **5.** 2000

Exercises 3.6

1(a) 7 **1(b)** 12

2(a) 8 **2(b)** 10 **2(c)** 13

3. $v(t) = e^{-2t} - 2te^{-2t}\,,\; a(t) = 4te^{-2t} - 4e^{-2t}$

Exercises 3.7

1. $y' = ky$ **2.** $y' = -ky^2$

3. $y = 2 + \dfrac{15}{t}$ **6.** E **7.** All

8. $\dfrac{dy}{dt} = -k(y - 18)\,.$ $18 + 80e^{-kt}$

9. $2 + 7e^{4t}$ **10.** $t \ln t + 5t$ **11.** $\left(2t + \frac{1}{9}\right)^{-\frac{1}{2}}$

Exercises 3.8

1. $A = -2$ **2.** $3e^{4t} - 3e^{-4t}$

3. $y(t) = 6 \sin 2t + 5 \cos 2t - \sin 4t$

4. e^{6t} and e^{-6t} **5.** e^{-3t} and e^{2t} **6.** $A = -7$