Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 5

1. Consider the illness-death model in Figure 1. A life age x takes out a policy with a term of n years that pays an annuity of £1 per year continuously during any period of illness, and a sum assured of £100 on death. A premium of $\bar{P}$ per annum is payable continuously while healthy.

   Let $V^j(t)$ be the prospective policy value at age $x+t$, conditional on then being in state j $(j = 0, 1, 2)$.

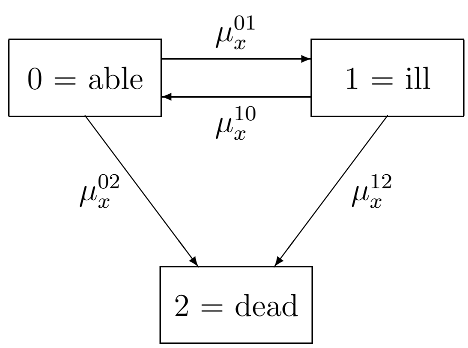


Figure 1: An illness-death model

   (a) State and prove the Kolmogorov equations for ${}_tp_x^{00}$ and ${}_tp_x^{01}$.
   (b) Write down Thiele's differential equations for this model.
   (c) State what are the boundary conditions you would use in solving Thiele's differential equations.

2. (Excel exercise): Consider the following (simplified) disability model, in which mortality has been ignored.

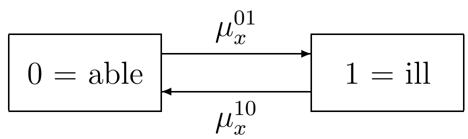


Figure 2: A simplified illness-death model

The transition intensities are given by:

$$\begin{aligned}\mu_x^{01} &= 0.01 \times 1.09^x \\ \mu_x^{10} &= 0.5.\end{aligned}$$

(a) Write down the Kolmogorov equations for ${}_tp_x^{00}$ and ${}_tp_x^{01}$.

(b) Using an Euler scheme with step size $h = 0.01$ years, solve the Kolmogorov equations to find approximate values of ${}_tp_{20}^{00}$ and ${}_tp_{20}^{01}$ for $0 \le t \le 10$.

(c) An insurance company sells a 10-year disability insurance policy to someone age 20. The benefit is a disability annuity at rate £1 per annum, payable while ill. Write down Thiele's equations for the statewise policy values $V^0(t)$ and $V^1(t)$.

(d) The actuary uses the above transition intensities and a force of interest $\delta = 0.05$ per annum to calculate premiums and policy values. By trial and error, or otherwise, find the annual rate of premium such that $V^0(0) = 0$.

(e) Explain what is happening to $V^1(t)$ as t approaches 10 years. Is this good or bad?

3. Consider the model and disability insurance policy in Question 2. Define T_i to be the random time of the ith transition in the policyholder's life history during the policy term, and let S_i be the state entered in the ith transition ($i = 1, 2, \ldots$). If the policyholder dies or the 10-year term ends before reaching the ith transition, define $T_i = \infty$ and $S_i = -1$.

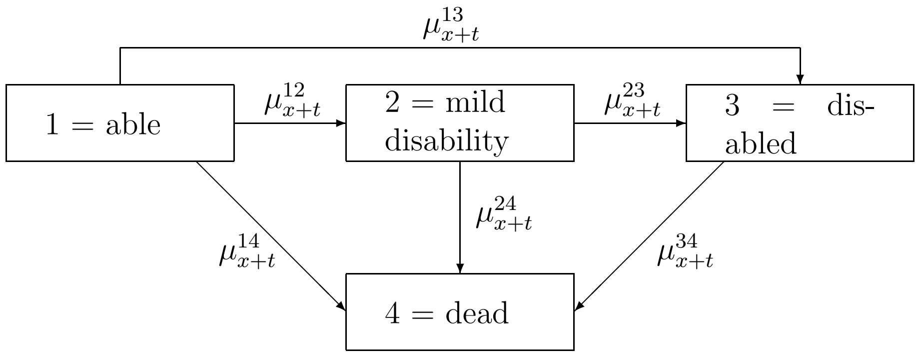
(a) List all possible life histories in which the policyholder makes 0, 1, 2 or 3 transitions during the policy term.

(b) Write down expressions for the present value at $t = 0$ of the premiums paid during each life history listed in (a).

(c) Hence, write down the first four terms in an expression for the EPV at $t = 0$ of the premiums paid.

(d) Comment on how practical it is to use the random times at which events take place to compute EPVs in this model, compared with the numerical solution of Thiele's equations.

4. An insurance company uses the multiple state model shown below to calculate premiums for long-term care contracts:



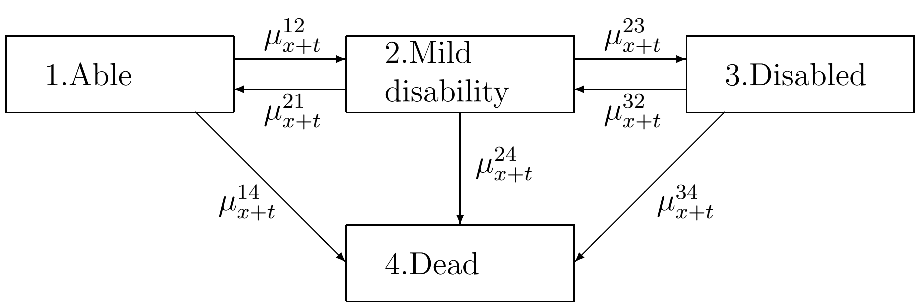
(a) Derive a differential equation for the occupancy probability ${}_tp_x^{\overline{11}}$.

(b) Show by direct substitution that that the following satisfies the differential equation derived in (a) as well as the boundary condition ${}_0p_x^{\overline{11}} = 1$:

$$ {}_tp_x^{\overline{11}} = \exp\left\{ -\int_0^t \mu^{12}_{x+r} + \mu^{13}_{x+r} + \mu^{14}_{x+r} dr \right\}. $$

5. An insurance company uses the multiple state model shown below to calculate premiums for the following long-term care contract:

Premiums:	Payable continuously while in state 1
Benefits:	50% SA per annum payable continuously while in state 2
	100% SA per annum payable continuously while in state 3
Waiver:	Premiums are waived while benefits are payable.



The company has estimated the transition probabilities for the above model as:

$$ {}_tp_x^{11} = \begin{cases} \left[\frac{100-x-t}{100-x}\right]^3 & x+t \leq 100 \\ 0 & \text{otherwise} \end{cases} \qquad {}_tp_x^{21} = \begin{cases} \frac{t\,(100-x-t)}{6000} & x+t \leq 100 \\ 0 & \text{otherwise} \end{cases} $$

$$ {}_tp_x^{12} = \begin{cases} \frac{t\,(100-x-t)}{4000} & x+t \leq 100 \\ 0 & \text{otherwise} \end{cases} \qquad {}_tp_x^{22} = \begin{cases} \left[\frac{100-x-t}{100-x}\right]^4 & x+t \leq 100 \\ 0 & \text{otherwise} \end{cases} $$

$$ {}_tp_x^{13} = \begin{cases} \frac{t\,(100-x-t)}{2000} & x+t \leq 100 \\ 0 & \text{otherwise} \end{cases} \qquad {}_tp_x^{23} = \begin{cases} \frac{t\,(100-x-t)}{1000} & x+t \leq 100 \\ 0 & \text{otherwise} \end{cases} $$

where ${}_tp_x^{ij}$ is the probability that a life aged x in state i will be in state j at age $x+t$.

The company uses the following basis for calculating premiums:

Basis:

Transition probabilities:	as given
Force of interest:	5% per annum continuously
Expenses:	8% of all premiums paid
Indexation:	Premiums and benefits indexed at 5% per annum continuously.

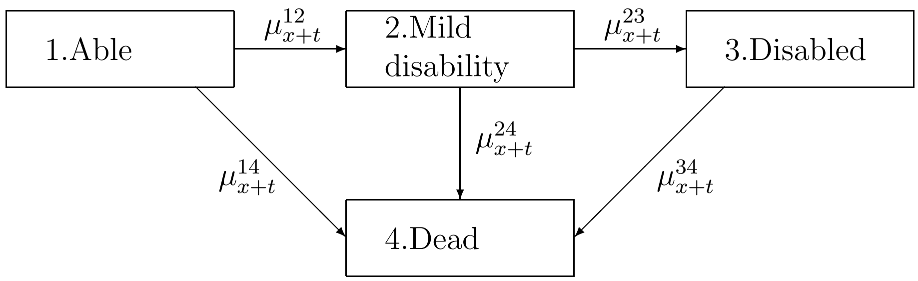
Show that the equation of value for the above long-term care policy for a life aged x and with an initial sum assured of rate $\bar{B}$ per annum is:

$$ 0.92\,\bar{P} \int_0^{100-x} {}_tp_x^{11}\,dt = \frac{\bar{B}}{2} \int_0^{100-x} {}_tp_x^{12}\,dt + \bar{B} \int_0^{100-x} {}_tp_x^{13}\,dt $$

where $\bar{P}$ is the initial annual premium rate.

Hence, calculate the annual premium rate at outset for a life aged 60 for a contract with an initial sum assured of £10,000 per annum on the above basis.

6. An insurance company uses the multiple state model shown below to calculate premiums for long-term care contracts:



The insurance company uses the following constant values for the transition intensities:

$$\mu^{14}_x = 0.01 \quad \mu^{24}_x = 0.02 \quad \mu^{34}_x = 0.04$$
$$\mu^{12}_x = 0.025 \quad \mu^{23}_x = 0.05.$$

(a) Show that Kolmogorov's differential equation for the occupancy probability ${}_tp^{12}_x$ is:

$$\frac{d}{dt}{}_tp^{12}_x = {}_tp^{11}_x\,\mu^{12}_{x+t} - {}_tp^{12}_x\,(\mu^{23}_{x+t} + \mu^{24}_{x+t}).$$

(b) Show, using an Euler scheme with step-size s and the formula above:

$${}_sp^{12}_x \approx s\,\mu^{12}_x.$$

Hence, setting $s = 1$ year, calculate an approximate value of ${}_1p^{12}_x$.

(c) Taking the next step in the Euler scheme, derive an expression for ${}_{2s}p^{12}_x$. You are given that:

$${}_tp^{11}_x = \exp\left\{-\int_0^t \mu^{12}_{x+r} + \mu^{14}_{x+r}dr\right\}.$$

Hence, setting $s = 0.5$ year, calculate another approximate value of ${}_1p^{12}_x$.

(d) Calculate ${}_1p^{12}_x$ exactly given that, if the transition intensities are constant, then:

$${}_tp^{12}_x = \frac{\mu^{12}_x}{\mu^{12}_x + \mu^{14}_x - \mu^{23}_x - \mu^{24}_x}\left[\mathrm{e}^{-(\mu^{23}_x+\mu^{24}_x)\,t} - \mathrm{e}^{-(\mu^{12}_x+\mu^{14}_x)\,t}\right]$$

and compare it to your answers for (b) and (c) and comment. How could you improve the accuracy of Euler's method to get a more accurate solution? (No further calculations are required.)

Numerical Solutions: Q.5 7,246.38 Q.6(b) 0.025 Q.6(c) 0.024346 Q.6(d) 0.023723.