Solutions to Exercises 1.1

1. $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\,b + \binom{n}{2}a^{n-2}\,b^2 + \cdots + \binom{n}{n-1}a\,b^{n-1} + \binom{n}{n}b^n$

2 $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + x^n$

3 We use the binomial formula with $a = x$, $b = y$ and $n = 3$. Then

$$\begin{aligned}(x+y)^3 &= \binom{3}{0}x^3 + \binom{3}{1}x^2\,y + \binom{3}{2}x\,y^2 + \binom{3}{3}y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

4 $(3+y)^4 = 81 + \binom{4}{1}(27)y + \binom{4}{2}(9)y^2 + \binom{4}{3}(3)y^3 + y^4$. Hence

$$(3+y)^4 = 81 + 108y + 54y^2 + 12y^3 + y^4$$

5

$$\begin{aligned}(x+\frac{1}{x})^3 &= x^3 + \binom{3}{1}x^2\left(\frac{1}{x}\right) + \binom{3}{2}x\left(\frac{1}{x^2}\right) + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\end{aligned}$$

6 $(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^6$.

7 Up to terms in x^2, $(1+x)^{20} = 1 + 20x + \binom{20}{2}x^2 = 1 + 20x + 190x^2$.

8 Up to terms in p^4, $(1+p^2)^{10} = 1 + 10p^2 + \binom{10}{2}p^2 = 1 + 10p^2 + 45p^4$.

9 Up to terms in x^2, $(1-x)^{10} = 1 - 10x + \binom{10}{2}x^2 = 1 - 10x + 45x^2$.

10 The first two terms in $(1+x)^6$ are $1 + 6x$.

If p increases by 2% then the new value of $y = p^6$ is

$$p^6(1+0.02)^6 \approx p^6(1+0.12)$$

Hence y increases by 12% approximately.

Solutions to Exercises 1.2

1(a) $r = 3$, next terms $3 \times 189 = 567$ and $3 \times 567 = 1701$

1(b) $r = 0.1$, next terms 0.0017 and 0.00017

2(a) Owe $(1.02)P, (1.02)^2P, (1.02)^3P, \ldots$ at the end of each month, a geometric sequence with $r = 1.02$

2(b) The interest rate charged by the company is not the same as an annual rate of $12 \times 2 = 24\%$ because interest is paid on the interest.

After 12 months, $(1.02)^3P = 1.268P$ is owed so the annual percentage rate is 26.8%.

3 Using the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ with $a = 1$, $r = 3$, $n = 8$ the sum is

$$S = \frac{3^8 - 1}{2} = 3280$$

4 Working in pounds and using the formula with $a = 0.01, r = 2, n = 30$ the sum is

$$\frac{0.01(2^{30} - 1)}{2 - 1} = 10.737 \times 10^6$$

or approximately eleven million pounds.

5 The required sum is

$$1000(1.08) + 1000(1.08)^2 + \ldots + 1000(1.08)^7$$

Using the formula with $a = 1000(1.08)$, $r = 1.08$, $n = 7$ the total amount saved after 7 years is

$$\frac{1000(1.08)\,[1.08^7 - 1]}{0.08} = 9636.6$$

6 Profits are $60000, (.95) \times 60000, (.95)^2 \times 60000, \ldots$.

This is a geometric sequence with $r = 0.95$.

Using the formula with $a = 60000$, $r = 0.95$, $n = 6$ the total profits after 6 years is

$$\frac{60000(1.08)\,[1 - 0.95^6]}{1 - 0.95} = 317900$$

7 $\dfrac{x + 11}{x + 1} = \dfrac{x + 41}{x + 11}$ so that $(x + 11)^2 = (x + 1)(x + 41)$

Hence $x^2 + 22x + 121 = x^2 + 42x + 41$ and $20x = 80$ so $x = 4$.

Solutions to Exercises 1.3

1. To express $\dfrac{2}{(x+7)(x+9)}$ in terms of partial fractions write:

$$\frac{2}{(x+7)(x+9)} = \frac{A}{(x+7)} + \frac{B}{(x+9)}$$

Multiplying by $(x+7)(x+9)$

$$2 = A(x+9) + B(x+7)$$

Put $x = -7$ then $2 = A(-7+9) + 0$ so $A = 1$.

Put $x = -9$ then $2 = 0 + B(-9+7) + 0$ so $B = -1$. Hence

$$\frac{2}{(x+7)(x+9)} = \frac{1}{x+7} - \frac{1}{x+9}$$

2 Let

$$\frac{2x+8}{(2x+5)(x+3)} = \frac{A}{(2x+5)} + \frac{B}{(x+3)}$$

Multiplying by $(2x+5)(x+3)$

$$2x + 8 = A(x+3) + B(2x+5)$$

Put $x = -3$ then $-6 + 8 = B(-6+5) + 0$ so $B = -2$.

Put $x = -5/2$ then $-5 + 8 = A(-5/2 + 3) + 0$ so $A = 6$. Hence

$$\frac{2x+8}{(2x+5)(x+3)} = \frac{6}{(2x+5)} - \frac{2}{(x+3)}$$

3 Let

$$\frac{5x+6}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Multiplying by $(x-1)^2$

$$5x + 6 = A(x-1) + B$$

Setting $x = 1$ gives $B = 11$ and equating coefficents of x gives $A = 5$. Hence

$$\frac{5x+6}{(x-1)^2} = \frac{5}{(x-1)} + \frac{11}{(x-1)^2}$$

4 Let

$$\frac{2x-1}{(2x+1)(x-3)} = \frac{A}{(2x-1)} + \frac{B}{(x-3)}$$

Multiplying by $(2x+1)(x-3)$

$$2x-1 = A(x-3) + B(2x+1)$$

Put $x = 3$ then $6 - 1 = B(6+1) + 0$ so $B = 5/7$.

Put $x = -1/2$ then $-2 = A(-1/2-3) + 0$ so $A = 4/7$. Hence

$$\frac{2x-1}{(2x+1)(x-3)} = \frac{4}{7(2x+1)} + \frac{5}{7(x-3)}$$

5 Let

$$\frac{10x+7}{(2x+3)^2} = \frac{A}{(2x+3)} + \frac{B}{(2x+3)^2}$$

Multiplying by $(2x+3)^2$

$$10x+7 = A(2x+3) + B$$

Setting $x = -3/2$ gives $B = -8$ and equating coefficents of x gives $A = 5$. Hence

$$\frac{10x+7}{(2x+3)^2} = \frac{5}{(2x+3)} - \frac{8}{(2x+3)^2}$$

Solutions to Exercises 1.4

1(a) $10 = 10^1$ so $\log(10) = 1$.

1(b) $100 = 10^2$ so $\log(100) = 2$.

1(c) $0.001 = 10^{-3}$ so $\log(0.001) = -3$.

2. $t = -\dfrac{\ln(12/2)}{10} = -\dfrac{\ln(6)}{10} = -0.179$

3. $\ln(7 - x)$ makes sense if $7 - x > 0$ that is $x < 7$.

4. $\ln(2x - 6)$ makes sense if $2x - 6 > 0$, that is $x > 3$.

$\ln A = 0$ if $A = 1$.

Hence $\ln(2x - 6) = 0$ if $2x - 6 = 1$, that is $x = 7/2$.

5. When $r = 10$,

$$v = \frac{25}{r^2} \ln\left(\frac{r}{5}\right) = \frac{25}{10^2} \ln\left(\frac{10}{5}\right) = 0.173$$

Solutions to Exercises 1.5

1 $\log 25 = \log 5^2 = 2\log 5 = 2 \times 0.6989 = 1.3979$

2(a) $\ln(xy^2) - 3\ln(y) + \ln(x) = \ln(xy^2) - \ln(y^3) + \ln(x) = \ln\left(\frac{x^2y^2}{y^3}\right) = \ln\left(\frac{x^2}{y}\right)$

2(b) $\ln x - 2\ln(3x+5) = \ln x - \ln(3x+5)^2 = \ln\left(\frac{x}{(3x+5)^2}\right)$

3 $(1.1)^x = 1999$ so $x\ln(1.1) = \ln(1999)$ and $x = 79.743$

4 Subtracting the equations

$$-Q\ln r = mx \qquad \text{and} \qquad -Q\ln s = my$$

gives

$$-Q\ln r + Q\ln s = mx - my$$

Hence

$$Q(\ln s - \ln r) = m(x-y)$$

Using $\ln s - \ln r = \ln(s/r)$ we get

$$Q = \frac{m(x-y)}{\ln\left(\frac{s}{r}\right)}$$

Solutions to Exercises 1.6

1 $f(0) = 2e^0 = 2$ and $f(0.5) = 2\exp(-0.25) = 1.5576$

2 $e^x = 666$ so $x = \ln(666) = 6.501$

3 $(e^x - e^{-x})^2 = (e^x)^2 - 2e^x\, e^{-x} + (e^{-x})^2 = e^{2x} - 2 + e^{-2x}$

4 When $t = 1.5$ we have that

$$Ae^{(1.5)k} = 2s(0) = 2A$$

Hence $e^{(1.5)k} = 2$ so $(1.5)k = \ln 2$ and $k \approx 0.46$.

5 The main points are (a) $y = 2$ when $t = 0$, (b) $y \approx 0$ for large t, (c) y is rapidly decreasing. The graph is shown below in Figure 1.

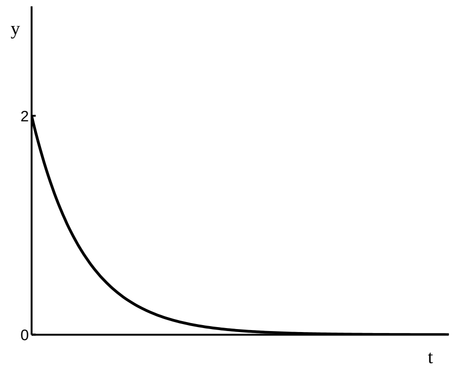

Figure 1

6 The main points are (a) $y = 3 - 2 = 1$ when $t = 0$, (b) $y \approx 3$ for large t, (c) y is increasing. The graph is shown below in Figure 2. Also, $2e^{-4t} = 3 - y$ so

$$t = -\frac{1}{4}\ln\left(\frac{3-y}{2}\right)$$

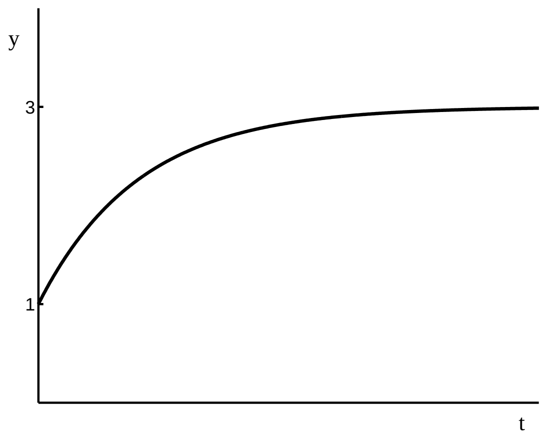

Figure 2

Solutions to Exercises 1.6 continued

7(a) $t = \ln x + \ln 2 = \ln(2x)$ and $3t = 3\ln(2x) = \ln(2x)^3$. Hence $e^{3t} = (2x)^3 = 8x^3$.

7(b) $3t = 6 - 3\ln x = 6 - \ln x^3$ so

$$e^{3t} = \exp(6 - \ln x^3) = e^6 \exp(-\ln x^3) = e^6 x^{-3}$$

8 If $x = me^{-at}$ and $y = me^{-bt}$ then

$$\frac{x}{y} = \frac{me^{-at}}{me^{-bt}} = \frac{e^{-at}}{e^{-bt}} = e^{-at}e^{bt} = e^{(b-a)t}$$

Hence, $z = \ln(x/y) = (b-a)t$.

9 When $x = 33$ the intensity is $I = (0.01)\, I_0$. Hence

$$e^{-33k} = 0.01$$

and $-33k = \ln(0.01)$ so that $k = 0.14$.

10 $N = N_0\, e^{-kt}$, and $N_0 = 100, k = 1/3$, so

$$N = 100\, e^{-\frac{t}{3}}$$

After 6 days

$$N = 100\, e^{-2} = 13.53$$

so he's (fairly!) safe.

11 $p = 760\, e^{-kh}$ so if $p = 670$ when $h = 1$, we have $670 = 760\, e^{-k}$. Then

$$e^{-k} = \frac{670}{760}$$

and

$$k = -\ln\left(\frac{670}{760}\right) = 0.126$$

Using this for $h = 2$ we have $p = 760\, e^{-0.252} = 590.7$.

Solutions to Exercises 1.7

1 From the definition of $\cosh x$, $\cosh 7x = \frac{1}{2}(e^{7x} + e^{-7x})$

2 From the definition of $\sinh x$, $\sinh 6x = \frac{1}{2}(e^{6x} - e^{-6x})$

3 From the definitions of $\cosh x$ and $\sinh x$,

$$\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = e^x$$

and

$$\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = e^{-x}$$

Then

$$9e^x + 2e^{-x} = 9\,(\cosh x + \sinh x) + 2\,(\cosh x - \sinh x) = 11\cosh 3x + 7\sinh 3x$$

4 From the definitions of $\cosh 3x$ and $\sinh 3x$,

$$\cosh 3x + \sinh 3x = \frac{1}{2}(e^{3x} + e^{-3x}) + \frac{1}{2}(e^{3x} - e^{-3x}) = e^{3x}$$

and

$$\cosh 3x - \sinh 3x = \frac{1}{2}(e^{3x} + e^{-3x}) - \frac{1}{2}(e^{3x} - e^{-3x}) = e^{-3x}$$

Then

$$8e^{3x} - 5e^{-3x} = 8\,(\cosh 3x + \sinh 3x) - 5\,(\cosh 3x - \sinh 3x) = 3\cosh 3x + 13\sinh 3x$$

5

$$\cosh^2 x = \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2}{4}$$

so

$$2\cosh^2 x - 1 = \frac{e^{2x} - e^{-2x}}{2} = \cosh 2x$$

6 $(1 + e^{-kx})^2 = 1 + 2e^{-kx} + e^{-2kx}$ so

$$F = \frac{e^{-kx}}{(1 + e^{-kx})^2} = \frac{e^{-kx}}{1 + 2e^{-kx} + e^{-2kx}}$$

Multiplying the top and bottom by e^{kx} gives

$$F = \frac{1}{e^{kx} + 2 + e^{-kx}}$$

Since $2\cosh kx = e^{kx} + e^{-kx}$

$$F = \frac{1}{2 + 2\cosh kx}$$

Solutions to Exercises 1.8

1 T is proportional to $\sqrt{L}$, so that T^2 is proportional to L and T^{-2} is proportional to L^{-1}. Hence **(a)** , **(c)** and **(f)** gives straight line graphs, the rest do not.

2(i) Plot x^2 against y.

2(ii) Plot $\dfrac{1}{x}$ against y.

3(i) Plot x^3 against y.

3(ii) Rewriting the formula as $ax + b = \frac{1}{y}$ we see that x plotted against $\frac{1}{y}$ gives a straight line.

3(iii) Rewriting the formula as $yx = ax^3 + b$ we see that x^3 plotted against yx gives a straight line.

4 Put $t = \sqrt{x}$ so that $y = at + b$.

The gradient of the line is 2 so $y = 2t + 4$.

Hence, $y = 2\sqrt{x} + 4$.

5 Put $t = x^{\frac{3}{2}}$ so that $y = at + b$.

The gradient of the line is -2 so $y = -2t + 2$.

Hence, $y = -2x^{\frac{3}{2}} + 2$.

6 We have that $\ln y = 2\ln x + 1.6 = \ln x^2 + 1.6$.

Taking exp of each side :

$$y = \exp(\ln x^2 + 1.6) = \exp(\ln x^2)\ \exp(1.6) = (4.95)\, x^2$$