Solutions to Exercises 2.1

1. With $f(x) = x^3$, so we get

$$\frac{f(1.3)-f(1)}{0.3} = \frac{(1.3)^3-1}{0.3} = 3.99$$

$$\frac{f(1.2)-f(1)}{0.2} = \frac{(1.2)^3-1}{0.2} = 3.64$$

$$\frac{f(1.1)-f(1)}{0.2} = \frac{(1.1)^3-1}{0.1} = 3.31$$

which is converging (slowly) to 3.

Solutions to Exercises 2.2

1(a) Using the formula for the derivative of x^n, if $f(x) = x^3$ then $f'(x) = 3x^2$.

1(b) If $f(x) = x^5$ then $f'(x) = 5x^4$.

1(c) $f(x) = \dfrac{1}{x} = x^{-1}$ so $f'(x) = -x^{-2}$.

1(d) $f(x) = \dfrac{1}{x^3} = x^{-3}$ so $f'(x) = -3x^{-4}$.

2 The gradient at x is $f'(x) = \dfrac{3}{2}x^{1/2}$.

At $x = 16$ the gradient is $f'(16) = \dfrac{3}{2}\sqrt{16} = 6$

3 The gradient at x is $f'(x) = -\dfrac{1}{x^2}$

At $x = 1/3$ the gradient is $f'(1/3) = -\dfrac{1}{1/9} = -9$

4(a) If $f(x) = 3x^2$ then $f'(x) = 6x$.

4(b) $f(x) = \dfrac{12}{x^3} = 12x^{-3}$ so $f'(x) = -36x^{-4}$.

4(c) $f(x) = 3x^2 + 4x + 7$ so $f'(x) = 6x + 4$

5 $f(x) = x^3 + x^2 + 5x + 6$ so $f'(x) = 3x^2 + 2x + 5$

6 $f(t) = 1 + 4t + \dfrac{3}{t^2} + \dfrac{4}{t}$ so $f'(t) = 4 - 6t^{-3} - 4t^{-2}$

7 $T = \dfrac{2\pi}{\sqrt{g}}L^{1/2}$ so

$$\frac{dT}{dL} = \frac{\pi}{\sqrt{Lg}}$$

Solutions to Exercises 2.3

1(a) Using the table with $a = 1$, if $f(x) = 7\cos x$ then $f'(x) = -7\sin x$.

1(b) Using the table with $a = 2$, if $f(x) = 2\sin 4x$ then $f'(x) = 8\cos 4x$

2 Using the table with $a = 3$, $f'(x) = 18\cos 3x + 3\sin 3x$.

3 If $f(x) = 5\cos 4x - 7\sin 2x$ then $f'(x) = -20\sin 4x - 14\cos 2x$ and $f'(0) = -14$.

4 $f'(x) = 2a\cos 2x - 2b\sin 2x$. Then $f(0) = b = 7$ and $f'(0) = 2a = -6$ so $a = -3$.

5(a) Using the formula for the derivative of e^{ax} with $a = 3$,
if $f(x) = 5e^{3x}$ then $f'(x) = 15e^{3x}$.

5(b) If $f(x) = 6e^{-5x}$ then $f'(x) = -30e^{-5x}$.

6 $f(x) = x - 3\ln x$ so $f'(x) = 1 - \dfrac{3}{x}$

7 $f(x) = 3e^{-2x}$ so $f'(x) = -6e^{-2x}$. Hence $f(0) = 3$ and $f'(0) = -6$.

8 $\dfrac{dN}{dt} = 20e^{2t}$ and $\dfrac{dN}{dt} - 2N = 20e^{2t} - (2)10e^{2t} = 0$.

9 $y = 4e^{-5t} - 2$ so $y' = -20e^{-5t}$. Since

$$-5(y+2) = -5\left(4e^{-5t} - 2 + 2\right) = -20e^{-5t}$$

we have that $y' = -5(y+2)$.

10 Using the table for the derivatives of $\sinh ax$ and $\cosh ax$ with $a = 2$, if

$$f(x) = 5\cosh 2x - 7\sinh 2x$$

then

$$f'(x) = 10\sinh 2x - 14\cosh 2x$$

Solutions to Exercises 2.4

1 $f'(x) = 9x^2 - 4x + 4$, so $f''(x) = 18x - 4$.

2 $f'(x) = -20\sin 5x$, so $f''(x) = -100\cos 5x$.

3 $f'(x) = -6e^{-2x} - 4x^{-1}$, so $f''(x) = 12e^{-2x} + 4x^{-2}$.

4 $y = \sin 2x$, so $y' = 2\cos 2x$ and $y'' = -4\sin 2x$.

$$y'' + 4y = -4\sin 2x + A\sin 2x = 0$$

if $A = 4$.

5 $y = e^{3x}$, so $y' = 3e^{3x}$ and $y'' = 9e^{3x}$. Then

$$y'' + Ay' + 3y = 9e^{3x} + 3Ae^{3x} + 3e^{3x} = e^{3x}\ (12 + 3A) = 0$$

if $A = -4$.

6 $y = e^{-2x}$, so $y' = -2e^{-2x}$ and $y'' = 4e^{-2x}$. Then

$$y'' - 3y' + Ay = 4e^{-2x} + 6e^{-2x} + Ae^{-2x} = e^{-2x}\,[10 + A] = 0$$

if $A = -10$.

For $y = e^{5x}$, we have that $y' = 5e^{5x}$ and $y'' = 25e^{5x}$. Then

$$y'' - 3y' - 10y = 25e^{5x} - 15e^{5x} - 10e^{5x} = 0$$

as required.

Solutions to Exercises 2.5

1(a) $y = f(x) = (7x-2)^3 = u^3$ where $u = 7x - 2$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2\,(7) = 21(7x-2)^2$$

1(b) $y = f(x) = (5x^2-1)^4 = u^4$ where $u = 5x^2 - 1$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3\,(10x) = 40x(5x^2-1)^3$$

1(c) $y = f(x) = (5x-3)^{-2} = u^{-2}$ where $u = 5x - 3$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5\,(-2u^{-3}) = -10(5x-3)^{-3}$$

2(a) $y = f(x) = 4\cos(2x - \pi/2) = \cos u$ where $u = 2x - \pi/2$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -8\sin(2x - \pi/2)$$

2(b) $y = f(x) = \ln(3x-7) = \ln u$ where $u = 3x - 7$;. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{3}{u} = \frac{3}{3x-7}$$

2(c) $y = e^u$ where $u = -5x^2$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -10xe^u = -10xe^{-5x^2}\,.$$

3 $y = u^2$ where $u = \sin 7x$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 7\cos 7x(2u) = 14\sin 7x\,\cos 7x\,.$$

Solutions to Exercises 2.5 continued

4 $f(x) = 3\ln(x) + 5\ln(4x-2)$. Using the chain rule for $5\ln(4x-2)$ we get

$$f'(x) = \frac{3}{x} + \frac{20}{4x-2}$$

5 $y = u^{1/2}$ where $u = x^2 + 7$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-1/2}\,(2x) = x(x^2+7)^{-1/2}$$

Then

$$f'(3) = 3(3^2+7)^{-1/2} = 3(16)^{-1/2} = \frac{3}{4}$$

6 $y = f(x) = \ln(x^2+8) = \ln u$ where $u = x^2 + 8;$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2x}{u} = \frac{2x}{x^2+8}$$

7 Let

$$\frac{4x-7}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Multiplying by $(x-1)^2$

$$54x - 7 = A(x-1) + B$$

Setting $x = 1$ gives $B = -3$ and equating coefficents of x gives $A = 4$. Hence

$$\frac{4x-7}{(x-1)^2} = \frac{4}{(x-1)} - \frac{3}{(x-1)^2}$$

Using the chain rule,

$$f'(x) = \frac{6}{(x-1)^3} - \frac{4}{(x-1)^2}$$

Solutions to Exercises 2.6

1(a) Using the product rule, $f'(x) = 2e^{-7x} - 14e^{-7x}$.

1(b) $f'(x) = 6\sin 4x + 24x\cos 4x$.

1(c) $f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x$.

2 Using the product rule, $f'(x) = e^{-2x} - 2xe^{-2x}$.

Differentiating again and using the product rule for the second term gives

$$f''(x) = -2e^{-2x} - 2e^{-2x} + 4xe^{-2x} = -4e^{-2x} + 4xe^{-2x} .$$

3 $f'(x) = \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x$ so $f''(x) = \frac{1}{x}$

4 $y = x\cos 3x$, so $y' = \cos 3x - 3x\,\sin 3x$ and

$$y'' = -3\sin 3x - 3\sin 3x - 9x\cos 3x = -6\sin 3x - 9x\cos 3x$$

Then $y'' + 9y = -6\sin 3x$ so $A = -6$.

5 $y = e^{4x}$, so $y' = 4e^{4x}$ and $y'' = 16e^{4x}$. Then

$$y'' + Ay' + 16y = 16e^{4x} + 4A\,e^{4x} + 16e^{4x} = e^{4x}\,(32 + 4A) = 0$$

if $A = -8$.

If $y = x\,e^{4x}$, then $y' = e^{4x} + 4xe^{4x}$ and

$$y'' = 4e^{4x} + 4e^{4x} + 16xe^{4x} = 8e^{4x} + 16xe^{4x}.$$

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8e^{4x} - 32xe^{4x} + 16xe^{4x} = 0 .$$

6 $y = e^{-kt}\cos t$, so $y' = -ke^{-kt}\cos t - e^{-kt}\sin t$

Solutions to Exercises 2.7

1 The quotient rule is :

$$if \quad f(x) = \frac{g(x)}{h(x)} \qquad \text{then} \qquad f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{h^2(x)}$$

Using the quotient rule with $g(x) = 4x + 3$ and $h(x) = 2x + 1$,

$$f'(x) = \frac{4(2x+1) - 2(4x+3)}{(2x+1)^2} = \frac{8x + 4 - 8x - 6}{(2x+1)^2} = -\frac{2}{(2x+1)^2}$$

2(a) $$f'(x) = \frac{2(5x+3) - 5(2x)}{(5x+3)^2} = \frac{6}{(5x+3)^2}$$

2(b) $$f'(x) = \frac{(2x)(1+x^2) - (2x)x^2}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

2(c) $$f'(x) = \frac{4x(x^3+1) - (3x^2)2x^2}{(1+x^3)^2} = \frac{4x - 2x^4}{(1+x^3)^2}$$

3(a) $$f'(x) = \frac{1 + \sin x - x\cos x}{(1+\sin x)^2} = \frac{1 + \sin x - x\cos x}{(1+\sin x)^2}$$

3(b) $$f'(x) = \frac{-4e^{4x}}{(1+e^{4x})^2} \, .$$

4 $$f'(x) = \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \, .$$

Hence, using $\cos^2 x + \sin^2 x = 1$,

$$f'(x) = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x} \, .$$

5 $$\frac{dN}{dt} = \frac{6e^{-2t}}{(2+3e^{-2t})^2} \, .$$

6 Using $\tan x = \dfrac{\sin x}{\cos x}$ and the quotient rule ,

$$d\tan x \qquad \cos x\cos x - \sin x(-\sin x) \qquad \cos^2 x + \sin^2 x \qquad 1 \qquad \sec^2 x$$

Solutions to Exercises 2.8

1 $x = t + 2t^2$ and $y = 3t$ so $\dfrac{dx}{dt} = 1 + 4t\,,\ \dfrac{dy}{dt} = 3$. The magic formula sez...

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}}$$

so here

$$\frac{dy}{dx} = \frac{3}{1+4t}$$

2 $x = 2\sin t\,,\ y = 6\cos t\,$, so $\dfrac{dx}{dt} = 2\cos t$ and $\dfrac{dy}{dt} = -6\sin t$. Putting these together

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{-6\sin t}{2\cos t} = -3\tan t$$

3 $\dfrac{dx}{dt} = -\dfrac{1}{t^2} = -t^{-2}$ and $\dfrac{dy}{dt} = 10t$ so

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = 10t\,(-t^2) = -10t^3$$

4 $\dfrac{dx}{dt} = -(t-1)^{-2}$ and

$$\frac{dy}{dt} = \frac{2t(t-1) - t^2}{(t-1)^2} = \frac{t^2 - 2t}{(t-1)^2}$$

so

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = 2t - t^2$$

Solutions to Exercises 2.9

1 Differentiating the equation $y^2 + x^2 = 18$, we get $2yy' + 2x = 0$ so $y' = -x/y$.

2 We take $x^2 + 2xy + y^3 = 0$ and differentiate everything to get:

$$2x + 2y + 2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

where we have used the product rule to differentiate $2xy$ and the chain rule to differentiate y^3. In both cases we remember that y depends inmplicitly on x, so when we differentiate it we must write $\frac{dy}{dx}$. We can now solve to get

$$\frac{dy}{dx} = -\frac{2(x+y)}{2x+3y^2}$$

3 Differentiating the equation $5y^2 + 2y + xy = x^4$,

$$10y\frac{dy}{dx} + 2\frac{dy}{dx} + y + x\frac{dy}{dx} = 4x^3$$

$$(10y + 2 + x)\frac{dy}{dx} = 4x^3 - y$$

$$\frac{dy}{dx} = \frac{4x^3 - y}{10y + 2 + x}$$

4 Differentiating the equation $x^2 + 5y^2 = 9$,

$$2x + 10y\frac{dy}{dx} = 0$$

When $x = 2$ and $y = -1$,

$$4 - 10\frac{dy}{dx} = 0$$

so $\frac{dy}{dx} = \frac{2}{5}$.

5 Differentiating the equation $3x^2 - xy - 2y^2 + 12 = 0$,

$$6x - y - x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$$

When $x = 2$ and $y = 3$,

$$12 - 3 - 2\frac{dy}{dx} - 12\frac{dy}{dx} = 0$$

so $\frac{dy}{dx} = \frac{9}{}$.

6(i) If $x^2 + y^2 = 1$, solving for y gives $y = \sqrt{1-x^2}$ (we take the positive square root since $y = \dfrac{1}{\sqrt{2}}$ is positive).

We can differentiate this to get

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

At $x = \dfrac{1}{\sqrt{2}}$

$$\sqrt{1-x^2} = \sqrt{1-1/2} = \sqrt{1/2} = \frac{1}{\sqrt{2}}$$

so at $x = \dfrac{1}{\sqrt{2}}$ we have that $\dfrac{dy}{dx} = -1$.

6(ii) Differentiate both sides of $x^2 + y^2 = 1$ to get $2x + 2y\dfrac{dy}{dx} = 0$, or

$$\frac{dy}{dx} = -\frac{x}{y}$$

so once again we find $\dfrac{dy}{dx} = -1$.

7 $g(t) = (y'(t))^2 + (y(t))^2 + (y(t))^4$ so

$$g'(t) = 2y'y'' + 2yy' + 4y^3y'$$

Using $y'' = -y - 2y^3$

$$g'(t) = 2y'\left(-y - 2y^3\right) + 2yy' + 4y^3y' = 0$$