Solutions to Exercises 3.1

1. Equation of the tangent line is $y = mx + c$.

$f'(x) = 2x - 9$ and $f'(2) = -5$. Hence $m = 5$ and $y = 5x + c$.

$f(2) = 2^2 - 9(2) + 8 = -6$.

Putting this in $y = 5x + c$ gives $-6 = -10 + c$ so $c = 4$ and the equation of the tangent is $y = -5x + 4$.

2 $f(x) = x^3 + 4x^2 - 2x - 1$ so $f'(x) = 3x^2 + 8x - 2$ and $f'(1) = 9$.

Hence the equation of the tangent line is $y = 9x + c$.

Since $f(1) = 2$, $2 = 9 + c$ so $c = -7$ and the equation of the tangent is $y = 9x - 7$.

3 $f'(x) = e^x$ so $f(0) = 1$ and $f'(0) = 1$.

Hence the equation of the tangent line is $y = x + c$ and $c = 1$ so $y = x + 1$.

4 $f'(x) = x^{-1}$ so $f(1) = \ln 1 = 0$ and $f'(1) = 1$.

Hence the equation of the tangent line is $y = x - 1$.

5 $f(x) = \frac{1}{x}$ so $f'(x) = -\frac{1}{x^2}$.

$f'(a) = -\frac{1}{a^2}$ so the equation of the tangent line is $y = -\frac{x}{a^2} + c$.

Putting $x = a$ gives $\frac{1}{a} = -\frac{1}{a} + c$ so $c = \frac{2}{a}$ and $y = -\frac{x}{a^2} + \frac{2}{a}$.

Multiplying by a^2 gives $a^2 y = 2a - x$.

Using $a^2 y = 2a - x$, when $x = 0$, $y = \frac{2}{a}$.

Using $a^2 y = 2a - x$, when $y = 0$, $x = 2a$.

Hence the area of the triangle is

$$\frac{1}{2} \times \frac{2}{a} \times 2a = 2$$

Solutions to Exercises 3.2

1. $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

$f(x) = \sqrt{1+x} = (1+x)^{1/2}$ so

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad\qquad f''(x) = -\frac{1}{4}(1+x)^{-3/2}$$

and

$$f(0) = 1 \qquad\qquad f'(0) = \frac{1}{2} \qquad\qquad f''(0) = -\frac{1}{4}$$

Hence $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \ldots$

2 $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$$f(x) = \ln(1+x) \qquad f'(x) = \frac{1}{1+x} \qquad f''(x) = -\frac{1}{(1+x)^2} \qquad f''(x) = \frac{2}{(1+x)^3}$$

$$f(0) = \ln(1) = 0 \qquad f'(0) = 1 \qquad f''(0) = -1 \qquad f'''(0) = 2$$

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots$

3 $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$

$$f(x) = \cos x \qquad f'(x) = -\sin x \qquad f''(x) = -\cos x \qquad f'''(x) = \sin x \qquad f^{(4)}(x) = \cos x$$

Using $\sin 0 = 0$, $\cos 0 = 1$, $\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + \ldots$

4 $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

$f(x) = \ln(1+e^x)$ so using the chain rule,

$$f'(x) = \frac{e^x}{1+e^x}$$

Using the quotient rule,

$$f''(x) = \frac{e^x(1+e^x) - e^x\, e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

Using $e^0 = 1$, $\ln(1+e^x) = \ln 2 + \dfrac{x}{2} + \dfrac{x^2}{8} + \ldots$

Solutions to Exercises 3.3

1. $f(x) = 1 - 10x^{-2}$ and $f'(x) = 20x^{-3}$. Then

$$x - \frac{f(x)}{f'(x)} = x - \frac{1 - 10x^{-2}}{20x^{-3}} = x - \frac{x^3}{20} + \frac{x}{2} = (1.5)x - (0.05)x^3$$

Hence the Newton-Raphson recurrence relation is

$$x_{n+1} = (1.5)x_n - (0.05)x_n^3$$

2 $f(x) = 1 - \frac{1}{cx}$ and $f'(x) = \frac{1}{cx^2}$. Then

$$x - \frac{f(x)}{f'(x)} = x - \frac{1 - \frac{1}{cx}}{\frac{1}{cx^2}} = 2x - cx^2$$

Hence the Newton-Raphson recurrence relation is

$$x_{n+1} = 2x_n - cx_n^2$$

Taking $x_0 = 0.1$ and $c = 7$,

$$x_1 = 2(0.1) - 7(0.1)^2 = 0.13 \qquad \text{and} \qquad x_2 = 2(0.13) - 7(0.13)^2 = 0.142$$

3 $f(x) = x^3 - c$ and $f'(x) = 3x^2$. Then

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - c}{3x^2} = x - \frac{x}{3} + \frac{c}{3x^2} = \frac{2x}{3} + \frac{c}{3x^2}$$

The Newton-Raphson iteration is given by

$$x_{n+1} = \frac{1}{3}(2x_n + \frac{c}{x_n^2})$$

With $c = 10$, the iteration is

$$x_{n+1} = \frac{1}{3}(2x_n + \frac{10}{x_n^2}).$$

Starting with $x_0 = 2$,

$$\begin{aligned} x_1 &= \frac{1}{3}(2 \times 2 + \frac{10}{2^2}) = 2.16667 \\ x_2 &= \frac{1}{3}(2 \times 2.16667 + \frac{10}{2.16667^2}) = 2.15450 \\ x_3 &= \frac{1}{3}(2 \times 2.15450 + \frac{10}{2.15450^2}) = 2.15443 \end{aligned}$$

4 $f(x) = e^x - 4x$. Then $f(2) = e^2 - 8 = -0.61$ and $f(3) = e^3 - 12 = 8.09$ so the equation $f(x) = 0$ has a solution in the interval $[2, 3]$. The Newton-Raphson iteration is given by

$$x_{n+1} = x_n - \frac{e^{x_n} - 4x_n}{e^{x_n} - 4}$$

Starting with $x_0 = 2$,

$$x_1 = 2 - \frac{e^2 - 8}{\cdots} = 2.18 \qquad \text{and} \qquad x_2 = 2.18 - \frac{e^{2.18} - 4.18}{\cdots} = 2.15$$

Solutions to Exercises 3.4

1. $f'(x) = 1 - \dfrac{9}{x^2}$ and $f''(x) = \dfrac{18}{x^3}\cdot$

If x is a stationary point then $f'(x) = 0$, that is $\dfrac{9}{x^2} = 1$

so $x^2 = 9$ and $x = 3$.

$f''(x) = \dfrac{18}{3^3} > 0$ so $x = 3$ is a local minimum .

2 $f'(x) = 3x^2 - 12$ and $f''(x) = 6x$.

If x is a stationary point then $f'(x) = 0$, that is $x^2 = 4$ so $x = \pm 2$.

$f''(-2) = -12 < 0$ and $f''(2) = 12 > 0$.

Hence $x = 2$ is a local minimum and $x = -2$ is a local maximum.

3 $f'(x) = 3x^2 + 6x - 45$ and $f''(x) = 6x + 6$.

If x is a stationary point then $f'(x) = 0$,

so $3x^2 + 6x - 45 = 0$, that is $x^2 + 2x - 15 = 0$.

Hence $(x+5)(x-3) = 0$ and $x = -5$ or $x = 3$.

Since $f''(-5) = -30 + 6 = -24$, $x = -5$ is a local maximum.

Since $f''(3) = 18 + 6 = 24$, $x = 3$ is a local minimum.

4 $f(x) = e^{3x} - 3e^x$ so

$$f'(x) = 3e^{3x} - 3e^x \qquad\qquad f''(x) = 9e^{3x} - 3e^x$$

$f'(0) = 3e^0 - 3e^0 = 0$ so $x = 0$ is a stationary point .

Since $f''(0) = 9e^0 - 3e^0 = 6 > 0$, $x = 0$ is local minimum.

5 $f'(x) = \dfrac{1}{x} - 1$ and $f''(x) = -\dfrac{1}{x^2}\cdot$

$f'(1) = 1 - 1 = 0$ so $x = 1$ is a stationary point .

Since $f''(1) = -1 < 0$, $x = 1$ is local maximum.

Solutions to Exercises 3.4 continued

6 $f'(x) = 6x - 4\sin 4x$ and $f''(x) = 6 - 16\cos 4x$.

$f'(0) = 0 - 0 = 0$ so $x = 0$ is a stationary point .

Since $f''(0) = 6 - 16 = -10 < 0$, $x = 0$ is local maximum.

7 $f'(x) = 2x - \dfrac{54}{x^2} = \dfrac{2x^3 - 54}{x^2} = \dfrac{2(x^3 - 27)}{x^2}$.

For $x > 3$, $x^3 > 3^3 = 27$ and $f'(x) > 0$.

It follows that $f(x) = x^2 + \dfrac{54}{x}$ is increasing for $x > 3$.

8 Let

$$\frac{3x+2}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

Multiplying by $(x+1)(x+2)$

$$3x + 2 = A(x+2) + B(x+1)$$

Put $x = -2$ then $-6 + 2 = B(-2+1)$ so $B = 4$.

Put $x = -1$ then $-3 + 2 = A(-1+2) + 0$ so $A = -1$. Hence

$$f(x) = \frac{3x+2}{(x+1)(x+2)} = \frac{4}{(x+2)} - \frac{1}{(x+1)}$$

$$f'(x) = \frac{1}{(x+1)^2} - \frac{4}{(x+2)^2} \qquad f''(x) = \frac{8}{(x+2)^3} - \frac{2}{(x+1)^3}$$

$f'(0) = 1 - \dfrac{4}{2^2} = 1 - 1 = 0$ so $x = 0$ is a stationary point .

$$f''(0) = \frac{8}{2^3} - \frac{2}{1^3} = 1 - 2 = -1 < 0$$

so $x = 0$ is local maximum.

Solutions to Exercises 3.5

1. If the sides are x and y then the perimeter is $P = 2x + 2y$.

Since the area is $49 = xy$, $y = \dfrac{49}{x}$ and $P = P(x) = 2x + \dfrac{98}{x}$.

$P'(x) = 2 - \dfrac{98}{x^2}$ and at stationary points $P'(x) = 0$.

Hence $x^2 = 49$ and $x = 7$ is the stationary point.

Since $P''(x) = \dfrac{196}{x^3}$, $P''(7) > 0$ so $x = 7$ is a minimum.

Hence the is perimeter a minimum when it is a square of side 7.

2. Let h be the height and x the length of the side of the base. The volume V is given by $V = x^2 h$.

The surface area S is $2x^2 + 4xh = 150$ so $x^2 + 2xh = 75$ and $h = \dfrac{75 - x^2}{2x}$.

Using this in $V = x^2 h$ we obtain

$$V = V(x) = x^2 \left[\frac{75 - x^2}{2x}\right] = \frac{75x - x^3}{2}$$

$V'(x) = \dfrac{75 - 3x^2}{2}$ and at stationary points $V'(x) = 0$.

Hence $3x^2 = 75$ and $x = 5$.

Since $V''(x) = -\dfrac{6x}{2}$ we have that $V''(5) < 0$ so $x = 5$ is a maximum.

Using $x = 5$ in $h = \frac{75-x^2}{2x}$ we get that $h = 5$.

3. Let h be the height and x the length of the side of the base. The surface area S is the area of the four walls and base, that is $S = x^2 + 4xh$.

Since the volume is 32 we have $x^2 h = 32$ so

$$h = \frac{32}{x^2}$$

and

$$S(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$$

Then

$$S'(x) = 2x - \frac{128}{x^2} = 0$$

if $x^3 = 64$. Hence $x = 4$ is the stationary point.

Since $S''(x) = 2 + 128x^{-3}$, $S''(4) > 0$ and $x = 4$ is a minimum.

Solutions to Exercises 3.5 continued

4. Let h be the height and x the length of the side of the base. The area of the four walls is $S = 4xh$ and the area of the roof is x^2.

The heat loss is proportional to $Q = 4xh + 4x^2$. Since $x^2h = 2000$

$$h = \frac{2000}{x^2}$$

and

$$Q = Q(x) = 4x^2 + 4x\left(\frac{2000}{x^2}\right) = 4x^2 + \frac{8000}{x}$$

Then

$$Q'(x) = 8x - \frac{8000}{x^2} = 0$$

if $x^3 = 1000$. Hence $x = 10$ is the stationary point.

Since $Q''(x) = 8 + 16000x^{-3}$, $Q''(10) > 0$ and $x = 10$ is a minimum.

Using $x = 10$ in $x^2h = 2000$ we get $h = 20$.

5. Let the field have dimensions x along the road and y. The area is $A = xy$.

The cost of the fence is

$$3x + 2x + 2y + 2y = 5x + 4y$$

Hence $5x + 4y = 400$ so $y = 100 - \dfrac{5x}{4}$ and

$$A = A(x) = xy = 100x - \frac{5x^2}{4}$$

$$A'(x) = 100 - \frac{10x}{4} = 0$$

if $x = 40$. Hence $x = 40$ is the stationary point.

Since $A''(x) < 0$, $x = 40$ is a maximum.

Using $x = 40$ in $5x + 4y = 400$ we get $y = 50$.

The maximum area that can be fenced off is $50 \times 40 = 2000$.

Solutions to Exercises 3.6

1. The distance from **P** after 1 second is $s(1) = 1 - 6 + 12 = 7$.

$v(t) = s'(t) = 3t^2 - 12t$ and $a(t) = v'(t) = 6t - 12$.

If $v(t) = 0$ then $3t^2 - 12t = 0$. Dividing by $3t$ gives $t = 4$.

At $t = 4$ its acceleration is $a(4) = 24 - 12 = 12.$

2. The height from which the ball is thrown is $s(0) = 8$.

$v(t) = s'(t) = 10 - 10t$. The initial velocity of the ball is $v(0) = 10$.

At maximum height $v(t) = 0$ so $10 - 10t = 0$ and $t = 1$.

The maximum height is $s(1) = 13$.

3. $v(t) = s'(t) = e^{-2t} - 2te^{-2t}$.

$a(t) = v'(t) = -2e^{-2t} - 2e^{-2t} + 4te^{-2t} = 4te^{-2t} - 4e^{-2t}$.

Solutions to Exercises 3.7

1. The rate of increase of $y(t)$ is $\dfrac{dy}{dt}$ so $\dfrac{dy}{dt} = ky$.

2. The rate of decrease of $y(t)$ is $-\dfrac{dy}{dt}$ so $-\dfrac{dy}{dt} = ky^2$. Hence $\dfrac{dy}{dt} = -ky^2$.

3. $y = 2 + Ct^{-1}$ so $\dfrac{dy}{dt} = -Ct^{-2}$.

$$t\frac{dy}{dt} + y = t\left(-Ct^{-2}\right) + 2 + Ct^{-1} = -Ct^{-1} + 2 + Ct^{-1} = 2$$

If $y = 7$ when $t = 3$. then $7 = 2 + \dfrac{C}{3}\cdot$

Hence $C = 15$ and $y = 2 + \dfrac{15}{t}\cdot$

4. $y(t) = 1 - t + 4e^{-t}$ so $\dfrac{dy}{dt} = -1 - 4e^{-t}$. Also,

$$-t - y = -t - 1 + t - 4e^{-t} = -1 - 4e^{-t} = \frac{dy}{dt}$$

Putting $t = 0$ in $y(t) = 1 - t + 4e^{-t}$ gives $y(0) = 1 + 4 = 5$ as required.

5. $y(t) = (1 - 8t)^2$ so $y'(t) = -16(1 - 8t)$.

$-16\sqrt{y(t)} = -16(1 - 8t)$ so $y'(t) = -16\sqrt{y(t)}$.

Putting $t = 0$ in $y(t) = (1 - 8t)^2$ gives $y(0) = 1$ as required.

6. $y(t) = E(1 - e^{-t})$ so $y'(t) = Ee^{-t}$.

$$E - y = E - E(1 - e^{-t}) = Ee^{-t} = \frac{dy}{dt}$$

Putting $t = 0$ in $y(t) = E(1 - e^{-t})$ gives $y(0) = E(1 - e^{-0}) = E(1 - 1) = 0$ as required.

For large t , $e^{-t} \approx 0$ so $y(t) \approx E$.

7. Using the chain rule, $y(t) = (1 + u)^{-1}$ with $u = 1 + e^{-t}$ so

$$y'(t) = \frac{e^{-t}}{(1 + e^{-t})^2}$$

Note that we could of used the quotient rule to find the derivative.

$$1 - y = 1 - \frac{1}{1 + e^{-t}} = \frac{1 + e^{-t} - 1}{1 + e^{-t}} = \frac{e^{-t}}{1 + e^{-t}}$$

and

Putting $t = 0$ in $y(t)$ gives

$$y(0) = \frac{1}{1 + e^{-0}} = \frac{1}{2}$$

as required.

For large t , $e^{-t} \approx 0$ so $y(t) \approx 1$.

8. The rate of decrease of $y(t)$ is $-\dfrac{dy}{dt}$ so $-\dfrac{dy}{dt} = k(y - 18)$. Hence $\dfrac{dy}{dt} = -k(y - 18)$.

For $y(t) = 18 + Ce^{-kt}$, $y'(t) = -Cke^{-kt}$.

$$-k(y - 18) = -k(18 + Ce^{-kt} - 18) = -Cke^{-kt} = y'(t)$$

Putting $t = 0$ in $y(t)$ gives $y(0) = 18 + C$ so $98 = 18 + C$ and $C = 80$.

Hence $y(t) = 18 + 80e^{-kt}$.

9. $\dfrac{dy}{dt} = 4Ce^{4t}$ and

$$4(y - 2) = 4(2 + Ce^{4t} - 2) = 4Ce^{4t} = \frac{dy}{dt}$$

Putting $t = 0$ in $y(t)$ gives $9 = 2 + C$ so $y = 2 + 7e^{4t}$.

10. $y = t \ln t + Ct$ so

$$\frac{dy}{dt} = \ln t + t\left(\frac{1}{t}\right) + C = \ln t + 1 + C$$

$$t\frac{dy}{dt} = t \ln t + t + Ct = t + y$$

as required.

Putting $t = 1$ in $y(t)$ gives $5 = \ln 1 + C = C$ so $y = t \ln t + 5t$.

11. Using the chain rule, $y(t) = u^{-1/2}$ with $u = 2t + C$ so

$$\frac{dy}{dt} = y'(t) = (-1/2)(2)(2t + C)^{-3/2} = -(2t + C)^{-3/2}$$

and $-y^3 = -(2t + C)^{-3/2}$ so $y'(t) = -y^3$.

Putting $t = 0$ in $y(t)$ gives $3 = C^{-1/2}$. Hence $C = 1/9$ and

$$y(t) = \left(2t + \frac{1}{9}\right)^{-\frac{1}{2}}$$

Solutions to Exercises 3.8

1. $y' = 3Ae^{3t}$ and $y'' = 9Ae^{3t}$. Then

$$y'' - 6y' + 3y = 9Ae^{3t} - 18Ae^{3t} + 3Ae^{3t} = -6Ae^{3t} = 12e^{3t}$$

if $A = -2$.

2. $y(t) = Ae^{4t} + Be^{-4t}$ so $y'(t) = 4Ae^{4t} - 4Be^{-4t}$ and $y''(t) = 16Ae^{4t} + 16Be^{-4t}$.

$$\frac{d^2y}{dt^2} - 16y = 16Ae^{4t} + 16Be^{-4t} - 16Ae^{4t} - 16Be^{-4t} = 0$$

as required.

Putting $t = 0$ in $y(t) = Ae^{4t} + Be^{-4t}$ and using $e^0 = 1$ gives $A + B = 0$ so $A = -B$.

Putting $t = 0$ in $y'(t) = 4Ae^{4t} - 4Be^{-4t}$ gives $4A - 4B = 24$.

Using $A = -B$ in $4A - 4B = 24$ gives $8A = 24$.

Hence $A = 3$, $B = -3$ and $y(t) = 3e^{4t} - 3e^{-4t}$.

3. $y(t) = A\sin 2t + B\cos 2t - \sin 4t$ so

$$\frac{dy}{dt} = 2A\cos 2t - 2B\sin 2t - 4\cos 4t$$

$$\frac{d^2y}{dt^2} = -4A\sin 2t - 4B\cos 2t + 16\sin 4t$$

$$\frac{d^2y}{dt^2} + 4y = -4A\sin 2t - 4B\cos 2t + 16\sin 4t + 4A\sin 2t + 4B\cos 2t - 4\sin 4t = 12\sin 4t$$

as required.

Putting $t = 0$ in $y(t) = A\sin 2t + B\cos 2t - \sin 4t$ gives $5 = B$.

Putting $t = 0$ in $y'(t) = 2A\cos 2t - 2B\sin 2t - 4\cos 4t$ gives $8 = 2A - 4$ so $A = 6$. Hence

$$y(t) = 6\sin 2t + 5\cos 2t - \sin 4t$$

4. $y(t) = e^{mt}$ so $y' = me^{mt}$ and $y'' = m^2e^{mt}$. Then

$$y'' - 36y = m^2e^{mt} - 36e^{mt} = (m^2 - 36)e^{mt} = 0$$

if $m^2 - 36 = 0$. Hence $m = 6$ or $m = -6$.

The solutions of the differential equation are $y(t) = e^{-6t}$ and $y(t) = e^{6t}$.

Solutions to Exercises 3.8 continued

5. $y(t) = e^{mt}$ so $y' = me^{mt}$ and $y'' = m^2e^{mt}$. Then

$$y'' + y' - 6y = m^2e^{mt} + me^{mt} - 6e^{mt} = (m^2 + m - 6)e^{mt} = 0$$

if $m^2 + m - 6 = 0$. Solving

$$m^2 + m - 6 = (m + 3)(m - 2) = 0$$

to get $m = -3$ and $m = 2$.

The solutions of the differential equation are $y(t) = e^{-3t}$ and $y(t) = e^{2t}$.

6. $y(t) = A\sin 2t$ so

$$y'(t) = 2A\cos 2t$$
$$y''(t) = -4A\sin 2t$$

Then

$$y'' + y = -4A\sin 2t + A\sin 2t = -3A\sin 2t = 21\sin 2t$$

if $A = -7$.